SUPPLEMENTAL MATERIAL
CASE STUDY: NATURAL FREQUENCIES OF TALL BUILDINGS

This advanced case study points toward more sophisticated analyses that build from elementary models but from which deeper meanings can be extracted. Engineers and researchers have long endeavored to model overall building behavior in terms of the response of beams or assemblages of beams. We might wonder whether, since modern high-rise buildings ("skyscrapers," as illustrated in Figure SCS.1) are such tall and slender structures, we could estimate their basic natural frequencies by modeling the vibration behavior of elementary beams. Unfortunately, as tempting and "intuitive" as it may seem, this notion is undermined by a simple, yet unpleasant, fact: empirical measurements of the natural frequency of tall buildings show that these frequencies vary inversely with their height, which is not the dependency shown by the familiar models of beam behavior. So, let's start there.

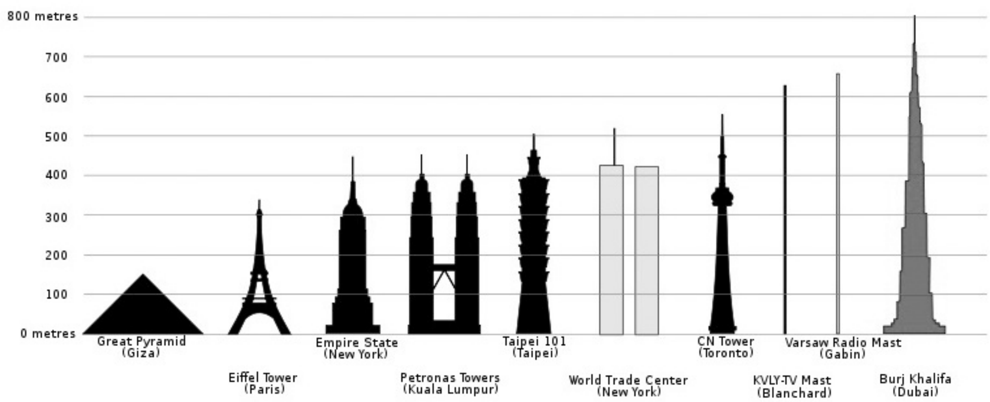

Figure SCS.1 Range of modern skyscrapers, shown in context of other tall built structures.

We will shortly show that the natural frequency of an elementary cantilever beam can be expressed in terms of the beam's usual properties as:

$$\omega_{cant} = 3.52\sqrt{\frac{EI}{\rho A L^4}} \sim \frac{1}{L^2} \tag{SCS.1}$$

We note in particular in equation (SCS.1) that the fundamental frequency is inversely proportional to the square of the beam's length—or of the building's height if it were modeled as a simple cantilever. But as it turns out, the literature features several empirically derived estimates of the fundamental frequency of a tall building as a function of L. One early and notable example is shown in Figure SCS.2, which is a plot of resonant frequencies of some 163 buildings of various heights L (measured in meters) and rectangular layouts.

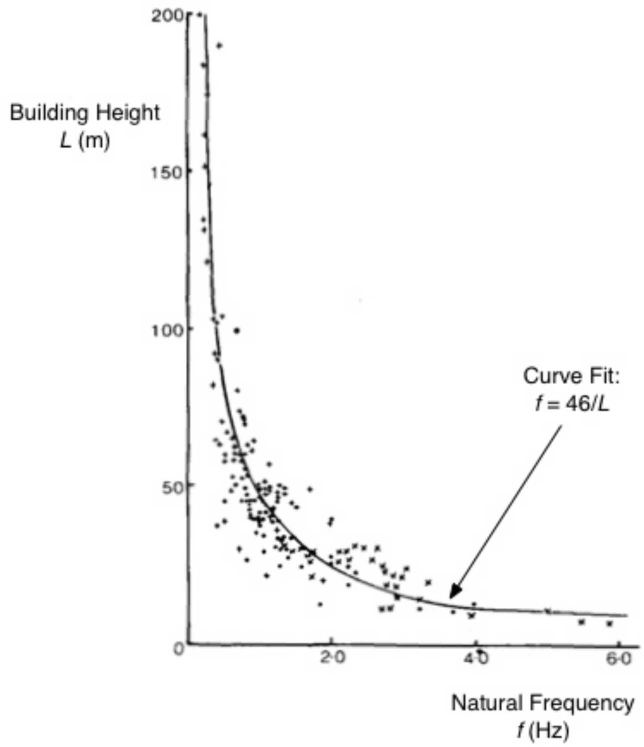

Figure SCS.2 An empirical study of the natural frequencies of 163 tall buildings showing frequency measurements from a variety of sources, as well as a curve fit of those data. (*Adapted from* B. R. Ellis, *Proceedings of the Institution of Civil Engineers*, 1980.)

Also shown in Figure SCS.2 is a curve fit through all the data, which we rewrite here in terms of the frequency *f,* with a subscript acknowledging the researcher who performed the curve fit, as:

$$f_{Ellis} = \frac{46}{L} \quad (\text{Hz}, L \text{ in m}) \tag{SCS.2a}$$

This result can also be written in terms of the angular frequency ω as:

$$\omega_{Ellis} = 2\pi f_{Ellis} \cong \frac{289}{L} \quad (\text{rad/s, m}) \tag{SCS.2b}$$

The structural engineering literature contains similar empirical results for various kinds of structural types, with frequencies expressed in terms of a building's length *L* or number of stories *N*, which tracks closely with the height (or length) *L*! We show some of these results in Table SCS.1.

Table SCS.1. Empirical formulas for the natural frequencies for various tall building types.

Building Types	**Frequency**
Braced steel frames and reinforced concrete shear walls.	$\omega_1 = \dfrac{20\pi}{N}$ (rad/s, N)
Braced steel frames and reinforced concrete shear walls and accounting for building depth D (m).	$\omega_2 \cong \dfrac{69\sqrt{D}}{L}$ (rad/s, m)
Steel moment-resisting frames.	$\omega_3 \propto L^{-0.80}$ (rad/s, m)
Reinforced concrete buildings.	$\omega_4 \propto L^{-0.90}$ (rad/s, m)

Note that equations (SCS.2) and the data in Table SCS.1 share two important features. First, they are all *unit-dependent* equations, that is, each has a numerical constant that is specifically linked to or dependent on the units in which its variables are measured. Second, every one of these results suggests a dependence of frequency ~ $1/L$, which clearly differs from the elementary (i.e., Euler-Bernoulli) beam model prediction in equation (SCS.1). Thus, it is quite appropriate to ask whether a tall building's frequency *can* be estimated from an elementary beam theory calculation, or whether the empirical formulas commonly used indicate a different behavior. Perhaps another ("higher-order") beam theory can be invoked to resolve this apparent conflict?

This line of questions is typically viewed in the context of extending the Euler-Bernoulli model of beam behavior (developed in Chapters 7 and 9) to account for the shear deformation that is not included in the elementary theory. There are, in fact, two different ways to incorporate shear based on two different models. The approach we could take here is to couple an Euler-Bernoulli beam to a shear beam in a *parallel* formulation in which each shares the same (total) deflection (see Figure SCS.3). This parallel coupling enables a model with a frequency dependence on beam length (or building height) that is consistent with measured data. We will also see that the model's behavior is quite dependent on a dimensionless parameter that relates the beam's shear and bending stiffness, i.e., $\alpha^2 = GAL^2 / EI$, so we'll have to pay close attention to how we identify these stiffness terms in our model.

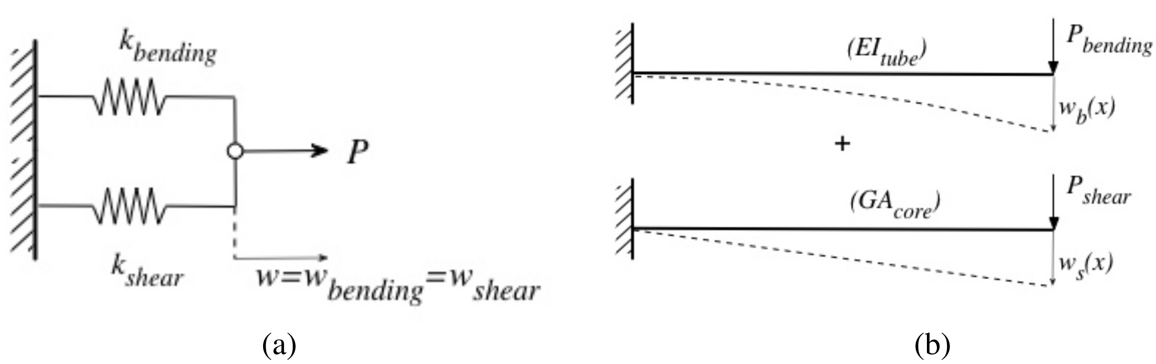

Figure SCS.3 Scehmatic drawings of (a) discrete and (b) continuous models of an elementary Euler-Bernoulli beam coupled with a shear beam, a combination referred to as a coupled "Euler-Shear" beam.

MODELING TALL BUILDINGS WITH BEAM THEORY

In the light of the foregoing discussion, we now want to ascertain whether the natural frequency of a vertical cantilever beam is a reasonable estimate of the frequency of a tall building and, if it is not, to identify a model that is consistent with the empirical data. The standard equation of motion for a vibrating beam of length L with bending stiffness EI, cross-section area A, and mass density ρ can be derived from textbook equations (22.13) and (22.14). It is:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{SCS.3}$$

This partial differential equation is not that difficult to solve, although the details are well beyond our scope. Rather, we are going to make some heuristic guesses to ascertain the behavior of the solution to equation (SCS.1). We start by introducing a dimensionless coordinate along the beam's length (in our case, the building's height) $\xi = x/L$, and by also assuming a separable variables solution in which the beam deflection can be written as the product of: a displacement amplitude W_0, a function of the (dimensionless) spatial coordinate $f(\xi)$, and a harmonic or circular function in time:

$$w(\xi,t)=W_0\hat{w}(\xi)\cos\omega t \tag{SCS.4}$$

If we substitute the solution form (SCS.4) into the differential equation (SCS.3) we get:

$$\left[\frac{d^4\hat{w}(\xi)}{d\xi^4}-\left(\frac{\rho A L^4\omega^2}{EI}\right)\hat{w}(\xi)\right]\left(\frac{EI}{L^4}\right)\left(W_0\cos\omega t\right)=0 \tag{SCS.5}$$

Now, equation (SCS.5) is the starting point for a classical *eigenvalue problem* that has a non-trivial solution (i.e., the displacement is itself non-zero) if and only if:

$$\frac{d^4\hat{w}(\xi)}{d\xi^4}-\left(\frac{\rho A L^4\omega^2}{EI}\right)\hat{w}(\xi)=0 \tag{SCS.6}$$

Equation (SCS.6) is the eigenvalue problem, which in mathematical terms is: The homogeneous ordinary differential equation (SCS.6) for $\hat{w}(\xi)$, together with a complete set of homogeneous boundary conditions (here four), has a non-vanishing solution *only* for *certain numerical values* of the (dimensionless) coefficient $\rho A L^4\omega^2/EI$. In fact, it is the exact solution to equation (SCS.6) satisfying a cantilever's boundary conditions ($\hat{w}(0)=\hat{w}'(0)=\hat{w}''(1)=\hat{w}'''(1)=0$) that produces the numerical coefficient 3.52 in equation (SCS.1)—but for now you'll have to take our word for it! The complete solution of this and other eigenvalue problems is also beyond our scope—and the solutions are algebraically complex and tedious even for simple beams, so we now describe another way to get to the main features of free vibration frequencies.

Mathematicians call this solution to the eigenvalue problem the *strong form* solution because the differential equation is satisfied at each and every point within the beam's domain, and each boundary condition is identically satisfied at the beam's ends. So imagine a *weak form* of the solution where instead of a point-by-point solution, we solve the eigenvalue problem by multiplying equation (SCS.10) by our yet-to-be-determined average solution $\tilde{w}(\xi)$, and then forcing the integral of that product over the beam's length to vanish:

$$\int_{\xi=0}^{\xi=1}\left[\frac{d^4\tilde{w}(\xi)}{d\xi^4}-\left(\frac{\rho A L^4\omega^2}{EI}\right)\tilde{w}(\xi)\right]\tilde{w}(\xi)\,d\xi=0 \qquad \text{(SCS.7)}$$

This effectively satisfies our original differential equation *on average* over the beam length, rather than point-by-point. This is a weaker form of the solution, although as a practical matter it is very powerful. Further, if the approximate solution $\tilde{w}(\xi)$ is chosen carefully, the weak form will produce the same exact solution as the strong form. Now this may seem a bit abstract and perhaps it doesn't look very exciting just yet, but it is not hard to show that by integrating equation (SCS.7) by parts and rearranging the results (see Problem SCS.2), the frequency we want to find is just:

$$\omega^2=\frac{EI\displaystyle\int_{\xi=0}^{\xi=1}\left(\frac{d^2\tilde{w}(\xi)}{d\xi^2}\right)^2 d\xi}{\rho A L^4\displaystyle\int_{\xi=0}^{\xi=1}\left(\tilde{w}(\xi)\right)^2 d\xi} \qquad \text{(SCS.8)}$$

Equation (SCS.8) is known as a *Rayleigh Quotient*, named after the eminent physicist John William Strutt (1842–1919), 3rd Baron Rayleigh, known as Lord Rayleigh. (In addition to his other achievements, he shared the 1904 Nobel Prize for Physics, for co-discovery of the element argon.) The Rayleigh quotient offers a simple way to calculate very good approximations for the natural frequencies of beams, and for any other vibrating solids described by an eigenvalue problem. (The Rayleigh quotient can also be exercised in the buckling eigenvalue problems we introduced in Chapter 11; see Problems SCS.10–11) below.) Even for a simple cantilever, the true mode shape $w(\xi)$ is a complex function, and the frequency we seek is found only by solving a transcendental equation. Little wonder Rayleigh and his contemporaries were looking for more manageable approximations: this was long before the age of computers. And nowadays, even with modern computational power, we engineers continue to seek simpler approximations or estimates.

Before seeking an approximate frequency with the Rayleigh quotient (SCS.8), we recast that quotient as:

$$\omega^2 = \frac{EI}{\rho A L^4}\mu_{bending}^{EB} \tag{SCS.9}$$

In equation (SCS.9) we have defined a new *number* $\mu_{bending}^{EB}$ as:

$$\mu_{bending}^{EB} = \frac{\int_{\xi=0}^{\xi=1}\left(\tilde{w}''(\xi)\right)^2 d\xi}{\int_{\xi=0}^{\xi=1}\left(\tilde{w}(\xi)\right)^2 d\xi} \tag{SCS.10}$$

We emphasize that $\mu_{bending}^{EB}$ is a number that is basically of order unity (i.e., $\sim O(1)$). As a result, the dimensions of equation (SCS.9) are evident and we can confirm the dimensional argument we made earlier about equation (SCS.1):

$$\omega = \sqrt{\frac{EI\mu_{bending}^{EB}}{\rho A}}\frac{1}{L^2} \tag{SCS.11}$$

To get an estimate of the fundamental frequency using the Rayleigh quotient, we can approximate the (unknown, exact) mode shape by assuming an algebraic form that either satisfies—or can be forced to satisfy—the boundary conditions. For example, for the elementary cantilever, this text's Table 9.1 suggests that we might try an approximate mode shape based on the deflection of a cantilever under a uniform load, that is,

$$\tilde{w}_1(\xi) \sim \left(6\xi^2 - 4\xi^3 + \xi^4\right) \quad \text{or} \quad \tilde{w}_1(\xi) = W_1\left(6\xi^2 - 4\xi^3 + \xi^4\right) \tag{SCS.12}$$

(Note that if we use the second form of equations (SCS.12), the amplitude W_1 simply cancels out when it is substituted into equation (SCS.10).) For the approximation (SCS.12), we can calculate that $\mu_{bending1}^{EB} = 12.46$, and so we find an approximate frequency $\omega_1 = 3.53\sqrt{EI/\rho A L^4}$, which is very close to the exact frequency (SCS.1).

MODELING TALL BUILDINGS AS "SHEAR BEAMS"

Returning to our primary focus, neither the exact nor approximate frequencies just calculated provide for the physical behavior seen in the measured building resonant frequencies. But we can improve our model by coupling our elementary bent beam to a shear beam, as illustrated in both discrete and continuous forms in Figure SCS.3. Why this particular model? For immediate motivation, we look at tall buildings configured as flexible tubes built around dense stiff cores. Such "tube-&-core" structures were pioneered by the late engineer Fazlur Rahman Kahn (1929–82), a partner at Skidmore Owings & Merrill. Kahn was the structural designer of both the John Hancock Center and the Sears (now Willis) Tower (Figures SCS.4 (a, b)). Another famous tube-and-core design was Leslie R. Robertson's original World Trade Center in New York City. So, if we picture a (very!) tall milk carton as our tube and imagine within it a concrete tower core with a much smaller footprint, and we model the tube as an elementary cantilever beam and the core as a shear beam, we have a basis for simple models of these structures. (We note too that this design concept was incredibly popular with the developers who paid for these buildings because it enabled open and easily changeable floor plans, which made the buildings much easier to market as office spaces.)

(a) (b)

Figure SCS.4 Two of the prioneering tubular designs pioneered by Fazlur R. Kahn: (a) The John Hancock Center in Chicago (1965) is a trussed tube system, nowadays called a diagrid system. (Photo courtesy Antoine Taveneaux.) (b) The Willis Tower in Chicago, originally known as the Sears Tower (1974), is actually a bundle of nine tubes.

The governing equation of motion for the coupled Euler-Shear beam can be derived as:

$$\left(\frac{EI}{L^4}\right)\frac{\partial^4 w_{ES}(\xi,t)}{\partial\xi^4}-\left(\frac{GA}{L^2}\right)\frac{\partial^2 w_{ES}(\xi,t)}{\partial\xi^2}+(\rho_b A_b+\rho_s A_s)\frac{\partial^2 w(\xi,t)}{\partial t^2}=0 \qquad \text{(SCS.13)}$$

We are not going to solve equation (SCS.13) either exactly or approximately. Rather, since our focus is on the behavior of the natural frequencies of tube-&-core buildings, we present the

Rayleigh quotient for the coupled Euler-Shear beam. Following exactly the same steps we took for the elementary beam, we can find that:

$$\omega_{ES}^2 = \frac{EI_{tube}}{\rho AL^4}\mu_{tube}^{ES} + \frac{GA_{core}}{\rho AL^2}\mu_{core}^{ES} = \frac{EI_{tube}}{\rho AL^4}\left[\mu_{tube}^{ES} + \alpha^2\mu_{core}^{ES}\right] \qquad \text{(SCS.14)}$$

The two numbers of order unity and the "slenderness" parameter α^2 in equation (SCS.14) are:

$$\mu_{tube}^{ES} = \frac{\int_{\xi=0}^{\xi=1}\left(\tilde{w}_{ES}''(\xi)\right)^2 d\xi}{\int_{\xi=0}^{\xi=1}\left(\tilde{w}_{ES}(\xi)\right)^2 d\xi}, \quad \mu_{core}^{ES} = \frac{\int_{\xi=0}^{\xi=1}\left(\tilde{w}_{ES}'(\xi)\right)^2 d\xi}{\int_{\xi=0}^{\xi=1}\left(\tilde{w}_{ES}(\xi)\right)^2 d\xi}, \quad \alpha^2 = \frac{GA_{core}L^2}{EI_{tube}} \qquad \text{(SCS.15)}$$

Equation (SCS.14) clearly shows a difference in how the tube-&-core fundamental frequency is influenced by relevant variables, especially the beam length (or building height) L and the slenderness ratio. We see that when $\alpha^2 > 1$ the limiting behavior of equation (SCS.14) becomes:

$$\omega_{ES}\Big|_{\alpha^2>1} \cong \sqrt{\mu_{core}^{ES}}\sqrt{\frac{GA_{core}}{\rho AL^2}} \sim \frac{1}{L} \qquad \text{(SCS.15)}$$

This limiting behavior contrasts sharply with that of the elementary beam (equation SCS.1), but is quite consistent with the behavior measured on actual buildings.

DISCUSSION

How well do the frequency predictions for the Euler-Shear beam models compare with known results? Unfortunately, we can't make this comparison easily because even for those buildings for which frequencies have been measured, there are no corresponding measurements or statements of their cross-sectional properties. However, Rayleigh quotient approximate frequencies for three 40-story buildings have been compared with frequencies predicted by exact beam theory solutions, as well as with frequencies calculated using the SAP2000 finite element program. We present just a few of those comparisons in Table SCS.2, and they look quite good.

Table SCS.2. Comparison of approximate fundamental frequencies (rad/s) calculated with Rayleigh quotients and with the SAP2000 computer program for three 40-story buildings.

Building type	α^2	$\omega_{Rayleigh}$	ω_{Beam}	$\omega_{SAP2000}$
Framed tube	3.76	1.94	1.92	1.98
Framed tube and core	6.55	2.29	2.23	2.03
Framed tube and core with outrigger	6.59	2.26	2.22	2.38

Several interesting points emerge from these data. First, the Rayleigh quotient data used the same mode shape derived for the uniformly loaded cantilever, and we note that the results agree very well with both the exact beam theory solution and the computer modeling numbers. Second, the Raleigh-based approximations are always higher than their exact beam theory results. This is one of the classic features of such approximations: the eigenvalues calculated by approximate techniques will always bound their corresponding exact results from above. The philosophical argument for this is that the approximate solutions are constrained to be (somewhat) different than the exact results, hence suggesting that the elastic body is somehow "stiffer" than the ideal body being modeled. Since Rayleigh quotients for frequency are ratios of stiffness-to-mass participation, and increase in stiffness should produce an increase in the quotient, and so the frequency. (And, by the way, the same Rayleigh quotient can be used to obtain reasonable approximations of the buckling eigenvalues in elastic stability problems.)

Third and last we observe that the final building type listed in Table SCS.2 is identified as having an *outrigger*. Figure SCS.5 shows the partially built Shimao International Plaza (2006) building in Shanghai. It has two very large truss-like structures, one near the middle of the building, the other near the top. However, while they are truss-like in appearance, their joints are not truly the pinned, moment-free joints of a typical truss. On the other hand, they are very much like the large frame-like "trusses" developed by the Belgian engineer Arthur Vierendeel in 1896, his idea being to create larger openings without the many diagonals of ordinary trusses. These outriggers or *outrigger belts* are typically used to reinforce and enclose the floors that house air conditioning and heating units, elevator motors and other heavy utilities, so their loads can be distributed along a building's height. The outriggers' mass provides some—but not very much!—added stiffness to the overall building behavior. It is also worth noting that these truss-like stiffening frames should not be confused with the *diagrid* trusses we saw, for example, in the John Hancock Center (Figure SCS.4 (a)). Such diagrid structures serve a very different purpose and achieve very different effects than do the "local" outrigger belts.

Figure SCS.5 The Shimao International Plaza building under construction in Shanghai. Note the two very prominent Vierendeel trusses, called outrigger belts, in the middle and near the top of the building.

Another modern development in skyscraper design that has had an effect on the vibration behavior of the structures is the use of *tuned mass damper systems*. These systems, often large masses on springs or pendula positioned high in the structures, act to lessen the amplitude of building vibrations, and can "tune" the building's vibration frequency away from potentially dangerous structural resonances. In both these ways, tuned mass dampers enhance the structural stability of very tall buildings. One prominent example of a tuned mass damper is the pendulum in Taipei 101, in Taiwan, shown in Figure SCS.6.

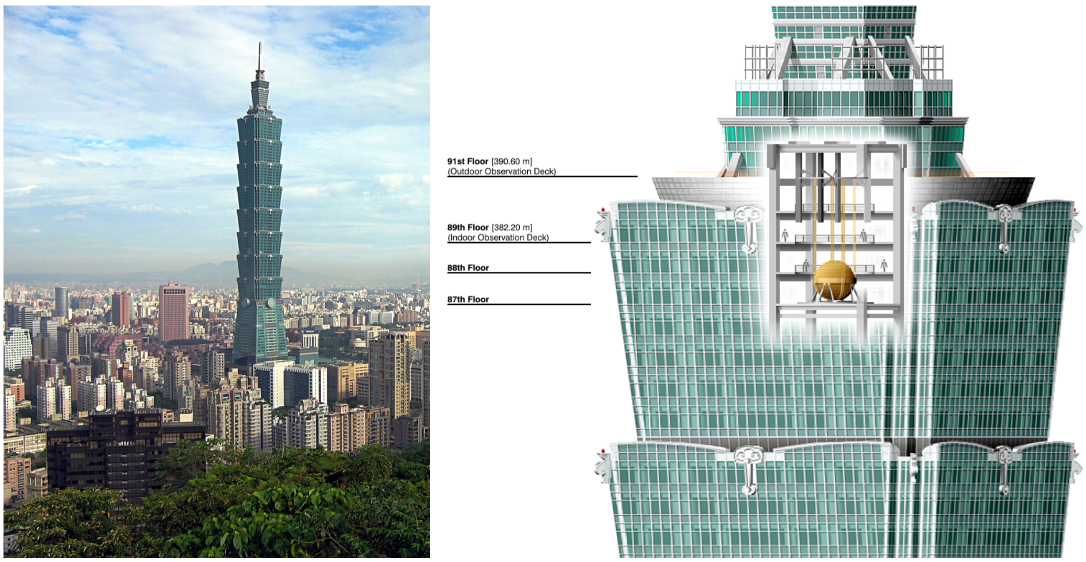

Figure SCS.6. Taipei 101: Diagram at right highlights position and scale of tuned mass damper pendulum.

We set out in this case study to answer the question of how we might simply model the natural frequency behavior of tall buildings. We have covered a lot of ground in the course of answering this question, including: describing the kinds of measurements that are central to an engineering understanding of physical behaviors; examining different ways of interpreting and assessing such data; describing a means of generating simple approximations and estimates of some important physical behaviors; and showing simple, "back of the envelope" calculations can be used to verify and validate our own understanding and any comparable results from corresponding computer modeling.

PROBLEMS

SCS.1 How would the units-dependent equation (SCS.2b) need to be changed if the building height were measured in feet (ft) rather than meters (m)?

SCS.2 Confirm that the integration by parts shown immediately below is correct, and identify and state the boundary condition choices that must be satisfied to make it so.

$$\int_{\xi=0}^{\xi=1} \tilde{w}(\xi)\tilde{w}^{iv}(\xi)d\xi = \int_{\xi=0}^{\xi=1} \left(\tilde{w}''(\xi)\right)^2 d\xi$$

SCS.3 What are the physical dimensions of the radicand (i.e., $\sqrt{EI\mu_{bending}^{EB} / \rho A}$) of equation (SCS.11)?

SCS.4 Consider the trial approximation $\tilde{w}_{P4}(\xi) = W_{P4}\left(1 - \cos\pi\xi / 2\right)$ for an elementary cantilever: (a) What beam boundary conditions does it satisfy? (b) Calculate its Rayleigh quotient frequency approximation. (c) Compare the result of (b) with the exact and approximate answers given in this case study.

SCS.5 Consider the trial approximation $\tilde{w}_{P5}(\xi) = W_{P5}\left(1 - \cos\pi\xi\right)$ for an elementary cantilever: (a) What beam boundary conditions does it satisfy? (b) Calculate its Rayleigh quotient frequency approximation. (c) Compare the result of (b) with the exact and approximate answers given in this case study.

SCS.6 Consider the trial approximation $\tilde{w}_{P6}(\xi) = W_{P6}\xi^2$ for an elementary cantilever: (a) What beam boundary conditions does it satisfy? (b) Calculate its Rayleigh quotient frequency approximation. (c) Compare the result found in (b) with the exact and approximate answers given in the case study above.

SCS.7 Suppose a cantilever beam carries a mass M at its tip (i.e, at $\xi = 1$). Confirm that the corresponding Rayleigh quotient would be:

$$\omega^2 = \left(\frac{EI}{\rho AL^4}\right)\frac{\int_{\xi=0}^{\xi=1}\left(\frac{d^2\tilde{w}(\xi)}{d\xi^2}\right)^2 d\xi}{\int_{\xi=0}^{\xi=1}\left(\tilde{w}(\xi)\right)^2 d\xi + \frac{M}{\rho AL}\left(\tilde{w}(1)\right)^2}$$

SCS.8 Using the trial function (SCS.12), calculate a Rayleigh quotient approximation for the natural frequency of a cantilever with a mass M at its tip.

SCS.9 How would the Rayleigh quotient found in Problem SCS.7 change if the mass M was at some arbitrary location $\xi = \xi^*$ along the beam's axis?

SCS.10 Find the Rayleigh quotient for the buckling problem of a simply supported column by integrating-by-parts the product of the differential equation (11.2) with an approximate solution $\tilde{w}(\xi)$ as follows:

$$\int_{\xi=0}^{\xi=1}\left[\tilde{w}''(\xi) + \left(P / EI\right)\tilde{w}(\xi)\right]\tilde{w}(\xi)d\xi = 0 \quad \Rightarrow \quad P = \left(\frac{EI}{L^2}\right)\frac{\int_{\xi=0}^{\xi=1}\left(\tilde{w}'(\xi)\right)^2 d\xi}{\int_{\xi=0}^{\xi=1}\left(\tilde{w}(\xi)\right)^2 d\xi}$$

SCS.11 Find the Rayleigh quotient for the fourth-order buckling problem (thus allowing the full range of support boundary conditions) column by integrating-by-parts the product of the correct fourth order differential equation with an approximate solution $\tilde{w}(\xi)$ as follows:

$$\int_{\xi=0}^{\xi=1}\left[\tilde{w}^{iv}(\xi)+\left(P/EI\right)\tilde{w}''(\xi)\right]\tilde{w}(\xi)d\xi=0 \quad \Rightarrow \quad P=\left(\frac{EI}{L^2}\right)\frac{\int_{\xi=0}^{\xi=1}\left(\tilde{w}''(\xi)\right)^2 d\xi}{\int_{\xi=0}^{\xi=1}\left(\tilde{w}'(\xi)\right)^2 d\xi}$$

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