Chapter 3
Auctions

Chapter 2 described several implementations of adversarial risk analysis (ARA), and compared those with solution concepts used in traditional game theory. This chapter extends that discussion through an in-depth treatment of auctions, a classic problem of practical importance. Specifically, we consider continuous asymmetric, first-price, independent-value, sealed-bid auctions with risk-neutral bidders.

Continuity means that the bids may take any value in an interval, in contrast with the discrete games treated in Chapter 2. Symmetric auctions assume that the values opponents have for the item on offer are randomly drawn from the same (known) distribution, whereas asymmetric auctions allow different opponents to draw from different (known) distributions. In first-price auctions, the highest bidder wins, and pays the amount of that bid. The independent-value condition implies that the private value that one bidder has for an object is not influenced by the private value that other bidders have for that object. The sealed-bid condition ensures that whatever initial information a bidder has about the value distributions of his opponents does not change as the auction proceeds (in contrast with, say, an English auction, where opponents place increasingly higher bids until all but one has dropped out). Finally, risk neutrality implies that each bidder attempts to maximize his expected monetary profit. For concision, we shall use the term "auction" to refer to a continuous asymmetric, first-price, independent-value, sealed-bid auction among risk-neutral bidders.

Auctions of this kind are common, and are especially popular when the commercial value of the item on offer is difficult to determine. They are sometimes used by auction houses, such as Christie's and Sotheby's. In the defense industry, companies make sealed bids on federal contracts, and the *lowest* qualified bidder prevails. But this is a distinction without a difference: the defense contractor's decision problem is formally equivalent to the situation in which the highest bidder wins.

In this chapter, for concreteness, we assume a high-bid auction. Specifically, a lady named Bonnie is bidding against a man named Clyde for a first edition of *The Theory of Games and Economic Behavior*. Also, this auction does not have a reservation price (a secret lower bound set by the owner—if no bid exceeds the reservation price, the book will not be sold).

Auctions easily illustrate aleatory, epistemic, and concept uncertainty. Aleatory uncertainty arises when Bonnie does not have full knowledge of the condition of the book—it could be damaged, which would lower its value, or it might contain marginalia by Lloyd Shapley, which would increase its value. Epistemic uncertainty about the private value of an opposing bidder (i.e., Clyde) can appear in several ways. Perhaps Clyde has better knowledge of the condition of the book; or perhaps the book had been owned by Clyde's thesis advisor, and thus has sentimental value to him. Finally, concept uncertainty occurs when Bonnie does not know what kind of strategic analysis Clyde will perform when calculating his bid.

The following sections consider bidding strategies from several different perspectives, using both classical and ARA techniques. The intent is to highlight the assumptions that are needed and the kinds of solutions that result. When possible, for simplicity, the analysis treats two-person auctions, but the last section discusses three-person auctions, which is sufficient to understand the n-person case.

3.1 Non-Strategic Play

Suppose Bonnie believes that Clyde is non-strategic. In that case, the rule Clyde uses to select his bid for the first edition does not depend upon his analysis of Bonnie's situation. For example, Clyde's rule might be to bid 90% of his true value.

If Bonnie has a distribution F over Clyde's bid, then, under the assumption that her utility function for money is linear, she will maximize her expected utility in a first-price auction by bidding

$$x^* = \text{argmax}_{x \in \mathbb{R}^+} (x_0 - x) F(x), \tag{3.1}$$

where x_0 is Bonnie's true value for the book. To see this, note that her utility (or profit) from a successful bid of x is $(x_0 - x)$, and her personal probability that a bid of x wins the two-person auction is $F(x)$. Thus the right-hand side of (3.1) is just her expected utility when she bids x (cf. Raiffa, Richardson and Metcalfe, 2002). Figure 3.1 illustrates this situation.

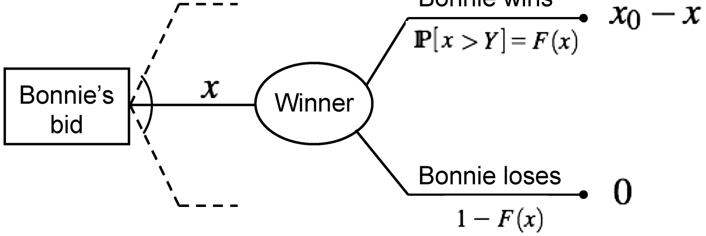

Fig. 3.1 A decision tree in which the possible bids are continuous. The distribution of Clyde's bid is F, so the probability that Bonnie wins with a bid of x is $F(x)$.

As a subjective Bayesian, Bonnie finds the distribution F through introspection, during which she reconciles everything she knows about Clyde with everything she knows about auctions, the market value of first editions of *The Theory of Games and Economic Behavior*, and all other relevant data. Often this is formalized by considering an infinite sequence of hypothetical wagers (De Finetti, 1974). In practice, such precision is impossible, and humans use cognitive shortcuts and approximations when eliciting personal probabilities (cf. O'Hagan et al., 2006).

In this example, a straightforward protocol for Bonnie to obtain her F is to divide the introspection into two parts. First, she puts a subjective distribution G_1 over the value of the first edition to Clyde. Then, she places a subjective distribution G_2 over the fraction of his true value that he bids. The distribution G_1 for Clyde's true value might be approximated by considering the sales prices of other first editions in recent auctions, or the appraisal value by experts, and so forth. Bonnie could adjust G_1 upward if she believes that Clyde puts special value on the book (e.g., she knows that it had been owned by Clyde's thesis advisor). Similarly, to find G_2, Bonnie's distribution for the fraction Clyde bids, she might draw upon knowledge of his success record in previous auctions, or statements he has made in the past, or empirical work in economics on the distribution of underbidding (cf. Case, 2008; Keefer, 1991).

For example, suppose Bonnie describes her epistemic uncertainty about Clyde's true value by a random variable V with distribution G_1 on $(0, \infty)$. She assumes he bids an unknown (and thus, to Bonnie, random) fraction P of that value, for which she has distribution G_2 with support in $[0, 1]$. Then her subjective distribution over $Y = PV$, the amount of Clyde's bid, can be found through a double integral over the shaded region shown in Fig. 3.2. Specifically, when G_1 and G_2 have densities g_1 and g_2, respectively, then

$$F(y) = \mathbb{P}[PV \le y] = int_0^y \int_0^1 g_1(v)\, g_2(p)\, dp\, dv + \int_y^\infty \int_0^{y/v} g_1(v)\, g_2(p)\, dp\, dv$$
$$= G_1(y) + \int_y^\infty g_1(v)\, G_2(y/v)\, dv. \tag{3.2}$$

This formulation assumes that the true value and the proportional reduction are independent. But if Bonnie thinks that Clyde's non-strategic rule is more complicated (e.g., the proportion P increases as the true value increases), then the analysis is still straightforward, although Bonnie would need to solve a more difficult integral.

It is worth emphasizing that this decomposition of the calculation into a value and a proportion is simply a device for helping Bonnie to develop her personal probability over the fundamental quantity of interest, the bid Y that Clyde makes. She may have other ways to discover F, perhaps through the advice of an informant, or data on Clyde's previous bids.

The following example shows, for the situation in which Clyde bids an unknown fraction of his true value, how the distribution of his bid Y is determined from the distributions G_1 and G_2 that Bonnie needs to assess for her beliefs about the value that the book has to Clyde and the fraction that he will bid, respectively.

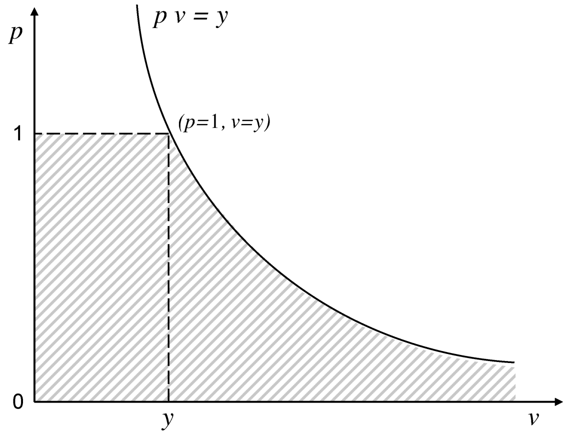

Fig. 3.2 Region of integration when the bid is a proportion of the true value.

Example 3.1: Suppose Bonnie's personal value for the book on auction is $x_0 = \$150$. She models Clyde's value for the book as a random variable taking values between \$0 and \$200 with uniform distribution $G_1(v) = v/200$. And she models the distribution for the proportion of his value that he bids as $G_2(p) = p^9$ for $0 \leq p \leq 1$. Then the distribution F of Clyde's bid y is

$$F(y) = G_1(y) + \int_y^{200} g_1(v) G_2(y/v)\, dv = \frac{9}{8}\frac{y}{200} - \frac{1}{8}\frac{y^9}{200^9}$$

for $\$0 \leq y \leq \$200$. Thus Bonnie finds the bid x^* that maximizes her expected utility by solving (3.1). She takes the derivative and sets it to 0, obtaining

$$0 = \frac{d}{dx}\left[(x_0 - x)F(x)\right] = 675 - 9x - \frac{675}{200^8}x^8 + \frac{5}{200^8}x^9.$$

Numerical solution shows her bid should be about half of x_0, or $x^* = \$75$.

In this discussion, concept uncertainty is absent since Bonnie is assumed to believe that Clyde is non-strategic. More precisely, she believes he is the kind of non-strategic player whose bid is proportional to his true value. Her epistemic uncertainty is expressed through the distributions G_1 and G_2, which leads her to F. As posed, there is no aleatory uncertainty, since in this scenario Bonnie knows the value, x_0, that she has for the book.

Now suppose that Bonnie does not know her true value x_0 for the book (e.g., it has not been appraised, or its provenance is uncertain). In that case, its value is a random variable, say X_0, with distribution H. Bonnie wants to make the bid x that maximizes her expected utility $\mathbb{E}_H[(X_0 - x)F(x)] = (\mu - x)F(x)$, where μ is the expected value of X_0. Conveniently, she need not completely specify H; in order to maximize expected utility, all she requires is its mean. (This is a consequence of the risk neutrality assumption, which implies that her utility for money is linear.)

In many situations, aleatory uncertainty is the dominant concern. A defense contractor who bids on a project probably does not know all the costs and difficulties that will arise, and thus does not know exactly what profit would be realized from his bid. This uncertainty can be more important than uncertainty about solution concepts used by opponents, or epistemic uncertainty about the valuations of opponents, especially as the number of opponents increases. In contrast, when each bidder knows the value of the item on offer (e.g., an auction for opera tickets, where each opponent knows his personal utility for opera), then epistemic and concept uncertainty become important. This situation is typical in private value auctions, where each bidder knows his own value, but not those of other bidders. But in common value auctions, for which all bidders have the same (possibly unknown) value for the item on offer, concept uncertainty is likely to dominate the analysis.

3.2 Minimax Perspectives

In a first-price private values auction, the minimax (technically, the maximin) perspective is unhelpful. Bonnie seeks to maximize her minimum utility against the worst possible bid by Clyde. If her preferences are linear in money, and if her true value for the *Theory of Games and Economic Behavior* is x_0, then Bonnie's utility function when she bids x and Clyde bids y is

$$u(x,y) = \begin{cases} x_0 - x & \text{if } x > y \\ 0 & \text{else.} \end{cases}$$

Thus Bonnie solves

$$\max_x \min_y \; u(x,y)$$

which is 0 for all $x \leq x_0$ (since Clyde could bid more than x_0).

If Bonnie knew that Clyde's true value was y_0 with $y_0 < x_0$, then her maximin solution is to bid just a little bit more than y_0. And in the pessimistic limit, as y_0 increases to x_0, Bonnie's bid increases and her profit diminishes to zero. At that limit, any bid x such that $0 \leq x \leq x_0$ fails to achieve the maximin solution against the worst possible bid by Clyde.

This line of thinking is untenable. Empirical evidence shows that people bid less than their true value (Case, 2008). One of those bidders wins the auction and realizes a positive profit. So a different solution concept is required to analyze bidding.

For example, an ARA solution concept properly leads Bonnie to underbid. She models Clyde's bidding strategy, and then maximizes her expected utility under that model. If she believes that Clyde's bid will have some distribution F, then Bonnie maximizes her expected utility by solving (3.1). When F is continuous with $F(x_0) > 0$, one can show that Bonnie's bid is strictly less than x_0.

Because traditional auctions encourage underbidding, Vickrey (1961) proposed an alternative—the second-price auction, which incentivizes participants to bid their true values. In a second-price auction, the highest bidder wins but pays the amount of the second-highest bid.

To see the underlying logic for the second-price auction, suppose Bonnie bids x and Clyde bids y. Then the gain for Bonnie is $x_0 - y$ if $x > y$, and 0 otherwise. If Bonnie were to bid a value $x^- < x_0$, then one of three outcomes must occur in a second-price auction:

(i) If $y > x_0$, then Bonnie would gain 0 from both a truthful bid x_0 and from her underbid.
(ii) If $y < x^-$, then Bonnie would gain $x_0 - y$ from both a truthful bid x_0 and from her underbid.
(iii) If $x^- \leq y \leq x_0$, then Bonnie would fail to gain $x_0 - y$; she can maximize her minimum gain by increasing her bid to x_0.

Thus Bonnie should not underbid. Similarly, if Bonnie were to bid $x^+ > x_0$, then one of three things must happen:

(i) If $y < x_0$, then Bonnie would gain $x_0 - y$ from both a truthful bid x_0 and from her overbid.
(ii) If $y > x^+$, then Bonnie would gain 0 from both a truthful bid x_0 and from her overbid.
(iii) If $x_0 \leq y \leq x^+$, then Bonnie would lose $y - x_0$; she can minimize her loss by decreasing her bid to x_0.

Thus Bonnie should not overbid. Taken together, these prove that Bonnie ought to bid her true value in a second-price private values auction.

This conclusion for the second-price auction is true whether or not Bonnie adopts an ARA perspective. If Bonnie bids x and Clyde bids y, then Bonnie gains $(x_0 - y)$ if $x > y$ and otherwise gets 0. Bonnie does not know y, so ARA views it as a random variable Y to which she assigns a subjective distribution F. It is convenient to assume that F has density f (but this may be relaxed). Then Bonnie's expected utility from a bid of x^- such that $x^- < x_0$ is

$$\mathbb{E}_F[x^- - Y] = \int_0^{x^-} (x_0 - y) f(y)\, dy$$

and

$$\mathbb{E}_F[x^- - Y] \leq \int_0^{x^-} (x_0 - y) f(y)\, dy + \int_{x^-}^{x_0} (x_0 - y) f(y)\, dy = \mathbb{E}_F[x_0 - Y].$$

The inequality is strict if f is positive on any region in (x^-, x_0). Similarly, if Bonnie bids $x^+ > x_0$, then

$$\mathbb{E}_F[x^+ - Y] = \int_0^{x_0} (x_0 - y) f(y)\,dy + \int_{x_0}^{x^+} (x_0 - y) f(y)\,dy \leq \mathbb{E}_F[x_0 - Y],$$

since the term $x_0 - y$ in the second integral is negative. Again, the inequality is strict if $f(y)$ is positive in (x_0, x^+).

However, in some second-price auctions, it is in Bonnie's interest to overbid, but only if there is the prospect of repeated play against the same opponent. Specifically, if Bonnie were confident that Clyde's true value is greater than hers, then overbidding will increase the price he must pay and thus reduce his profit, which will advantage her in future auctions. For example, Boeing knows it will compete with Lockheed Martin for future military contracts. So Boeing wants to ensure that Lockheed Martin makes the smallest possible profit on each contract. If the U.S. Department of Defense awarded contracts through second-price auctions, then Boeing's bidding strategy should treat this game as part of a larger multi-game with repeated play against the same opponent (Camerer, 2003).

It deserves emphasis that in both the first-price and second-price auctions, there are concerns about the applicability of the Nash equilibrium solution concept. Even in the second-price auction there is evidence that real people do not bid as the solution concept directs; instead, they often irrationally underbid (Rothkopf, 2007).

This analysis assumed that Bonnie had no aleatory uncertainty: she knows x_0, her value for the first edition. But, as previously discussed, the logic still holds when the value of the book is a random variable with known mean μ, provided that Bonnie is risk neutral. Since Bonnie seeks to maximize her expected utility, she can replace x_0 by μ in the preceding inequalities, which leads her to bid μ, her expected value for the book.

3.3 Bayes Nash Equilibrium

Much of the research on first-price, sealed-bid, independent-value, rational-bidder auctions has focused on Bayes Nash Equilibrium (BNE) solutions (Krishna, 2010; Klemperer, 2004). There are two cases:

- In the symmetric case, it is assumed that the value each bidder has for the item on auction is a random draw from the same commonly known distribution.
- In the asymmetric case, it is assumed that the value each bidder has for the item on auction is a random draw from a distinct distribution, and that those distributions are known to all bidders.

For the symmetric case, the general solution was found by Vickrey (1961). For the asymmetric case, there remain many open questions; in particular, no current algorithm provably converges to the solution (Fibich and Gavious, 2011).

In the symmetric case, suppose there are n bidders and that the ith bidder has value V_i for the item on offer, where $V_1, \ldots, V_n$ are independent with distribution G. It is assumed that each bidder knows his own value, but not those of his opponents. The strong common knowledge assumption needed for the BNE is that each bidder knows G, and knows that all other bidders also know G. Additionally, in this discussion, assume that all bidders are risk neutral—each seeks to maximize his or her expected profit.

The objective in the symmetric case is to find the bidding function $b(v)$ that maps a value v into a corresponding bid. If one assumes that G is continuous, increasing and differentiable, with support on a compact interval $[L,U]$, then the BNE solution exists and is unique, and $b(v)$ is also continuous, increasing and differentiable (Myerson, 1991, Chap. 3).

Symmetry implies that the equilibrium bidding function $b(\cdot)$ is the same for all players. Suppose that Bonnie values the object at v and, consequently, bids $b(v)$. Then her random profit has expected value

$$[v - b(v)]\mathbb{P}[b(v) \text{ wins}].$$

This is a winning bid if and only if all other players place bids $b(v_i)$ that are less than $b(v)$. Note that G is continuous, so one may ignore the possibility of ties. Since $b(v)$ is strictly increasing, it wins if and only if the values v_i for all other players are less than Bonnie's v. The values are independent, so this happens with probability $G(v)^{n-1}$. Thus her expected profit from bidding $b(v)$ is

$$[v - b(v)]G(v)^{n-1} \tag{3.3}$$

and the same expression holds for all of the other bidders, with their own personal values v_i substituted for Bonnie's value v.

By definition, the equilibrium bid for the ith bidder should be $b(v_i)$. Since this is an optimum of (3.3) and $b(\cdot)$ is continuous, it follows that

$$b(v) = \text{argmax}_w \, [v - b(w)]G(w)^{n-1}$$

for w in some ball around v. Thus, the derivative of this expression at $w = v$ must be zero:

$$\begin{aligned} 0 &= \frac{d}{dw}[v - b(w)]G(w)^{n-1} \\ &= (v - b(v))(n-1)G'(v)G(v)^{n-2} - b'(v)G(v)^{n-1}. \end{aligned}$$

Solving this differential equation shows that for any v in $[L,U]$,

$$b(v) = \frac{\int_0^v z(n-1)G'(z)G(z)^{n-2}\,dz}{G(v)^{n-1}}. \tag{3.4}$$

In general, this expression requires numerical solution.

Example 3.2: Suppose that the value each bidder holds is an independent draw from the distribution $G(v) = v^q$ for $0 \leq v \leq 1$ and $q > 0$. In that case one can find a closed form solution for (3.4):

$$b(v) = v^{-q(n-1)} \int_0^v q(n-1)z^{qn-q}\,dz = \frac{qn-q}{qn-q+1}v.$$

This result shows that as the number n of bidders increases, Bonnie should bid a larger fraction of her true value v, as one would expect. Also, as q increases, Bonnie must bid a larger fraction of her true value.

The ARA perspective has a more natural justification for the BNE solution than the common knowledge assumption. Instead of assuming that everyone knows the common distribution G on the values, it is easier to imagine that Bonnie believes that each of her $n-1$ opponents will draw his true value from G. Bonnie then finds exactly the same solution to the symmetric auction.

The asymmetric auction is more difficult. In general, no expression such as (3.4) exists. When bids are discrete (e.g., they must be whole dollars), then tie-breaking rules are needed to ensure the existence of an equilibrium solution (Lebrun, 1996; Maskin and Riley, 2000a). But when the distributions for the values of the bidders are continuous and differentiable, and one of several possible additional regularity conditions is satisfied, then the bidding functions are unique, continuous, and differentiable (Lebrun, 2006; Maskin and Riley, 2000b). The two most practical regularity conditions are: (1) all value distributions have common support with density that is strictly positive at the lower limit of the support (Lebrun, 1999); or (2) the valuation distributions are locally log-concave at the largest of the lower bounds of the non-common support sets (Lebrun, 2006).

Figure 3.3 shows the MAID that describes the two-person asymmetric auction. The double circle around "Winner" denotes a deterministic node: once both Bonnie's and Clyde's bids are declared, the outcome is non-random.

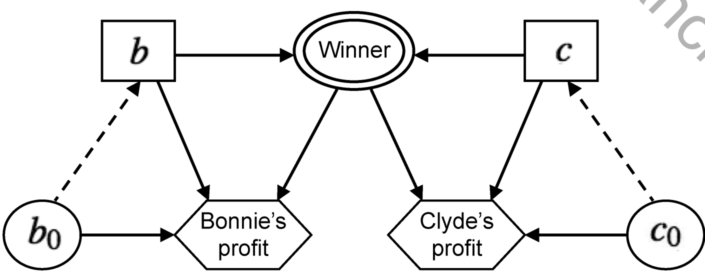

Fig. 3.3 The MAID for a two-person auction. Rectangular decision nodes show the bid that each party makes. Hexagons show the outcome for each bidder given the bids that are placed. Circular nodes indicate that, from the opponent's perspective, the true value is a random variable.

To discuss the asymmetric auction, let F_{IJ} denote what bidder I thinks is the bid that bidder J will place, and let G_{IJ} denote what bidder I thinks is the distribution of bidder J's value. Thus, if Bonnie thinks that Clyde's bid has distribution F_{BC}, her bid should be $b^* = \text{argmax}_{b\in\mathbb{R}^+}(b_0 - b)F_{BC}(b)$ where b_0 is her known true value. And if Clyde thinks that Bonnie's bid has distribution the F_{CB}, then his bid should be $c^* = \text{argmax}_{c\in\mathbb{R}^+}(c_0 - c)F_{CB}(c)$, where c_0 is Clyde's true value. Since neither knows the true value of their opponent, the BNE approach puts commonly known distributions over those values, and solves

$$\begin{aligned} B^* &= \text{argmax}_{b\in\mathbb{R}^+}(B-b)F_{BC}(b) \sim F_{CB} \\ C^* &= \text{argmax}_{c\in\mathbb{R}^+}(C-c)F_{CB}(c) \sim F_{BC}, \end{aligned} \tag{3.5}$$

where $B \sim G_{CB}$ is what Clyde believes is the distribution for Bonnie's true value and $C \sim G_{BC}$ is what Bonnie believes is the distribution for Clyde's true value, and both know what distribution the other has and knows that this is known. If this system of equations has a unique solution, then it determines the F_{BC} that Bonnie needs to calculate her optimal bid, and the F_{CB} that Clyde needs to find his optimal bid. Rarely does (3.5) have a closed-form solution. Kaplan and Zamir (2012) describe some special cases.

Example 3.3: Suppose $B \sim \text{Unif}(0,1)$ and $C \sim \text{Unif}(0,2)$. Then the unique solution to (3.5) is

$$F_{BC}(x) = 4x/(4-3x^2) \qquad F_{CB}(y) = 8y/(4+3y^2)$$

for $0 \le x, y \le \frac{2}{3}$ (both distributions are 0 for $x, y < 0$ and 1 for $x, y > \frac{2}{3}$).

To verify this, one can find the maximizing B^* and C^* in (3.5) by differentiating and setting the results to 0:

$$B = \frac{F_{BC}(B^*)}{f_{BC}(B^*)} + B^* \qquad C = \frac{F_{CB}(C^*)}{f_{CB}(C^*)} + C^*.$$

For the F_{BC} and F_{CB} that are given, solve for B^* and C^*, obtaining

$$B^* = \frac{4-2\sqrt{4-3B^2}}{3B} \qquad C^* = \frac{4-2\sqrt{4+3C^2}}{-3C}.$$

These are both monotone increasing functions, which is logical since the optimum bid should increase with the personal value.

If a random variable W has distribution $H(w)$ and $\theta(\cdot)$ is a monotone increasing transformation with inverse $\theta^{-1}(\cdot)$, then the distribution of $\theta(W)$ is $H(\theta^{-1}(w))$. Since $G_{CB}(b) = b$ for $0 \le b \le 1$ and $G_{BC}(c) = c/2$ for $0 \le c \le 2$, then a little algebra confirms that F_{BC} and F_{CB} solve the system.

When there is no closed-form solution to (3.5) one must use numerical methods. The standard approach is the backshooting algorithm, developed by Marshall et al. (1994). It was later refined by Bajari (2001), Li and Riley (2007), and Gayle and Richard (2008). But Fibich and Gavious (2011) proves that no backshooting algorithm can converge in an epsilon ball around zero. There are additional issues when the value distributions G_{BC} and G_{CB} have one or more crossings (i.e., when one distribution does not stochastically dominate the other, so neither $G_{BC}(x) \geq G_{CB}(x)$ for all x nor $G_{BC}(x) \leq G_{CB}(x)$ for all x). Hubbard, Kirkegaard and Paarsch (2011) proposed a corroborative "visual test" for the accuracy of the numerical solution in the case of multiple crossings, but Au (2014) found errors in the argument. Au proposes a new algorithm, the Backwards Indifference Derivation (BID) scheme, which is successful in cases with known solutions and passes the visual test that backshooting methods sometimes fail. The BID algorithm forms a mesh that discretizes the values of possible bids for each bidder, finds adjacent values between which each bidder is indifferent, and then refines the mesh.

This BNE framework relies upon the common knowledge assumption. For two bidders, the ARA formulation finds the same result through an alternative logic. Instead of common knowledge, Bonnie might reasonably believe that Clyde draws his value from G_{BC} and she also thinks that he believes her value is a draw from G_{CB}. Then Bonnie is led to solve (3.5). But when there are more than two bidders, the ARA perspective opens a larger class of equilibrium problems. Bonnie can model not only what she thinks are the distributions of her opponents' values, but also what she believes are the distributions each opponent has for the values of the other bidders. This topic is further developed in Section 3.6.

3.4 Level-*k* Thinking

Bayesian level-*k* thinking is an important family of ARA strategies. The family is diverse, since, at each level, the analyst has many choices regarding how to model the epistemic and aleatory uncertainties. This section applies level-*k* thinking to auctions (cf. Banks, Petralia and Wang, 2011).

If Bonnie is a level-0 thinker, she bids non-strategically, making no attempt to model her opponents. She might bid 90% of her true value, or place the bid that won a similar book at a recent auction. And the case in which Bonnie is a level-1 thinker was addressed in Section 3.1. She assumed Clyde was non-strategic, and found her best bid given her distribution over his actions.

Things are more interesting when Bonnie is a level-2 thinker. She models Clyde as a level-1 thinker, who believes that Bonnie is a level-0 thinker. Bonnie begins her ARA by developing a subjective distribution F_{CB} to describe what Bonnie believes Clyde thinks is the distribution for her bid. She also needs a subjective distribution G_{BC} for what she believes is Clyde's true value. Finally, she needs to know her own true value b_0 (or, if there is aleatory uncertainty regarding, say, the condition of the book, she needs her expected value, μ).

In this framework, suppose Bonnie believes that Clyde seeks to maximize his expected utility. Clyde knows his true value c_0, and will make the bid c^* such that

$$c^* = \text{argmax}_{c \in \mathbb{R}^+} (c_0 - c) F_B^*(c),$$

where F_B^* is the distribution that Clyde has for Bonnie's bid. Since Bonnie knows neither F_B^* nor c_0, she uses her subjective beliefs to solve the analogous problem:

$$C^* = \text{argmax}_{c \in \mathbb{R}^+} (C_0 - c) F_{CB}(c),$$

where $C_0 \sim G_{BC}$ and F_{CB} is Bonnie's belief about Clyde's belief about the distribution of her bid. Since C_0 is a random variable, then so is C^*; denote its distribution by F_{BC}.

Bonnie has now obtained her belief F_{BC} about the distribution of Clyde's bid, enabling her to solve (3.1). The result is the bid that maximizes her expected utility, where the expectation takes proper account of her uncertainty about both Clyde's true value and his belief about her bid.

Example 3.4: Suppose Bonnie, a level-2 thinker, thinks Clyde believes that her value for the book is a random variable with the uniform distribution on [\$100, \$200]. And further suppose that she thinks Clyde believes that the proportion of her value that she will bid is a random variable with distribution p^9 for $0 \leq p \leq 1$. From Example 3.1 (with the roles reversed), she thinks Clyde's distribution on her bid is $F_{CB}(b) = \frac{9}{8}(b/200) - \frac{1}{8}(b/200)^9$ on [\$0, \$200], and thus his optimal bid is approximately half of his true value.

Bonnie does not know Clyde's true value, but she has a distribution G_{BC} that describes her subjective judgment. Suppose that judgment is that his true value has the triangular distribution on [\$252, \$360] with peak at \$300. Since Clyde should bid 50% of his true value, Bonnie believes that his bid will be a random variable with triangular distribution $F_{BC}(c)$ that is supported on [\$126, \$180] with peak at \$150.

Recall that Bonnie's true value for the book is $b_0 = \$150$. She seeks the bid b^* that maximizes her expected profit, or $(150 - b)F_{BC}(b)$. Simple calculus shows Bonnie should bid \$141.67.

This level-2 solution raises three questions. First, how can one calculate the solution when the assumed distributions are non-trivial? Second, is it reasonable for Bonnie to have such precise opinions about Clyde's beliefs? And third, when should Bonnie proceed to higher levels of thinking?

How Can One Calculate Solutions?

In general there will not be a closed-form solution. The optimum bid must be found numerically, through the following algorithm:

Repeat from $j = 1$ to J:

Sample $c_0^j \sim G_{BC}$

Solve $c_j^* = \text{argmax}_{c \in \mathbb{R}^+} (c_0^j - c) F_{CB}(c)$.

Set $\hat{F}_{BC}(b) = \frac{1}{J} \sum_{j=1}^{J} I(c_j^* \le b)$.

Solve $b^* = \text{argmax}_{b \in \mathbb{R}^+} (b_0 - b) \hat{F}_{BC}(b)$.

The first step simulates from what Bonnie thinks is Clyde's value distribution and finds his optimal bid for that random draw. The second step finds the empirical cumulative distribution of his optimal bid. The third solves Bonnie's optimization problem, using the empirical cumulative distribution function of his optimal bid. As J increases, the approximation becomes arbitrarily accurate. If the distributions G_{BC} or F_{CB} are discrete, then the solution may not be unique; adjacent support points (i.e., neighboring values) can have equal expected utility.

Often, F_{CB} is not explicitly available. In that case, the algorithm must be extended. For example, suppose Bonnie believes that Clyde thinks her true value has distribution $G_1(b)$ and that she will bid a random proportion of that value, where the proportion has distribution $G_2(p)$. In that case she can modify the algorithm as follows:

Repeat from $k = 1$ to K:

Sample $v_k \sim G_1$

Sample $p_k \sim G_2$

Set $c_k = p_k v_k$.

Set $\hat{F}_{BC}(c) = \frac{1}{K} \sum_{k=1}^{K} I(c_k \le c)$.

Repeat from $j = 1$ to J:

Sample $c_0^j \sim G_{BC}$

Solve $c_j^* = \text{argmax}_{c \in \mathbb{R}^+} (c_0^j - c) \hat{F}_{CB}(c)$.

Set $\hat{F}_{BC}(b) = \frac{1}{J} \sum_{i=1}^{J} I(y_j^* \le b)$.

Solve $b^* = \text{argmax}_{b \in \mathbb{R}^+} (b_0 - b) \hat{F}_{BC}(b)$.

As before, the empirical cumulative distribution function $\hat{F}_{BC}$ converges to the distribution F_{BC} as K and J increase. More complicated rules for generating F_{BC} (such as assuming some dependence between V_k and P_k) can be accommodated through modifications of this algorithm.

How Does Bonnie Have Precise Opinions about Clyde's Beliefs?

In real applications, Bonnie would encounter substantial cognitive difficulty in developing her ideas about F_{BC}, and perhaps even G_{BC}. The preceding discussion teased apart that development into two pieces: her belief about Clyde's true value and her belief about the proportion of that value that he bids. But it avoided serious consideration of how to model both distributions. Also, the discussion did not address concept uncertainty. How certain can Bonnie be that Clyde is non-strategic, and that the particular instantiation of his non-strategy is to bid a fraction of his true value?

This difficulty is fundamental—people do not think clearly enough to have fully coherent Bayesian beliefs that incorporate all of their information and intuition. The literature on elicitation of subjective probabilities is extensive and discouraging: experts are overconfident, the framing of the problem matters, mutually contradictory opinions are held, and so forth. O'Hagan et al. (2006), Kahneman (2003), and Garthwaite, Kadane and O'Hagan (2005) are prominent voices in this discussion. Fortunately, in many situations the solution is insensitive to minor errors in the specification of subjective opinion, and sensitivity analysis can flag the cases when greater reflection is required.

There are sensible strategies that can improve Bonnie's assessments. For example, she may not know F_{CB} with confidence, but it is not intellectually overwhelming for her to consider, say, ten fairly distinct choices for it. In examining those ten alternatives for F_{CB}, Bonnie will find that some of them seem more likely to her than others, and she should give those distributions higher probabilities. Then she can combine those by taking the weighted sum of the distributions, with weights corresponding to her probabilities. If done thoughtfully, the resulting distribution will capture much of her judgment. And, of course, the same procedure could be used to formulate her belief about the distribution of Clyde's true value for the book, G_{BC}.

A more sophisticated approach enables Bonnie to express her uncertainty about F_{CB} through a Dirichlet process with central measure F and concentration parameter α. The effect of this would be to increase the variance of F_{BC} above that found from a derivation based on a single F_{CB}. There is a large literature on Bayesian nonparametrics using Dirichlet processes. Müller and Rodriguez (2013) is a short introduction, and Ghosh and Ramamoorthi (2008) provides a more mathematical treatment.

What Level Should One Use?

As previously mentioned, choosing the correct value of k for level-k thinking is an issue. The auction example considered $k = 0, 1, 2$, but one could certainly go higher. Bonnie might attempt to model what Clyde believes Bonnie believes about Clyde, leading to a level-3 analysis (which is straightforward, but the layered reasoning becomes tedious).

Ideally, Bonnie wants to think just one level higher than does Clyde. If she goes beyond that, she is solving the wrong problem, and in general the resulting bidding strategy will be inferior. Thus, selecting the value of k entails uncertainty about precisely which version of the level-k solution concept Clyde is using.

Let p_k denote Bonnie's personal probability that Clyde is a level-k thinker, for $k = 0, 1, \ldots$, such that $\sum p_k = 1$. If $p_k > 0$, Bonnie should do a level-$k+1$ analysis to determine F_{BC}^k, her distribution for Clyde's bid. Then Bonnie combines all of these distributions as a mixture distribution, so that $F_{BC} = \sum p_k F_{BC}^k$. This F_{BC} is the expression of Bonnie's full belief about Clyde's bid, and incorporates her uncertainty about the depth of his strategic thinking. She uses this F_{BC} in (3.1), solving to find her optimal bid.

At some point in the potentially infinite ascent, Bonnie will feel that she no longer has relevant information about Clyde's beliefs. For that value of k, she should assign a uniform distribution over all unknown quantities. A discussion of the use of non-informative distributions to terminate the hierarchy is given in Ríos Insua, Rios and Banks (2009).

3.5 Mirror Equilibria

For auctions, the mirror equilibrium approach is similar to the BNE solution concept. With just two bidders, the key calculation is the same, although the perspective and assumptions are different (cf.0 Banks, Petralia and Wang, 2011). One solves (3.5), but instead of assuming that the distributions are common knowledge, the G_{BC} is Bonnie's belief about the distribution of Clyde's value and the G_{CB} is what she thinks is Clyde's distribution for her value. Then, after deriving F_{BC}, her distribution for Clyde's bid, Bonnie uses her known value b_0 and solves (3.1).

But the mirror equilibrium solution becomes interestingly different from the BNE formulation when the number of bidders is greater than two.

3.6 Three Bidders

Essentially all of the previous discussion addressed games in which there are only two opponents. But an important advantage of ARA is that it enables a more nuanced treatment of many-player games. Specifically, the ARA formulation allows one to frame fresh problems in auction theory when there are more than two bidders, by permitting asymmetric models for how each opponent views the others. We develop the ARA solutions in cases with three opponents for both the level-k thinking solution concept and the mirror equilibrium solution concept. In the following discussion, we now assume that Bonnie is bidding against both Alvin and Clyde to obtain a first edition of the *Theory of Games and Economic Behavior*.

3.6.1 Level-k Thinking

If Bonnie is a level-1 thinker, then she assumes that Alvin and Clyde are non-strategic, and there is no novelty in the analysis. She has distributions over the non-strategic bids of each, and chooses her bid according to the maximum of those. Specifically, she has a subjective distribution F_A over Alvin's bid A and a subjective distribution F_C over Clyde's bid C, and she calculates the distribution F of $\max\{A, C\}$. Then she makes the bid

$$b^* = \text{argmax}_{b \in \mathbb{R}^+} (b_0 - b) F(b),$$

where b_0 is her true value for the book.

But now suppose Bonnie is a level-2 thinker. She thinks that Alvin has a belief about the distribution of her bid and also Clyde's bid; similarly, she thinks Clyde has a distribution for her bid and for Alvin's. Recall the previous notation: $F_{IJ}(x)$ is what Bonnie thinks player I thinks is the distribution for player J's bid, and $G_{IJ}(x)$ is her belief about what player I thinks is the distribution for player J's value. Since her level-2 analysis assumes both Alvin and Clyde are level-1 thinkers who believe their opponents are level-0 thinkers, then knowing F_{IJ} directly determines G_{IJ}, as in Example 3.1, where Clyde bids a fraction P of his value V.

The level-2 ARA formulation means that Bonnie thinks Alvin will make the bid $a^* = \max\{a_B^*, a_C^*\}$ for

$$\begin{aligned} a_B^* &= \text{argmax}_{a \in \mathbb{R}^+} (a_0 - a) \mathbb{P}[B^* < a] \\ a_C^* &= \text{argmax}_{a \in \mathbb{R}^+} (a_0 - a) \mathbb{P}[C^* < a], \end{aligned}$$

where a_0 is Alvin's true value, B^* is a random variable whose distribution is Alvin's opinion about Bonnie's bid, and C^* is a random variable whose distribution is Alvin's opinion about Clyde's bid. Bonnie does not know a_0, and she does not know Alvin's distributions for the bids, but as a Bayesian, she has a subjective opinion about these. She regards a_0 as a random variable with distribution G_{BA}, and her best guess is that B^* and C^* have distributions F_{AB} and F_{AC}, respectively.

In order to find F_{AB}, Bonnie uses the fact that Alvin thinks she is a level-0 thinker. He views her as non-strategic, and thus thinks her bid follows some probability distribution, perhaps an unknown proportion of her unknown true value, where both the unknown proportion and the unknown true value can be modeled as random variables. (Of course, she could think that he thinks she follows some other kind of rule, e.g., bidding the last winning bid for similar first editions, or using a random number generator, but she will still always have a subjective distribution over what he believes about the distribution of her bid.)

Thus, Bonnie's opinion about the distribution of Alvin's bid is found by solving

$$A_B^* = \text{argmax}_{a\in\mathbb{R}^+}(A_0 - a)F_{AB}(a)$$
$$A_C^* = \text{argmax}_{a\in\mathbb{R}^+}(A_0 - a)F_{AC}(a)$$

and then assuming that Alvin bids the larger of those two random variables. So his bid is $A^* = \max\{A_B^*, A_C^*\}$.

Similarly, Bonnie belief about Clyde's bid C^* is that it has the distribution of $\max\{C_A^*, C_B^*\}$, where

$$C_A^* = \text{argmax}_{c\in\mathbb{R}^+}(C_0 - c)F_{CA}(c)$$
$$C_B^* = \text{argmax}_{c\in\mathbb{R}^+}(C_0 - c)F_{CB}(c)$$

and C_0 is Clyde's true value, with distribution G_{BC}, since it is unknown to Bonnie. Just as before, Bonnie uses her beliefs about what Clyde thinks about Alvin's non-strategy and her non-strategy to identify F_{CA} and F_{CB}, respectively, and thus finds the distribution of C^*.

Bonnie has calculated her distribution for Alvin's bid A^* and Clyde's bid C^*. Now she should place the bid

$$b^* = \text{argmax}_{b\in\mathbb{R}^+}(b_0 - b)\mathbb{P}[\max\{A^*, C^*\} < b].$$

Generally, this ARA solution will require extensive numerical computation.

One can go further. If Bonnie does a level-3 analysis, she requires two replicates of the level-2 analysis, where Bonnie imagines each opponent is solving his own system of level-2 equations. The nested thinking is complex but straightforward, and the notation must be extended. Let $G_{IJ}(x)$ represent what Bonnie thinks bidder I thinks is the distribution of the value of the book to bidder J, and let G_{IJK} represent what Bonnie thinks bidder I thinks is the distribution that bidder J has for bidder K's value for the book. Similarly, let F_{IJ} represent what Bonnie thinks is the distribution that bidder I has for bidder J's bid, and F_{IJK} represent what Bonnie thinks bidder I thinks is the distribution that bidder J has for bidder K's bid.

Bonnie thinks the level-2 Alvin will reason as follows. First, he thinks Bonnie will make the bid $b^* = \max\{b_A^*, b_C^*\}$ for

$$b_A^* = \text{argmax}_{b\in\mathbb{R}^+}(b_0 - b)\mathbb{P}[A^* < b]$$
$$b_C^* = \text{argmax}_{b\in\mathbb{R}^+}(b_0 - b)\mathbb{P}[C^* < b],$$

where b_0 is Bonnie's true value, A^* is a random variable with distribution F_{BA}, and C^* is a random variable with distribution F_{BC}. Since b_0 is unknown to Alvin, he treats it as a random variable B_0 with distribution G_{AB}. Alvin also does not know know F_{BA} or F_{BC}, but Bonnie believes he thinks A^* has distribution F_{ABA}, and C^* has distribution F_{ABC}. This means that, to Alvin, Bonnie's solutions B_A^* and B_C^* are random variables, and he thinks Bonnie's bid B^* has the distribution of their maximum.

In order to find F_{ABA}, Bonnie thinks level-2 Alvin will model her as a level-1 thinker. That means that he thinks she thinks that his starting point in the level-k

hierarchy is non-strategic. He will have some distribution over what she thinks will be his non-strategic bid, which Alvin must elicit from his personal beliefs. Bonnie does not know what that distribution is, but suppose her subjective belief is that it is H_{ABA}. In that case, Bonnie's best opinion about what Alvin thinks a level-1 Bonnie would bid in order to beat him is

$$B_A^* = \text{argmax}_{b \in \mathbb{R}^+}(B_0 - b)H_{ABA}(b),$$

where B_0 has distribution G_{AB}. Solving this gives F_{ABA}.

Similarly, Bonnie thinks Alvin thinks that her starting point in the level-k reasoning is that Clyde is non-strategic, and thus Alvin must have a distribution over Bonnie's belief about Clyde's bid. Denote Bonnie's best guess about Alvin's distribution for Bonnie's belief about Clyde's bid by H_{ABC}. So Alvin thinks a level-1 Bonnie solves

$$B_C^* = \text{argmax}_{b \in \mathbb{R}^+}(B_0 - b)H_{ABC}(b),$$

where B_0 has distribution G_{AB}, as before. Solving this gives F_{ABC}.

Finally, Alvin should think that Bonnie will bid the maximum of B_A^* and B_C^*. This maximum has distribution F_{AB}.

Similarly, Bonnie thinks Alvin thinks Clyde will bid $C^* = \max\{C_A^*, C_B^*\}$ such that

$$\begin{aligned} C_A^* &= \text{argmax}_{c \in \mathbb{R}^+}(c_0 - c)\mathbb{P}[A^* < c] \\ C_B^* &= \text{argmax}_{c \in \mathbb{R}^+}(c_0 - c)\mathbb{P}[B^* < c] \end{aligned}$$

where c_0 is Clyde's true value, which is unknown to Alvin, and for which Bonnie believes he has distribution G_{AC}. Also, A^* is a random variable that Bonnie thinks has distribution F_{ACA}, and B^* is a random variable that she thinks has distribution F_{ACB}.

Now Bonnie has calculated what she believes Alvin thinks is the distribution of her bid B^* and the distribution of Clyde's bid C^*. So her best guess is that Alvin will make the bid

$$a^* = \text{argmax}_{a \in \mathbb{R}^+}(a_0 - a)\mathbb{P}[\max\{B^*, C^*\} < a].$$

She does not know his value a_0, and thus replaces it with the random variable A_0 with distribution G_{BA}. Solving this new equation provides her distribution F_{BA} for Alvin's bid A^*.

She repeats this reasoning for Clyde instead of Alvin, ultimately obtaining F_{BC}, her distribution for Clyde's bid C^*. Now, Bonnie should make the bid

$$b^* = \text{argmax}_{b \in \mathbb{R}^+}(b_0 - b)\mathbb{P}[\max\{A^*, C^*\} < b].$$

Obviously, implementing the ARA paradigm for the level-k solution concept is intricate—the nested reasoning is difficult for humans to describe, much less perform. But the logic is actually simple, and one can write software that automatically performs these recursions, and thus handles many more than three opponents.

As a final note, when there are more than two bidders, it is possible for different bidders to think at different levels. For example, if Bonnie thinks Alvin is a level-2 thinker but Clyde is only a level-1 thinker, then her analysis might be denoted as level-(3,2) thinking.

3.6.2 Mirror Equilibrium

Now consider the use of the mirror equilibrium solution concept when there are three bidders. This concept assumes that all bidders are solving the problem in the same way, but with possibly different subjective distributions over all unknown quantities.

The two-person system in (3.5) extends so that the basic problem is to solve

$$\begin{aligned} A^* &= \text{argmax}_{a\in\mathbb{R}^+} (A_0 - a)F_A^*(a) \\ B^* &= \text{argmax}_{b\in\mathbb{R}^+} (B_0 - b)F_B^*(b) \\ C^* &= \text{argmax}_{c\in\mathbb{R}^+} (C_0 - c)F_C^*(c) \end{aligned} \tag{3.6}$$

from the perspective of each of the players, where $F_I^*(x)$ is what bidder I thinks is the chance that a bid of x will win. Bonnie does not know F_I^*, but she can use ARA to find F_I, which is her belief about what each opponent thinks is the chance that a given bid is successful.

Figure 3.4 may be helpful in following the reasoning. It shows the notation that describes what Bonnie thinks each person believes about the distributions for each of the other bidders' true values. As indicated previously, G_{IJ} is what Bonnie thinks bidder I believes is distribution of the true value for bidder J, and G_{IJK} is the distribution that Bonnie thinks bidder I thinks bidder J has for the true value of the book to bidder K.

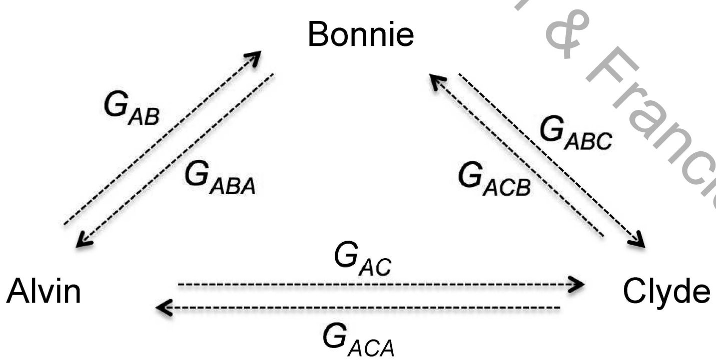

Fig. 3.4 A representation of what Bonnie believes about the opinions held by each of the bidders regarding the value of the book to each the other bidders.

First, she models Alvin's logic. Bonnie thinks he obtains his distribution for her bid by solving (3.6) with $A_0 \sim G_{ABA}$, $B_0 \sim G_{AB}$, and $C_0 \sim G_{ABC}$. Since he, like Bonnie, does not know the true F_I^*, he must develop his own beliefs about them. Here, his F_A is the distribution of the maximum of B^* and C^*, F_B is the distribution of the maximum of A^* and C^*, and F_C is the distribution of the maximum of B^* and C^*. After numerical computation to find the equilibrium solution, he obtains F_{AB}, his belief about the distribution of Bonnie's bid.

Next, Alvin considers Clyde. Bonnie thinks he solves (3.6) with $A_0 \sim G_{ACA}$, $B_0 \sim G_{ACB}$, and $C_0 \sim G_{AC}$. He proceeds as before, and obtains F_{AC}, his belief about the distribution of Clyde's bid. From this, Bonnie thinks his distribution for the probability of winning with a bid of a is F_A, where F_A is the distribution of the maximum of $B \sim F_{AB}$ and $C \sim F_{AC}$.

Bonnie's analysis for Clyde is analogous. To find Clyde's distribution for Bonnie's bid, she thinks he solves (3.6) with $A_0 \sim G_{CBA}$, $B_0 \sim G_{CB}$, and $C_0 \sim G_{CBC}$ to obtain F_{CB}. Similarly, to find Clyde's distribution for Alvin's bid, he uses $A_0 \sim G_{CA}$, $B_0 \sim G_{CAB}$, and $C_0 \sim G_{CAC}$ to obtain F_{CA}. Putting these together, Bonnie thinks that Clyde thinks the probability that a bid of c will win is $F_C(c)$, which is the distribution of the maximum of $A \sim F_{CA}$ and $B \sim F_{CB}$.

Based on this reasoning, Bonnie thinks that Alvin's bid will be

$$A^* = \text{argmax}_{a \in \mathbb{R}^+} (A_0 - a) F_A(a) \sim F_{BA},$$

where $A_0 \sim G_{BA}$. And she thinks Clyde's bid will be

$$C^* = \text{argmax}_{c \in \mathbb{R}^+} (C_0 - c) F_C(c) \sim F_{BC},$$

where $C_0 \sim G_{BC}$. From this, the chance that a bid of b will win is $F_B(b)$, where F_B is the distribution of the maximum of $A^* \sim F_{BA}$ and $C^* \sim F_{BC}$. Now Bonnie uses her known value b_0 and solves

$$b^* = \text{argmax}_{b \in \mathbb{R}^+} (b_0 - b) F_B(b)$$

to obtain her best bid under the mirror equilibrium solution concept.

Some readers may question whether a solution is guaranteed to exist in the mirror equilibrium analysis. The answer is that it must, because at each step, one solves a well-posed Nash equilibrium problem for an asymmetric n-person auction. Lebrun (1999) shows that an equilibrium solution always exists, and Lebrun (2006) proves that, under a mild log concavity condition, the equilibrium is unique.

Exercises

3.1. Suppose n people are bidding to own a Miró in a first-price sealed-bid auction. Each participant has a private valuation for the painting v_i, $i = 1, \ldots, n$: it represents how much the ith bidder is willing to pay. Assume all bidders believe that the others have private valuations that are independent and Unif$[0,1]$, and that all bidders know each bidder makes this assumption. Consider only bidding strategies of the form $s(v) = \alpha v$ for $\alpha \in [0,1]$, so that each participant bids a fraction of his valuation. Find a symmetric Bayes Nash equilibrium in this family of strategies.

3.2. Suppose Bonnie is certain that Clyde will bid a fraction P of his true value V for the book *Theory of Games and Economic Behavior*. Also, suppose her distribution over Clyde's value V has a density function with support on $[a,b]$, $0 \le a < b$, and P has distribution supported on $[0,1]$ and is independent of V. Describe how Bonnie should obtain her distribution for Clyde's bid when $a > 0$. Find Bonnie's optimal bid when her value for the book is $x_0 = \$160$ and she models V with an uniform distribution between \$100 and \$200 and P with the distribution p^2 for $0 \le p \le 1$.

3.3. In Exercise 3.3, suppose Bonnie models Clyde's value as a triangular distribution supported between \$100 and \$200 with peak at \$150, and his proportional reduction P as a Beta$(20,10)$. Approximate Bonnie's beliefs about the distribution of Clyde's bid, and obtain her optimal bid when $x_0 = 200$.

3.4. Suppose Bonnie believes there is a positive probability that her opponent's bid will be lower than her value x_0 for the item on offer. Prove that Bonnie's optimal bid x^* against a non-strategic opponent is strictly lower than x_0. Assume that Bonnie's distribution F over her opponent's bid is continuous with $F(x_0) > 0$.

3.5. A Dutch auction (or open-outcry descending-price auction) is an auction in which the seller starts off asking a high price for the item on offer. Then, the price is gradually reduced until a bidder accepts the last announced price. The first bid wins and pays the last price called by the seller. Prove that a Dutch auction has the same optimal bidding strategy as a sealed-bid first-price auction. (Dutch auctions are used when one wants to sell quickly; e.g., bidding on a fishing boat's catch.)

3.6. Bonnie and Clyde are the only bidders for a Juan Gris painting. The auctioneer thinks the item is of high value to both, but he also thinks that each believes the other is an amateur collector who does not value the painting highly. Specifically, the auctioneer thinks Bonnie's and Clyde's valuations are greater than \$10M but each believes the other's valuation is less than \$1M. From the auctioneer's perspective, is it smart to have a sealed-bid first-price auction? What auction mechanisms might be better?

3.7. Suppose a \$100 bill is offered in a first-price sealed-bid auction between Bonnie and Clyde. Assume bids must be integer multiples of pennies (\$0.01). (If both bid the same amount, a coin determines the winner.) Find the Nash equilibrium of this auction. Now suppose Bonnie knows that in real auctions of this kind, participants' bids have the discrete uniform distribution between \$60 and \$100. How much should she bid?

3.8. In Exercise 3.8, suppose Bonnie is bidding against both Alvin and Clyde. Bids many now be continuous, and she knows that with three bidders, bids are uniformly distributed between \$70 and \$100. What should Bonnie bid?

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