Chapter 2

Sets and Logic

2.1 Introduction and Summary

Sets and logic are the fundamentals that underlie all of mathematics, not just discrete mathematics. However, a discrete mathematics course is a customary place to address them directly. Sets are collections of objects. Logic is a formal way of describing reasoning. We will both describe and construct sets, and we will develop truth tables as a way to use logic on compound statements. Logical tools are available for when we have trouble figuring out how to reason precisely using English.

Both sets and logic come with a lot of notation. In order to do anything interesting with either sets or logic, you need to be familiar with that notation. (In the case of logic, we will not use the notation very often after this chapter.) Hence, this chapter has a lot of reading that you must complete before you can get on with the discovery and *doing* of related mathematics. It may feel a bit tedious; sorry. Break it up into smaller chunks to aid focus and retention.

This chapter also contains our first introduction to the interesting proof technique of contradiction (and to the less interesting, but super-useful, proof technique of double-inclusion). Proof by contradiction basically works by hypothesizing that a theorem is false (say "suppose not!") and then obtaining a statement that is clearly false (such as $0 = 1$).

Try not to be intimidated by the amount of unfamiliar material in this chapter. We will be working with logical thinking and proof techniques all semester, and you are not expected to fully grasp them yet. The intent of this chapter is to give you the ideas and terminology so you can work to master the ideas as you use them in context. You will probably want to reread parts of this material later in the course to assist in that endeavor.

2.2 Sets

Sets are ubiquitous in mathematics (and in life!). The definition of the word *set* has a long and sordid history, full of confusions such as whether a set is allowed to contain itself. We will be a bit imprecise here and give more of a description than a definition.

Definition 2.2.1 (of set). A *set* contains *elements*. The elements must be distinct, but their order does not matter. There may be finitely many or infinitely many elements in a set. Elements can be words, objects, numbers, or other sets (i.e., basically anything).

When an element a is a member of a set A, we denote this by $a \in A$ (and read it aloud as "a is in A" or "a is an element of A"). The notation $a_1, a_2 \in A$ means that both a_1 and a_2 are elements of A. Often, sets are denoted by capital letters, and their elements are denoted by related lowercase letters.

Example 2.2.2 (of your favorite sets). The sets most commonly used in discrete math are

- the natural numbers, $\mathbb{N} = \{1,2,3,\ldots\}$,
- the binary digits, $\mathbb{Z}_2 = \{0,1\}$,
- the integers, $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$.

Beware that some people (many computer scientists and some mathematicians) think that $0 \in \mathbb{N}$, perhaps because computer scientists often start counting with zero instead of with one. In order to have consistency with mathematical induction (see Chapter 4), we disagree with this view. Instead, we refer to the set $\{0,1,2,3,\ldots\} = \mathbb{W}$ as the whole numbers (but we refer to it rarely).

Example 2.2.3 (of other sets). The set $\{1,2,3\}$ is the same set as $\{2,3,1\}$. Similarly, $\{\ldots,-6,-4,-2,0,2,4,\ldots\}$ is the same infinite set as $\{0,2,-2,4,-4,\ldots\}$. (The dots indicate that the established pattern keeps on going.) By some definitions, $\{1,1,2,3\}$ is not a set because elements are repeated, but in this text we will simply consider $\{1,1,2,3\}$ as an inefficient expression of the set $\{1,2,3\}$. On the other hand, $\{1,\{1,2,3\},3\}$ is a perfectly fine (and well-expressed) set. The set with no elements $\{\}$ is often denoted $\emptyset$ and called the *empty set* or the *null set*. It is different from $\{\{\}\} = \{\emptyset\}$, which contains one element (the empty set). A set of four duck heads is shown in Figure 2.1.

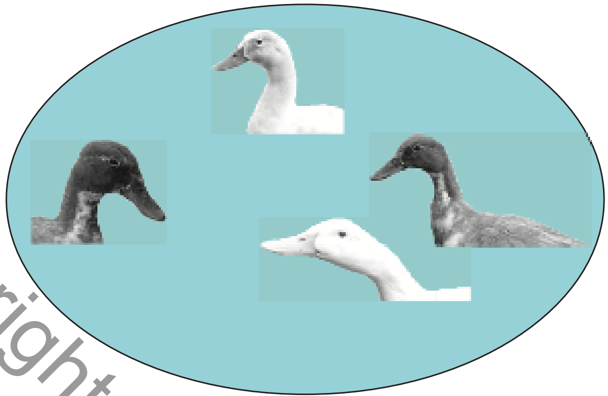


Figure 2.1. The elements of the set $\{dh_1, dh_2, dh_3, dh_4\}$ are duck heads.

This is an appropriate moment to recall that $|A|$ denotes the number of elements in a set, also called its *size* or its *cardinality*. We will only consider the cardinality of finite sets here, and if you are interested in infinite sets, you should look at Chapter 15. Here are a few examples: $|\{1,2,3\}| = 3$; $|\{\{1,2,3\}\}| = 1$; $|\{\{1,2,3\},\mathbb{N}\}| = 2$. Do not confuse set cardinality with absolute value, even though they use the same notation; one applies to sets and the other to numbers, so there is no conflict.

2.2.1 Making New Sets from Scratch

So far, we have described a set by listing all its elements. Most of the time we instead describe the pattern that the elements follow. For example, $2\mathbb{Z} = \{k \in \mathbb{Z} \mid k$ is even$\} = \{\ldots,-4,-2,0,2,4,\ldots\}$. The first expression is read as "two zee is the set of k in zee such that k is even," or as "two zee is the set of integers k such that k is even," or as "two zee is the set of all integers that are even." Another way of writing this same set dispenses with the word "even": $2\mathbb{Z} = \{k \in \mathbb{Z} \mid k = 2\ell$ for some $\ell \in \mathbb{Z}\}$. Here we have substituted the definition of *even* for the word "even."

Example 2.2.4. The set $\{a_1a_2a_3 \mid a_i \in \mathbb{Z}_2\}$ is the set of all three-digit binary strings $\{000,001,010,011,100,101,110,111\}$. Similarly, $\{a_1a_2a_3a_4 \mid a_i \in \mathbb{Z}_2, a_1 = 1, a_3 = 0\}$ is the set of all four-digit binary strings with first digit 1 and third digit 0, or $\{1000,1001,1100,1101\}$. The set $\{(a,b) \mid a \in 2\mathbb{Z}, b \in \{0,1,2\}\}$ is the set of all ordered pairs where the first component is an even integer and the second component is $0, 1$, or 2.

Basically, we write sets in the form $\{\text{type of elements} \mid \text{condition(s)}\}$. Often the type of elements will include a restriction to some set.

2.2.2 Finding Sets inside Other Sets

Recall from Chapter 1 that if we have two sets A and B, then A is a subset of B if every element of A is also an element of B. Let's say it again:

Definition 2.2.5. If A and B are sets, then A is a *subset* of B if every element of A is also an element of B. We denote this relationship as $A \subset B$.

Technically, the symbol $\subset$ means that A is a *proper* subset, so that there is at least one element in B that is not in A, but we will be loosey-goosey with our usage and allow $A \subset B$ to mean that A is perhaps equal to B. The symbol $\subseteq$ is used to indicate that perhaps A and B are equal, and the symbol $\subsetneq$ indicates that A and B are definitely not equal. (Do not confuse $\subsetneq$ with $\not\subset$, which means that A is *not* a subset of B!) Notice that $\emptyset \subset A$ for any set A—because all zero of the elements in $\emptyset$ are also elements of A! Every set contains some nothingness.

Example 2.2.6 (of flavors of subsets and non-subsets). We start with $A = \{2k \mid k > 0, k \in \mathbb{Z}\}$, the even natural numbers; $A \subset \mathbb{N}$ and, in fact, $A \subsetneq \mathbb{N}$. In binary land, $\{1\} \subset \mathbb{Z}_2$ and $\{0,1\} \subseteq \mathbb{Z}_2$ but $\{2\} \not\subset \mathbb{Z}_2$. Less commonly seen are the equivalent statements $\mathbb{Z}_2 \supset \{1\}$, $\mathbb{Z}_2 \supseteq \{0,1\}$, and $\mathbb{Z}_2 \not\supset \{2\}$. We could have instead written $1 \in \mathbb{Z}_2, 2 \notin \mathbb{Z}_2$ for the first and last of those statements (do you see why?).

A related concept is that of the *power set* $\mathscr{P}(A)$ of a set A. It is the set of all subsets of A. (You know from Theorem 1.5.2 that if A is finite, then $|\mathscr{P}(A)| = 2^{|A|}$.) We will not use this concept very often, but it is worth mentioning because other sources you encounter in your mathematical life will expect you to recognize it.

The notion of subset allows us to define the idea of set complement. We denote the complement of A by $\overline{A}$, though other people use notations like A^C or A' (that last one is silly because the symbol $'$ is used for so many other things, but still, you should be warned).

Definition 2.2.7. If $A \subset B$, then $\overline{A} = B \setminus A$, all the elements of B that are not in A, is called the *complement* of A relative to B. (This is sometimes written as $B - A$.)

So if you see the symbol $\overline{A}$, know that there is secretly a B out there that you must know about in order to understand what $\overline{A}$ is. Sometimes the *universe* is temporarily redefined as a particular set (instead of the universe we live in) and it takes the place of B for all sets $A_1, A_2, \ldots, A_n$ in a discussion. (By the way, if there are several sets under discussion, we may refer to them as the first set or A_1 (pronounced “A-one”), the second set, the nth set, etc.). We can think of a set complement as a way of removing one set from another.

Example 2.2.8 (of complements). As a small example, note that $\{1,3,5,7\} \setminus \{1,5\} = \{3,7\}$. Now let B be the set of four-digit binary strings. Then $B \setminus \{a_1a_2a_3a_4 \mid a_i \in \mathbb{Z}_2, a_1 = 1, a_3 = 0\} = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1010, 1011, 1110, 1111\}$.

The notation $B \setminus A$ can be extended to situations where A is not a subset of B; in these cases, we interpret $B \setminus A$ to mean $B \setminus (\textit{elements of } A \textit{ in } B) = B \setminus (A \cap B)$. For example, $\{1,3,5,7\} \setminus \{1,5,6\} = \{3,7\}$. We simply remove any elements of B that are elements of A.

2.2.3 Proof Technique: Double-Inclusion

There is a simple way to show that two sets are equal (if in fact they are), and it has a special name because it is used so frequently. You may deduce that name from the title of this section. To show that $A = B$, show first that $A \subset B$ and then show that $B \subset A$. This means that A is included in B and B is included in A and thus arises the term *double-inclusion*.

Of course, it might be useful to understand how to show that $A \subset B$ (or $B \subset A$) in order to execute a double-inclusion proof. A technical way to think about $A \subset B$ is with the statement *if $a \in A$, then $a \in B$*. So a formal inclusion proof proceeds as follows:

- Let a be any element of A.
- (Reasoning, statements.)
- Therefore, $a \in B$, and so $A \subset B$.

Example 2.2.9. Two different expressions can describe the same set. Let us show that two descriptions of the set of even numbers are equivalent. To that end, let $E_1 = \{k \in \mathbb{Z} \mid k = 2\ell \text{ for some } \ell \in \mathbb{Z}\}$ and let $E_2 = \{2r+6 \mid r \in \mathbb{Z}\}$. First, we will show that $E_1 \subset E_2$. Let e be any element of E_1. Then $e = 2\ell$ for some $\ell \in \mathbb{Z}$. If we let $r = \ell - 3$, then $e = 2\ell = 2(r+3) = 2r+6$, where $r \in \mathbb{Z}$, and therefore $e \in E_2$. Now, we will show that $E_2 \subset E_1$. Let t be any element of E_2. Then $t = 2r+6$, where $r \in \mathbb{Z}$. Setting $\ell = r+3$, we have that $t = 2r+6 = 2(r+3) = 2\ell$, where $\ell \in \mathbb{Z}$, and therefore $t \in E_1$. Because $E_1 \subset E_2$ and $E_2 \subset E_1$, we conclude that $E_1 = E_2$.

2.2.4 Making New Sets from Old

The most common operations on sets are the three defined here.

Definition 2.2.10. The *union* of sets A and B is a set $A \cup B$ containing all the elements in A and all the elements in B (with any duplicates removed). Similarly, the union of sets $A_1, A_2, \ldots, A_n$ is $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$ and contains all elements in the A_i (with any duplicates removed). Dealing with infinitely many sets is a little bit trickier and depends on how many there are (see Chapter 15 for more on this), but for now we'll say that $\bigcup_{i=1}^{\infty} A_i$ and $\bigcup_{i \in \mathbb{N}} A_i$ are the same.

Example 2.2.11. Let $A = \{egg, duck, 3, 4\}$ and let $B = \{duck, goose, 7, 8\}$. Then $A \cup B = \{egg, duck, goose, 3, 4, 7, 8\}$.

Let $A_i = \{i\}$. Then $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Definition 2.2.12. The *intersection* of sets A and B is a set $A \cap B$ containing every element that is in both A and B. Similarly, the intersection of sets $A_1, A_2, \ldots, A_n$ is $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$ and contains only elements that are in all of the A_i. We may sometimes take infinite intersections as in $\bigcap_{i=1}^{\infty} A_i$ and $\bigcap_{i \in \mathbb{N}} A_i$.

Example 2.2.13. With A and B and A_i defined as in Example 2.2.11, $A \cap B = \{duck\}$ and $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

Two sets A and B are called *disjoint* if $A \cap B = \emptyset$. We now have enough notation to give a super-formal way of restating the sum principle.

Theorem 2.2.14. *If $A_1, \ldots, A_n$ are disjoint finite sets, then $|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$.*

That is perhaps the most boring way to state the sum principle (can you think of a more boring way?), so we will not generally use it. It is, however, worth noting that almost every mathematical statement can be rewritten to use formal set language; and, it is also worth noting that this often borifies a given statement. (**Definition**: *bor · i · fy*, to intensify the level of boringness something has.) At the same time, we informally use set theory in our daily lives; for example, red-headed women are the intersection of the set of redheads and the set of women. Most of the time, we don't even notice that we're using set theory, but if you listen to conversations and look in the media, it's all over the place (albeit implicitly).

Definition 2.2.15. The *Cartesian product* of sets A and B is a set $A \times B$ containing all possible ordered pairs where the first component is an element of A and the second component is an element of B. In other words, $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$. Likewise, the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$.

Example 2.2.16 (of Cartesian products). The set $\{duck, goose\} \times \{egg\} = \{(duck, egg), (goose, egg)\}$. When the empty set is involved, there's a trick; $\{5,7,9,11\} \times \emptyset = \emptyset$ because there are no possible ordered pairs with the second component from the empty set. Binary strings of length two are formally $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{0,1\} \times \{0,1\} = \{(0,0),(1,0),(0,1),(1,1)\}$. This is sometimes abbreviated as $(\mathbb{Z}_2)^2$. Likewise, binary strings of length n are formally $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 = (\mathbb{Z}_2)^n$.

2.2.5 Looking at Sets

The most common example of a Cartesian product is that the real plane $\mathbb{R}^2$ is secretly $\mathbb{R} \times \mathbb{R}$, as shown in Figure 2.2. ($\mathbb{R}$ is shorthand for the real numbers.)

Figure 2.3 shows two other examples of Cartesian products.

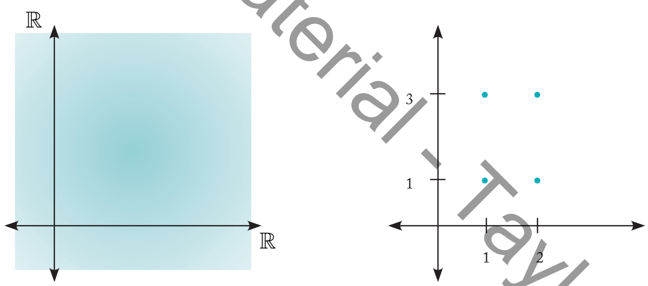


Figure 2.2. At left, $\mathbb{R}^2$; at right, $\{1,2\} \times \{1,3\}$.

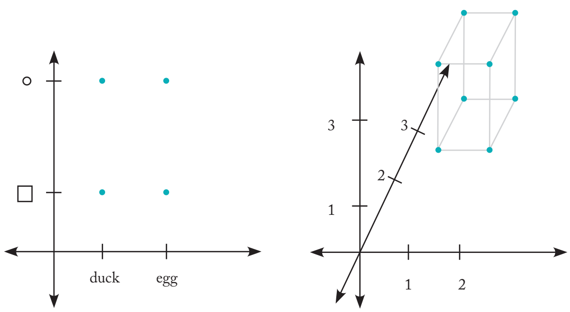


Figure 2.3. At left, $\{duck, egg\} \times \{\square, \circ\}$. At right, $\{1,2\} \times \{1,3\} \times \{2,3\}$. Although the set looks as though it is misplaced, it is not. (Grey lines are added to help locate the points in space but are not part of the set.)

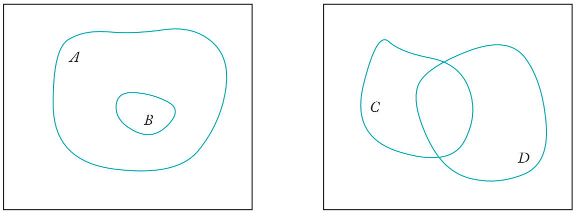


Figure 2.4. Each Venn diagram shows the relationship between two sets. Note that $B \subset A$ but no subset relationship exists between C and D.

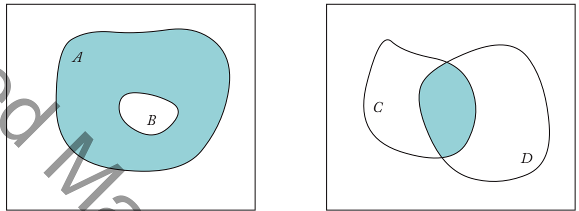


Figure 2.5. At left, $A \setminus B$, the part of A that does not include B, is shaded; at right, $C \cap D$, the overlap between C and D, is shaded.

We need ways of visualizing larger and more abstract sets. The usual method is called a *Venn diagram*, in which we draw a big box to denote the universe and then blobs to represent sets. Here are a couple of examples, shown in Figure 2.4. Those are pretty boring because they simply show two sets each. The information provided by the Venn diagrams is what kind of subset relationship (if any) exists between the two sets. Let's indicate some new sets that are derived from the old sets—in Figure 2.5 we shade the results of performing set operations on our old sets. This process extends to some fancy shaded diagrams when we have three sets and multiple set operations, as in Figure 2.6. In Figures 2.7–2.9, we show how to find these same sets using hatching.

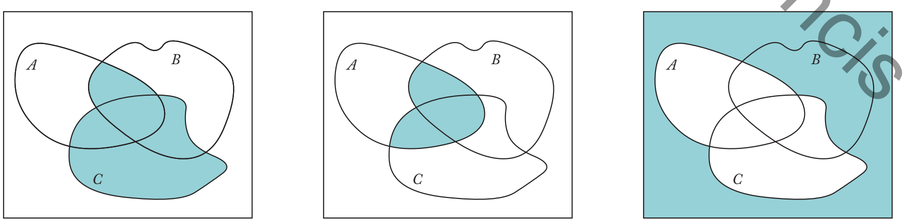


Figure 2.6. From left to right, $(A \cap B) \cup C$, $A \cap (B \cup C)$, and $\overline{A \cup C}$.

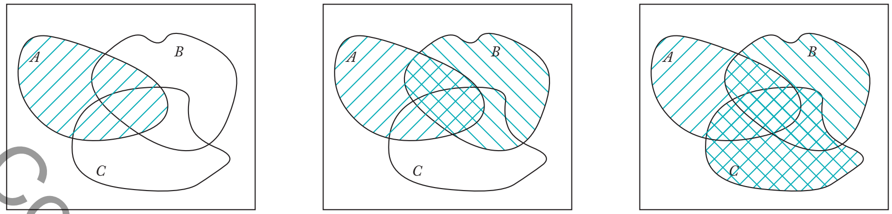


Figure 2.7. From left to right, A, $A \cap B$, and $(A \cap B) \cup C$.

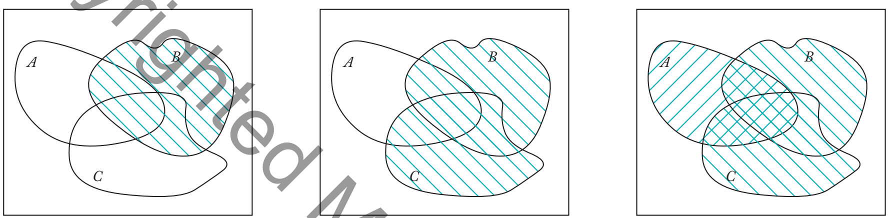


Figure 2.8. From left to right, B, $B \cup C$, and $A \cap (B \cup C)$.

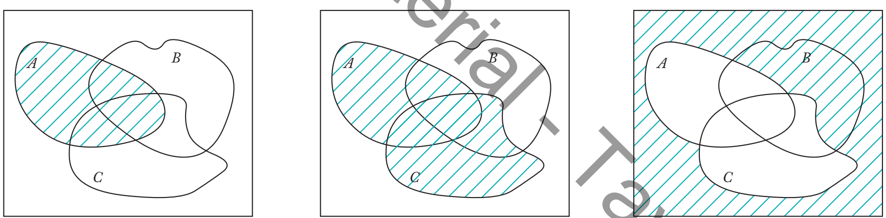


Figure 2.9. From left to right, A, $A \cup C$, and $\overline{A \cup C}$.

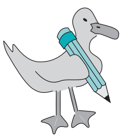
To exhibit $(A \cap B) \cup C$, we look within the parentheses. We start at left in Figure 2.7 by hatching A. Because we want $A \cap B$, we use a different hatching for B so that $A \cap B$ is crosshatched. Then, to demonstrate the union with C, we crosshatch C to match.

To exhibit $A \cap (B \cup C)$ in Figure 2.8, we again look within the parentheses. We start by hatching B. Because we want $B \cup C$, we use the same hatching on C as on B. In contrast, we want to intersect this set with A, so we use a different hatching on A so that the intersection is crosshatched.

The first step in showing $\overline{A \cup C}$ is to hatch A at left in Figure 2.9. To show $A \cup C$, we use the same hatching on C as on A. Finally, what we want to exhibit is $\overline{A \cup C}$, so we apply hatching on the remainder of the diagram and erase the previously applied hatching.

If you would like to practice using Venn diagrams, here are three online resources that you will likely find helpful.

- http://demonstrations.wolfram.com/InteractiveVennDiagrams/: This software lets you click on parts of a two- or three-set Venn diagram to shade them, and then it shows the set notation for the corresponding set and its complement.
- http://randomservices.org/random/apps/VennGame.html: This applet has you click on a set-notation description and then shades the corresponding regions of a Venn diagram.
- http://math.uww.edu/~mcfarlat/143venn.htm: This "quiz" applet has 15 different symbolic descriptions of sets. You have to figure out which regions on the corresponding Venn diagrams should be shaded, and mousing over a nearby diagram will show the correct shading.



Check Yourself

There may seem to be a lot of these problems, but each one is quick to do.

1. List the elements of $\{z \in \mathbb{Z} \mid -10 \leq z < 10\}$.
2. Write the set $\{2,4,6,8,10\}$ as a set of elements subject to a condition.
3. What is the cardinality of the set $\{duck, \emptyset, \{duck, egg\}, \{duck, \{duck, egg, \emptyset\}\}\}$?
4. Is $\{3,6,13,67\} \subset \{67,4,53,5,13,6\}$?
5. List the elements of $\mathscr{P}(\{-1,5,20\})$.
6. Let $A = \{5,6,7,8,9,23\}$, $B = \{6,7,9,456,3.142\}$, and $C = \{7,4,8,2.3,\pi,6\}$. List the elements of …
   (a) … $A \cup B$.
   (b) … $B \cap C$.
   (c) … $A \setminus C$.
7. Let $D = \{6.53, 42, 1, hat\}$ and $F = \{0, -2\}$. List the elements of …
   (a) … $D \times F$.
   (b) … $F \times D$.
   (c) … $D \times D$.
   (d) … $\emptyset \times F$.

8. Draw a visual representation of the set $\{1,2,3\} \times \{4,5\}$.
9. Make a Venn diagram that represents $\{1,2,3,4,5,6\} \cap \{4,5,6,7,8,9\}$.
10. **Challenge:**
    (a) Invent three sets of your own.
    (b) Find a different way to write each of the sets (for example, list the elements, or describe what the elements have in common using set notation).
    (c) Make a Venn diagram showing the relationships between your three sets.

2.3 Logic

Regular old English communication is not very precise, and many sentences have more than one interpretation. The reason logical notation and language have developed is so that there can be no question as to what a statement is intended to convey. The word "logic" is used to refer to an area of mathematics as well as a type of thinking. In all of mathematics, we use logical thinking, and we use the notation and language of the area of mathematics known as logic when less formal communication does not serve us well.

The basic component of logical language is the *statement*, which is a sentence that is either true or false. (To say that in a snooty way, a statement has a truth value from the set $\{true, false\}$.) Here is a non-statement: "Be a blue-footed booby." That sentence is an imperative; likewise, questions are not statements. Similarly, "$\{-3,0,2\} \setminus \{0,1\}$" is not a statement because it lacks a verb; it is only an expression.

Example 2.3.1 (of statements). Here are a few statements.

- The December 2009 issue of *Mathematics Magazine* has 78 pages.
- $32-6=16$.
- $\{1,5,7\} \cap \{1,2,8\} = \{2\}$.
- There is a one-to-one correspondence between four-digit binary strings and the corners of a four-dimensional cube.

In logic, we don't care about whether a statement is true or whether it is false. (Reread Example 2.3.1 with this in mind!) Our intent will be to examine the relationships between statements when they are combined in certain ways. We care about the roles that statements play rather than their validity or truth value. Thus, logical language omits the details of statements by referring to them with variables (usually P or Q or R), so that one can stick *any* statements into the templates that result. This simultaneously makes logical language useful and more difficult to read.

2.3.1 Combining Statements

There are just a few constructions used in logic to combine statements, called *connectives*. They are as follows:

- *and* is the verbal analogue to set intersection, so P-and-Q is only true if both P and Q are true;
- *or* is the verbal analogue to set union, so P-or-Q is true whenever either P or Q is true;
- *not* makes a true statement false and makes a false statement true; it gives a statement its opposite meaning;
- *implies* means that one statement is a consequence of the other; it is also written as *if-then* and is called a *conditional* statement.

Example 2.3.2 (of a very compound statement). Consider the statement *if* $x \in \mathbb{Z}$ *and* $x < 2.7$ *then* x *is negative or* $x \in \{0,1,2\}$. The implication combines the substatements $x \in \mathbb{Z}$ *and* $x < 2.7$ and x *is negative or* $x \in \{0,1,2\}$. Each of those has two substatements of its own; the *and* has substatements $x \in \mathbb{Z}$ and $x < 2.7$, and the *or* has sub-statements x *is negative* and $x \in \{0,1,2\}$. Then, note that the statement under consideration is true. (If we changed $x \in \{0,1,2\}$ to $x \in \{0,1\}$, then it would be false.)

Example 2.3.3 (of ambiguity without parentheses). Consider the statement $x \in \mathbb{Z}$ *and* $x < 3.6$ *or* $x > 628.3$. Does it mean ($x \in \mathbb{Z}$ *and* $x < 3.6$) *or* $x > 628.3$, or does it mean $x \in \mathbb{Z}$ *and* ($x < 3.6$ *or* $x > 628.3$)? The number $x = 1{,}002.7$ is described by the first statement but not the second statement. The number $x = -23$ is described by both statements. When we combine statements, we must be careful that the resulting statements are unambiguous, and so we must use enough parentheses.

Now we will be completely precise: we will define each of the connective terms using a *truth table*. As the name indicates, a truth table is a table that lists the truth values of a statement. Here is a silly and useless truth table:

P	P
T	T
F	F

This can be read aloud as *when P is true, P is true; when P is false, P is false.* See? It is indeed useless.

We will now define *and* (denoted $\wedge$), *or* (denoted $\vee$), and *not* (denoted $\neg$) using serious and useful truth tables.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	$\neg P$
T	F
F	T

P	Q	P xor Q
T	T	F
T	F	T
F	T	T
F	F	F

Looking at these truth tables, we can see that there is a difference between the usual English use of *or* and the formal logical use of *or*. After dinner, a host might ask, "Would you like coffee or tea?" (The answer "neither" corresponds to the line in the truth table where P and Q are both false.) The intent is to offer *either* coffee *or* tea, not both—regular English *or* is actually *exclusive or*, abbreviated xor. We have given a bonus truth table for xor above. Notice that the number of rows in a truth table depends on the number of statements involved. We need 2 rows for P, 4 for P, Q, 8 for P, Q, R, 16 for P, Q, R, S, and so forth, so that we can have all possible combinations of true and false.

Example 2.3.4. We will make a truth table for $(P \wedge Q) \vee R$.

P	Q	R	$P \wedge Q$	$(P \wedge Q) \vee R$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Sometimes we can ignore a few rows of a truth table: if we have particular statements corresponding to $P, Q, R, \ldots$, and we know that one of the statements is true (or, likewise, false), then we only need the rows of the truth table corresponding to that truth (or falsehood). Let us suppose that Q stands for the statement *the sun is plaid*. This is clearly false, so we could just write

P	Q	R	$P \wedge Q$	$(P \wedge Q) \vee R$
T	F	T	F	T
T	F	F	F	F
F	F	T	F	T
F	F	F	F	F

Translating between logic and set notations. There is a correspondence between set and logic notations, particularly when the logical statements are about sets. The elements for which the statement $P \wedge Q$ holds are those in the set $A = \{x \mid P \text{ is true for } x\}$ *and* the set $B = \{x \mid Q \text{ is true for } x\}$, and together those elements form the set $A \cap B$. Similarly, the elements for which the statement $P \vee Q$ holds are those in the set $A = \{x \mid P \text{ is true for } x\}$ *or* the set $B = \{x \mid Q \text{ is true for } x\}$, and together those elements form the set $A \cup B$. In this sense, $\wedge$ (or *and*) for statements corresponds to $\cap$ for sets, and $\vee$ (or *or*) for statements corresponds to $\cup$ for sets. The analogy for the connective *not* is a bit subtler; elements for which $\neg P$ holds are those not in the set $A = \{x \mid P$ is true for $x\}$, but then where *are* they? For this to make sense, we must make reference to a universe set U so that the elements not in A are those in $\overline{A}$, the complement of A relative to U.

Example 2.3.5 (of combining set and logic notations). We can describe the set $A_1 \cap (A_2 \cup A_3)$ as $\{x \mid x \in A_1 \cap (A_2 \cup A_3)\}$. Via a set of equivalences, we can turn it into another set:

$$\begin{aligned}
\{x \mid x \in A_1 \cap (A_2 \cup A_3)\} &= \{x \mid x \in A_1 \text{ and } x \in (A_2 \cup A_3)\} \\
&= \{x \mid x \in A_1 \text{ and } x \in (A_2 \text{ or } A_3)\} \\
&= \{x \mid (x \in A_1 \text{ and } x \in A_2) \text{ or } (x \in A_1 \text{ and } x \in A_3)\} \\
&= \{x \mid (x \in A_1 \cap A_2) \text{ or } (x \in A_1 \cap A_3)\} \\
&= \{x \mid (x \in A_1 \cap A_2) \cup (x \in A_1 \cap A_3)\} \\
&= (A_1 \cap A_2) \cup (A_1 \cap A_3).
\end{aligned}$$

Cool!

Next is *implies* (denoted by $\Rightarrow$). We read $P \Rightarrow Q$ as "P implies Q" or as "If P, then Q." Implication can be seen from different perspectives; when we are writing a proof, $P \Rightarrow Q$ needs justification, and we consider P and Q as separate statements, with $\Rightarrow$ standing in for the chain of argumentation that forms the bulk of a proof. In a logical context, $P \Rightarrow Q$ is a single statement that has truth values defined by the following truth table.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

This might seem a little weird. Or, more precisely, the last two lines of the table might seem a little bit weird. How can $P \Rightarrow Q$ be true if P is false? Consider a practical and pleasant example, namely, the statement *if you go to the party, then you will get some candy.* If you don't go to the party, you don't expect to get any candy, but you might get some anyway from some other source. But it's still true that if you *did* go, you'd get candy, so even though you don't go to the party, the implication still holds; the promise made to you is true.

There are many equivalent ways of writing implication, which is lovely but sometimes confusing. The statement $P \Rightarrow Q$ is usually read as *P implies Q* or as *if P then Q* but can also be read as *P only if Q* and *P is sufficient for Q to hold*. On the other hand, $Q \Rightarrow P$ can also be read as *P if Q* (see, if Q then …) and *P is necessary for Q*. Let's look again at the statement *if you go to the party, then you will get some candy.* Here, P is *you go to the party* and Q is *you will get some candy*. We could restate the statement as *going to the party is sufficient for getting some candy*, or as *you go to the party only if you get some candy*, or also as *getting some candy is necessary when you go to the party*.

Now, check this out: we can combine truth tables. (Note that arrows do work the way they should, so $P \Leftarrow Q$ means "If Q, then P.")

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \Leftarrow Q$
T	T	T
T	F	T
F	T	F
F	F	T

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$	$(P \Leftarrow Q) \wedge (P \Rightarrow Q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

It may not surprise you to learn that we abbreviate $(P \Leftarrow Q) \wedge (P \Rightarrow Q)$ as $P \Leftrightarrow Q$ and read that as "P if and only if Q." This is a fairly common kind of mathematical statement, used to show that two statements P and Q are logically equivalent. (More generally, any time two compound statements have the same truth tables, they are considered logically equivalent.) Some people are irritated by having to write out the words "if and only if" and abbreviate the phrase to *iff*. This statement type is called a *biconditional*. Additionally, even though we're not talking about proofs at the moment, it's worth pointing out that if you want to prove a biconditional statement you almost always have to split it into the two implications and prove them separately. (It's possible to string together a bunch of biconditionals, but that's hard. Don't bother.) We often write $(\Rightarrow)$ to indicate we'll prove that P implies Q and then write $(\Leftarrow)$ to indicate we'll prove that Q implies P ... and we start a new paragraph for each.

Advice. If you're new to the mathematical uses of *and*, *or*, *not*, and *implies*, then you might want to carry their truth tables around with you for a while until you internalize them.

Logic is related to our goal of learning proof crafting because there we need to produce rigorous and airtight reasoning. Well, using logical language certainly does that! When we aren't sure whether we're being rigorous enough, logic is here for us to fall back on. However, we don't want to resort to formal logic too often because it kills ease of communication. Plus, logical language is devoid of context—it doesn't care whether a given statement is true or false, but we do. And we want to convince others of that truth or falsehood.

On the other hand, logical notation is used in writing computer code, especially in creating conditionals (that's code-speak for if-then statements). For example, `If[ (a==b || a==0) && c < 5, c, 0]` says *if $a = b$ or $a = 0$, and if c is less than* 5*, then return the value of c; otherwise, return* 0. It may seem like the major use of logic for computer scientists is knowing the notation so that code can be written, but it is important to understand logical equivalence so that code can be refined for speed increases. Hardware designers use circuitry that corresponds to logical connectives, so minimizing their number can have positive consequences for power consumption and manufacturing cost.

2.3.2 Restriction of Variables via Quantifiers

One can make—and in fact we have already made—statements that include variables, such as *k is even* or $x^2 - 3 = 1$. In these cases, whether or not the statement is true depends on what value the variable (here, k or x) has. The statement *k is* *even* is true only when k is even (duh) and $x^2 - 3 = 1$ is true only when $x = 2$ or $x = -2$ (slightly less duh). Notice that statements always have verbs in them (is this a "duh"?) so they differ from functions like k or $x^2 - 3$ that merely produce numbers. Sometimes people will refer to a variable-including statement as $P(k)$ instead of just P; we won't do that here because it confuses us and, therefore, potentially you as well.

The *quantifiers* "for all" (denoted $\forall$, which is sometimes colloquially referred to as "the upside-down A" by students who forget what it stands for) and "there exists" (denoted $\exists$, which is similarly sometimes colloquially referred to as "the backwards E") restrict the variables referred to in a statement. We can rewrite our two example variable statements using these quantifiers.

Example 2.3.6. The statement *for all even k, k is even* is certainly true, though *for all k, k is even* is false and *there exists k such that k is even* is true. Similarly, *for all x,* $x^2 - 3 = 1$ is false, whereas *there exists x such that* $x^2 - 3 = 1$ is true.

We can prove that last statement. Consider $x = 2$ and note that $2^2 - 3 = 1$, so there does exist an x such that $x^2 - 3 = 1$. This technique generalizes. Existence proofs can be done simply by giving an example: you've shown that the desired object exists! But this is the *only* time an example works as a proof.

Sometimes it would be more convenient if people used quantifiers in ordinary English. For example, in the common statement *every duck wants a cookie*, the speaker could mean that given any duck, it desires some cookie ($\forall\, d \in Ducks, \exists\, c \in Cookies$ such that d wants c), or the speaker could mean there exists a cookie that every duck wants ($\exists\, c \in Cookies$ such that $\forall\, d \in Ducks, d$ wants c). Notice that this exemplifies not only the vagueness of English but that placing quantifiers in different orders changes the meaning of a statement. So be careful!

Example 2.3.7. Consider the statement $\forall n \in 2\mathbb{Z}, \exists\, a, b \in \mathbb{Z}$ *such that* $a = 2k_1 + 1$, $b = 2k_2 + 1$, *and* $n = b - a$. This basically says that for every even integer, there exist two odd integers such that the even integer is the difference of the odd integers. This is a true statement; given any even integer n, the integer $a = n - 1$ will be odd, as will $b = n - 1 + n = 2n - 1$, and $b - a = 2n - 1 - (n - 1) = n$.

If we change the order of the quantifiers, we may obtain $\exists\, a, b \in \mathbb{Z}$ such that $a = 2k_1 + 1, b = 2k_2 + 1$, and such that $\forall n \in 2\mathbb{Z}, n = b - a$. This says that there exist two odd integers such that for every even integer, that even integer is the difference of the two odd integers. This is a false statement; no matter which two odd integers a, b are considered, they have a single difference $b - a$ that is even. Any other even integer, such as $b - a + 2$, cannot be the difference of a and b, so the statement does not hold for most even integers (let alone for *every* even integer).

2.3.3 Negation Interactions

Even professional mathematicians sometimes find negating statements to be somewhat challenging. To be safe, take the English-mathematics version of a statement, substitute quantifiers (but don't go to full logic-speak), and then use the rules we will see here.

Example 2.3.8. $\neg$(*All ducks like cookies*) is logically equivalent to *there exists a duck who does not like cookies*. Unsurprisingly then, $\neg$(*some duck likes cookies*) is logically equivalent to *all ducks dislike cookies*. More mathematically, $\neg$(*for all integers k, k* $= 2.5$) is equivalent to *there exists an integer k such that* $k \neq 2.5$.

Basically, if you have the statement $\neg(\forall$ stuff), that converts to $\exists\neg$(stuff), and if you have the statement $\neg(\exists$ stuff), that converts to $\forall\neg$(stuff). At least this reduces the problem of negating to a shorter statement, though (stuff) might have some more quantifiers hidden within it.

Example 2.3.9 (of wacky negations). Let's negate a couple of statements. Consider *for all ducks, there exists a cookie such that a tree weeps*. In logic notation, this becomes $\forall$ *ducks,* $\exists$ *a cookie such that a tree weeps*. Thus, the negation proceeds as $\neg$($\forall$ *ducks,* $\exists$ *a cookie such that a tree weeps*), which becomes $\exists$ *a duck,* $\neg$($\exists$ *a cookie such that a tree weeps*), and then $\exists$ *a duck, such that* $\forall$ *cookies* $\neg$(*a tree weeps*), ending with *there exists a duck such that for all cookies, no tree weeps*. Consider now *there exists an egg such that it cracks for all cooks*. Its negation is slightly simpler. We translate first to logical notation to achieve $\exists$ *an egg, such that it cracks* $\forall$ *cooks*. Its negation is $\neg$($\exists$ *an egg, such that it cracks* $\forall$ *cooks*), which becomes $\forall$ *eggs,* $\neg$($\forall$ *cooks it cracks*), then $\forall$ *eggs,* $\exists$ *cooks* $\neg$(*it cracks*) and finally $\forall$ *eggs,* $\exists$ *cooks it does not crack*. This doesn't make much grammatical sense, so we reword it to read *for any egg, there exists a cook who cannot crack it.*

Negation plays nicely with other connectives, as follows.

DeMorgan's laws (logic version). $(\neg P) \vee (\neg Q)$ is logically equivalent to $\neg(P \wedge Q)$, and $(\neg P) \wedge (\neg Q)$ is logically equivalent to $\neg(P \vee Q)$.

Example 2.3.10. *No ducks and no chickens* is the same as *no ducks or chickens*.

People often think DeMorgan's laws are pretty obvious, but we have stated them here for completeness (as well as because sometimes they are needed when statements P and Q are elaborate). We will investigate another form of DeMorgan's laws in Section 2.4.

Negation and implication. The statement $P \Rightarrow Q$ is logically equivalent to the statement $\neg Q \Rightarrow \neg P$.

Definition 2.3.11 (of implication relatives). We sometimes call $P \Rightarrow Q$ the *original statement* and always call $\neg Q \Rightarrow \neg P$ the *contrapositive* statement. Along these lines, $Q \Rightarrow P$ is the *converse* statement, and $\neg P \Rightarrow \neg Q$ is the *inverse* statement, and also the contrapositive of the converse statement. All four of these statements are known as *implications*.

Notice that an implication and its converse are usually not both true at the same time. For example, *if I am at the combination Pizza Hut and Taco Bell, then I am at the Pizza Hut* is always true, but *if I am at the Pizza Hut, then I am at the combination Pizza Hut and Taco Bell* is often false.

Should you wish to practice the use of logic notation, logical thinking, and truth tables, here are some resources.

- http://demonstrations.wolfram.com/PropositionalLogicPuzzleGenerator/: You are shown some polygons along with a list of statements in logic notation. (The logic notation is not quite the same as used in this book, but there is a help option that explains it.) Each statement is marked as true or false. The challenge is that the polygons are not labeled but referred to in the statements as A, B, C, etc., and you get to match the labels with the polygons.

- http://demonstrations.wolfram.com/LogicWithLetters/ and http://demonstrations.wolfram.com/2DLogicGameWithLetters/ and http://demonstrations.wolfram.com/LogicWithLogicians/: These puzzles do not use formal logic notation but give practice in logical thinking.

- http://www.cs.utexas.edu/~learnlogic/truthtables/: After typing in a logical statement, you are given a corresponding blank truth table to fill in—it just has headers and a few beginning columns. You can choose whether to have your work checked entry by entry, or when you're done filling in the table. Warning: this applet uses a single arrow for *implies* instead of the double arrow we use in this text.

Check Yourself

These problems take less time to do than they at first appear to take.

1. Let P represent the statement *Ximena is pretty*, Q represent *Ximena is quizzical*, and R represent *Ximena is a rugby player*. Write $(P \vee Q) \wedge R$ as an English sentence.
2. Write *Miyuki does not like kumquats, but ze likes pickles or daikon* in logic notation.
3. Rewrite *every cat drinks beer* as an implication.
4. **Challenge:** Come up with two examples of mathematical statements and two examples of mathematical non-statements.
5. Using truth tables, verify that the converse of a statement is not logically equivalent to the original statement. (Suggestion: make the columns P, Q, $P \Rightarrow Q$, and $Q \Rightarrow P$, and compare the last two columns.)
6. Write the contrapositive of the statement *if the maple tree is orange, then the scissors are closed.*
7. Using truth tables, verify that the statement *if I am at the combination Pizza Hut and Taco Bell, then I am at the Pizza Hut* is always true.
8. Negate the statement *there exists an even number n such that* $n < 10$.

2.4 Try This! Problems on Sets and Logic

These problems are intended to be discussed with peers. Some students find these problems quite challenging and others find them easy. Your eventual success in discrete mathematics is unlikely to be related to your feelings about this particular collection of problems.

1. What is the cardinality of $\{0, cat, \{dog\}, \{2.1, 6\}\}$? List all its subsets. (How many should there be?)
2. Formally negate the statement "You can fool all of the people all of the time."
3. List several elements of the set $E = \{x \in \mathbb{Z} \mid \frac{1}{2}x \in \mathbb{Z}\}$ and then give a simpler description of E.
4. Here are DeMorgan's laws, given in logic notation: $\neg(P \vee Q)$ is logically equivalent to $(\neg P) \wedge (\neg Q)$ and $\neg(P \wedge Q)$ is logically equivalent to $(\neg P) \vee (\neg Q)$.

(a) Express DeMorgan's laws using set notation.

(b) Prove DeMorgan's laws using truth tables.

(c) Prove DeMorgan's laws using Venn diagrams.

(d) Prove DeMorgan's laws using set-element notation. (Suggestion: use double-inclusion.)

(e) Can you state DeMorgan's laws for three or more sets?

(f) Does that give you any ideas for stating, using logic notation, DeMorgan's laws for three or more statements?

5. Let A be the set of even numbers from -6 to 6 (inclusive), and let B be the set of odd numbers from -6 to 6 (inclusive), living in the universe of integers from -10 to 10 (inclusive).

(a) List the elements of $\overline{B}$.

(b) What is $\overline{A \cup B}$?

(c) Describe $A \setminus B$ using fewer symbols.

6. Is $\neg(P \Rightarrow Q)$ logically equivalent to $P \wedge \neg Q$?

7. Let $A_k = \{0, 1, \ldots, k\}$. What is $\bigcup_{i=1}^{n} A_i$? How about $\bigcap_{i=0}^{n} A_i$?

8. Draw a Venn diagram representing $(A \cap B) \cap (A \cup C)$.

9. Is it true that $\exists m \in \mathbb{Z} \mid \forall n \in \mathbb{Z}, m = n + 5$?

2.5 Proof Techniques: Not!

After all that boring reading, you probably are sighing at the thought of dealing with more material in this chapter. But fear not! This is shorter and more interesting (really!).

We already know how to do a straightforward proof, by directly proving an implication $P \Rightarrow Q$: we assume P is true and then deduce that Q is therefore true. We already know one way to disprove $P \Rightarrow Q$: find a counterexample. Now we will use a single fact from logic to burst wide open the clouds surrounding proof and shine glowing rays of truth on the situation.

Remember from Section 2.3.3 that the contrapositive of a statement is logically equivalent to the statement itself. That means we could prove $(\neg Q) \Rightarrow (\neg P)$

instead! This is but a tiny step removed from doing a direct proof: here we assume $\neg Q$ and deduce that $\neg P$ is therefore true. In fact, you can use the template from Section 1.4 (page 12) by simply inserting $\neg Q$ for P and inserting $\neg P$ for Q.

Example 2.5.1. Let $n, m \in \mathbb{N}$. We will prove that if $n \cdot m$ is odd, then an $n \times m$ grid cannot be tiled with dominoes. (A grid is *tiled* if every square is covered exactly once.) The contrapositive of this statement is *if an $n \times m$ grid can be tiled with dominoes, then $n \cdot m$ is not odd.* So, suppose an $n \times m$ grid can be tiled with dominoes. There are a total of $n \cdot m$ squares, and every domino covers two squares. Therefore, the tiling uses $\frac{n \cdot m}{2}$ dominoes, and so $n \cdot m$ must be even. Therefore, $n \cdot m$ is not odd.

There is a related technique we can use—it is called *proof by contradiction* and it proceeds by assuming the statement we want to prove is false and obtaining a logical problem of some kind. For an oversimplified example, if we want to prove that $P \Rightarrow Q$, we would assume P is true and Q is false, and if we can show that Q false implies P false, then this contradicts our assumption that P was true. (Read *that* aloud three times...) You may astutely notice that this is actually proving the contrapositive. In this case, we might start by drafting a proof by contradiction, continue by discovering that we've proven the contrapositive, and write the clean version of the proof as a contrapositive proof.

More commonly when using proof by contradiction, the P in $P \Rightarrow Q$ is a compound statement containing several conditions (e.g., *if k is an integer, ℓ is even, and the moon is green*), and we will only contradict one part of P rather than proving the negation of P as a whole (e.g., showing that the moon is not green and thus deriving a contradiction).

Less common but still useful is assuming Q is false and deriving a contradiction unrelated to the statements under consideration—for example, showing that Q is false implies that 2 is an odd number.

Template for a proof by contradiction:

1. Restate the theorem in the form *if (conditions) are true, then (conclusion) is true*.

2. On a scratch sheet, write *suppose not*. Then write out (conditions) and the negation of (conclusion).

3. Try to simplify the statement of ¬(conclusion) and see what this might mean.

4. Attempt to derive a contradiction of some kind—to one or more of (conditions) or to a commonly known mathematical truth.

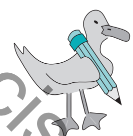
5. Repeat attempts until you are successful.

6. Write up the results on a clean sheet, as follows.

   - Theorem: (State theorem here.)
   - Proof: Suppose not. That is, suppose (conditions) are true but (conclusion) is false.
   - (Translate this to a simpler statement if applicable. Derive a contradiction.)
   - Contradiction!
   - Therefore, (conclusion) is true. (Draw a box or checkmark or write Q.E.D. to indicate that you're done.)

Example 2.5.2. We will prove that there are infinitely many powers of 2, i.e., $2^0, 2^1, 2^2, \ldots$. Suppose not. Then there are finitely many powers of 2; let the number of them be n. Therefore, we can sort them in increasing order of size. Consider the largest of these, k. Then 2^k is not one of the n powers of 2; it is larger than any of them because $2^k > k$. Therefore, there are at least $n+1$ powers of 2, which contradicts the supposition that there were only n of them.

Contradiction can also be used to disprove false statements. In this case, assume the statement is true and derive a contradiction.

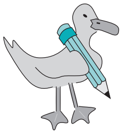


Check Yourself

1. Prove that if n^2 is odd, then n is odd. (Suggestion: try proving the contrapositive.)

2. Prove that if there are ten ducks paddling in four ponds, then some pond must contain at least three paddling ducks. (Suggestion: try contradiction.)

3. **Challenge:** Develop your own statement that can be proved by contradiction.



2.6 Try This! A Tricky Conundrum

Consider the following argument: *You must learn about sets or learn about logic if you go on to the next chapter. You did not learn about sets and did not go on to the next chapter. Therefore, you must not have learned about logic.*

1. Decide for yourself whether or not the conclusion is correct (that you must not have learned about logic). Make a note of this decision.

2. In a small group, exchange your decisions and share your reasoning (justify your decisions). Please collaborate from here on out.

3. Let's check our logic formally.

   (a) Dissect the first sentence and find three statements within it that you can label with letters.

   (b) Turn the first sentence into an expression using formal logic symbols.

   (c) Express the second and third sentences in formal logic symbols, too.

   (d) Make a (big) truth table that includes parts for each of the sentences and for the argument as a whole.

4. Compare the result of this truth table to your original idea. If they agree, explain how they are compatible. If they do not agree, find the source of the error.

5. If you have some time left over, work on these proofs.

   (a) For $n \in \mathbb{N}$, prove that if $n^3 + 6n^2 - 2n$ is even, then n is even.

   (b) Let $x \in \mathbb{R}$. Show that if $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$, then $x \geq 0$.

   (c) Prove that an 8×8 chessboard with a square missing cannot be tiled with dominoes.

   (d) Prove that for n odd, an $n \times n$ chessboard missing its lower-right-hand corner can be tiled with dominoes.

2.7 Additional Examples

Example 2.7.1 (of manipulating set notation). Let $S_1 = \{q+1 \in \mathbb{Z} \mid q = 2k \text{ for some } k \in \mathbb{Z}\}$, and let $S_2 = \{2r+5 \mid r \in \mathbb{Z}\}$; we want to show that $S_1 = S_2$. First, we will show that $S_1 \subset S_2$. Let s be any element of S_1. Then $s = 2k+1$ for some

$k \in \mathbb{Z}$. If we let $r = k - 2$, then $s = 2k + 1 = 2(r+2) + 1 = 2r + 5$, where $r \in \mathbb{Z}$, and therefore $s \in S_2$. Now, we will show that $S_2 \subset S_1$. Let t be any element of S_2. Then $t = 2r + 5$, where $r \in \mathbb{Z}$. Setting $k = r + 2$, we have that $t = 2r + 5 = 2(k-2) + 5 = 2k + 1$, where $k \in \mathbb{Z}$, and therefore $t \in S_1$. Because $S_1 \subset S_2$ and $S_2 \subset S_1$, we conclude that $S_1 = S_2$.

Example 2.7.2 (of Venn diagrams). We will exhibit $(A \cap \overline{B}) \cup (\overline{A} \cap B)$ using Venn diagrams.

We begin by looking within the parentheses. The first set of parentheses contains $A \cap \overline{B}$. We start at left in Figure 2.10 by hatching A. Because we want $A \cap \overline{B}$, we use a different hatching for $\overline{B}$ and then combine these so that $A \cap \overline{B}$ is crosshatched.

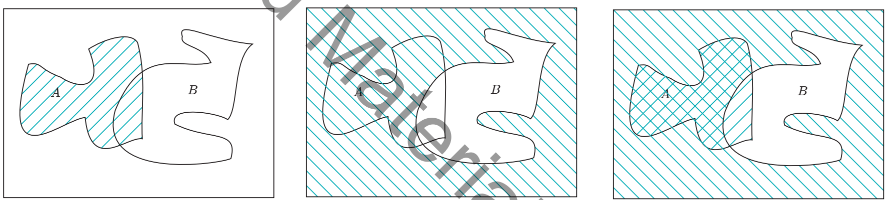


Figure 2.10. At left, A; in the middle, $\overline{B}$; at right, $A \cap \overline{B}$.

The second set of parentheses contains $\overline{A} \cap B$. We start at left in Figure 2.11 by hatching $\overline{A}$. Because we want $\overline{A} \cap B$, we use a different hatching for B and then combine these so that $\overline{A} \cap B$ is crosshatched.

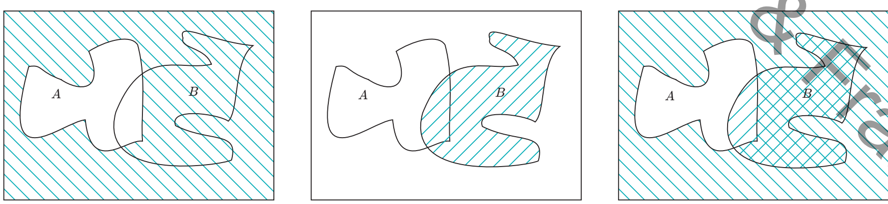


Figure 2.11. At left, $\overline{A}$; in the middle, B; at right, $\overline{A} \cap B$.

Finally, we combine these sets. We start at left in Figure 2.12 by showing $A \cap \overline{B}$, and in the middle we show $\overline{A} \cap B$. Because we want $(A \cap \overline{B}) \cup (\overline{A} \cap B)$, we display both at once using the same type of hatching.

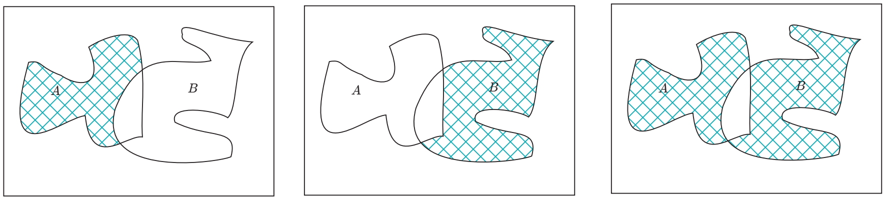


Figure 2.12. At left, $A \cap \overline{B}$; in the middle, $\overline{A} \cap B$; at right, $(A \cap \overline{B}) \cup (\overline{A} \cap B)$.

Example 2.7.3 (of breaking down a very compound statement). Consider the statement *if* $x \in \mathbb{Z}$ *and* $x > -7.2$ *then* x *is positive or* $x \in \{0, -1, -2, -3, -4, -5, -6, -7\}$. The largest logical substructure is the if-then implication, which combines the substatements $\langle x \in \mathbb{Z}$ *and* $x > -7.2\rangle$ and $\langle x$ *is positive or* $x \in \{0, -1, -2, -3, -4, -5, -6, -7\}\rangle$. Each of those has two substatements of its own; the *and* has substatements $\langle x \in \mathbb{Z}\rangle$ and $\langle x > -7.2\rangle$, and the *or* has substatements $\langle x$ *is positive*$\rangle$ and $\langle x \in \{0, -1, -2, -3, -4, -5, -6, -7\}\rangle$.

Example 2.7.4 (of evaluating statements with truth tables). Here is an argument someone might make: *The jelly bean is blue. Blue things are tasty. Therefore, the jelly bean is tasty.* Is this argument correct? We will represent *jelly bean* as J, *blue* as B, and *tasty* as T. Then *the jelly bean is blue* is really *if it is a jelly bean, then it is blue* or $J \Rightarrow B$. We can similarly write the other statements as $B \Rightarrow T$ and $J \Rightarrow T$. Surely, if $J \Rightarrow B$ and $B \Rightarrow T$, then $J \Rightarrow T$, right? Let's see…

J	B	T	$J \Rightarrow B$	$B \Rightarrow T$	$(J \Rightarrow B) \wedge (B \Rightarrow T)$	$J \Rightarrow T$	$((J \Rightarrow B) \wedge (B \Rightarrow T)) \Rightarrow (J \Rightarrow T)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Yup, it's all true! Literally, all entries in the last column of the truth table are T—this means the implication, and therefore the argument, is correct.

Example 2.7.5 (of quantifier order mattering). Let $d, e \in \mathbb{Z}$. Consider the statement $\forall e, \exists d$ *such that* $d < e$. This true statement basically says that given an integer, we can find a smaller one. For example, given $e = -32$, we can find $d = -4{,}389$.

If we change the order of the quantifiers, our new statement is $\exists d, \forall e$ such that $d < e$. This statement says there is some integer such that every other integer is larger. That's not true!

(If you are (or have been) a student of calculus, compare this example to the formal (ε-δ) definition of limit.)

Example 2.7.6 (of wacky negations). Consider the statement *for all futons, there exists a duck such that stripes are in fashion*. In logic notation, this becomes ∀ *futons*, ∃ *a duck such that stripes are in fashion*. Thus, the negation proceeds as ⟨¬(∀ *futons*, ∃ *a duck such that stripes are in fashion*)⟩; ⟨∃ *a futon*, ¬(∃ *a duck such that stripes are in fashion*)⟩; ⟨∃ *a futon, such that* ∀ *ducks* ¬(*stripes are in fashion*)⟩; … and finally, ⟨*there exists a futon such that for all ducks, stripes are not in fashion*⟩.

2.8 Where to Go from Here

Commandment. Go back and reread the material on proof in Section 1.4. And (*grin*) reread Section 3 on how to read mathematics.

We will apply the concepts introduced in this chapter throughout the text, but logic will be particularly important in Chapter 5 when we study the construction of algorithms. The type of basic set theory introduced in this chapter is pervasive in and essential for all of mathematics and has a somewhat different flavor when used in courses based in continuous as opposed to discrete mathematics, such as real analysis and topology. If after working through the material in this chapter, you want to see more examples and have more elementary exercises to work, consult *Book of Proof* by Richard Hammack [12].

Venn diagrams are a source of much interesting investigation. If you try to draw a Venn diagram that represents four or more sets, you will quickly run into trouble showing all possible intersections. For a good survey of approaches to this problem, see http://www.combinatorics.org/Surveys/ds5/VennEJC.html, which also tells you more than you ever wanted to know about Venn diagrams—and includes a zillion references.

Set theory and logic are subfields of mathematics on their own, so there is a great deal to learn about each of these. (Sometimes they are lumped together as *foundations of mathematics*.) We will address a small bit of set theory in Chapter 15. You can take upper-level undergraduate courses on set theory and on logic; if you wish to self-study, *Sweet Reason: A Field Guide to Modern Logic* by Tom

Tymoczko and Jim Henle and *An Outline of Set Theory* by Jim Henle should be the resources you use first.

Within mathematics, set theory and logic are small subfields but are quite active. For example, the Association for Symbolic Logic sponsors sessions of research talks at national mathematics conferences. One famous result in the area is Gödel's incompleteness theorem, which basically says that in any logical system there are statements that cannot be proven to be true or shown to be false. Classical problems in foundations of math were often related to what set of axioms (assumptions or rules) is needed, or is best, for various statements to be true. Modern logic research involves making formal abstract models of other parts of mathematics in order to prove more powerful theorems.

Credit where credit is due: The first activity in Section 2.6 was adapted from an example in [8]; the first puzzle and the project in Section 2.10 were adapted from exercises in [1]. The example on page 45 references a song by Das Racist (find it on YouTube). Problem 12 in Section 2.12 includes a phrase from "Song for a Future Generation" by the B-52s. Four problems in the latter part of Section 2.12 were donated or inspired by Heather Ames Lewis.

2.9 Chapter 2 Definitions

set: A mathematical object that contains distinct unordered elements. There may be finitely many or infinitely many elements in a set.

element: Elements can be words, objects, numbers, or sets (i.e., basically anything).

empty set: The set with no elements. Also called the null set.

null set: The empty set.

cardinality: The number of elements in a set.

size: The cardinality of a set.

subset: A is a subset of B if every element of A is also an element of B.

proper subset: A is a proper subset of B there is at least one element in B that is not an element of A.

power set: The set of all subsets of A, denoted $\mathscr{P}(A)$.

set complement: If $A \subset B$, then $\overline{A} = B \setminus A$, all the elements of B that are not in A, is called the complement of A relative to B.

union: The union of sets A and B is a set $A \cup B$ containing all the elements in A and all the elements in B (with any duplicates removed). The union of many sets A_i contains all elements in the A_i (with any duplicates removed).

intersection: The intersection of sets A and B is a set $A \cap B$ containing every element that is in both A and B. The intersection of many sets A_i contains only elements that are in all of the A_i.

disjoint: Two sets A and B are called disjoint if $A \cap B = \emptyset$.

Cartesian product: The Cartesian product of sets A and B is a set $A \times B$ containing all possible ordered pairs where the first component is an element of A and the second component is an element of B. In other words, $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$. Likewise, the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$. The name *Cartesian* is derived from René Descartes (1596–1650).

Venn diagram: A picture in which a big box denotes the universe of things under consideration and blobs represent sets. Venn diagrams are used to show relationships between sets. Named after John Venn (1834–1923), who wrote influential works on logic and probability/statistics.

statement: A sentence that is either true or false; it is the basic component of logical language. (To say that in a snooty way, a statement has a truth value from the set $\{true, false\}$.)

connective: A logical construction used to combine statements.

truth table: A table that lists the truth values of a statement.

and: The verbal analogue to set intersection, so P-and-Q is only true if both P and Q are true; denoted $\wedge$. The corresponding truth table is shown in Figure 2.13.

or: The verbal analogue to set union, so P-or-Q is true whenever either P or Q is true; denoted $\vee$. The corresponding truth table is shown in Figure 2.13.

xor: "Exclusive or" means that one statement or the other is true, but not both. The corresponding truth table is shown in Figure 2.13.

not: This gives a statement its opposite meaning; denoted by $\neg$, it makes a true statement false and makes a false statement true. The corresponding truth table is shown in Figure 2.13.

implies: This means that one statement is a consequence of the other; denoted $\Rightarrow$. The corresponding truth table is shown in Figure 2.13.

if-then: A statement involving implication.

conditional: An if-then statement.

if and only if: "P if and only if Q" is denoted $P \Leftrightarrow Q$ and means that the statements P and Q are logically equivalent. The corresponding truth table is shown in Figure 2.13.

DeMorgan's laws: The logical rules for how *not* interacts with *or* and *and*. Named after Augustus DeMorgan (1806–1871).

iff: If and only if.

biconditional: An if-and-only-if statement.

quantifier: Quantifiers such as "for all" and "there exists" restrict the variables referred to in a statement.

implication: A statement of the form $P \Rightarrow Q$.

contrapositive: When $P \Rightarrow Q$ is the original statement, $\neg Q \Rightarrow \neg P$ is the contrapositive statement.

converse: When $P \Rightarrow Q$ is the original statement, $Q \Rightarrow P$ is the converse statement.

inverse statement: When $P \Rightarrow Q$ is the original statement, $\neg P \Rightarrow \neg Q$ is the inverse statement.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	P xor Q
T	T	F
T	F	T
F	T	T
F	F	F

P	$\neg P$
T	F
F	T

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 2.13. The truth tables for *and*, *or*, xor, *not*, *implies*, and *if and only if*.

2.10 Bonus: Truth Tellers

One application of logical thinking is the class of truth-teller puzzles. The basic format for these is that some statements are made, and each speaker either always tells the truth or always lies. Your assignment is to figure out what's going on (either who is telling the truth or what the truth of the matter is). Such puzzles can be unraveled using truth tables or simply by using logical reasoning. Here we will give a few examples of how to use truth tables to resolve these puzzles.

Suppose you meet some ducks. It is known that a given duck either always tells the truth or always lies. (This is theorized to be the origin of the common expression "Ducks usually lie." See [21].)

Example 2.10.1. One duck says, "I am a truth-telling duck." Another duck quacks, "I am a lying duck." Can we determine anything about either duck's nature? Let us make a truth table to investigate. Let D represent the duck; it gets the value T if it is a truth-telling duck and the value F if it is a lying duck. The statement "D tells the truth" is true exactly when D is a truth-telling duck; the statement "D lies" is true exactly when D is a lying duck.

D	D tells the truth	D lies
T	T	F
F	F	T

That's the unvarnished truth of the situation. But, of course, a lying duck lies (duh)... and our truth table doesn't take that into account. So we modify the truth table to reveal what each type of duck would say in each situation—we swap T and F for the lying duck:

D	D tells the truth	D lies
T	T	F
F	~~F~~ T	~~T~~ F

We can now see that either sort of duck would say that it tells the truth, so we can determine nothing about the first duck. We also see that neither sort of duck would say that it lies, so the second "duck" must not be a duck at all.

Example 2.10.2. A pair of ducks approaches. One quacks, "Exactly one of us is a liar." The other says, "Both of us tell the truth." Huh! What is going on? Let's look at a truth table.

D_1	D_2	D_1 xor D_2 lies	$D_1 \wedge D_2$ tell the truth
T	T	F	T
T	F	T	F
F	T	T	F
F	F	F	F

Again, we modify the table to account for what lying ducks say, and remember that D_1 made the statement in the third column, whereas D_2 made the statement in the fourth column:

D_1	D_2	D_1 xor D_2 lies	$D_1 \wedge D_2$ tell the truth
T	T	F	T
T	F	T	~~F~~ T
F	T	~~T~~ F	F
F	F	~~F~~ T	~~F~~ T

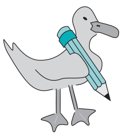
Interestingly, we can only conclude that D_2 is a liar—the statements are consistent whether D_1 is a truth teller or a liar!

Puzzle 1. Amy finds a present on hir doorstep. Ze suspects it was left by either Rachel, Tess, or Nicol. Ze confronts each one.

> *Rachel*: Not me! Tess knows you, and Nicol is your BFF.
>
> *Tess*: I don't know you, and besides, I've been on vacation in Europe for the last several weeks. I didn't leave you a present.
>
> *Nicol*: It wasn't me, but I did happen to see Tess and Rachel walking along the river together last week. It must have been one of them.

Let us assume that the present-giver is lying and the other two individuals are telling the truth. Who left Amy the present?

Puzzle 2. In *Math Curse* [22], the main character has a strange experience at dinner. “While passing the mashed potatoes, Mom says, ‘What your father says is false.’ Dad helps himself to some potatoes and says, ‘What your mother says is true.’ … Can that be true?” Figure out what is going on here… and if you have not already done so, read *Math Curse*. Your local public library surely has it in the picture-book section.



Project: You are walking about and see some tasty-looking berries. You also meet a duck, which, like any duck, always lies or always tells the truth. You may ask the duck exactly one question. Explain why you will not definitely learn whether the tasty-looking berries are safe to eat by asking any of the following questions:

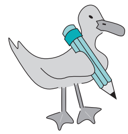
- Are these tasty-looking berries safe for a human to eat?
- Do you tell the truth?
- Do you tell the truth and are these tasty-looking berries safe for a human to eat?
- Do you tell the truth or are these tasty-looking berries safe for a human to eat?
- If you tell the truth, then are these tasty-looking berries safe for a human to eat?
- If these tasty-looking berries are safe for a human to eat, then do you tell the truth?
- Do you tell the truth if and only if you lie?

Design a single question to ask the unknown duck such that the answer will tell you whether the tasty-looking berries are safe to eat.

If you want to play with many, many, many more puzzles of this sort, consult a book by Raymond Smullyan. He has written lots of logic puzzle books—perhaps the first was *What Is the Name of This Book?*—and they are easy to find. If you prefer an electronic playground, here are a few sources of logic puzzles:

http://demonstrations.wolfram.com/KnightsKnavesAndNormalsPuzzleGenerator/,
http://demonstrations.wolfram.com/KnightsAndKnavesPuzzleGenerator/,
http://demonstrations.wolfram.com/AnotherKnightsAndKnavesPuzzleGenerator/.

All generate collections of statements. You decide which speakers are knights (who tell the truth) and which are knaves (who lie). The software has options to translate each statement into logic notation and to reveal the solution to each puzzle.



2.11 Bonus Check-Yourself Problems

Solutions to these problems appear starting on page 595. Those solutions that model a formal write-up (such as one might hand in for homework) are to Problems 7 and 9.

1. On an October 2014 visit to the CVS Minute Clinic, the check-in kiosk asked the question, "If you have a copay for today's visit, will you be paying for it with a credit or debit card?"
   (a) Identify the formal logic quantifiers and structure in this question.
   (b) The visit in question was for a flu vaccine, which does not require a copay. The kiosk gave options of *Yes* and *No*. How should the visitor have answered?
   (c) Can you find a simpler way to word the question clearly? (In other words, what *should* the kiosk question ask?)
2. There was a recent campaign slogan heard on the radio: *Not just Blue Cross Blue Shield of Massachusetts, but Blue Cross Blue Shield ... of you.* Why is this mathematically nonsensical for residents of Massachusetts?

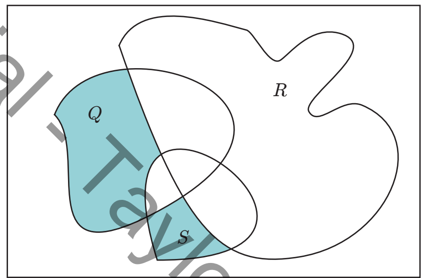


Figure 2.14. A Venn diagram of mystery.

3. Consider the Venn diagram in Figure 2.14.
   (a) Express the shaded area as a set using unions, intersections, and/or complements of the sets Q, R, and S.
   (b) Let $Q = \{k \in \mathbb{Z} \mid |k| \leq 10\}$, $R =$ even numbers, and $S = \{n \in \mathbb{N} \mid n \text{ is a perfect square}\}$. List the elements of the shaded area.

4. Let $A =$ multiples of 4, and $B =$ multiples of 6. Write $A \cap B$ as a set in the form $\{$ sets $|$ conditions $\}$.

5. Negate the statement $\forall\, n \in \mathbb{Z}, \exists\, y \in 2\mathbb{N}$ *such that* $n = y \cdot k$ *for some* $k \in \mathbb{Z}$. Is either the statement or its negation true?

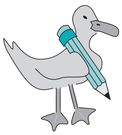
6. Prove that $k \in \mathbb{Z}$ is positive if and only if k^3 is positive.

7. Make a truth table for $\neg(P \wedge Q) \wedge ((P \vee Q) \wedge R)$. Can you express this statement (henceforth referred to as *aaaaaa!*) more simply?

8. Let $A = \{0,1,2\}$ and $B = \{1,3,5,7\}$.
   (a) List the elements of $(A \times B) \cap (B \times A)$.
   (b) List the elements of $(A \setminus B) \times (B \setminus A)$.

9. Show that $(A \times B) \cup (C \times B) = (A \cup C) \times B$.

10. Show that $\{2k \mid k \in \mathbb{N}\} \cup \{4k+1 \mid k \in \mathbb{W}\} \cup \{4k+3 \mid k \in \mathbb{W}\} = \mathbb{N}$.



2.12 Problems about Sets and Logic

1. List the elements of $\{n \in \mathbb{N} \mid n^2 = 4\}$.

2. An excerpt from a 2010 Blue Cross Blue Shield survey: "Do **not** include care you got when you stayed overnight in a hospital. Do **not** include the times you went for dental care visits … In the last 12 months, **not** counting the times you needed care right away, how often did you get an appointment for your health care at a doctor's office or clinic as soon as you thought you needed?" What type of needed care is the question asking about? What is excluded? Can you find a simpler way to word the question clearly?

3. Another excerpt from a 2010 Blue Cross Blue Shield survey: "In the last 12 months, how often did your doctor or health provider discuss or provide methods and strategies other than medication to assist you with quitting smoking or using tobacco?" Analyze the connectives in the question. Are any or all of them used in the same way we use them in mathematics?

4. Compute $|\{z \in \mathbb{Z} \mid z > -10, z^3 < 0\}|$.

5. Make a truth table for $P \wedge (\neg P \vee Q)$.

6. Write the set $\{1,2,4,8,\dots\}$ without using dots.

7. Use Venn diagrams to indicate the even numbers less than ten.

8. Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$. List the elements of …
   (a) … $(A \times A) \cap (B \times B)$.
   (b) … $(A \times B) \cup (B \times A)$.
   (c) … $A \times (A \setminus B)$.

9. Using truth tables, verify that the contrapositive and original statement are logically equivalent.

10. Again using truth tables, verify that the converse and inverse statements are logically equivalent.

11. Give a counterexample to the statement $|A \cup B| = |A| + |B|$.

12. Is the statement *if the moon is made of green cheese, then Aristotle is the President of Moscow* true or false?
13. Draw a Venn diagram that indicates $(A \cup B) \setminus C$.
14. Decide whether or not it is true that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$. If true, give a proof. If false, give a counterexample.
15. Show that if A and B are sets, then if $A \setminus B = \emptyset$, then $B \neq \emptyset$ (unless $A = \emptyset$).
16. Suppose R is false but that $(P \Rightarrow Q) \Leftrightarrow (R \wedge S)$ is true. Is P true or false? What about Q?
17. Could we rewrite the conditional `((c > 5 && b == a) || c >= 5)` in a simpler way? If so, what is it? (Suggestion: use a truth table.)
18. Write this in English: $\forall k \in 3\mathbb{Z}, \exists S \subseteq \mathbb{N}, |S| = k$. (Is it true?) What is the negation of this statement? (Is the negation true?)
19. Prove that $n \in \mathbb{N}$ is odd if and only if n^2 is odd.
20. Prove that $\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\} \cup \{3k+1 \mid k \in \mathbb{Z}\} \cup \{3k+2 \mid k \in \mathbb{Z}\}$.
21. Prove that there are infinitely many prime numbers. (Suggestion: try using contradiction.)
22. Show that $n \in \mathbb{N}$ is not divisible by 4 if and only if the binary representation of n ends in 1 or in 10. (Suggestion: use the contrapositive.)
23. Express $P \Rightarrow Q$ using $\neg$ and $\vee$ but not $\Rightarrow$. (Suggestion: play around with truth tables.)
24. Some of the pigeonhole principle proofs in Chapter 1 are secretly proofs by contradiction or proofs that use the contrapositive. Which ones?
25. On route I-91 near Springfield, MA, there was once a sign that said "WASH YOUR BOAT" (pause) "AFTER USE" (pause). Explain why you are complying with the sign if you do not own a boat. How does this relate to truth tables?
26. Compute the cardinality of the set …
    (a) … $\{wiggle, worm, wiggle\ worm\}$.
    (b) … $\{wiggle, \{wiggle\}, \{worm\}, worm\}$.
    (c) … $\{\{\{wiggle, worm\}\}\}$.
27. Let $A = \{(2,5), (-3,1), (4,2), (1,1), (0,1)\}$. List the elements in each of the following sets (or write $\emptyset$ if appropriate).
    (a) $\{(a_1, a_2) \in A \mid a_1 < a_2\}$.
    (b) $\{a_1 \mid (a_1, a_2) \in A \text{ and } a_1 > a_2\}$.
    (c) $\{a_2 \mid (a_1, a_2) \in A \text{ and } a_2 = 0\}$.
28. Let the universe be $U = \{x \in \mathbb{N} \mid x \leq 10\}$, and let $A = \{1,2,3,4,5\}$, $B = \{5,6,7\}$, and $C = \{1,6,9\}$. List the elements of …
    (a) … $\overline{A} \cup C$.
    (b) … $(B \setminus C) \setminus A$.
    (c) … $(A \cap B) \times C$.
29. Write the negation of *x is prime or $x < 52$*. (Don't say, "It's not true that ….")
30. Use a truth table to show that $((\neg p) \wedge q) \wedge (p \vee (\neg q))$ is a contradiction.
31. Write the negation of *for all integers x and y, the number $\frac{x-y}{5}$ is an integer.* (Don't say, "It's not true that ….")
32. Write each of the following statements using formal logic notation.
    (a) Even numbers are never prime.
    (b) Triangles never have four sides.
    (c) There are no integers a, b such that $a^2/b^2 = 2$.

(d) No square number immediately follows a prime number.

33. Write the contrapositive of *if* $x^2 > 100$*, then y has a sister.*

34. Carefully write out some of your results from Problem 4 of Section 2.4: State DeMorgan's laws for two sets using set notation, and prove them using set-element notation. Now state DeMorgan's laws for n sets.

35. Carefully write out more of your results from Problem 4 of Section 2.4: Prove DeMorgan's laws for two statements using Venn diagrams, being sure to include intermediate steps and complete sentences. Now state DeMorgan's laws for n statements.

36. Prove that if a natural number n is even, then $n-1$ is odd …

(a) … using a direct proof.

(b) … by proving the contrapositive.

(c) … using proof by contradiction.

37. Prove that x is even if and only if $4x^2-3x+1$ is odd.

38. **Challenge**: Try to rewrite *this* sentence as a logical statement!! (That is, write it as a collection of short statements joined by logical connectives and quantifiers.) Can you write a simplified version of the next statement? *The following two categories of charitable organizations are not required to have a "Certificate of Solicitation": An organization that is primarily religious in purpose and falls under the regulations 940 CMR 2.00; or An organization that does not raise or receive contributions from the public in excess of $5,000 during a calendar year or does not receive contributions from more than ten persons during a calendar year, if all of their functions, including fundraising activities, are performed by persons who are not paid for their services and if no part of their assets or income inures to the benefit of, or is paid to, any officer or members (M.G.L. c. 68, s. 20).* (Source: http://www.mass.gov/ago/doing-business-in-massachusetts/public-charities-or-not-for-profits/soliciting-funds/overview-of-solicitation.html)

39. Write the set $\{\dots,-8,-4,0,4,8,\dots\}$ without using dots.

40. Evaluate the statement $A \cap B = A \setminus B$. Is it true? If so, prove it. If not, find a counterexample and determine whether it is *always* false or whether there exist A, B for which the statement is true.

2.13 Instructor Notes

This chapter is written with the intent that students will read Sections 2.1, 2.2, and 2.3 and attempt the Check Yourself problems before the first class of the week. You may look at the amount of text/material in the chapter and think, "There's no way we can get through this much material in a week." If you expect mastery from the students, then yes, there's no way. But if you expect that the students will get the gist of the material, with little immediate recall and some details filled in over time, then a week is enough time (says the author from experience). The point of dumping all this material on the students at once, and quickly, is to de-emphasize background material while giving them surface familiarity with the concepts; they can then develop deeper familiarity over time as they use sets and logical thinking in other contexts. The practical effect is that students will need to look up notation and terminology and facts/theorems/truth tables all week and for some weeks to come.

Because set theory and logic involve so much new notation, and because different sources use different notation, it is worth exposing students to variances. Examples include denoting *such that* as s.t. or $|$ or :, denoting the set $\{1, \ldots, n\}$ as $[n]$, using $-$ or $\setminus$ for set subtraction, noting that $|$ can mean *divides* as well as *such that*, and denoting complementation by an overline versus a superscripted C versus a prime. Whatever notation you like to use, point it out to the students. Of course, you may not prefer the notation used in this book, and students are likely to encounter other notations in their mathematical lives; you may as well warn them now.

Such a discussion of notation is a good warmup for the first class of the week. There is a lot of reading in Sections 2.2 and 2.3, so it makes sense to follow a short warmup with a request for any questions over the reading or Check Yourself problems. After such a discussion, break students into groups to work on Section 2.4. The DeMorgan's law exercise is likely to take them quite a while, so it is unlikely that they will complete these problems in the remaining class time.

Ask the students to read Section 2.5 for the next class. You may want to devote some class time to further work on Section 2.4 before embarking on the activity in Section 2.6, and it's always good to ask whether there are questions over the reading or the Check Yourselfs. (Should those be pluralized as Check Yourselves?) It is likely that this activity will take most of a class period, if not all of it. My experience is that much of a third class meeting is needed to fully address all the problems.

A cheery warmup for a third day of class is to project the Greek alphabet (Google Images will produce a table to your liking) and go through the pronunciations and uses of the letters. Some are listed on page 641. Students like to share their prior knowledge as part of this discussion.

If you choose to include the Bonus Section 2.10 material in class, you might show your class a *Doctor Who* clip (from "The Pyramids of Mars") containing a truth-tellers problem; it is available at https://www.youtube.com/watch?v=W90s58LtYhk. (This tip courtesy of Tom Hull!) Beware that this may provide savvy students with significant clues for solving the final question of the Section 2.10 Project.

Finally, please remember that this chapter is an *overview* of set theory and logic and proof techniques. Students will practice using these ideas throughout the course and need not have mastered them just yet. Should you want to supplement this material with some additional basic proof problems, a few are provided in Section TI.2.