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Understanding Darcian flow leads to understanding aquifer mechanics

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ABSTRACT: Darcy's law controls laminar groundwater flow. Empirically this fact represents "firm ground." Stress and strain relations control ground deformation. This complicated truth also stands empirically on "firm ground." Unfortunately, an allegorical "river" separates these two disciplines. The equation that bridges these two and establishes a common or shared "terra firma" is Gauss' divergence theorem. Ever since Biot's pioneering work, the problem has been that this mathematical "bridge" has been placed at perhaps the most inconvenient location along the "river"—along the stream of logical development. Flow and strain had already become distinct academic disciplines before Biot's attempt to merge them into a unified hydro-geotechnical discipline. Four locations exist for a mathematical crossing, one being the traditional geotechnical location. All are investigated from the perspective of physics and mathematical rigor. The present paper crosses at a more geohydrological location. The resulting flow equations of this new unified theory of aquifer mechanics have many practical implications for engineering geologists and hydrogeologists. For example, with a small increase in CPU time any reliable computer code for transient three-dimensional groundwater flow, which like MODFLOW already computes Darcian specific discharge rates across each face of every cell, can be transformed directly into an equally reliable code for three-dimensional transient deformation of saturated sedimentary material.

1 INTRODUCTION

Groundwater flow through sedimentary material has a more intimate relation with the deformation of this sedimentary material than most engineering geologists realize. How precise can we become in quantifying this intimate connection? The answer is: Very precise.

Darcy's law controls laminar groundwater flow. Empirically this fact represents "firm ground." Stress and strain relations control ground deformation. This complicated truth also stands empirically on "firm ground." Unfortunately, a treacherous allegorical "river" separates these two disciplines of groundwater flow and ground deformation. Fortunately, a fundamental equation bridges this divide. Gauss' divergence theorem establishes a commonality between flow and strain. Ever since Biot's (1941) pioneering work on flow and strain, geotechnical engineers have tried to cross this allegorical river at perhaps the most inconvenient location (actually, at the most difficult place along the stream of logical development). Flow and strain had already become distinct academic disciplines before Biot's attempt to merge them into a unified hydro-geotechnical discipline. Four locations exist for a mathematical crossing, one being the traditional geotechnical location. The present paper crosses at a more geohydrological location. All are investigated from the perspective of physics and mathematical rigor.

The guiding conceptual equation for this new unified theory of aquifer mechanics is a bulk flow equation. One practical implication is that any reliable computer code of transient three-dimensional groundwater flow, which like MODFLOW already computes Darcian specific discharge across each face of every cell, can be transformed directly into an equally reliable code for three-dimensional transient deformation of saturated sedimentary material.

2 GROUNDWATER FLOW AND INDUCED AQUIFER DEFORMATION

When stressed, an aquifer of saturated sedimentary material tends itself to flow as a deforming particulate structure. Though the grains behave somewhat like a rigid skeletal frame, they actually move not only in response to the motion of the surrounding water but also to a critical extent they move relative to one another. Flowing water passes through pore spaces. This water flows past

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a granular structure that frames the pore spaces. Nonlinear water-to-grain viscosity tends to impel the granular structure to move whereas nonlinear grain-to-grain friction tends to constrain movement. Relative flow occurs along the boundary of a given pore space and defines specific discharge $\boldsymbol{q}$ (Gersevanov, 1934; Biot, 1941; DeWiest, 1965; Bear, 1972; Helm, 1979), namely

$$\boldsymbol{q} \triangleq \boldsymbol{n}(\boldsymbol{v}_w - \boldsymbol{v}_s) \tag{1}$$

where n ($\triangleq V_v/V$) is the porosity of bulk volume V, $\boldsymbol{v}_w$ is the averaged velocity field of water within the saturated pore volume V_v, and $\boldsymbol{v}_s$ is the averaged velocity field of the skeletal structure that frames the pore spaces that comprise V_v. Specific discharge is further controlled mathematically by Darcy's law,

$$\boldsymbol{q} = -K\nabla h \tag{2}$$

where K is hydraulic conductivity. In a nutshell: the gradient of hydraulic head h is the dimensionless driving force that impels Darcy's specific discharge (Hubbert, 1940).

Helm (1979) introduced a straightforward definition of bulk flux $\boldsymbol{q}_b$ of aggregate material within V, which when saturated holds volumetrically a mixture of both sedimentary material and water,

$$\boldsymbol{q}_b \triangleq n\boldsymbol{v}_w + (1-n)\boldsymbol{v}_s \tag{3}$$

The resulting theory of aquifer movement (Helm 1979, 1984) has been corroborated empirically in the laboratory (Helm, 1987, 1998) and indirectly in the field (Helm, 1994a; Burbey, 2002, 2006).

As used here, the equality symbol "$\triangleq$" can be translated to mean "The variable on the left is defined by the expression on the right." It should, however, not be confused with the equality symbol "$\equiv$" that appears first in equation 6. The latter is used in this paper to link two expressions that are essentially identical, in that they simply indicate alternative ways to portray mathematically the same physical event.

3 THE DIVERGENCE THEOREM

The classic equation for transient groundwater flow (Jacob, 1950) stems from the continuity equation. In turn, the roots of any continuity equation grow from within the divergence theorem of Gauss.

Let us clarify the above two statements. MODFLOW is an example of an internationally respected computer code for three-dimensional groundwater flow. Like other models, it is based on Jacob's development of the continuity equation. As used in MODFLOW, McDonald and Harbaugh's (1988) algebraic formulation of the continuity equation is

$$\sum Q_i = S_s(\Delta h/\Delta t)V \tag{4}$$

where the sum Q_i is the volume rate of water flowing into a cell orthogonally across the total surface of the cell, V is the volume of the cell, and Δh is the change in head (averaged within V) over a time interval of duration Δt. S_s is called specific storage and functions as a compressibility type of coefficient.

The basis for all computer models that link groundwater flow to strain through a continuity equation similar to (4) is Gauss' theorem. Gauss' theorem features a purely mathematical truth. It states (Kreyszig, 1983) that for any vector, $\boldsymbol{v}$

$$\iint_S (\boldsymbol{v}\cdot\boldsymbol{n})\,dS = \iiint_V (\nabla\cdot\boldsymbol{v})\,dV \tag{5}$$

where $\nabla\cdot\boldsymbol{v}$ indicates the divergence of $\boldsymbol{v}$, S is the closed surface that bounds a specified bulk volume V, and $\boldsymbol{n}$ is the outer unit normal vector of S. The vector $\boldsymbol{n}$ (unit normal vector) should not be confused with the scalar n (porosity). What the divergence theorem says is essentially that when the outward normal components of any specified vector $\boldsymbol{v}$ are summed over an entire closed surface S, the result can be transformed into the sum of all the divergences of the field generated by vector $\boldsymbol{v}$ within the interior of the enclosed volume V of interest. The reverse transformation is also true. This is a mathematical theorem and hence is valid for each and every vector $\boldsymbol{v}$ so long as both it and its first partial derivatives can be shown to express continuous mathematical functions.

In other words, one can substitute any physically meaningful vector, whether $\boldsymbol{v}_s$, $\boldsymbol{q}$, $\boldsymbol{q}_b$, or any other, for mathematical vector $\boldsymbol{v}$ in equation (5) so long as the differentiability requirements are satisfied. In doing so, equation (5) immediately reveals an unarguable relationship between such a physically meaningful vector and its divergence.

4 BASIC EQUATIONS THAT LINK GROUND-WATER FLOW TO AQUIFER DEFORMATION

4.1 *Physical expressions of the divergence theorem*

Physical expressions of Gaussian equation (5) that are of interest to engineering geologists, hydrogeologists, subsidence modelers, and others are given below by equations (6), (8), (11), and (13).

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First we write

$$\iint_S (\boldsymbol{u}_s \cdot \boldsymbol{n}) dS = \iiint_V (\nabla \cdot \boldsymbol{u}_s) dV \equiv \iiint_V \varepsilon dV \tag{6}$$

as a physical application of Gauss' divergence theorem, where the bulk volume V of a cell of interest has a surrounding surface S that is fixed within the deforming skeletal frame in accordance with Cooper's (1966) fundamental analysis of groundwater flow. The vector $\boldsymbol{u}_s$ represents the transient displacement field of saturated sedimentary material and ε is defined as being the divergence of $\boldsymbol{u}_s$, namely

$$\varepsilon \triangleq \nabla \cdot \boldsymbol{u}_s \tag{7}$$

Definition (7) has been invoked within the parentheses that extend the right hand side of equation (6). Divergence ε represents the trace (first invariant) of the transient total strain tensor of the porous granular structure. It signifies a change in volume of this saturated porous structure. This contrasts to the deviatoric component of the total structural strain tensor, which signifies a change in shape.

Second, in terms of the velocity field of the granular structure, the Gaussian divergence theorem becomes

$$\begin{aligned}\iint_S (\boldsymbol{v}_s \cdot \boldsymbol{n}) dS &= \iiint_V (\nabla \cdot \boldsymbol{v}_s) dV \\ &\equiv \iiint_V \left(\frac{\partial \varepsilon}{\partial t} + \boldsymbol{v}_p \cdot \nabla \varepsilon \right)_{Eulerian} dV \\ &\equiv \iiint_V \left(\frac{\partial \varepsilon}{\partial t} \right)_{Lagrangian} dV \end{aligned} \tag{8}$$

where $\boldsymbol{v}_s$ is defined by

$$\boldsymbol{v}_s \triangleq \frac{D\boldsymbol{u}_s}{Dt} \equiv \frac{\partial \boldsymbol{u}_s}{\partial t} + (\boldsymbol{v}_p \cdot \nabla)\boldsymbol{u}_s \tag{9}$$

Hence from the two definitions, (7) and (9), one can write

$$\frac{D\varepsilon}{Dt} \equiv \nabla \cdot \boldsymbol{v}_s \tag{10}$$

which is invoked within parentheses on the right-hand side of equation (8).

4.2 *Eulerian and Lagrangian perspectives*

The term $\boldsymbol{v}_p$ in equation (8) indicates the rate of movement of a specified bulk volume's centroid—where averaged interior values of scalars, vectors, or divergences become focused in a smeared sort of way. The moving surface S deforms relative to a moving centroid. Movement is expressed in terms of a selected coordinate system. The term "Eulerian" depicts deformation from the perspective of a coordinate system fixed by a specified "initial" distribution of matter, which is usually called being fixed in "space." When one describes motion in terms of such a non-deforming or Eulerian coordinate system, $\boldsymbol{v}_p$ in equation (8) is set equal to $\boldsymbol{v}_s$ for a volume element that moves and deforms with the skeletal frame. An Eulerian perspective contrasts to a "Lagrangian" perspective. The latter depicts the deformation of S from the perspective of a co-deforming and co-moving coordinate system that is itself fixed in, say, the skeletal frame (Cooper, 1966). In terms of such a co-moving frame, $\boldsymbol{v}_p$ inherently cannot be expressed. As a result, Lagrangian $\boldsymbol{v}_p$ is universally set equal to zero (Li and Helm, 1998).

In a nutshell: Jacob's (1950) standard transient groundwater flow equation to be valid, a Lagrangian coordinate system must be invoked that is fixed within the deforming skeletal frame (Cooper, 1966). Such a coordinate system is inherently presumed within all standard groundwater-flow models, such as MODFLOW, that make use of the standard transient groundwater flow equation. Accordingly, it is also tacitly assumed by every user of all such models.

4.3 *Additional groundwater manifestations of Gauss' divergence theorem*

In terms of specific discharge, Gauss' theorem (5) becomes

$$\begin{aligned}\iint_S (\boldsymbol{q} \cdot \boldsymbol{n}) dS &= \iiint_V (\nabla \cdot \boldsymbol{q}) dV \\ &= \iiint_V (\nabla \cdot \boldsymbol{q}_b - \nabla \cdot \boldsymbol{v}_s) dV \end{aligned} \tag{11}$$

The all-important flow relation

$$\boldsymbol{q} \equiv \boldsymbol{q}_b - \boldsymbol{v}_s \tag{12}$$

and its two alternative expressions

$$\boldsymbol{v}_s \equiv \boldsymbol{q}_b - \boldsymbol{q} \tag{12a}$$

$$\boldsymbol{q}_b \equiv \boldsymbol{q} + \boldsymbol{v}_s \tag{12b}$$

follow directly from definitions (1) and (3). Relation (12) is used to extend the right-hand side of equation (11).

Finally, for bulk flux, Gauss' divergence theorem (5) becomes

$$\iint_S (\boldsymbol{q}_b \cdot \boldsymbol{n}) dS = \iiint_V (\nabla \cdot \boldsymbol{q}_b) dV \tag{13}$$

where based on mass conservation (Helm, 1987)

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$$\nabla \cdot \boldsymbol{q}_b = (n/\rho_w)\left(\frac{\partial \rho_w}{\partial t} + \boldsymbol{v}_w \cdot \nabla \rho_w\right) + (1-n)/\rho_s\left(\frac{\partial \rho_s}{\partial t} + \boldsymbol{v}_s \cdot \nabla \rho_s\right) \tag{14}$$

ρ_w is the density of water, and ρ_s is the density of individual solid grains. The divergence $\nabla \cdot \boldsymbol{q}_b$ is given rigorously by equation (14) in terms of material derivatives of a mixture of moving water particles and solid grains. When one incorporates the standard simplifying assumption made by geohydrologists that individual grains are much less compressible than the interstitial water, the divergence of bulk flux reduces to

$$\nabla \cdot \boldsymbol{q}_b = \left(\frac{n}{\rho_w}\right)\left(\frac{\partial \rho_w}{\partial t} + \boldsymbol{v}_w \cdot \nabla \rho_w\right) \tag{15}$$

When equations (10) and (15) are substituted into the right-hand extension of equation (11), namely when

$$\iint_S (\boldsymbol{q} \cdot \boldsymbol{n})\, dS = \iiint_V \left[\left(\frac{n}{\rho_w}\right)\left(\frac{\partial \rho_w}{\partial t} + \boldsymbol{v}_w \cdot \nabla \rho_w\right) - \frac{\partial \varepsilon}{\partial t}\right] dV \tag{16}$$

then two strain and compressibility conditions that underlie MODFLOW and other groundwater flow equations have been incorporated into the more general divergence equation (11). A third condition or assumption is that change in the density of water ρ_w and change in the bulk volume strain ε of the porous granular structure can both be expressed in terms of a corresponding change in hydraulic head, namely that $\rho_w = \rho_w(h)$ and $\varepsilon = \varepsilon(h)$ Including this third condition yields

$$\iint_S (\boldsymbol{q} \cdot \boldsymbol{n}) dS = -\iiint_V S_s \frac{\partial h}{\partial t} dV \tag{17}$$

Hydrogeology's specific storage S_s can accordingly be defined as

$$\begin{aligned} S_s &\triangleq \left[\left(\frac{n}{\rho_w}\right)\left(\frac{\partial \rho_w}{\partial h}\right)\right] - \frac{\partial \varepsilon}{\partial h} \\ &= \left(\frac{n}{\rho_w}\right)\left(\frac{\partial \rho_w}{\partial p}\frac{\partial p}{\partial h}\right) - \frac{\partial \varepsilon}{\partial \sigma_e}\frac{\partial \sigma_e}{\partial h} \\ &= \rho_w g\left[\left(\frac{n}{\rho_w}\right)\left(\frac{\partial \rho_w}{\partial p}\right) - \frac{\partial \varepsilon}{\partial \sigma_e}\right] \end{aligned} \tag{18}$$

where p is pore-water pressure, σ_e is Terzaghi's (1925) effective stress (namely, $\sigma_e \triangleq \sigma_T - p$), and it is presumed that $\partial p/\partial h \cong \rho_w g$ where g is gravitational acceleration. The rightmost approximate equality in equation (18) tacitly requires there to be no change in total load σ_T within or along the boundary of the geohydrologic region of interest. This latter restriction separates geohydrology from geotechnical engineering.

When an additional simplifying assumption that water is significantly less compressible than the porous granular structure, equations (14) and (15) further simplify to

$$\nabla \cdot \boldsymbol{q}_b = 0 \tag{19}$$

This assumption is universally made by geotechnical engineers. When equations (10) and (19) are substituted into the right-hand side of equation (11), namely when

$$\iint_S (\boldsymbol{q} \cdot \boldsymbol{n}) dS = -\iiint_V \frac{\partial \varepsilon}{\partial t} dV \tag{20}$$

then two geotechnical strain and compressibility requirements that undergird soil consolidation equations have been incorporated into divergence theorem (11).

Equation (17) can be considered geohydrology's interpretation of Gauss' divergence theorem (11). Contrastingly, equation (20) can be considered geotechnical engineering's interpretation of Gauss' divergence theorem (11). The right-hand sides of all divergence theorem applications in this paper pertain to strain whereas the left-hand sides pertain to Darcian flow in accordance with equations (2) and (12).

5 TWO SIGNIFICANT POINTS TO NOTICE ARE

i. The left-hand side of MacDonald and Harbaugh's (1988) algebraic continuity equation (4) is similar to the left-hand side of Gaussian equation (11) and its geohydrologic special case (17).
ii. The right-hand side of algebraic continuity equation (4) can be expressed rigorously and more generally by the right-hand side of Gaussian equation (11) and its geohydrologic special case (17).

Points i and ii indicate that geohydrologic divergence equation (11) can be employed as a "bridge" across the allegorical river described in the Introduction that separates Darcian "flow" interrelations on the left from Biot's "strain" interrelations on the right. The strategy introduced in the present paper is to cross the "river" at equation (11). From

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thence one can leave behind much of the complexity and ambiguity associated with strain.

Up to this point we have considered the complexity and ambiguity associated with strain in some detail. The purpose for doing so is twofold. First, it demonstrate that either side of the divergence equation yields an identical physical result on the opposite side wherever and whenever real world events can be applied to the divergence theorem. One can cross over from one side to the other with impunity at any convenient Gaussian "bridge," namely at equations (6), (8), (11), and even at equation (13). Second, it demonstrates that if one chooses to cross at geohydrologic equation (11) what is important is to examine the confidence one has in the calculated flow values of specific discharge $\boldsymbol{q}$ that appear on the left-hand side. Whether one assumes the traditional linear poroelastic constitutive law (Biot, 1941) to express stress and structural strain on the right hand side of equation (11) or one chooses a nonlinear poroviscous constitutive law (Helm, 1998) that more closely matches empirical behavior, whenever the calculated values for specific discharge $\boldsymbol{q}$ are deemed sufficiently acceptable for the real-world problem at hand, one has successfully crossed to the left-hand side.

6 THE GOAL OF SUBSIDENCE MODELERS

The goal of subsidence modelers is to reach accurately and with computational speed the left-hand side of Gaussian equation (6), namely to evaluate the displacement field of the skeletal frame $\boldsymbol{u}_s$. Groundwater subsidence modelers have appropriately attempted to reach their goal by starting from continuity equation (11), or its algebraic counterpart (4). They must choose, however, which side of the river to trek on their way towards the left-hand side of equation (6).

Flow equation (12) is the equation that directly couples together the left-hand sides of equations (8), (11), and (13). Indirectly, it also couples together the right-hand sides of equations (8), (11), and (13). The all-important left-hand side of equation (6) can be reached from the left-hand side of equation (8) simply by integrating definition (9). This yields

$$\boldsymbol{u}_s = \int_0^t \boldsymbol{v}_s dt + \boldsymbol{u}_{s0} \tag{21}$$

where $\boldsymbol{u}_{s0}$ is the cumulative displacement field that has occurred prior to when a proverbial "stopwatch" is started at time $t = 0$. In MODFLOW, this initial moment would coincide with the beginning of a new "stress event."

7 A NEW STRATEGY FOR CALCULATING THE MOVEMENT OF GEOLOGIC MATERIAL

Nearly all solution strategies for calculating both transient groundwater flow $\boldsymbol{q}$ and transient displacements $\boldsymbol{u}_s$ of saturated geologic material begin from Gaussian equation (11), which is a rigorous expression of groundwater continuity equation (4).

7.1 *The traditional strategy*

To find the displacement field, the traditional geotechnical strategy is to remain conceptually on the strain side of equation (11) and eventually to find the transient value of ε on the right-hand side of equation (6). Following such a strategy, one is forced as a final step to cross over at equation (6) from the right-hand side to the left-hand side. Such a final step requires the Herculean task of integrating strain thrice (once over each of the three dimensions of space) from a known or "anchored" location (say, a three-dimensionally fixed benchmark).

Such Herculean calculations can, of course, be done by a computer. However, to model ground movement even for a small sedimentary basin a great many CPUs are required for calculating afresh at every time step each direction of the three-dimensional displacement field for every specified node or cell.

The GDM model for transient three-dimensional land subsidence (Burbey, 1994; Burbey and Helm, 1999) is an example of this "right-hand" or strain-based strategy of reaching both sides of equation (6) starting essentially from MacDonald and Harbaugh's continuity equation (4). Unfortunately, the process within the computer of going from transient volume strains ε to transient displacements $\boldsymbol{u}_s$ in three dimensions for each cell at every time step, though accurate, takes so much computer time that GDM becomes rendered less than a practical computational tool when applied at the field scale.

Burbey (1994) accounted in three dimensions for another major consideration, which is beyond the scope of the present paper to discuss, and helped Zhang (2009) do the same. This is the well-known distinction in soil mechanics between the behavior of material within a cell when the cell changes from being in a condition of overconsolidation to becoming normally consolidated (and vice versa) as local water levels fluctuate. These two behavioral conditions are controlled by each cell's ongoing history of effective stress. The challenge in simulating this fundamental and characteristic nonlinear stress-dependent behavior of granular and platelet structures was solved decades ago (Helm, 1972, 1975, 1976). Empirically corroborated stress

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dependency (Helm, 1976) of both hydraulic conductivity K and the grain-to-grain structural component of specific storage S_s has been refined to include creep (Helm, 1998). The curious reader is referred to these publications.

7.2 *The new strategy*

The present paper "cuts the Gordian knot." It crosses from the right-hand side of Gauss' equations to the left-hand side immediately and completely at equation (11) rather than at equation (6), the traditional location. The new strategy is simply to remain on the flow side (the left-hand side) of all the Gaussian equations (6), (8), (11), and (13) with the help of equations (12) and (21). The breakthrough that allows one to remain entirely on the flow or Darcian side of the allegorical river is equation (12). This strategy represents conceptually the most direct and computationally the quickest and most efficient way to reach the goal of all subsidence modelers, which is to solve accurately in three-dimensions the transient displacement field $\boldsymbol{u}_s$.

8 CONCLUSION

8.1 *An outline for writing an aquifer mechanics computer code*

How would one construct a computer code for calculating structural deformation that is based on Strategy 7.2 described above? Zhang (2009) has done so as a module within MODFLOW. The practical procedure followed by Zhang is outlined here.

From equation (12), it is evident that when any two of the three vectors $\boldsymbol{q}$, $\boldsymbol{v}_s$, and $\boldsymbol{q}_b$ are known, the third can be calculated.

8.2 *Consider vector* $\boldsymbol{q}$

At every time step, MODFLOW already calculates the directional components of specific discharge q_x, q_y, q_z across every cell's six faces (See equation (4)). Subscripts x, y, and z indicate mutually orthogonal Cartesian coordinates, which are specified at $t = 0$ whether one chooses a Lagrangian or an Eulerian approach. These components of specific discharge $\boldsymbol{q}$ can be retrieved from within the current calculations of MODFLOW. This means that one of the three vectors that appear in equation (12) is already available. One more of the three flow vectors is required in order for all three to become known.

8.3 *Consider vector* $\boldsymbol{q}_b$

In a region where either three-dimensional movement along the land surface or critical deformation at depth has already become or may yet become a hazard, the programmer can reasonably assert the standard geotechnical assumption that

$$\partial\varepsilon/\partial t \gg (1/\rho_w)(\partial\rho_w/\partial t) \gg (1/\rho_s)(\partial\rho_s/\partial t). \quad (22)$$

Another way to write condition (22) is

$$\partial\varepsilon/\partial t \gg \partial\ln(\rho_w/\rho_{w0})/\partial t \gg \partial\ln(\rho_w/\rho_{w0})/\partial t \quad (23)$$

where ln (X) is the Naperian logarithm of X. When condition (23) is physically valid for a region of interest, equation (19) can correspondingly be assumed.

For steady boundary conditions and steady-flow sinks and sources, equation (19) indicates steady incompressible flow that is in equilibrium with such conditions. These conditions correspond to a "stress event" in MODFLOW. MODFLOW already can or does calculate steady flow across each face of every cell for any stress event. The programmer needs merely re-label and re-interpret these steady flow solutions as q_{bx}, q_{by}, and q_{bz}. Together they express $\boldsymbol{q}_b$, which is the steady-flow (equilibrium) asymptote that transient (non-equilibrium) specific discharge $\boldsymbol{q}$ is approaching with time until the model user designates another stress event, which changes the asymptote. Bulk flux $\boldsymbol{q}_b$ can be considered to be that step-wise changing asymptote. In most real world situations that are usually being modeled, transient $\boldsymbol{q}$ does not reach its current asymptote $\boldsymbol{q}_b$ before this asymptote is changed by a new stress event.

8.4 *Consider vector* $\boldsymbol{v}_s$

Knowing the Cartesian directional components of these two vectors ($\boldsymbol{q}$ and $\boldsymbol{q}_b$), one can immediately calculate v_{sx}, v_{sy}, and v_{sz} from

$$v_{si} = q_{bi} - q_i, \quad i = x, y, z. \quad (24)$$

where equation (24) follows directly from equation (12a) and vice versa.

Equation (21) furnishes the final step in computing the cumulative displacement field $\boldsymbol{u}_s$ of the aquifer system. A practical method for calculating reliably and swiftly in three dimensions the transient deformation of saturated sedimentary material has been outlined. It is based on Darcian flow conditions.

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