Chapter One

Section 1.1.4

1.1.4.1. $\left[\begin{array}{rr|r} 1 & -1 & -7 \\ 3 & -4 & 11 \end{array}\right] \underset{-3R_1+R_2\to R_2}{\sim} \left[\begin{array}{rr|r} 1 & -1 & -7 \\ 0 & -1 & 32 \end{array}\right] \Rightarrow -y = 32 \quad \Rightarrow y = -32 \quad \Rightarrow x = -7+y = -39$

$\Longrightarrow$ unique solution: $(x, y) = (-39, -32)$

1.1.4.3. $\left[\begin{array}{rrr|r} 0 & -1 & 4 & 1 \\ -1 & 3 & 2 & 0 \\ 2 & 0 & -1 & -2 \end{array}\right] \underset{\substack{R_1\leftrightarrow R_2 \\ -R_1\to R_1 \\ -2R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrr|r} 1 & -3 & -2 & 0 \\ 0 & -1 & 4 & 1 \\ 0 & 6 & 3 & -2 \end{array}\right] \underset{\substack{-R_2\to R_2 \\ -6R_2+R_3\to R_3}}{\sim} \left[\begin{array}{rrr|r} 1 & -3 & -2 & 0 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 27 & 4 \end{array}\right]$

$\Longrightarrow$ unique solution $\mathbf{x} = \begin{bmatrix} -25/27 \\ -11/27 \\ 4/27 \end{bmatrix}$.

1.1.4.5. (a) $r = 6 \Rightarrow$ last row is $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \neq 0 \end{array}\right] \Rightarrow$ there is no solution.

(b) $r = 0 \Rightarrow \left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 2 & 3 & -1 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{-3R_2+R_1\to R_1}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & -11 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\Longrightarrow$ Solutions are: $\mathbf{x} = \begin{bmatrix} -11 - 2c_1 + c_2 \\ c_1 \\ 5 \\ c_2 \end{bmatrix}$, arbitrary scalars c_1, c_2

1.1.4.7. Ex: Here are four (the question asked for only three) such possible matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1.1.4.9.

Plant	*#Hours Run per day*	*#Cars Built*	*#Trucks Built*
I	x_1	4	4
II	x_2	4	1
III	x_3	2	3

Currently, numbers of hours run per day are $x_1 = 7, x_2 = 6, x_3 = 9$
$\Longrightarrow$ Each day, $4(7) + 4(6) + 2(9) = 70$ cars produced and $4(7) + 1(6) + 3(9) = 61$ trucks produced

Shutting down Plant I $\Longrightarrow$ We want to solve $4x_2 + 2x_3 = 70$ and $x_2 + 3x_3 = 61$.

$$\begin{bmatrix} 4 & 2 & | & 70 \\ 1 & 3 & | & 61 \end{bmatrix} \underset{\substack{\frac{1}{4}R_1 \to R_1 \\ -R_1 + R_2 \to R_2}}{\sim} \begin{bmatrix} 1 & 0.5 & | & 17.5 \\ 0 & 2.5 & | & 43.5 \end{bmatrix} \Rightarrow \left\{ \begin{array}{l} x_3 = \frac{43.5}{2.5} = 17.4 \\ \\ x_2 = 17.5 - 0.5(17.4) = 8.8 \end{array} \right\}$$

In whole hours, run Plant II for 9 hours/day and Plant III for 17 hours/day, or, if trucks are more profitable than cars, run Plant II for 8 hours/day and Plant III for 18 hours a day.

1.1.4.11.

Food	*#Grams (in units of* 100)	*mg Calcium*	*mg Potassium*	*mg Magnesium*
#1	x_1	40	20	40
#2	x_2	70	10	30
#3	x_3	50	40	60
Meal		120	30	70

$\Longrightarrow$ We want to solve, for example, $120 = 40x_1 + 70x_2 + 50x_3$ in order to have the correct amount of calcium in the meal:

$$\begin{bmatrix} 40 & 70 & 50 & | & 120 \\ 20 & 10 & 40 & | & 30 \\ 40 & 30 & 60 & | & 70 \end{bmatrix} \underset{\substack{R_1 \leftrightarrow R_2 \\ \frac{1}{20}R_1 \to R_1 \\ -40R_1 + R_2 \to R_2 \\ -40R_1 + R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & 0.5 & 2 & | & 1.5 \\ 0 & 50 & -30 & | & 60 \\ 0 & 10 & -20 & | & 10 \end{bmatrix} \underset{\substack{R_2 \leftrightarrow R_3 \\ \frac{1}{10}R_2 \to R_2 \\ -0.5R_2 + R_1 \to R_1 \\ -50R_2 + R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & 0 & 3 & | & 1 \\ 0 & 1 & -2 & | & 1 \\ 0 & 0 & 70 & | & 10 \end{bmatrix}$$

$\Rightarrow x_3 = \frac{1}{7} \quad \Rightarrow x_2 = 1 + 2 \cdot \frac{1}{7} = \frac{9}{7} \quad \Rightarrow x_1 = 1 - 3 \cdot \frac{1}{7} = \frac{4}{7}$

The meal should consist of $\frac{4}{7} \times 100\ g \approx 57.14\ g$ of food #1, $\frac{9}{7} \times 100\ g \approx 128.57\ g$ of food #2, and $\frac{1}{7} \times 100\ g \approx 14.29\ g$ of food #3.

1.1.4.13. (a) May be true or may be false: Ex. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{bmatrix}$

(b) May be true or may be false: Ex. $\begin{bmatrix} 1 & 1 & | & 1 \\ 2 & 2 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & | & 1 \\ 2 & 2 & | & 2 \\ 3 & 3 & | & 3 \end{bmatrix}$, (c), Must be true,

(d) Must be true, (e) Must be true.

1.1.4.15. (a) $\underset{\substack{R_2 \leftrightarrow R_3 \\ \frac{1}{2}R_2 \to R_1 \\ -4R_2 + R_4 \to R_4}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 & | & 5 \\ 0 & 1 & 1.5 & -2 & | & 3.5 \\ 0 & 0 & 1 & 0 & | & 6 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \underset{-1.5R_3 + R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 & | & 5 \\ 0 & 1 & 0 & -2 & | & -5.5 \\ 0 & 0 & 1 & 0 & | & 6 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

$= \mathsf{RREF}\,([A \mid \mathbf{b}])$

$$\mathsf{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $\mathsf{RREF}\,([A \mid \mathbf{b}]) \Longrightarrow$ The solutions are: $\mathbf{x} = \begin{bmatrix} 5-2c_1 \\ -5.5+2c_1 \\ 6 \\ c_1 \end{bmatrix}$, arbitrary constant c_1

(c) Both $\mathsf{RREF}\,([A \mid \mathbf{b}])$ and $\mathsf{RREF}(A)$ have rank three.

Section 1.2.5

1.2.5.1. <u>Ex</u>. $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Longrightarrow AB = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ and

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \neq AB$$

1.2.5.3. Here are two examples of non-zero A and B for which $AB = 0$:

<u>Ex. 1</u> $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, and <u>Ex. 2</u> $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$.

1.2.5.5. $A^2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

$$A^3 = A \cdot A^2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

1.2.5.7. Assume $A = [a_{ij}]$, where $a_{ij} = 0$ for $i > j$ and $B = [b_{jk}]$, where $b_{jk} = 0$ for $j > k$. We will explain why $AB = C = [c_{ik}]$ is upper triangular, that is, why $c_{ik} = 0$ for $i > k$.

Suppose $i > k$: We calculate

$$c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = \sum_{j \le i} a_{ij}b_{jk} + \sum_{j>i} a_{ij}b_{jk} = \sum_{j \le i} 0 \cdot b_{jk} + \sum_{j>i} a_{ij}b_{jk} = \sum_{j=i+1}^{n} a_{ij}b_{jk} = a_{i,i+1}b_{i+1,k} + \ldots + a_{i,n}b_{n,k}.$$

But, $i > k$ implies that all of $i+1 > k$, $i+2 > k$, ... , $n > k$ are true, so the assumption that $b_{jk} = 0$ for $j > k$ implies $0 = b_{i+1,k} = \ldots = b_{n,k}$. So,

$$c_{ik} = a_{i,i+1}b_{i+1,k} + \ldots + a_{i,n}b_{n,k} = a_{i,i+1} \cdot 0 + \ldots + a_{i,n} \cdot 0 = 0.$$

So, $AB = C$ is upper triangular.

1.2.5.9. False. <u>Ex</u>. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has rank 1, but $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has rank 0.

1.2.5.11. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, because

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -2a_{11} + a_{31} & -2a_{12} + a_{32} & -2a_{13} + a_{33} \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, because $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \frac{1}{2}a_{21} & \frac{1}{2}a_{22} & \frac{1}{2}a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, because $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

1.2.5.13. Suppose $A = [\, \mathbf{A}_{*1} \;\vdots\; \mathbf{A}_{*2} \;\vdots\; ... \;\vdots\; \mathbf{A}_{*n} \,]$ and $D = diag(d_{11}, d_{22}, ..., d_{nn})$. We will explain why

$$AD = \left[\; d_{11}\mathbf{A}_{*1} \;\vdots\; d_{22}\mathbf{A}_{*2} \;\vdots\; ... \;\vdots\; d_{nn}\mathbf{A}_{*n} \;\right] :$$

First, by Theorem 1.9 in Section 1.2, rewriting $D = [\, \mathbf{D}_{*1}, \mathbf{D}_{*2}, ..., \mathbf{D}_{*n} \,]$ in terms of its columns, we have

$$(\star) \quad AD = A \left[\, \mathbf{D}_{*1} \;\vdots\; \mathbf{D}_{*2} \;\vdots\; ... \;\vdots\; \mathbf{D}_{*n} \,\right] = \left[\, A\mathbf{D}_{*1} \;\vdots\; A\mathbf{D}_{*2} \;\vdots\; ... \;\vdots\; A\mathbf{D}_{*n} \,\right].$$

Using Lemma 1.2 in Section 1.2 and writing the $j-th$ column of D as

$$\mathbf{D}_{*j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_{jj} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

we have that the $j-$th column of AD is

$$A\mathbf{D}_{*j} = 0 \cdot \mathbf{A}_{*1} + ... + 0 \cdot \mathbf{A}_{*(j-1)} + d_{jj}\mathbf{A}_{*j} + 0 \cdot \mathbf{A}_{*(j+1)} + ... + 0 \cdot \mathbf{A}_{*n} = d_{jj}\mathbf{A}_{*j}.$$

This and $(\star)$ imply

$$AD = \left[\, A\mathbf{D}_{*1} \;\vdots\; A\mathbf{D}_{*2} \;\vdots\; ... \;\vdots\; A\mathbf{D}_{*n} \,\right] = \left[\; d_{11}\mathbf{A}_{*1} \;\vdots\; ... \;\vdots\; d_{jj}\mathbf{A}_{*j} \;\vdots\; ... \;\vdots\; d_{nn}\mathbf{A}_{*n} \;\right],$$

as was desired.

Section 1.3.1

1.3.1.1. $[A \mid \mathbf{0}] \underset{-2R_1+R_2\to R_2}{\sim} \left[\begin{array}{rrrr|r} 1 & 3 & -1 & 1 & 0 \\ 0 & -2 & 2 & 1 & 0 \end{array}\right] \underset{\substack{-\frac{1}{2}R_2\to R_2 \\ -3R_2+R_1\to R_1}}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 5/2 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \end{array}\right]$

$\Rightarrow x_3, x_4$ are free variables: Let $x_3 = c_1, x_4 = c_2$ $\Rightarrow$ General solution is $\mathbf{x} = c_1 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -5/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$, where c_1, c_2 =arbitrary constants

1.3.1.3. $[A \mid \mathbf{0}] \underset{\substack{R_1\leftrightarrow R_2 \\ R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 5 & 5 & 5 & 0 \end{array}\right] \underset{\substack{R_2\leftrightarrow R_3 \\ \frac{1}{5}R_2\to R_2 \\ -2R_2+R_1\to R_1}}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$

$\underset{\substack{-R_3+R_2\to R_2 \\ -R_3+R_1\to R_1}}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow$ only x_4 free: $c_1 = x_4$

$\Rightarrow$ General Solution is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$, where c_1 =arbitrary constant

1.3.1.5. For some scalars c_1, c_2, α_1, α_2, and β_1, β_2, $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, $\mathbf{v}_1 = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$, and $\mathbf{v}_2 = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2$.
So,

$$\mathbf{w} = (c_1\alpha_1 + c_2\beta_1)\mathbf{u}_1 + (c_1\alpha_2 + c_2\beta_2)\mathbf{u}_2$$

is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$.

1.3.1.7. (a) Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $B = I - A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow [B \mid \mathbf{0}\,] \underset{\substack{-R_1\to R_1 \\ -R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrr|r} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1, x_2$ free; $x_3 = 0$

$\Longrightarrow$ solutions of $B\mathbf{x} = 0$ are $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, where c_1, c_2 are arbitrary constants

Section 1.4.1

1.4.1.1. $\left[\begin{array}{rrr|r} 1 & 0 & 2 & -1 \\ 1 & 1 & 4 & 1 \\ -1 & 0 & -2 & 1 \end{array}\right] \underset{\substack{-R_1+R_2\to R_2 \\ R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrr|r} ① & 0 & 2 & -1 \\ 0 & ① & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\Rightarrow x_1 = -1-2x_3, x_2 = 2-2x_3, x_3 =$ free

$$\Rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h, c_1 = \text{arbitrary constant, where } \mathbf{x}_p = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{x}_h = c_1 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

1.4.1.3. $\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 & -5 \\ 1 & 0 & -2 & 5 & -6 \\ 0 & 1 & 1 & -1 & 5 \end{array}\right] \underset{-R_1+R_3\to R_3}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 & -5 \\ 0 & -2 & -2 & 2 & -10 \\ 0 & 1 & 1 & -1 & 5 \end{array}\right] \underset{-R_2+R_4\to R_4}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & 1 & -5 \\ 0 & -2 & -2 & 2 & -10 \\ 0 & 0 & 0 & 0 & 10 \end{array}\right]$

$\Longrightarrow$ last row is $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \neq 0 \end{array}\right] \Rightarrow$ there is no solution.

1.4.1.5. (a) $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_1-c_2 \\ -c_1-c_2 \\ c_1 \\ c_2 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_h = \begin{bmatrix} c_1-c_2 \\ -c_1-c_2 \\ c_1 \\ c_2 \end{bmatrix}$,

c_1, c_2 are arbitrary constants

(b) The general solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{x}_h = \begin{bmatrix} c_1-c_2 \\ -c_1-c_2 \\ c_1 \\ c_2 \end{bmatrix}$, where c_1, c_2 are arbitrary constants

(c) $\mathbf{x}_h = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, so $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}^{(2)} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ are the basic solutions of $A\mathbf{x} = \mathbf{0}$.

1.4.1.7. Hint: $A\mathbf{x} = \mathbf{b} \Longrightarrow \mathbf{b}^T = \mathbf{x}^T A^T$. Using the hint and the assumption that $A^T\mathbf{z} = \mathbf{0}$, we find that

$$\mathbf{b}^T\mathbf{z} = \left(\mathbf{x}^T A^T\right)\mathbf{z} = \mathbf{x}^T\left(A^T\mathbf{z}\right) = \mathbf{x}^T\mathbf{0} = 0.$$

Note: If $A\mathbf{x} = \mathbf{b}$ has no solution $\mathbf{x}$ then we should not multiply $A\mathbf{x} = \mathbf{b}$ on both sides by $\mathbf{z}$ because that would implicitly assume that $A\mathbf{x} = \mathbf{b}$ is true for some $\mathbf{x}$.

1.4.1.9. No. One can find a $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution. [Hint: Use the elementary matrices that row reduce A to its RREF.]

Because A has only 5 columns, RREF has a ① in at most 5 rows. So, at least the 6th row of $\mathsf{RREF}(A)$ is $[0 \ldots 0]$.

Let $E_k \ldots E_2E_1$ be a sequence of elementary matrices that row reduce A to RREF. Let

$$\mathbf{b} = \left(E_k \ldots E_2E_1\right)^{-1}\mathbf{e}^{(6)} = \left(E_k \ldots E_2E_1\right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Then, multiplication by $E_k \dots E_2 E_1$ on both sides of $A\mathbf{x} = \mathbf{b}$ gives

$$(E_k \dots E_2 E_1)\, A = (E_k \dots E_2 E_1)\, (E_k \dots E_2 E_1)^{-1}\, \mathbf{e}^{(6)} = \left(E_k \dots E_2 E_1 E_1^{-1} E_2^{-1} \dots E_k^{-1}\right) \mathbf{e}^{(6)} = \mathbf{e}^{(6)}.$$

The last row of the system $(E_k \dots E_1)\, A = \mathbf{e}^{(6)}$ is $[0 \dots 0 \mid 1]$ hence, there was no solution of $A\mathbf{x} = \mathbf{b}$. So this $\mathbf{b} \triangleq (E_k \dots E_2 E_1)^{-1}\, \mathbf{e}^{(6)}$ is a $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

Section 1.5.3

1.5.3.1. $$\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \underset{2R_1+R_2\to R_2}{\sim} \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 11 & 2 & 1 \end{array}\right] \underset{\substack{\frac{1}{11}R_2\to R_2 \\ -4R_2+R_1\to R_1}}{\sim} \left[\begin{array}{rr|rr} 1 & 0 & 3/11 & -4/11 \\ 0 & 1 & 2/11 & 1/11 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}^{-1} = \tfrac{1}{11}\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix} \text{ exists.}$$

1.5.3.3. $$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{2R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -4 & 3 & 2 & 1 & 0 \\ 0 & 3 & -4 & -2 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{-\frac{1}{4}R_2\to R_2 \\ 2R_2+R_1\to R_1 \\ -3R_2+R_3\to R_3}}{\sim} \left[\begin{array}{rrr|rrr} 1 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & 1 & -0.75 & -0.5 & -0.25 & 0 \\ 0 & 0 & -1.75 & -0.5 & 0.75 & 1 \end{array}\right] \underset{\substack{-\frac{4}{7}R_3\to R_3 \\ \frac{3}{4}R_3+R_2\to R_2 \\ -\frac{1}{2}R_3+R_1\to R_1}}{\sim} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1/7 & -2/7 & 2/7 \\ 0 & 1 & 0 & -2/7 & -4/7 & -3/7 \\ 0 & 0 & 1 & 2/7 & -3/7 & -4/7 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}^{-1} = \tfrac{1}{7}\begin{bmatrix} -1 & -2 & 2 \\ -2 & -4 & -3 \\ 2 & -3 & -4 \end{bmatrix} \text{ exists.}$$

1.5.3.5. $$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 5/2 & -5/2 & 1/2 & -1/2 \end{array}\right] \underset{\substack{\frac{2}{5}R_3\to R_3 \\ \frac{3}{2}R_3+R_2\to R_2 \\ R_3+R_1\to R_1}}{\sim} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 1/5 & -1/5 \\ 0 & 1 & 0 & -1/2 & -1/5 & -3/10 \\ 0 & 0 & 1 & -1 & 1/5 & -1/5 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 1 \\ -3 & -2 & 1 \end{bmatrix}^{-1} = \tfrac{1}{10}\begin{bmatrix} 0 & 2 & -2 \\ -5 & -2 & -3 \\ -10 & 2 & -2 \end{bmatrix} \text{ exists.}$$

1.5.3.7. $\left(A^TB\right)^{-1} = B^{-1}\left(A^T\right)^{-1} = \left(A^{-1}\right)^{-1}\left(A^{-1}\right)^T = AB^T$

1.5.3.9. Yes: $$A^{-1} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} a_{11}^{-1} & -\frac{a_{12}}{a_{11}a_{22}} & \dots & & \\ & a_{22}^{-1} & -\frac{a_{23}}{a_{22}a_{33}} & \dots & \\ & & & \ddots & \vdots \\ 0 & & & & a_{nn}^{-1} \end{bmatrix}$$

Further information: Let $A^{-1} = B = [\, b_{ij}\,]$. The diagonal entries of B are easy to determine: $b_{ii} = a_{ii}^{-1}$. After that, we can determine the first superdiagonal entries, as indicated above:

$$b_{i,i+1} = -\frac{a_{i,i+1}}{a_{ii}a_{i+1,i+1}}.$$

In order to have $AB = I$, consider the $(1,3)$ entries of AB and I: We need

$$0 = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} + ... + a_{1n}b_{n3} = a_{11}b_{13} + a_{12}\left(-\frac{a_{23}}{a_{22}a_{33}}\right) + a_{13}\left(a_{33}^{-1}\right) + 0 + ... + 0$$

implies

$$b_{13} = a_{11}^{-1}\left(\frac{a_{12}a_{23}}{a_{22}a_{33}} - \frac{a_{13}}{a_{33}}\right).$$

Continuing in this way we can construct all of $B = A^{-1}$ and see that it is also upper triangular.

1.5.3.11. $\mathbf{e}^{(2)} = \frac{1}{2}\left(\mathbf{e}^{(2)} + \mathbf{e}^{(4)}\right) + \frac{1}{2}\left(\mathbf{e}^{(2)} - \mathbf{e}^{(4)}\right) = A\left(\mathbf{y}^{(3)} + \mathbf{y}^{(4)}\right), \quad \mathbf{e}^{(3)} = A\left(-\mathbf{y}^{(2)}\right),$

$\mathbf{e}^{(4)} = \frac{1}{2}\left(\mathbf{e}^{(2)} + \mathbf{e}^{(4)}\right) - \frac{1}{2}\left(\mathbf{e}^{(2)} - \mathbf{e}^{(4)}\right) = A\left(\mathbf{y}^{(3)} - \mathbf{y}^{(4)}\right) \;\Rightarrow A\left[\, \mathbf{y}^{(1)} \,\vdots\, \mathbf{y}^{(3)} + \mathbf{y}^{(4)} \,\vdots\, -\mathbf{y}^{(2)} \,\vdots\, \mathbf{y}^{(3)} - \mathbf{y}^{(4)}\right] = I$

$\Rightarrow A^{-1} = \left[\, \mathbf{y}^{(1)} \,\vdots\, \mathbf{y}^{(3)} + \mathbf{y}^{(4)} \,\vdots\, -\mathbf{y}^{(2)} \,\vdots\, \mathbf{y}^{(3)} - \mathbf{y}^{(4)}\right].$

1.5.3.13. $A\left(-\mathbf{y}^{(2)}\right) = \mathbf{e}^{(1)} \;\Rightarrow A\left(\mathbf{y}^{(1)} + \mathbf{y}^{(2)}\right) = \left(\mathbf{e}^{(1)} + \mathbf{e}^{(2)}\right) + \left(-\mathbf{e}^{(1)}\right) = \mathbf{e}^{(2)}$

$\Rightarrow A\left(\mathbf{y}^{(3)} - \left(\mathbf{y}^{(1)} + \mathbf{y}^{(2)}\right)\right) = \left(\mathbf{e}^{(2)} + \mathbf{e}^{(3)}\right) - \left(\mathbf{e}^{(2)}\right) = \mathbf{e}^{(3)}$

$\Rightarrow A\left[\, -\mathbf{y}^{(2)} \,\vdots\, \mathbf{y}^{(1)} + \mathbf{y}^{(2)} \,\vdots\, -\mathbf{y}^{(1)} - \mathbf{y}^{(2)} + \mathbf{y}^{(3)}\right] = I$

$\Rightarrow A^{-1} = \left[\, -\mathbf{y}^{(2)} \,\vdots\, \mathbf{y}^{(1)} + \mathbf{y}^{(2)} \,\vdots\, -\mathbf{y}^{(1)} - \mathbf{y}^{(2)} + \mathbf{y}^{(3)}\right]$

1.5.3.15. (a) If $A\mathbf{x} = \mathbf{0}$, then $(CA)\mathbf{x} = C(A\mathbf{x}) = C(\mathbf{0}) = \mathbf{0}$. So, $\mathbf{x}$ solves $CA\mathbf{x} = \mathbf{0}$.

(b) If $CA\mathbf{x} = \mathbf{0}$, then $C^{-1}(CA\mathbf{x}) = C^{-1}(\mathbf{0}) = \mathbf{0}$. Because $C^{-1}(CA\mathbf{x}) = A\mathbf{x}$, it follows that $\mathbf{x}$ solves $A\mathbf{x} = \mathbf{0}$.

1.5.3.17. Define $D = AC$ and $B = C^{-1}A^{-1}$. We will explain why $DB = I$, and that will imply AC is invertible and $(AC)^{-1} = C^{-1}A^{-1}$

$DB = (AC)\left(C^{-1}A^{-1}\right) = A(CC^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. Because $DB = I$. Theorem 1.21 in Section 1.5 implies $D = AC$ is invertible and $(AC)^{-1} = B = C^{-1}A^{-1}$.

1.5.3.19. (a) Must be false, by Theorem 1.25 in Section 1.5 $\left(\text{not (d)} \Longrightarrow \text{not (a)}\right)$

(b) Must be true, because of part (a) and "not invertible" and "singular" are synonymous.

(c) Must be false, by Theorem 1.25 in Section 1.5 $\left(\text{not (d)} \Longrightarrow \text{not (e)}\right)$

(d) Must be false, because $\mathbf{0}$ is in $\mathcal{W}$ no matter what A is.

1.5.3.21. (a) Yes, because $A(BC) = I$ implies A is invertible and $A^{-1} = BC$.

(b) Yes, $(AB)C = I$ and Theorem 1.21 in Section 1.5 implies C is invertible and $C^{-1} = AB$.

(c) Yes, because $ABC = I$ and part (a) implies $BC = A^{-1}(ABC) = A^{-1}I = A^{-1}$. Similarly part (b) says C invertible, hence $B = (BC)C^{-1} = A^{-1}C^{-1}$. But, A^{-1} and C^{-1} are invertible, so Theorem 1.23(c) in Section 1.5 implies $B = A^{-1}C^{-1}$ is invertible, as well as $B^{-1} = \left(A^{-1}C^{-1}\right)^{-1} = CA$.

1.5.3.23. (a) $X = AX + C \iff (I - A)X = X - AX = C \iff X = (I - A)^{-1}C = BC$.

(b) $AX = X + C \iff -C = -AX + X = (I - A)X \iff X = -(I - A)^{-1}C = -BC$.

(c) $XA = X + C \iff -C = X - XA = X(I - A) \iff X = -C(I - A)^{-1} = -CB$.

1.5.3.25. look for a matrix B in block form $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$ that we want to satisfy $AB = I$.

We calculate

$$\left[\begin{array}{c|c} I & O \\ \hline O & I \end{array}\right] = I =^{?} AB = \left[\begin{array}{c|c} A_{11} & O \\ \hline A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

that is,

$$(\star) \quad AB = \left[\begin{array}{c|c} A_{11}B_{11} + O\cdot B_{21} & A_{11}B_{12} + O\cdot B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} & A_{11}B_{12} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right].$$

The first row implies $O = A_{11}B_{12}$ and $I = A_{11}B_{11}$. Because we assumed that A_{11} and A_{22} are invertible, we get $B_{12} = O$ and $B_{11} = A_{11}^{-1}$. Substitute those into $(\star)$ to get

$$\left[\begin{array}{c|c} I & O \\ \hline O & I \end{array}\right] = I_n =^? AB = \left[\begin{array}{c|c} I & O \\ \hline A_{21}A_{11}^{-1} + A_{22}B_{21} & A_{22}B_{22} \end{array}\right],$$

hence we need $I = A_{22}B_{22}$ and $O = A_{21}A_{11}^{-1} + A_{22}B_{21}$, hence $B_{22} = A_{22}^{-1}$ and $B_{21} = -\,A_{22}^{-1}A_{21}A_{11}^{-1}$. So, we conclude that A is invertible and A^{-1} has the partitioned form

$$A^{-1} = \left[\begin{array}{c|c} A_{11}^{-1} & O \\ \hline -\,A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{array}\right].$$

Section 1.6.3

1.6.3.1 (a) for example, expanding along the first row,

$$\begin{vmatrix} 0 & 1 & 4 \\ -1 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -(1)\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} = -(-1-4)+4(0-6) = -19$$

(b) $\begin{vmatrix} 0 & 1 & 4 \\ -1 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 3 & 2 \\ 0 & 1 & 4 \\ 2 & 0 & 1 \end{vmatrix}$, after $R_1 \leftrightarrow R_2$

$= -\begin{vmatrix} -1 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 6 & 5 \end{vmatrix}$, after $2R_1 + R_3 \to R_3$

$= -\begin{vmatrix} -1 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & -19 \end{vmatrix}$, after $-6R_2 + R_3 \to R_3$

$= -(-1)(1)(-19) = -19$

1.6.3.3. <u>Ex</u>. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ has

$|A| + |B| = (-1) + (-1) = -2$ and $|A+B| = |O| = 0 \neq -2 = |A| + |B|$.

1.6.3.5 (a) After $-R_1 + R_2 \to R_2$, $\begin{vmatrix} 1 & 5 & 1 & 10 \\ 3 & 12 & 0 & 18 \\ 3 & 9 & -2 & 6 \\ 4 & 0 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 1 & 10 \\ 2 & 7 & -1 & 8 \\ 3 & 9 & -2 & 6 \\ 4 & 0 & -3 & 4 \end{vmatrix} = |A| = -132.$

(b) After $R_2 \leftarrow \frac{1}{3}R_2$, $\begin{vmatrix} 1 & 5 & 1 & 10 \\ 1 & 4 & 0 & 6 \\ 3 & 9 & -2 & 6 \\ 4 & 0 & -3 & 4 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 1 & 5 & 1 & 10 \\ 3 & 12 & 0 & 18 \\ 3 & 9 & -2 & 6 \\ 4 & 0 & -3 & 4 \end{vmatrix}$

and then using part (a), this is

$= \frac{1}{3}\begin{vmatrix} 1 & 5 & 1 & 10 \\ 2 & 7 & -2 & 8 \\ 3 & 9 & -2 & 6 \\ 4 & 0 & -3 & 4 \end{vmatrix} = \frac{1}{3}|A| = \frac{1}{3}(-132) = -44.$

1.6.3.7. After $-aR_1 + R_2 \to R_2$, $-a^2R_1 + R_3 \to R_3$,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

followed by $-(b+a)R_2 + R_3 \to R_3$, gives

$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{vmatrix} = (1)(b-a)(c-b)(c-a) = (c-a)(c-b)(b-a).$

1.6.3.9. (a) *Method 1*: $B\left(\alpha^{-1}A^{-1}\right) = \alpha A\left(\alpha^{-1}A^{-1}\right) = \alpha\alpha^{-1}\left(AA^{-1}\right) = I$. So, $B^{-1} = \alpha^{-1}A^{-1}$ exists.

Method 2: $B = \alpha A = A(\alpha I) \Longrightarrow B^{-1} = (\alpha I)^{-1}A^{-1} = (\alpha^{-1}I)A^{-1} = \alpha^{-1}A^{-1}$ exists.

(b) $adj(B) \cdot B = |B| = |\alpha\, A| = \alpha^n|A| \Longrightarrow adj(B)B \cdot B^{-1} = |B|\, B^{-1} = \alpha^n|A|\left(\alpha^{-1}A^{-1}\right) = \left(\alpha^{n-1}\right)|A|A^{-1}$

But, $adj(B)B \cdot B^{-1} = adj(B) \cdot I = adj(B)$, so $|A| = adj(A)A$ implies

$adj(B) = adj(B)B \cdot B^{-1} = \alpha^{n-1}(adj\,(A)\cancel{A})\,\cancel{A^{-1}} = \alpha^{n-1}adj\,(A)$.

1.6.3.11. $x_1 = \dfrac{\begin{vmatrix} s & 2 & -1 \\ 5 & 0 & 1 \\ t & -2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -3 & -2 & 1 \end{vmatrix}} = \dfrac{2s+2t}{-4} = -\dfrac{1}{2}\,s - \dfrac{1}{2}\,t, \quad x_2 = \dfrac{\begin{vmatrix} 1 & s & -1 \\ 2 & 5 & 1 \\ -3 & t & 1 \end{vmatrix}}{-4} = \dfrac{-5s-10-3t}{-4} = \dfrac{5}{2} + \dfrac{5}{4}\,s + \dfrac{3}{4}\,t,$

and $x_3 = \dfrac{\begin{vmatrix} 1 & 2 & s \\ 2 & 0 & 5 \\ -3 & -2 & t \end{vmatrix}}{-4} = \dfrac{-4s-20-4t}{-4} = 5 + s + t$

1.6.3.13. (a) $\begin{vmatrix} s & 1 \\ 4 & s \end{vmatrix} = s^2 - 4$ is non-zero for $|s| \neq 2$, that is, $2 \neq s \neq -2$, in which case the system has exactly one solution

(b) For $|s| \neq 2$, $x_1 = \dfrac{\begin{vmatrix} 3 & 1 \\ -6 & s \end{vmatrix}}{s^2-4} = \dfrac{3s+6}{s^2-4} = \dfrac{3\,\cancel{(s+2)}}{(s-2)\,\cancel{(s+2)}} = \dfrac{3}{s-2}$,

and $x_2 = \dfrac{\begin{vmatrix} s & 3 \\ 4 & -6 \end{vmatrix}}{s^2-4} = \dfrac{-6s-12}{s^2-4} = \dfrac{-6\,(s+2)}{s^2-4} = \dfrac{-6}{s-2}$.

For $|s| \neq 2$, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \dfrac{1}{s-2}\begin{bmatrix} 3 \\ -6 \end{bmatrix}$.

(c) For $s = 2$, the augmented matrix is $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 2 & -6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 0 & -12 \end{array}\right]$.

The bottom row is $[\,0 \quad 0 \;\; | \neq 0\,] \Longrightarrow$ There does not exist a solution

For $s = -2$, the augmented matrix is $\left[\begin{array}{cc|c} -2 & 1 & 3 \\ 4 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$\Longrightarrow$ There are $\infty-$ly many solutions $\quad \mathbf{x} = \begin{bmatrix} -\frac{3}{2} + \frac{1}{2}\,c_1 \\ c_1 \end{bmatrix}$, $c_1 =$ arbitrary constant

1.6.3.15. $x_2 = \dfrac{\begin{vmatrix} 1 & b_1 & 1 \\ 0 & b_2 & -1 \\ -1 & b_3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & -1 \end{vmatrix}} = \dfrac{b_1+b_3}{2} = \dfrac{1}{2}\,b_1 + \dfrac{1}{2}\,b_3 \Longrightarrow x_2 = \frac{1}{2}\,b_1 + \frac{1}{2}\,b_3$

1.6.3.17. $\quad 0 = \begin{vmatrix} 4 & 0 & k \\ 1 & k & 0 \\ k & 0 & 9 \end{vmatrix} = 4\begin{vmatrix} k & 0 \\ 0 & 9 \end{vmatrix} + k\begin{vmatrix} 1 & k \\ k & 0 \end{vmatrix} = 36k - k^3 = k\left(36 - k^2\right) \iff k = 0, -6, \text{ or } 6.$

If k is neither $0, -6$, nor 6, then the matrix is invertible.

1.6.3.19 (a) Suppose $R_i = R_j$ in matrix A. Then $|A| = |B|$, where B is obtained from A by $-R_i + R_j \to R_j$. But, B has its R_j being all zeroes. Expanding $|B|$ along R_j implies $|B| = 0$, so $|A| = 0$.

(b) If A has two equal columns, then A^T has two equal rows, hence $|A^T| = 0$ by part (a). So, $|A| = |A^T| = 0$.

(c) (i) $a_{i1}A_{j1} + a_{i2}A_{j2} + ... + a_{in}A_{jn}$ is the expansion along the $j-th$ row of B, where B is obtained from A by replacing the $j-$th row of A by the $i-th$ row of A, although the i-th row of B is the i-th row of A. If $i \neq j$, then the matrix B has two equal rows, namely the $i-$th and the $j-$th. By part (a), $|B|$=0. So, $0 = |B| = a_{i1}A_{j1} + a_{i2}A_{j2} + ... + a_{in}A_{jn}$.

(ii) $a_{1j}A_{1i} + a_{2j}A_{2i} + ... + a_{nj}A_{ni}$ is the expansion along the $i-th$ column of $|C|$, where C is obtained from A by replacing the $i-$th column of A by the $j-th$ column of A, although the i-th column of C is the i-th column of A. If $i \neq j$, then the matrix C has two equal columns, namely the $i-$th and the $j-$th. By part (b), $|C|$=0. So, $0 = |C| = a_{1j}A_{1i} + a_{2j}A_{2i} + ... + a_{nj}A_{ni}$.

1.6.3.21. $C \triangleq AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$. Because $0 \neq ad - bc = |A|$, A is invertible, and similarly $0 \neq eh - fg = |B|$ implies B is invertible. Because A and B are invertible, so is C. So, $0 \neq |C| = (ae+bg)(cf+dh) - (af+bh)(ce+dg)$.

1.6.3.23 (a) Yes, $|A| \neq 0$, because $(-1)^n = |-I_n| = |AB| = |A|\ |B|$ implies $|A| \neq 0$, because if $|A| = 0$, then $(-1)^n = 0 \cdot |B| = 0$, giving a contradiction.

(b) $(-1)^n = |-I_n| = |A|\ |B|$, so similarly to part (a), $|B| \neq 0$. It follows that B is invertible. So, it is not true that B is not invertible.

(c) By part (a), $|A| \neq 0$ and A is invertible. So, $-I_n = AB \Longrightarrow I_n = (-1)AB = (A\big((-1)B\big) = A(-B)$ $\Longrightarrow -A^{-1} = -B$ must be true.

(d) $|A^T| = 0$ is false because $|A^T| = |A| \neq 0$.

1.6.3.25. (a) A, B invertible $\Longrightarrow A^{-1} = \frac{1}{|A|}adj(A)$, $B^{-1} = \frac{1}{|B|}adjB)$, so

$$(AB)^{-1} = B^{-1}A^{-1} = \frac{1}{|B|}adj(B)\frac{1}{|A|}adj(A) = \frac{1}{|A||B|}adj(B)adj(A)$$

but also

$$(AB)^{-1} = \frac{1}{|AB|}adj(AB) = \frac{1}{|A||B|}adj(AB)$$

$\Rightarrow adj(AB) = adj(B)adj(A)$.

(b) No, not necessarily:

<u>Ex</u>. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ have $adj(A) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, $adj(B) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow adj(B)adj(A) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

versus $adj(AB) = adj\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq adj(B)adj(A)$.

1.6.3.27. The last $n-r$ columns of A are the last $n-r$ columns of the $n \times n$ identity matrix I_n. Why? First, note that the $(n-i)-$th column of $I_n = \left[\, \mathbf{e}^{(1)} \,\vdots\, \ldots \,\vdots\, \mathbf{e}^{(n)} \,\right]$ is

$$\mathbf{e}^{(n-i)} = [0 \quad \ldots \quad 0 \,\vdots\, 0 \quad \ldots \quad 0 \quad 1 \quad 0 \quad \ldots \quad 0]^T$$

where there are, first, $(n-r)$ entries of zero, followed by $\big((n-i)-(n-r)\big)$ zeros until the $(n-i)-th$ entry being a one is reached, followed by another $(i-1)$ zeros. Let I_{n-r} be the $(n-r)\times(n-r)$ identity matrix, and write it in terms of its columns as $I_{n-r} = \left[\, \mathbf{f}^{(1)} \,\vdots\, \ldots \,\vdots\, \mathbf{f}^{(n-r)} \,\right]$. We see that

$$\mathbf{e}^{(n-i)} = \left[\, \mathbf{0} \,\vdots\, \mathbf{f}^{\big((n-i)-(n-r)\big)} \,\right]^T,$$

where $\mathbf{0}$ is in $\mathbb{R}^{n-r}$.

So, expand the determinant of A along the last column to get

$$|A| = \begin{vmatrix} A_{11} & \vdots & O_{r,n-r} \\ -- & \vdots & -- \\ O_{n-r,r} & \vdots & I_{n-r} \end{vmatrix} = 1 \cdot \begin{vmatrix} A_{11} & \vdots & O_{r,n-r-1} \\ --- & \vdots & --- \\ O_{n-r-1,r} & \vdots & I_{n-r-1} \end{vmatrix}$$

and then expand the determinant along the last of the $(n-1)$ columns to get

$$|A| = \begin{vmatrix} A_{11} & \vdots & O_{r,n-r-1} \\ --- & \vdots & --- \\ O_{n-r-1,r} & \vdots & I_{n-r-1} \end{vmatrix} = 1 \cdot \begin{vmatrix} A_{11} & \vdots & O_{r,n-r-2} \\ --- & \vdots & --- \\ O_{n-r-2,r} & \vdots & I_{n-r-2} \end{vmatrix}.$$

Continue this way until we get the determinant of an $r \times n$ matrix, namely

$$|A| = 1 \cdot \quad \cdots \quad \cdot 1 \cdot |A_{11}|,$$

as was desired.

1.6.3.29. $|A| = \begin{vmatrix} A_{11} & \vdots & O \\ -- & \vdots & -- \\ O & \vdots & I \end{vmatrix} \begin{vmatrix} I & \vdots & A_{11}^{-1}A_{12} \\ -- & \vdots & --- \\ O & \vdots & A_{22} \end{vmatrix} = |A_{11}| \cdot |A_{22}|,$
by the results of problems 1.6.3.27 and 1.6.3.28.

Section 1.7.3

1.7.3.1. $\mathbf{b}$ is in the Span of those three given vectors $\iff$ $\mathbf{b} = A\mathbf{x}$, for some $\mathbf{x}$. The latter equation gives augmented matrix

$$\left[\begin{array}{rrr|r} 2 & -2 & 2 & b_1 \\ 1 & 2 & 4 & b_2 \\ 0 & 3 & 3 & b_3 \end{array}\right] \underset{\substack{\frac{1}{2}R_1 \to R_1 \\ -R_1+R_2 \to R_2}}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & 1 & b_1/2 \\ 0 & 3 & 3 & b_2-(b_1/2) \\ 0 & 3 & 3 & b_3 \end{array}\right] \underset{-R_2+R_3 \to R_3}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & 1 & b_1/2 \\ 0 & 3 & 3 & b_2-(b_1/2) \\ 0 & 0 & 0 & b_3-b_2+(b_1/2) \end{array}\right]$$

So, $\mathbf{b}$ is in that Span $\iff$ $0 = \frac{1}{2}b_1 - b_2 + b_3$.

1.7.3.3. $\begin{vmatrix} 1 & t & 0 \\ -1 & 2 & 1 \\ 2 & 3 & t \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & t \end{vmatrix} - t\begin{vmatrix} -1 & 1 \\ 2 & t \end{vmatrix} = 2t - 3 + t^2 + 2t = t^2 + 4t - 3.$

So, the vectors are linearly dependent exactly for $\iff$ $t = -2 \pm \sqrt{7}$.

1.7.3.5. (a) $\begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$ because $R_2 = R_3$. So, by Theorem 1.43 in Section 1.7, the set of three given vectors in $\mathbb{R}^3$ is not a basis for $\mathbb{R}^3$, is not linearly independent, and does not span $\mathbb{R}^3$.

(b) Using $-R_1 + R_2 \to R_2$, $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$

$\Rightarrow$ The set of three given vectors is linearly independent, does span $\mathbb{R}^3$, and is a basis for $\mathbb{R}^3$, all by Theorem 1.43 in Section 1.7.

(c) The set of two given vectors cannot span $\mathbb{R}^3$ and cannot be a basis for $\mathbb{R}^3$ because we need at least three vectors to span and need exactly three vectors in order to be a basis. This follows from the Goldilocks Theorem 1.42 in Section 1.7.

$$\left[\begin{array}{rr|r} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \underset{-R_1+R_2 \to R_2}{\sim} \left[\begin{array}{rrr} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right] \underset{\substack{-R_2 \to R_2 \\ -R_2+R_1 \to R_1 \\ -R_2+R_3 \to R_3}}{\sim} \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ the only solution of $-c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$ is $c_1 = c_2 = 0$, so yes, it is linearly independent.

To summarize, the set of two given vectors is linearly independent, does not span, and is not a basis.

(d) The set of four given vectors in $\mathbb{R}^3$ cannot be linearly independent and cannot be a basis for $\mathbb{R}^3$ because $4 > 3$ and the Goldilocks Theorem 1.42 in Section 1.7.

Does the set of four given vectors span $\mathbb{R}^3$? For any $\mathbf{b}$ in $\mathbb{R}^3$,

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 1 & b_1 \\ 2 & 1 & 0 & -3 & b_2 \\ 3 & 4 & 5 & 0 & b_3 \end{array}\right] \underset{\substack{-2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3}}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 1 & b_1 \\ 0 & 1 & 2 & -5 & b_2-2b_1 \\ 0 & 4 & 8 & -3 & b_3-3b_1 \end{array}\right]$$

$$\underset{-4R_2+R_3 \to R_3}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 1 & b_1 \\ 0 & 1 & 2 & -5 & b_2-2b_1 \\ 0 & 0 & 0 & 17 & b_3-3b_1-4b_2+8b_1 \end{array}\right]$$

$\Longrightarrow$ There exists a solution for any $\mathbf{b}$ in $\mathbb{R}^3$, that is, the set of four given vectors does span $\mathbb{R}^3$. To summarize, the set of four given vectors is not linearly independent, does span, and is not a basis.

1.7.3.7. (a) First, $rank(A) \geq 2$ because because there exists at least two pivot positions in $\left[\, \mathbf{v}^{(1)} \;\vdots\; \mathbf{v}^{(2)} \,\right]$. Why? Because, otherwise, either (i) $\mathbf{v}^{(1)} \neq 0$ but $\mathbf{v}^{(2)}$ is a multiple of $\mathbf{v}^{(1)}$, or (ii) $\mathbf{v}^{(1)} = \mathbf{0}$. In either case of (i) or (ii), the set $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ would be linearly dependent, giving a contradiction.

Second, the rank of a 4×5 matrix must be ≤ 4, because $rank(A) =$number of pivot positions in A, which is $\leq$ number of rows of A.

To summarize, $2 \leq rank(A) \leq 4$.

(b) $\nu(A) = n - rank(A)$ is always true. Here $n = 5$. From part (a), $2 \leq rank(A)$ so $-rank(A) \leq -2$ implies $\nu(A) = 5 + \big(-rank(A)\big) \leq 5 + (-2) = 3$.

Also, $-rank(A) \geq -4$ implies $\nu(A) = 5 + \big(-rank(A)\big) \geq 5 + (-4) = 1$.

To summarize, $1 \leq nullity(A) \leq 3$.

1.7.3.9. (a) $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \underset{-3R_1 + R_2 \to R_2}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -6 \end{bmatrix} \Rightarrow rank(A) = 2$

(b) $\nu(A) = 3 - rank(A) = 3 - 2 = 1$

(c) $\left[\begin{array}{rrr|r} 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right] \underset{-3R_1 + R_2 \to R_2}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 0 \\ 0 & 4 & -6 & 0 \end{array}\right] \underset{\substack{\frac{1}{4}R_2 \to R_2 \\ R_2 + R_1 \to R_1}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0.5 & 0 \\ 0 & 1 & -1.5 & 0 \end{array}\right]$

$\Rightarrow x_3 = \text{free} = c_1$, and solutions are $\mathbf{x} = \begin{bmatrix} -0.5c_1 \\ 1.5c_1 \\ c_1 \end{bmatrix} = c_1 \begin{bmatrix} -0.5 \\ 1.5 \\ 1 \end{bmatrix} \Longrightarrow \mathcal{W}$ has basis $\left\{ \begin{bmatrix} -0.5 \\ 1.5 \\ 1 \end{bmatrix} \right\}$.

1.7.3.11. (a) <u>Ex</u>: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has $rank(A) = 2 \quad \Longrightarrow n = 3$, $\nu(A) = 3 - rank(A) = 3 - 2 = 1$.

$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $rank(A^T) = 2$. So, A^T being 3×2 implies $\nu(A^T) = 2 - rank(A^T) = 2 - 2 = 0$. So, $\nu\left(A^T\right) = 0 \neq 1 = \nu(A)$.

(b) Suppose $m = n$. By Theorem 1.44 in Section 1.7, $rank(A) = rank\left(A^T\right)$, so $m = n \Rightarrow \nu(A) = n - rank(A)$, so

$$\nu(A^T) = n - rank(A^T) = n - rank(A) = \nu(A).$$

Chapter Two

Section 2.1.6

2.1.6.1. $0 = \begin{vmatrix} -2-\lambda & 7 \\ 1 & 4-\lambda \end{vmatrix} = (-2-\lambda)(4-\lambda) - 7 = \lambda^2 - 2\lambda - 15 = (\lambda-5)(\lambda+3)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -3, \lambda_2 = 5$

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 1 & 7 & 0 \\ 1 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -7 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -3$.

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -7 & 7 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-\frac{1}{7}R_1 \rightarrow R_1,\ -R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 5$.

2.1.6.3. $0 = \begin{vmatrix} -1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) - 4 = \lambda^2 - 5$

$\Rightarrow$ eigenvalues are $\lambda_1 = \sqrt{5},\ \lambda_2 = -\sqrt{5}$

$[A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -1-\sqrt{5} & 4 & 0 \\ 1 & 1-\sqrt{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1-\sqrt{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $(1+\sqrt{5})R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -1+\sqrt{5} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = \sqrt{5}$.

$[A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -1+\sqrt{5} & 4 & 0 \\ 1 & 1+\sqrt{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1+\sqrt{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $(1-\sqrt{5})R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} -1-\sqrt{5} \\ 1 \end{bmatrix}$, for any const. $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = -\sqrt{5}$.

2.1.6.5. We are given three distinct eigenvalues, so the only eigenvalues are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 4$. All we need to do is to find the eigenvectors.

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -1 & 3 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \underset{\substack{-R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\substack{\frac{1}{4}R_2 \rightarrow R_2 \\ -R_2 + R_1 \rightarrow R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = 0$.

$$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \underset{\substack{-R_1 \rightarrow R_1 \\ R_1 + R_2 \rightarrow R_2 \\ -R_1 + R_3 \rightarrow R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \underset{\substack{R_2 \leftrightarrow R_3 \\ \frac{1}{2}R_2 \rightarrow R_2 \\ R_2 + R_1 \rightarrow R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_2 = 2.$$

$$[\,A - \lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{rrr|r} -3 & 1 & 2 & 0 \\ -1 & -1 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_3 \\ R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3}}{\sim} \left[\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right] \underset{\substack{R_2 \leftrightarrow R_3 \\ \frac{1}{4}R_2 \to R_2 \\ -R_2 + R_1 \to R_1}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \mathbf{v}_3 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_3 = 4.$$

2.1.6.7. By expanding along the first row, we calculate

$$0 = \begin{vmatrix} -3-\lambda & 0 & 0 \\ 4 & -4-\lambda & -3 \\ -1 & 1 & -\lambda \end{vmatrix} = (-3-\lambda) \cdot \begin{vmatrix} -4-\lambda & -3 \\ 1 & -\lambda \end{vmatrix} = (-3-\lambda)\big((-4-\lambda)(-\lambda)+3\big) = (-3-\lambda)\big(\lambda^2+4\lambda+3\big)$$

$$= (-3-\lambda)(\lambda+3)(\lambda+1),$$

so the eigenvalues are $\lambda_1 = \lambda_2 = -3, \lambda_3 = -1$.

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 4 & -1 & -3 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_3 \\ -R_1 \to R_1 \\ -4R_1 + R_2 \to R_2}}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & -3 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\substack{\frac{1}{3}R_2 \to R_2 \\ R_2 + R_1 \to R_1}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the only eigenvectors corresponding to eigenvalue } \lambda_1 = -3.$$

So, $\lambda = -3$ is a defective eigenvalue.

Because $\lambda_2 = \lambda_1$, we get no further eigenvectors corresponding to eigenvalue λ_2.

$$[\,A - \lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{rrr|r} -2 & 0 & 0 & 0 \\ 4 & -3 & -3 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \underset{\substack{-\frac{1}{2}R_1 \to R_1 \\ -4R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{\substack{-\frac{1}{3}R_2 \to R_2 \\ -R_2 + R_3 \to R_3 \\ 2R_2 + R_3 \to R_3}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \mathbf{v}_3 = c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_3 = -1.$$

2.1.6.9. By expanding along the first row, we calculate

$$0 = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 1 \\ -1 & 2 & 5-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 2 & 5-\lambda \end{vmatrix} = (1-\lambda)\Big((3-\lambda)(5-\lambda)-2\Big) = (1-\lambda)(\lambda^2 - 8\lambda + 13),$$

so the eigenvalues are $\lambda_1 = 1, \lambda_2 = 4 + \sqrt{3}, \lambda_3 = 4 - \sqrt{3}$.

$$[\,A-\lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ -1 & 2 & 4 & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_3 \\ -R_1 \to R_1 \\ -2R_1 + R_2 \to R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 6 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\substack{\frac{1}{6}R_2 \to R_2 \\ 2R_2 + R_1 \to R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = 1$.

$$[A-\lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -3-\sqrt{3} & 0 & 0 & 0 \\ 2 & -1-\sqrt{3} & 1 & 0 \\ -1 & 2 & 1-\sqrt{3} & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_3 \\ -R_1 \to R_1 \\ -2R_1 + R_2 \to R_2 \\ (3+\sqrt{3})R_1 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -1+\sqrt{3} & 0 \\ 0 & 3-\sqrt{3} & 3-2\sqrt{3} & 0 \\ 0 & -2(3+\sqrt{3}) & 2\sqrt{3} & 0 \end{array}\right].$$

Note that $\dfrac{3-2\sqrt{3}}{3-\sqrt{3}} = \dfrac{3-2\sqrt{3}}{3-\sqrt{3}} \cdot \dfrac{3+\sqrt{3}}{3+\sqrt{3}} = ... = \dfrac{1-\sqrt{3}}{2}$. So,

$$[A-\lambda_2 I \mid \mathbf{0}\,] \underset{\substack{(3-\sqrt{3})^{-1}R_2 \to R_2 \\ 2R_2 + R_1 \to R_1 \\ 2(3+\sqrt{3})R_2 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1-\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 0 \\ \frac{-1+\sqrt{3}}{2} \\ 1 \end{bmatrix}$, for any const. $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 4+\sqrt{3}$.

$$[A-\lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -3+\sqrt{3} & 0 & 0 & 0 \\ 2 & -1+\sqrt{3} & 1 & 0 \\ -1 & 2 & 1+\sqrt{3} & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_3 \\ -R_1 \to R_1 \\ -2R_1 + R_2 \to R_2 \\ (3-\sqrt{3})R_1 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -1-\sqrt{3} & 0 \\ 0 & 3+\sqrt{3} & 3+2\sqrt{3} & 0 \\ 0 & -2(3-\sqrt{3}) & -2\sqrt{3} & 0 \end{array}\right].$$

Note that $\dfrac{3+2\sqrt{3}}{3+\sqrt{3}} = \dfrac{3+2\sqrt{3}}{3+\sqrt{3}} \cdot \dfrac{3-\sqrt{3}}{3-\sqrt{3}} = ... = \dfrac{1+\sqrt{3}}{2}$. So,

$$[A-\lambda_3 I \mid \mathbf{0}\,] \underset{\substack{(3-\sqrt{3})^{-1}R_2 \to R_2 \\ 2R_2 + R_1 \to R_1 \\ 2(3-\sqrt{3})R_2 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1+\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \mathbf{v}_3 = c_1 \begin{bmatrix} 0 \\ \frac{-1-\sqrt{3}}{2} \\ 1 \end{bmatrix}$, for any const. $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_3 = 4-\sqrt{3}$.

2.1.6.11. $A\mathbf{x} = \lambda B\mathbf{x}$, that is, $(A - \lambda B)\mathbf{x} = \mathbf{0}$, has a non-trivial solution for $\mathbf{x}$ if, and only if,

$$0 = |\,A - \lambda B\,| = \left| \begin{bmatrix} -1 & 0 & 5 \\ 2 & 1 & 4 \\ 3 & -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -1 & 0 & 5 \\ 2 & 1-\lambda & 4 \\ 3 & -2 & 3-\lambda \end{vmatrix}$$

$$= (-1)\cdot \begin{vmatrix} 1-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} + 5\cdot \begin{vmatrix} 2 & 1-\lambda \\ 3 & -2 \end{vmatrix} = -\big((1-\lambda)(3-\lambda)+8\big) + 5\big(-4-3(1-\lambda)\big),$$

$$= -\lambda^2 + 4\lambda - 11 - 35 + 15\lambda = -(\lambda^2 - 19\lambda + 46).$$

by expanding along the first row. So, the only such values are $\lambda = \dfrac{19 \pm \sqrt{177}}{2}$.

2.1.6.13. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ being an eigenvector of $A = \begin{bmatrix} 4 & 0 & 10 \\ -5 & -6 & -5 \\ 5 & 0 & -1 \end{bmatrix}$ leads us to calculate

$$A\mathbf{v} = \begin{bmatrix} 4 & 0 & 10 \\ -5 & -6 & -5 \\ 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ -9 \\ 9 \end{bmatrix} = 9 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \Rightarrow \lambda = 9 \text{ is an eigenvalue of } A$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ being an eigenvector of $A = \begin{bmatrix} 4 & 0 & 10 \\ -5 & -6 & -5 \\ 5 & 0 & -1 \end{bmatrix}$ leads us to calculate

$$A\mathbf{v} = \begin{bmatrix} 4 & 0 & 10 \\ -5 & -6 & -5 \\ 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix} = -6 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \Rightarrow \lambda = -6 \text{ is an eigenvalue of } A$$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ being an eigenvector of $A = \begin{bmatrix} 4 & 0 & 10 \\ -5 & -6 & -5 \\ 5 & 0 & -1 \end{bmatrix}$ leads us to calculate

$$A\mathbf{v} = \begin{bmatrix} 4 & 0 & 10 \\ -5 & -6 & -5 \\ 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix} = -6 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \Rightarrow \lambda = -6 \text{ is an eigenvalue of } A$$

Denote the eigenvalues we found by $\mu_1 = 9$ and $\mu_2 = -6$. Are there any other eigenvalues of A?

The latter two eigenvectors form a linearly independent set $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, so the nullity of $\big(A - (-6)I\big)$, that is, the geometric multiplicity, is $m_2 \geq 2$. This implies that α_2, the algebraic multiplicity of -6, is at least two.

α_1, the algebraic multiplicity of 9, is at least one. By Theorem 2.3(a) in Section 2.1, $\alpha_1 + \alpha_2 \leq 3$ But $\alpha_1 \geq 1$ and $\alpha_2 \geq 2$, so $\alpha_1 + \alpha_2 \geq 3$. It follows that $\alpha_1 + \alpha_2 = 3$ and the only eigenvalues of A are 9 and -6.

Because $\alpha_2 \geq 2$, $\alpha_1 \geq 1$, and $\alpha_1 + \alpha_2 = 3$, we conclude that $\alpha_1 = 1$ and $\alpha_2 = 2$. Because $m_2 \geq 2$ and $m_2 \leq \alpha_2 = 2$, we conclude that $m_2 = 2$.

Also, $\alpha_1 = 1$ and $1 \leq m_1 \leq \alpha_1$ imply $m_1 = 1$.

To summarize, the only eigenvalues of A are $\mu_1 = 9$, with algebraic multiplicity $\alpha_1 = 1$ and geometric multiplicity $m_1 = 1$, and eigenvalue $\mu_2 = -6$, with algebraic multiplicity $\alpha_2 = 2$ and geometric multiplicity $m_2 = 2$.

2.1.6.15. (a) Can $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$, as well as $A\mathbf{x} = \lambda_2 \mathbf{x}$, if $\lambda_1 \neq \lambda_2$? No, two unequal eigenvalues cannot have the same eigenvector because then we would have

$$\lambda_1 \mathbf{x} = A\mathbf{x} = \lambda_2 \mathbf{x} \quad \text{and} \quad \mathbf{x} \neq \mathbf{0},$$

hence

$$\lambda_1 \mathbf{x} = \lambda_2 \mathbf{x} \quad \text{and} \quad \mathbf{x} \neq \mathbf{0},$$

hence

$$(\lambda_1 - \lambda_2)\mathbf{x} = \lambda_1 \mathbf{x} - \lambda_2 \mathbf{x} = \mathbf{0} \quad \text{and} \quad \mathbf{x} \neq \mathbf{0}.$$

If the vector $\mathbf{x} \neq \mathbf{0}$ and the vector $(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$, then we *must have* that the scalar $(\lambda_1 - \lambda_2) = 0$, contradicting the assumption that λ_1 and λ_2 are unequal. [Why "*must have*?" Because $\mathbf{x} \neq \mathbf{0}$, at least one of its entries, say x_j, must be nonzero. If $(\lambda_1 - \lambda_2) \neq 0$, then the j-th entry in $(\lambda_1 - \lambda_2)\mathbf{x}$ would be $(\lambda_1 - \lambda_2)x_j \neq 0$, so $(\lambda_1 - \lambda_2)\mathbf{x}$ would be a nonzero vector, contradicting $(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$.]

(b) Yes, a nonzero vector $\mathbf{x}$ be an eigenvector for two unequal eigenvalues λ_1 and λ_2 corresponding to two different matrices A and B, respectively.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$ both have $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector corresponding to A's eigenvalue $\lambda_1 = 1$ and B's eigenvalue $\lambda_2 = 5$, respectively.

2.1.6.17. (a) Because $C = A + B$ and $\mathbf{x}$ is an eigenvector for *both* A and B, corresponding to eigenvalues λ and β, respectively, $\mathbf{x} \neq \mathbf{0}$ and

$$C\mathbf{x} = (A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \lambda \mathbf{x} + \beta \mathbf{x} = (\lambda + \beta)\mathbf{x},$$

which implies $\gamma \triangleq \lambda + \beta$ is an eigenvalue of C with corresponding eigenvector $\mathbf{x}$.

(b) If, in addition, $C = A^2$, then $\lambda^2 \mathbf{x} = A^2 \mathbf{x} = C\mathbf{x} = (\lambda + \beta)\mathbf{x}$, hence $\big(\lambda^2 - (\lambda + \beta)\big)\mathbf{x} = \mathbf{0}$. Because $\mathbf{x} \neq \mathbf{0}$, it follows that

$$\lambda^2 - \lambda - \beta = 0.$$

This quadratic equation for λ only has solutions $\lambda = \dfrac{1 \pm \sqrt{1 + 4\beta}}{2}$. So it follows that either $\lambda = \frac{1}{2}\big(1 + \sqrt{1 + 4\beta}\big)$ or $\lambda = \frac{1}{2}\big(1 - \sqrt{1 + 4\beta}\big)$.

2.1.6.19. Let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be two distinct solutions of $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{x}^{(1)} = \mathbf{b}$ and $A\mathbf{x}^{(2)} = \mathbf{b}$, so $A\big(\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\big) = A\mathbf{x}^{(1)} - A\mathbf{x}^{(2)} = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

(a) So, $\mathbf{x} \triangleq \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$ is a nonzero solution of $A\mathbf{x} = \mathbf{0}$. By definition, $\lambda = 0$ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$.

(b) The assumption that $\mathbf{b} = \mathbf{0}$ is actually irrelevant. By part (a), $A\mathbf{x} = \mathbf{0}$ has a nonzero solution $\mathbf{x}$. Linearity implies that $A\mathbf{x} = \mathbf{0}$ would have infinitely many solutions, namely $c\,\mathbf{x}$ for all scalars c.

2.1.6.21. The n distinct eigenvalues of A must each have algebraic multiplicity of one, by Theorem 2.3(a) in Section 2.1. It follows that each of the geometric multiplicities must be one, by Theorem 2.3(c) in Section 2.1.

Because

$$0 = |\, A - \lambda I \,| = \begin{vmatrix} a_{11} - \lambda & a_{12} & . \;\; . \;\; . & a_{1n} \\ 0 & a_{22} - \lambda & & a_{2n} \\ . & . & . & . \\ . & . & \;\; . & . \\ . & . & \;\;\;\; . & . \\ 0 & 0 & . \;\; . \;\; . & a_{nn} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda),$$

the eigenvalues of A are a_{11}, a_{22}, ..., a_{nn}.

To find the eigenvectors, note that the fact that the a_{jj} are distinct implies $a_{22} - a_{11} \neq 0$, so

$$[\,A - a_{11}I\,|\,\mathbf{0}\,] = \left[\begin{array}{ccccc|c} 0 & a_{12} & . \ . \ . & a_{1n} & & 0 \\ 0 & a_{22}-a_{11} & & a_{2n} & & 0 \\ . & 0 & . & . & & 0 \\ . & . & . & . & & 0 \\ . & . & . & . & & 0 \\ 0 & 0 & . \ . \ . & a_{nn}-a_{11} & & 0 \end{array}\right] \underset{\substack{\frac{1}{a_{22}-a_{11}}R_2 \to R_2 \\ \vdots \\ \frac{1}{a_{nn}-a_{11}}R_n \to R_n}}{\sim} \left[\begin{array}{ccccc|c} 0 & a_{12} & & & a_{1n} & 0 \\ 0 & 1 & . \ . \ . & & a_{2n} & 0 \\ . & 0 & 1 & & . & 0 \\ . & . & & . & . & 0 \\ . & . & & . & . & 0 \\ 0 & 0 & . \ . \ . & & 1 & 0 \end{array}\right].$$

We see that $\mathbf{e}^{(1)} = [\,1 \quad 0 \quad ... \quad 0\,]^T$ is an eigenvector corresponding to eigenvalue a_{11}.

Similarly,

$$[\,A - a_{22}I\,|\,\mathbf{0}\,] = \left[\begin{array}{ccccc|c} a_{11}-a_{22} & a_{12} & . \ . \ . & a_{1n} & & 0 \\ 0 & 0 & & a_{2n} & & 0 \\ . & 0 & . & . & & 0 \\ . & . & . & . & & 0 \\ . & . & . & . & & 0 \\ 0 & 0 & . \ . \ . & a_{nn}-a_{22} & & 0 \end{array}\right] \underset{\substack{\frac{1}{a_{11}-a_{22}}R_n \\ \vdots \\ \frac{1}{a_{nn}-a_{22}}R_n \to R_n}}{\sim} \left[\begin{array}{ccccc|c} 1 & a_{12} & . \ . \ . & & a_{1n} & 0 \\ 0 & 0 & & & a_{2n} & 0 \\ . & 0 & . & & . & 0 \\ . & . & . & & . & 0 \\ . & . & & . & . & 0 \\ 0 & 0 & . \ . \ . & & 1 & 0 \end{array}\right].$$

We see that $\mathbf{e}^{(2)} = [\,1 \quad 0 \quad ... \quad 0\,]^T$ is an eigenvector corresponding to eigenvalue a_{22}.

We see in that for an upper triangular matrix whose diagonal entries are distinct, the eigenvalues are the diagonal entries a_{11}, a_{22}, ..., a_{nn} and the corresponding eigenvectors are $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$,..., $\mathbf{e}^{(n)}$, respectively.

2.1.6.23. (a) If $(I - A^{-1})$ were not invertible, then there would be an $\mathbf{x} \neq \mathbf{0}$ with $(I - A^{-1})\mathbf{x} = \mathbf{0}$, hence $\mathbf{x} - A^{-1}\mathbf{x} = \mathbf{0}$, hence $\mathbf{x} = A^{-1}\mathbf{x}$, hence $A\mathbf{x} = AA^{-1}\mathbf{x} = I\mathbf{x} = \mathbf{x}$, hence $\lambda = 1$ would be an eigenvalue of A, contradicting the given information. So, $(I - A^{-1})$ is invertible.

Alternatively, we can calculate $|\,(A - I)\,| = |\,A(I - A^{-1})\,| = |\,A\,| \cdot |\,(I - A^{-1})\,|$, and use the given information...

(b) Using Theorem 1.23(c) in Section 1.5, $A^{-1}(I-A^{-1})^{-1} = \big((I-A^{-1})A\big)^{-1} = \big(I\,A - A^{-1}A\big)^{-1} = (A-I)^{-1}$.

(c) Using Theorem 1.23(c) in Section 1.5, $(I-A^{-1})^{-1}A^{-1} = \big(A(I-A^{-1})\big)^{-1} = \big(A\,I - AA^{-1}\big)^{-1} = (A-I)^{-1}$.

2.1.6.25. (a) We are given that $\mathbf{x} \neq \mathbf{0}$, which implies $\mathbf{y} \triangleq B\mathbf{x} \neq \mathbf{0}$. Why? Because, if not, then $\mathbf{0} = B^{-1}\mathbf{0} = B^{-1}\mathbf{y} = B^{-1}B\mathbf{x} = \mathbf{x}$, which would contradict $\mathbf{x} \neq \mathbf{0}$.

(b) We were given that $AB = BA$ and that $\mathbf{x}$ is an eigenvector of A corresponding to eigenvalue λ. It follows that

$$A\big(B\mathbf{x}\big) = (AB)\mathbf{x} = (BA)\mathbf{x} = B\big(A\mathbf{x}\big) = B\big(\lambda\mathbf{x}\big) = \lambda(B\mathbf{x}).$$

$\mathbf{y} \triangleq B\mathbf{x} \neq \mathbf{0}$ satisfies $A\mathbf{y} = \lambda\mathbf{y}$, hence by definition $\mathbf{y}$ is an eigenvector of A corresponding to eigenvalue λ.

(c) Suppose, in addition to all of the above assumptions, A's eigenvalue λ has geometric multiplicity equal to one. Then the additional facts that both $B\mathbf{x}$ and $\mathbf{x}$ are nonzero and eigenvectors of A corresponding to eigenvalue λ imply that $\{B\mathbf{x}, \mathbf{x}\}$ is linearly dependent. It follows from Theorem 1.35 in Section 1.7 that either $\mathbf{x}$ can be written as a linear combination of $B\mathbf{x}$ or $B\mathbf{x}$ can be written as a linear combination of $\mathbf{x}$. Because both $\mathbf{x}$ and $B\mathbf{x}$ are nonzero, in either case it follows that $B\mathbf{x} = \mu\mathbf{x}$, for some scalar μ. So, $\mathbf{x}$ is an eigenvector of B.

2.1.6.27. (a) A is invertible if, and only if, $|A| \neq 0$. Because $|A| = |\,A - 0 \cdot I\,| = \mathcal{P}(0)$, we see that A is invertible if, and only if, $\mathcal{P}(0) \neq 0$, which is true if and only if 0 is not an eigenvalue of A.

(b) If every eigenvalue of A is greater than 2, then $\mathcal{P}(\lambda) \neq 0$ for all $\lambda \leq 2$. In particular, $|\, A - 1 \cdot I \,| = \mathcal{P}(1) \neq 0$, so $(A - I)$ is invertible. It follows that $(I - A) = -(A - I)$ is invertible.

(c) By the result of part (a), because 0 is not an eigenvalue of A, it follows that A is invertible. Because every eigenvalue of A is greater than 2, there is no nonzero vector $\mathbf{x}$ for which $A\mathbf{x} = 1 \cdot \mathbf{x}$. It follows that there is no nonzero vector for which $\mathbf{x} = I\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{x}$. It follows that $(A^{-1} - I)$ is invertible, hence $(I - A^{-1}) = -(A^{-1} - I)$ is invertible.

2.1.6.29. (a) $(A - \lambda B)\mathbf{x} = \mathbf{0}$ has a non-trivial solution for $\mathbf{x}$ if, and only if,

$$0 = |\, A - \lambda B \,| = \left| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1+\lambda \\ -\lambda & -\lambda \end{vmatrix} = -\lambda(-\lambda) + \lambda(1+\lambda) = \lambda(2\lambda + 1),$$

if and only if $\lambda_1 = 0$ or $\lambda_2 = -\frac{1}{2}$.

(b) For generalized eigenvalue $\lambda_1 = 0$, $[\, A - \lambda_1 B \mid \mathbf{0} \,] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$ is already in RREF.

$\Rightarrow \mathbf{w}_1 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for any const. $c_1 \neq 0$, are the eigenvectors corresponding to generalized eigenvalue $\lambda_1 = 0$.

$$[\, A - \lambda_2 B \mid \mathbf{0} \,] = \left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ & & \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right], \text{ after } -R_1 + R_2 \to R_2,\ 2R_1 \to R_1$$

$\Rightarrow \mathbf{w}_2 = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, for any const. $c_1 \neq 0$, are the eigenvectors corresponding to generalized eigenvalue $\lambda_2 = -\frac{1}{2}$.

Section 2.2.3

2.2.3.1. $0 = \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)+3 = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4)$

$\Rightarrow$ eigenvalues $\lambda_1 = 4$, $\lambda_2 = 2$.

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-3R_1 + R_2 \rightarrow R_2$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_1 = 4$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{rr|r} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-R_1 + R_2 \rightarrow R_2$, $\frac{1}{3}R_1 \rightarrow R_1$.

$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_2 = 2$

The matrix $P = [\,\mathbf{p}^{(1)} \mid \mathbf{p}^{(2)}\,] = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ should diagonalize A.

2.2.3.3. $0 = \begin{vmatrix} 2-\lambda & 0 \\ -1 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 0 = (2-\lambda)(-1-\lambda)$

$\Rightarrow$ eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$.

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{rr|r} 0 & 0 & 0 \\ -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-R_1 \rightarrow R_1$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_1 = 6$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{rr|r} 3 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $\frac{1}{3}R_1 \rightarrow R_1$, $R_1 + R_2 \rightarrow R_1$.

$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_2 = -1$

The matrix $P = [\,\mathbf{p}^{(1)} \mid \mathbf{p}^{(2)}\,] = \begin{bmatrix} -3 & 0 \\ 1 & 1 \end{bmatrix}$ should diagonalize A.

2.2.3.5. $0 = \begin{vmatrix} -2-\lambda & \sqrt{2} \\ -\sqrt{2} & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) + 2 = \lambda^2 + \lambda = \lambda^2 - 2$

$\Rightarrow$ eigenvalues $\lambda_1 = \sqrt{2}$, $\lambda_2 = -\sqrt{2}$.

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{rr|r} -2-\sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & 2-\sqrt{2} & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 1-\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-\frac{1}{\sqrt{2}}R_1 \rightarrow R_1$, $(2+\sqrt{2})R_1 + R_2 \rightarrow R_2$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_1 = \sqrt{2}$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{rr|r} -2+\sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & 2+\sqrt{2} & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -1-\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-\frac{1}{\sqrt{2}}R_1 \rightarrow R_1$, $(2-\sqrt{2})R_1 + R_2 \rightarrow R_2$.

$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_2 = -\sqrt{2}$

The matrix $P = [\,\mathbf{p}^{(1)} \mid \mathbf{p}^{(2)}\,] = \begin{bmatrix} -1+\sqrt{2} & 1+\sqrt{2} \\ 1 & 1 \end{bmatrix}$ should diagonalize A.

2.2.3.7. $0 = \begin{vmatrix} 6-\lambda & 7 & 7 \\ -7 & -8-\lambda & -7 \\ 7 & 7 & 6-\lambda \end{vmatrix} \underset{R_2+R_3\to R_3}{=} \begin{vmatrix} 6-\lambda & 7 & 7 \\ -7 & -8-\lambda & -7 \\ 0 & -1-\lambda & -1-\lambda \end{vmatrix}$

$$\underset{R_3\leftarrow(-1-\lambda)R_3}{=} (-1-\lambda)\begin{vmatrix} 6-\lambda & 7 & 7 \\ -7 & -8-\lambda & -7 \\ 0 & 1 & 1 \end{vmatrix} = (-1-\lambda)\left(-\begin{vmatrix} 6-\lambda & 7 \\ -7 & -7 \end{vmatrix} + \begin{vmatrix} 6-\lambda & 7 \\ -7 & -8-\lambda \end{vmatrix}\right)$$

$= (-1-\lambda)\Big(7(6-\lambda) - \cancel{49} + (6-\lambda)(-8-\lambda) + \cancel{49}\Big) = (-1-\lambda)(6-\lambda)(7-8-\lambda) = (-1-\lambda)^2(6-\lambda)$ so the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 6$.

$$[\,A-\lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 7 & 7 & 7 & 0 \\ -7 & -7 & -7 & 0 \\ 7 & 7 & 7 & 0 \end{array}\right] \underset{\substack{R_1+R_2\to R_2 \\ -R_1+R_3\to R_3 \\ 7^{-1}R_1\to R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_2 = c_1$ and $x_3 = c_2$ are free variables and $\mathbf{v}_1 = \begin{bmatrix} -c_1-c_2 \\ c_1 \\ c_2 \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, for any constants c_1, c_2 with $|c_1| + |c_2| > 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = \lambda_2 = -1$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{p}^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors that span the eigenspace $\mathcal{E}_{\lambda=-1}$.

$$[A-\lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 0 & 7 & 7 & 0 \\ -7 & -14 & -7 & 0 \\ 7 & 7 & 0 & 0 \end{array}\right] \underset{\substack{R_1\leftrightarrow R_3 \\ R_1+R_2\to R_2 \\ 7^{-1}R_1\to R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right] \underset{\substack{R_2+R_3\to R_3 \\ (-7)^{-1}R_2\to R_2 \\ -R_2+R_1\to R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_3 = c_1$ is the only free variable and $\mathbf{v}_3 = \begin{bmatrix} c_1 \\ -c_1 \\ c_1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_3 = 6$.

The matrix $P = [\,\mathbf{p}^{(1)} \mid \mathbf{p}^{(2)} \mid \mathbf{p}^{(3)}\,] = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ should diagonalize A.

2.2.3.9. It turns out that we can do the problem in a straight forward way without the given information that -1 and 3 are eigenvalues. Expanding along the third row,

$$0 = \begin{vmatrix} 3-\lambda & 0 & -12 \\ 4 & -1-\lambda & -12 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 4 & -1-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)(-1-\lambda)$$

$\Rightarrow$ eigenvalues are $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 3$.

$$[\,A-\lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 4 & 0 & -12 & 0 \\ 4 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\substack{-R_1+R_2\to R_2 \\ 4^{-1}R_1\to R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_2 = c_1$ and $x_3 = c_2$ are free variables and $\mathbf{v}_1 = \begin{bmatrix} 3c_2 \\ c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, for any constants c_1, c_2 with $|c_1| + |c_2| > 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = \lambda_2 = -1$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{p}^{(2)} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors that span the eigenspace $\mathcal{E}_{\lambda=-1}$.

$$[A - \lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 0 & 0 & -12 & 0 \\ 4 & -4 & -12 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_2 \\ 4^{-1}R_1 \to R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \underset{\substack{(-12)^{-1}R_2 \to R_3 \\ 3R_2 + R_1 \to R_1 \\ 4R_2 + R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_3 = 0$ and $x_2 = c_1$ is the only free variable and $\mathbf{v}_3 = \begin{bmatrix} c_1 \\ c_1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_3 = 3$.

The matrix $P = [\,\mathbf{p}^{(1)} \mid \mathbf{p}^{(2)} \mid \mathbf{p}^{(3)}\,] = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ should diagonalize A.

A calculator gives $P^{-1} = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix}$ and

$$P^{-1}AP = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 & -12 \\ 4 & -1 & -12 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & -3 & 3 \\ -1 & 0 & 3 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D,$$

as we expected.

2.2.3.11. (a) Ex: $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is upper triangular so it has eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$.

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \underset{\substack{2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ -R_1 \to R_1}}{\sim} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 = c_1$ and $x_2 = c_2$ are free variables and $\mathbf{v}_1 = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, for any constants c_1, c_2 with $|c_1| + |c_2| > 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = \lambda_2 = 1$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{p}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are eigenvectors that span the eigenspace $\mathcal{E}_{\lambda=1}$.

$[A-\lambda_3 I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ is already in RREF.

$\Rightarrow x_3 = c_1$ is the only free variable and $\mathbf{v}_3 = \begin{bmatrix} c_1 \\ -2c_1 \\ c_1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_3 = 0$.

(b) $\{\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \mathbf{p}^{(3)}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

2.2.3.13. (a) must be true, by the definition of the word "eigenvector"

(b) may be true and may be false. E.g., if $A = A^T$ then $\mathbf{x}$ is an eigenvector of A^T. See also problem 2.1.6.28, where the eigenvectors of A are not eigenvectors of A^T.

(c) must be false, by the definition of the word "eigenvector"

(d) must be false. Note that $\mathbf{x}$ is a vector in $\mathbb{R}^n$ so it couldn't be a factor of any equation in $\mathbb{R}^n$ unless $n = 1$! But, we were given that $n \geq 2$.

(e) must be true, because $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A^2\mathbf{x} = \lambda^2\mathbf{x}$

(f) may be true and may be false. If the eigenspace of A in which $\mathbf{x}$ lies is one dimensional, then because $\mathbf{x} \neq \mathbf{0}$ it would follow that $\mathbf{x}$ is a basis for that eigenspace. But if that eigenspace has dimension two or higher then $\mathbf{x}$ can't be a basis for that eigenspace.

(g) may be true and may be false. Because B is similar to A there is an invertible matrix P with $B = P^{-1}AP$. If $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$, then $B\mathbf{x} = P^{-1}AP\mathbf{x}$. If $P\mathbf{x} = \beta\mathbf{x}$ and $\beta \neq 0$, then $P^{-1}\mathbf{x} = \beta^{-1}\mathbf{x}$ and

$$B\mathbf{x} = P^{-1}AP\mathbf{x} = P^{-1}A\beta\mathbf{x} = P^{-1}\lambda\beta\mathbf{x} = \lambda\beta P^{-1}\mathbf{x} = \lambda\beta\beta^{-1}\mathbf{x} = \lambda\mathbf{x},$$

so $\mathbf{x}$ would be an eigenvector of B, too. But, on the other hand, if $P\mathbf{x}$ is not an eigenvector of A then $\mathbf{x}$ is not an eigenvector of B, by Theorem 2.11 in Section 2.2.

2.2.3.15. $[A-(-2I) \mid \mathbf{0}] = \left[\begin{array}{ccc|c} -2 & -2 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_2 \\ 2R_1 + R_2 \rightarrow R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_2 = c_1$ and $x_3 = c_2$ are free variables and $\mathbf{v}_1 = \begin{bmatrix} -c_1 + c_2 \\ c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, for any constants c_1, c_2 with $|c_1| + |c_2| > 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = \lambda_2 = -2$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{p}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors that span the eigenspace $\mathcal{E}_{\lambda=-2}$.

$[A-(-3I) \mid \mathbf{0}] = \left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_2 \\ R_1 + R_2 \rightarrow R_2 \\ R_2 \leftrightarrow R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{\substack{R_2 + R_1 \rightarrow R_1 \\ -R_2 + R_3 \rightarrow R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_3 = 0$ and $x_2 = c_1$ is the only free variable and $\mathbf{v}_3 = \begin{bmatrix} -2c_1 \\ c_1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_3 = -3$.

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a set of three eigenvectors of A, and it is a basis for $\mathbb{R}^3$ because by expanding along the third row,

$$\begin{vmatrix} -1 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} = -1 \neq 0.$$

2.2.3.17. Because $n = 3$, the geometric multiplicity of A's eigenvalue $\mu_1 = 2$ is $m_1 = n - rank(A - 2I_3) = 3 - 2 = 1$ and the geometric multiplicity of A's eigenvalue $\mu_2 = 3$ is $m_2 = n - rank(A - 3I_3) = 3 - 1 = 2$. Because $m_1 + m_2 = n$, it follows that the corresponding algebraic multiplicities are $\alpha_1 = 1$ and $\alpha_2 = 2$.

To finish the work, we have a choice of two different methods:

Method I: The algebraic multiplicities imply both that the eigenvalues of A are $2, 3, 3$, counting multiplicity, and that the characteristic polynomial of A is $\mathcal{P}(\lambda) = (2 - \lambda)(3 - \lambda)^2$. Using the result of problem 2.1.6.20, or arguing directly from $\mathcal{P}(\lambda)$, we conclude that $|A| = \lambda_1 \lambda_2 \lambda_3 = 2 \cdot 3 \cdot 3 = 18$.

Method II: The geometric multiplicities of the eigenvalues of A add up to 3, which equals n, so A is diagonalizable with A being similar to $D = diag(2, 3, 3)$. By the result of problem 2.2.3.16(a), $|A| = |D| = 2 \cdot 3 \cdot 3 = 18$.

2.2.3.19. $S^2 = S \cdot S = (P\ diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ P^{-1})(P\ diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ P^{-1})$

$$= P\ diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ (P^{-1}P)\ diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ P^{-1}$$

$$= P\ diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ (I)\ diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ P^{-1}$$

$$= P(diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n}) \cdot diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ P^{-1} = P(diag\Big(\big(\sqrt{\lambda_1}\big)^2, ..., \big(\sqrt{\lambda_n}\big)^2\Big)P^{-1}$$

$$= Pdiag\big(\lambda_1, ..., \lambda_n\big)P^{-1} = A.$$

2.2.3.21. Use the two given eigenvectors to form the matrix $P = [\,\mathbf{p}^{(1)} \,\vdots\, \mathbf{p}^{(2)}\,] = \begin{bmatrix} -5 & 4 \\ 1 & 1 \end{bmatrix}$ that diagonalizes A. For example, if A is similar to the diagonal matrix $D = \begin{bmatrix} 9 & 0 \\ 0 & 27 \end{bmatrix}$, then the result of problem 2.2.3.20 gives

$$A = PDP^{-1} = \begin{bmatrix} -5 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} -5 & 4 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 27 \end{bmatrix} \left(\frac{1}{-9} \begin{bmatrix} 1 & -4 \\ -1 & -5 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -5 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 3 & 15 \end{bmatrix} = \begin{bmatrix} 17 & 40 \\ 2 & 19 \end{bmatrix}.$$

2.2.3.23. (a) A is 3×3 and three eigenvalues, $2, -2, \sqrt{3}$ are given, so those are the only eigenvalues and the characteristic polynomial is $\mathcal{P}_A(\lambda) = (2 - \lambda)(-2 - \lambda)(\sqrt{3} - \lambda)$.

(b) We were given that $A\mathbf{x}_1 = 2\mathbf{x}_1$, $A\mathbf{x}_2 = -2\mathbf{x}_2$, and $A\mathbf{x}_3 = \sqrt{3}\,\mathbf{x}_3$. Because these are eigenvectors corresponding to distinct eigenvalues of A, $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set.

It follows that $A^2\mathbf{x}_1 = 4\mathbf{x}_1$, $A^2\mathbf{x}_2 = 4\mathbf{x}_2$, and $A^2\mathbf{x}_3 = 3\mathbf{x}_3$. So, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are also eigenvectors of A^2 and we already know that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set. So, $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set of eigenvectors of A^2.

(c) Because $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set of eigenvectors of A^2 corresponding to eigenvalues 4, 4, and 3, respectively, it follows that the characteristic polynomial of A^2 is $\mathcal{P}_{A^2}(\lambda) = (4 - \lambda)(4 - \lambda)(3 - \lambda)$.

2.2.3.25. Use multiplication on the left by $(A-\mu_2 I)(A-\mu_3 I)\cdots(A-\mu_p I)$. Why? Suppose

$$\mathbf{0} = c_{1,1}\mathbf{x}^{1,1} + ... + c_{1,m_1}\mathbf{x}^{1,m_1} + c_{2,1}\mathbf{x}^{2,1} + ... + c_{2,m_2}\mathbf{x}^{2,m_2} + ... + c_{p,1}\mathbf{x}^{p,1} + ... + c_{p,m_p}\mathbf{x}^{p,m_p}.$$

If we multiply on the left by the matrix $(A-\mu_2 I)(A-\mu_3 I)\cdots(A-\mu_p I)$, we have

$$\begin{aligned}\mathbf{0} &= (A-\mu_2 I)(A-\mu_3 I)\cdots(A-\mu_p I)(c_{1,1}\mathbf{x}^{1,1} + ... + c_{1,m_1}\mathbf{x}^{1,m_1})+\\ &\quad +(A-\mu_2 I)(A-\mu_3 I)\cdots(A-\mu_p I)(c_{2,1}\mathbf{x}^{2,1} + ... + c_{2,m_2}\mathbf{x}^{2,m_2}) + ...+\\ &\quad +(A-\mu_2 I)(A-\mu_3 I)\cdots(A-\mu_p I)(c_{p,1}\mathbf{x}^{p,1} + ... + c_{p,m_p}\mathbf{x}^{p,m_p})\\ &= (\mu_1-\mu_2)\cdots(\mu_1-\mu_p)(c_{1,1}\mathbf{x}^{1,1} + ... + c_{1,m_1}\mathbf{x}^{1,m_1}) + (\mu_2-\mu_2)\cdots(\mu_2-\mu_p)(c_{2,1}\mathbf{x}^{2,1} + ... + c_{2,m_2}\mathbf{x}^{2,m_2})+\\ &\quad +... + (\mu_p-\mu_2)\cdots(\mu_p-\mu_p)(c_{p,1}\mathbf{x}^{p,1} + ... + c_{p,m_p}\mathbf{x}^{p,m_p})\\ &= (\mu_1-\mu_2)\cdots(\mu_1-\mu_p)(c_{1,1}\mathbf{x}^{1,1} + ... + c_{1,m_1}\mathbf{x}^{1,m_1}) + \mathbf{0} + ... + \mathbf{0}\end{aligned}$$

Because the $\mu_1, ..., \mu_p$ were assumed to be distinct, it follows that $c_{1,1}\mathbf{x}^{1,1} + ... + c_{1,m_1}\mathbf{x}^{1,m_1} = \mathbf{0}$.

But, in Theorem 2.7(a) in Section 2.2 we also assumed that $\{\mathbf{x}^{1,1}, .., \mathbf{x}^{1,m_1}\}$ is linearly independent. This implies that $c_{1,1} = \cdots = c_{1,m_1} = 0$.

2.2.3.27. By expanding along the first column,

$$0 = |\,A-\lambda I\,| = \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ 0 & 1 & 8-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 8-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 1 & 8-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((1-\lambda)(8-\lambda)+1\big) + \big(-(8-\lambda)+1\big) = (1-\lambda)\big(\lambda^2-9\lambda+9\big) - 7 + \lambda$$

$$= -\lambda^3 + 10\lambda^2 - 18\lambda + 9 - 7 + \lambda = -\lambda^3 + 10\lambda^2 - 17\lambda + 2 \triangleq \mathcal{P}(\lambda).$$

Standard advice for factoring polynomials suggests trying $\lambda = \pm 1, \pm 2$. We find that $\mathcal{P}(1) = -6$, $\mathcal{P}(-1) = 30$, $\mathcal{P}(2) = 0$. The latter implies 2 is an eigenvalue and enables factoring

$$\mathcal{P}(\lambda) = (2-\lambda)(\lambda^2 - 8\lambda + 1).$$

The quadratic equation $\lambda^2 - 8\lambda + 1 = 0$ has solutions $\lambda = \dfrac{8 \pm \sqrt{60}}{2} = 4 \pm \sqrt{15}$.

Because there are three distinct eigenvalues, $2, 4 \pm \sqrt{15}$, for this 3×3 matrix, theory in Section 2.2 guarantees that A has a set of three linearly independent eigenvectors.

Section 2.3.4

2.3.4.1. Denote $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. To start the Gram-Schmidt process, let

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{2},\ \mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \left(\frac{2}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \sqrt{2}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

According to Theorem 2.16 in Section 2.3, the o.n. set

$$\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

has span equal to the span of the given set of vectors, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$.

2.3.4.3. Denote $\mathbf{a}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. To start the Gram-Schmidt process, let

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{5},$$

and

$$\mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \bullet \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - 0 \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \sqrt{6}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Further, let

$$\mathbf{v}_3 \triangleq \mathbf{a}_3 - (\mathbf{a}_3 \bullet \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{a}_3 \bullet \mathbf{q}_2)\mathbf{q}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \bullet \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$- \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \bullet \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right) \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \left(\frac{-1}{\sqrt{5}}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - 0 \cdot \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix},$$

$r_{33} \triangleq ||\mathbf{v}_3|| = \frac{2\sqrt{30}}{5}$, and

$$\mathbf{q}_3 = r_{33}^{-1}\mathbf{v}_3 = \frac{1}{\sqrt{30}}\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}.$$

According to Theorem 2.16 in Section 2.3, the o.n. set

$$\left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix},\ \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},\ \frac{1}{\sqrt{30}}\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\}$$

has span equal to the span of the given set of vectors, $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

2.3.4.5. Denote $\mathbf{a}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$. To start the Gram-Schmidt process, let

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = 5,\ \mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} - \left(\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \bullet \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix} \right) \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} - \left(\frac{3-4\sqrt{2}}{5} \right) \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 16+12\sqrt{2} \\ 12+9\sqrt{2} \end{bmatrix} = \frac{4+3\sqrt{2}}{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{4+3\sqrt{2}}{5}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

According to Theorem 2.16 in Section 2.3, the o.n. set

$$\mathcal{S} = \left\{ \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix},\ \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$$

has span equal to the span of the given set of vectors, $\left\{ \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \right\}$.

2.3.4.7. <u>Ex</u>: G-S. on $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ yields $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, while G-S. on $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ yields $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

2.3.4.9. We calculate

$$\langle \mathbf{x}, \mathbf{x}+\mathbf{y}-\mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{z} \rangle = ||\,\mathbf{x}\,||^2 + \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{z} \rangle,$$

$$\langle \mathbf{y}, \mathbf{y}+\mathbf{z}-\mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle = ||\,\mathbf{y}\,||^2 + \langle \mathbf{y}, \mathbf{z} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle,$$

and

$$\langle \mathbf{z}, \mathbf{z}+\mathbf{x}-\mathbf{y} \rangle = \langle \mathbf{z}, \mathbf{z} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle - \langle \mathbf{z}, \mathbf{y} \rangle = ||\,\mathbf{z}\,||^2 + \langle \mathbf{z}, \mathbf{x} \rangle - \langle \mathbf{z}, \mathbf{y} \rangle.$$

So,

$$\langle \mathbf{x}, \mathbf{x}+\mathbf{y}-\mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{y}+\mathbf{z}-\mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{z}+\mathbf{x}-\mathbf{y} \rangle = ||\,\mathbf{x}\,||^2 + \cancel{\langle \mathbf{x}, \mathbf{y} \rangle} - \cancel{\langle \mathbf{x}, \mathbf{z} \rangle} + ||\,\mathbf{y}\,||^2 + \cancel{\langle \mathbf{y}, \mathbf{z} \rangle} - \cancel{\langle \mathbf{y}, \mathbf{x} \rangle} + ||\,\mathbf{z}\,||^2 + \cancel{\langle \mathbf{z}, \mathbf{x} \rangle} - \cancel{\langle \mathbf{z}, \mathbf{y} \rangle}$$

$$= ||\,\mathbf{x}\,||^2 + ||\,\mathbf{y}\,||^2 + ||\mathbf{z}||^2.$$

Yes,
$$\langle \mathbf{x}, \mathbf{x}+\mathbf{y}-\mathbf{z}\rangle + \langle \mathbf{y}, \mathbf{y}+\mathbf{z}-\mathbf{x}\rangle + \langle \mathbf{z}, \mathbf{z}+\mathbf{x}-\mathbf{y}\rangle = ||\,\mathbf{x}\,||^2 + ||\,\mathbf{y}\,||^2 + ||\mathbf{z}||^2$$
is true for all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

2.3.4.11. (a) To start the Gram-Schmidt process, let
$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{\mathbf{a}_1 \bullet \mathbf{a}_1} = \sqrt{2},$$
and
$$\mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{2}}\,\mathbf{a}_1.$$
Next, let
$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \mathbf{a}_2 - \Big(\mathbf{a}_2 \bullet \frac{1}{\sqrt{2}}\,\mathbf{a}_1\Big)\frac{1}{\sqrt{2}}\,\mathbf{a}_1 = \mathbf{a}_2 - \frac{1}{2}(\mathbf{a}_2 \bullet \mathbf{a}_1)\mathbf{a}_1 = \mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1.$$
So
$$r_{22}^2 \triangleq ||\mathbf{v}_2||^2 = \Big|\Big|\mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1\Big|\Big|^2 = \Big\langle \mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1,\ \mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1\Big\rangle = \langle \mathbf{a}_2,\ \mathbf{a}_2\rangle - 3\,\langle \mathbf{a}_2,\ \mathbf{a}_1\rangle + \frac{9}{4}\langle \mathbf{a}_1,\ \mathbf{a}_1\rangle = 5 - 3\cdot 3 + \frac{9}{4}\cdot 2 = \frac{1}{2}.$$
We have
$$r_{22} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \sqrt{2}\Big(\mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1\Big).$$
Next,
$$\begin{aligned}\mathbf{v}_3 \triangleq \mathbf{a}_3 - (\mathbf{a}_3 \bullet \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{a}_3 \bullet \mathbf{q}_2)\mathbf{q}_2 = \mathbf{a}_3 - \Big(\mathbf{a}_3 \bullet \frac{1}{\sqrt{2}}\,\mathbf{a}_1\Big)\frac{1}{\sqrt{2}}\,\mathbf{a}_1 \\ - \Big(\mathbf{a}_3 \bullet \sqrt{2}\Big(\mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1\Big)\Big)\ \sqrt{2}\Big(\mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1\Big) = \mathbf{a}_3 - \frac{1}{2}(\mathbf{a}_3 \bullet \mathbf{a}_1)\mathbf{a}_1 - 2\Big((\mathbf{a}_3 \bullet \mathbf{a}_2) - \frac{3}{2}(\mathbf{a}_3 \bullet \mathbf{a}_1)\Big)\Big(\mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1\Big) \\ = \mathbf{a}_3 - \frac{1}{2}\cdot 4\mathbf{a}_1 - 2\Big(6 - \frac{3}{2}\cdot 4\Big)\Big(\mathbf{a}_2 - \frac{3}{2}\,\mathbf{a}_1\Big) = \mathbf{a}_3 - 2\mathbf{a}_1.\end{aligned}$$
So, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, where $\mathbf{v}_1 = \mathbf{a}_1$, $\mathbf{v}_2 = \mathbf{a}_2 - \frac{3}{2}\mathbf{a}_1$, and $\mathbf{v}_3 = \mathbf{a}_3 - 2\mathbf{a}_1$.

(b) Continuing with the Gram-Schmidt process,
$$\begin{aligned} r_{33}^2 \triangleq ||\mathbf{v}_3||^2 &= ||\mathbf{a}_3 - 2\mathbf{a}_1||^2 = \langle \mathbf{a}_3 - 2\mathbf{a}_1,\ \mathbf{a}_3 - 2\mathbf{a}_1\rangle \\ &= \langle \mathbf{a}_3,\ \mathbf{a}_3\rangle - 4\,\langle \mathbf{a}_3,\ \mathbf{a}_1\rangle + 4\langle \mathbf{a}_1,\ \mathbf{a}_1\rangle \\ &= 9 - 4\cdot 4 + 4\cdot 2 = 1.\end{aligned}$$
We have
$$r_{33} = 1 \quad \text{and} \quad \mathbf{q}_3 = r_{33}^{-1}\mathbf{v}_3 = \mathbf{a}_3 - 2\mathbf{a}_1.$$
So, $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, where $\mathbf{q}_1 = \frac{1}{\sqrt{2}}\mathbf{a}_1$, $\mathbf{q}_2 = \sqrt{2}(\mathbf{a}_2 - \frac{3}{2}\mathbf{a}_1)$, and $\mathbf{q}_3 = \mathbf{a}_3 - 2\mathbf{a}_1$.

2.3.4.13. For all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, we have
$$||\mathbf{x}+\mathbf{y}||^2 = \langle \mathbf{x}+\mathbf{y},\ \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x},\ \mathbf{x}\rangle + 2\langle \mathbf{x},\ \mathbf{y}\rangle + \langle \mathbf{y},\ \mathbf{y}\rangle,$$
so
$$||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + 2\langle \mathbf{x},\ \mathbf{y}\rangle + ||\mathbf{y}||^2.$$
It follows that $||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$ if, and only if, $2\langle \mathbf{x},\ \mathbf{y}\rangle = 0$, that is, if and only if $\mathbf{x} \perp \mathbf{y}$.

2.3.4.15. We are given that $\mathbf{a}_n$ is a linear combination of $\{\mathbf{a}_1, ..., \mathbf{a}_{n-1}\}$, hence $\mathbf{a}_n$ is in the $Span\{\mathbf{a}_1, ..., \mathbf{a}_n\} = Span\{\mathbf{q}_1, ..., \mathbf{q}_n\}$. So, $\mathbf{a}_n$ is a linear combination of $\{\mathbf{q}_1, ..., \mathbf{q}_{n-1}\}$, that is, there exists scalars $c_1, ..., c_{n-1}$ such that
$$(\star) \quad \mathbf{a}_n = c_1\mathbf{q}_1 + ... + c_{n-1}\mathbf{q}_{n-1}.$$

Because $\{\mathbf{q}_1, ..., \mathbf{q}_{n-1}\}$ is an o.n. set of vectors, $c_j = \mathbf{a}_n \bullet \mathbf{q}_j$ for $j = 1, ..., n-1$. [This follows from taking the dot product of $(\star)$ with $\mathbf{q}_j$ to get

$$\mathbf{a}_n \bullet \mathbf{q}_j = c_1 \mathbf{q}_1 \bullet \mathbf{q}_j + ... + c_{n-1} \mathbf{q}_{n-1} \bullet \mathbf{q}_j$$

$$= c_1 \cdot 0 + ... + c_{j-1} \cdot 0 + c_j \cdot 1 + c_{j+1} \cdot 0 + ... + c_n \cdot 0 = c_j.]$$

It follows that

$$\mathbf{v}_n \triangleq \mathbf{a}_n - (\mathbf{a}_n \bullet \mathbf{q}_1)\mathbf{q}_1 - ... - (\mathbf{a}_n \bullet \mathbf{q}_{n-1})\mathbf{q}_{n-1}$$

$$= \mathbf{a}_n - (c_1 \mathbf{q}_1 + ... + c_{n-1} \mathbf{q}_{n-1}) = \mathbf{a}_n - \mathbf{a}_n = \mathbf{0}.$$

It follows that we cannot construct $\mathbf{q}_n$ from $\{\mathbf{a}_1, ..., \mathbf{a}_n\}$ and the Gram-Schmidt process fails at this step.

2.3.4.17. Assume $\mathbf{q}$ is a unit vector and define $P \triangleq \mathbf{q}\mathbf{q}^T$. Then

$$P^2 = (\mathbf{q}\mathbf{q}^T)(\mathbf{q}\mathbf{q}^T) = \mathbf{q}(\mathbf{q}^T\mathbf{q})\mathbf{q}^T = \mathbf{q}(||\mathbf{q}^T||^2)\mathbf{q}^T = \mathbf{q}(1)\mathbf{q}^T = \mathbf{q}\mathbf{q}^T = P$$

and

$$P^T = (\mathbf{q}\mathbf{q}^T)^T = (\mathbf{q}^T)^T\mathbf{q}^T = \mathbf{q}\mathbf{q}^T = P.$$

So, P satisfies the two properties of an orthogonal projection, that is, P is an orthogonal projection.

2.3.4.19. Define $P \triangleq P_1 P_2$. We calculate

$$P^2 = (P_1 P_2)(P_1 P_2) = P_1(P_2 P_1)P_2,$$

We were given that P_1 and P_2 are orthogonal projections and that $P_1 P_2 = P_2 P_1$, so

$$P^2 = P_1(P_2 P_1)P_2 = P_1(P_1 P_2)P_2 = P_1^2 P_2^2 = P_1 P_2 = P.$$

Also,

$$P^T = (P_1 P_2)^T = P_2^T P_1^T = P_2 P_1 = P_1 P_2 = P.$$

So, P satisfies the two properties of an orthogonal projection, that is, P is an orthogonal projection.

Section 2.4.3

2.4.3.1. Denote $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$. To start the Gram-Schmidt process, let

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{2},\ \mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{-1}{\sqrt{6}}\right)\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{\sqrt{66}}{6}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{\sqrt{66}}\begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix}.$$

Further, let

$$\mathbf{v}_3 \triangleq \mathbf{a}_3 - (\mathbf{a}_3 \bullet \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{a}_3 \bullet \mathbf{q}_2)\mathbf{q}_2$$

$$= \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \bullet \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \bullet \frac{1}{\sqrt{66}}\begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix}\right)\frac{1}{\sqrt{66}}\begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} - \left(\frac{4}{\sqrt{6}}\right)\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \left(\frac{22}{\sqrt{66}}\right)\frac{1}{\sqrt{66}}\begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

In using the G.-S. process, we arrive at $\mathbf{v}_3 = \mathbf{0}$, which cannot be used to create the third orthonormal vector $\mathbf{q}_3$.

So, no, we cannot use the given set of vectors to construct an o.n. basis for $\mathbb{R}^3$ using the G.-S. process. The underlying cause of the G.-S. process's failure is that the given set of three vectors is linearly dependent.

2.4.3.3. A is a real, orthogonal matrix exactly when $I_3 = A^T A =$

$$= \begin{bmatrix} a & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & b & 0 \end{bmatrix}\begin{bmatrix} a & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & b \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a^2 + \frac{1}{2} & 0 & (a+b)/\sqrt{2} \\ 0 & 1 & 0 \\ (a+b)/\sqrt{2} & 0 & b^2 + \frac{1}{2} \end{bmatrix},$$

which is true if, and only if, $a^2 + \frac{1}{2} = 1$, $a + b = 0$, and $b^2 + \frac{1}{2} = 1$.

The first equation is satisfied if, and only if, $a = \pm\frac{1}{\sqrt{2}}$, in which case $b = -a = -\pm\frac{1}{\sqrt{2}} = \mp\frac{1}{\sqrt{2}}$ satisfies the third equation.

So, there are two solutions for a and b: (1) $a = \frac{1}{\sqrt{2}}$ and $b = -\frac{1}{\sqrt{2}}$; (2) $a = -\frac{1}{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$.

2.4.3.5. *Method 1*: We can use the method in the Appendix, starting from the given vector

$$\mathbf{u}_1 = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

which is to be the first column of Q.

We will use the Gram-Schmidt process on the set of vectors $\{\mathbf{u}_1, \mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}\}$ to get an o.n. set of three vectors; we will stop the process after we have three vectors. Hopefully each of those vectors will have no zero entry; if not, we can try using the Gram-Schmidt process on a different set of vectors.

Let

$$\mathbf{v}_2 \triangleq \mathbf{e}^{(1)} - (\mathbf{e}^{(1)} \bullet \mathbf{u}_1)\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{14}} \right) \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{\sqrt{182}}{14}$, and

$$\mathbf{u}_2 = r_{22}^{-1} \mathbf{v}_2 = \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}.$$

Further, let

$$\mathbf{v}_3 \triangleq \mathbf{e}^{(2)} - (\mathbf{e}^{(2)} \bullet \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{e}^{(2)} \bullet \mathbf{u}_2)\mathbf{u}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix} \right) \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{2}{\sqrt{14}} \right) \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{-2}{\sqrt{182}} \right) \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix} = \frac{1}{182} \begin{bmatrix} 0 \\ 126 \\ -84 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 0 \\ 9 \\ -6 \end{bmatrix},$$

$r_{33} \triangleq ||\mathbf{v}_2|| = \frac{\sqrt{117}}{13}$, and

$$\mathbf{u}_3 = r_{33}^{-1} \mathbf{v}_3 = \frac{1}{\sqrt{117}} \begin{bmatrix} 0 \\ 9 \\ -6 \end{bmatrix}.$$

A desired orthogonal matrix is

$$Q = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{13}{\sqrt{182}} & 0 \\ \frac{2}{\sqrt{14}} & -\frac{2}{\sqrt{182}} & \frac{9}{\sqrt{117}} \\ \frac{3}{\sqrt{14}} & -\frac{3}{\sqrt{182}} & -\frac{6}{\sqrt{117}} \end{bmatrix}.$$

Method 2: It is easy to guess a column vector that is orthogonal to the first column of Q:

$$\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and then normalize it to get the second column of Q to be

$$\mathbf{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

After that, we can use the method of the Appendix to find a third column: Let

$$\mathbf{v}_3 \triangleq \mathbf{e}^{(1)} - (\mathbf{e}^{(1)} \bullet \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{e}^{1)} \bullet \mathbf{u}_2)\mathbf{u}_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{14}} \right) \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{-2}{\sqrt{5}} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{70} \begin{bmatrix} 9 \\ 18 \\ -15 \end{bmatrix} = \frac{3}{70} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix},$$

$r_{33} \triangleq ||\mathbf{v}_2|| = \frac{3\sqrt{70}}{70}$, and

$$\mathbf{u}_3 = r_{33}^{-1} \mathbf{v}_3 = \frac{1}{\sqrt{70}} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}.$$

A desired orthogonal matrix is

$$Q = Q = \begin{bmatrix} \frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{5}} & \frac{6}{\sqrt{70}} \\ \frac{3}{\sqrt{14}} & 0 & -\frac{5}{\sqrt{70}} \end{bmatrix}.$$

We could multiply the last column by (-1) to get another orthogonal matrix,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{5}} & -\frac{3}{\sqrt{70}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{5}} & -\frac{6}{\sqrt{70}} \\ \frac{3}{\sqrt{14}} & 0 & \frac{5}{\sqrt{70}} \end{bmatrix}.$$

2.4.3.7. We could start with the first column of Q being $\mathbf{e}^{(1)}$, which has two zeros, but then we would have left exactly one zero that we should have in the remaining two columns.

Instead, let's try the first column of Q to be

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

We will use the Gram-Schmidt process on the set of vectors $\{\mathbf{u}_1, \mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}\}$ to get an o.n. set of three vectors; we will stop the process after we have three vectors. Hopefully each of those vectors will have no zero entry; if not, we can try using the Gram-Schmidt process on a different set of vectors.

Let

$$\mathbf{v}_2 \triangleq \mathbf{e}^{(1)} - (\mathbf{e}^{(1)} \bullet \mathbf{u}_1)\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{\sqrt{2}}{2}$, and

$$\mathbf{u}_2 = r_{22}^{-1} \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Further, let

$$\mathbf{v}_3 \triangleq \mathbf{e}^{(2)} - (\mathbf{e}^{(2)} \bullet \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{e}^{(2)} \bullet \mathbf{u}_2)\mathbf{u}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{-1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is a dead end.

So, let's try yet again. Let $\mathbf{a}_3 = \mathbf{e}^{(3)}$, and

$$\mathbf{v}_3 \triangleq \mathbf{a}_3 - (\mathbf{a}_3 \bullet \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{a}_3 \bullet \mathbf{u}_2)\mathbf{u}_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \bullet \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{0}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{0}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mathbf{e}^{(3)}.$$

[We probably could have guessed the third column after looking at the first two columns!]

A desired orthogonal matrix is

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has exactly four zeros.

Alternatively, it's even easier to think geometrically to guess an example of Q, one of whose column vectors is $\mathbf{e}^{(3)}$, along the $z-axis$, and whose other two column vectors are in the plane $z = 0$.

2.4.3.9. Q_1 and Q_2 are both orthogonal matrices and $Q \triangleq Q_1Q_2$. Because Q_1 and Q_2 are both square, so is Q (otherwise Q doesn't even exist.) We have

$$Q^TQ = (Q_1Q_2)^T(Q_1Q_2) = \big(Q_2^TQ_1^T\big)(Q_1Q_2) = Q_2^T(Q_1^TQ_1)Q_2 = Q_2^T(I)Q_2 = Q_2^TQ_2 = I,$$

so, Q is an orthogonal matrix.

2.4.3.11. For all $\mathbf{x}, \mathbf{y}$,

$$\langle Q\mathbf{x}, Q\mathbf{y}\rangle = (Q\mathbf{x})^T(Q\mathbf{y}) = (\mathbf{x}^TQ^T)(Q\mathbf{y}) = \mathbf{x}^T(Q^TQ)\mathbf{y} = \mathbf{x}^T(I)\mathbf{y} = \langle \mathbf{x}, \mathbf{y}\rangle.$$

2.4.3.13. *Method 1*: Yes, because B is a real, orthogonal matrix and we have the result of Problem 2.4.3.12.

Method 2: Alternatively, we could explicitly calculate that for all $\mathbf{x} = [\,x_1 \;\; x_2 \;\; x_3\,]^T$ in $\mathbb{R}^3$,

$$||B\mathbf{x}||^2 = \left|\left|\begin{bmatrix} \frac{1}{\sqrt{3}}x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{6}}x_3 \\ \frac{1}{\sqrt{3}}x_1 - \frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{6}}x_3 \\ \frac{1}{\sqrt{3}}x_1 - \frac{2}{\sqrt{6}}x_3 \end{bmatrix}\right|\right|^2$$

$$= \left(\frac{1}{\sqrt{3}}x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{6}}x_3\right)^2 + \left(\frac{1}{\sqrt{3}}x_1 - \frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{6}}x_3\right)^2 + \left(\frac{1}{\sqrt{3}}x_1 - \frac{2}{\sqrt{6}}x_3\right)^2$$

$$= \frac{1}{3}\,x_1^2 + \frac{1}{2}\,x_2^2 + \frac{1}{6}\,x_3^2 + \frac{2}{\sqrt{6}}\,x_1x_2 + \frac{2}{\sqrt{18}}\,x_1x_3 + \frac{2}{\sqrt{12}}\,x_2x_3$$

$$+\frac{1}{3}\,x_1^2 + \frac{1}{2}\,x_2^2 + \frac{1}{6}\,x_3^2 - \frac{2}{\sqrt{6}}\,x_1x_2 + \frac{2}{\sqrt{18}}\,x_1x_3 - \frac{2}{\sqrt{12}}\,x_2x_3 + \frac{1}{3}\,x_1^2 + \frac{4}{6}\,x_3^2 - \frac{4}{\sqrt{18}}\,x_1x_3 + \frac{2}{\sqrt{12}}\,x_2x_3$$

$$= x_1^2 + x_2^2 + x_3^2 = ||\mathbf{x}||^2.$$

2.4.3.15. Suppose $\mathbf{x}$ and $\mathbf{y}$ are any vectors in $\mathbb{R}^n$ and $\{\mathbf{q}_1, ..., \mathbf{q}_n\}$ is an o.n. set in $\mathbb{R}^n$. Then from (2.19)(a) in Section 2.4,

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{q}_1 \rangle \mathbf{q}_1 + ... + \langle \mathbf{x}, \mathbf{q}_n \rangle \mathbf{q}_n,$$

so

$$\langle \mathbf{x}, \mathbf{y} \rangle = \Big\langle \langle \mathbf{x}, \mathbf{q}_1 \rangle \mathbf{q}_1 + ... + \langle \mathbf{x}, \mathbf{q}_n \rangle \mathbf{q}_n, \mathbf{y} \Big\rangle = \langle \mathbf{x}, \mathbf{q}_1 \rangle \langle \mathbf{q}_1, \mathbf{y} \rangle + ... + \langle \mathbf{x}, \mathbf{q}_n \rangle \langle \mathbf{q}_n, \mathbf{y} \rangle,$$

as we wanted to derive.

2.4.3.17. $Q^T = \left(I - 2\mathbf{q}\mathbf{q}^T\right)^T = I^T - 2\left(\mathbf{q}^T\right)^T \mathbf{q}^T = I - \mathbf{q}^T\mathbf{q}^T = Q$ and so $Q^TQ = Q^2 = \left(I - 2\mathbf{q}\mathbf{q}^T\right)^2 = I^2 - 4\mathbf{q}\mathbf{q}^T + \left(2\mathbf{q}\mathbf{q}^T\right)^2 = I - 4\mathbf{q}\mathbf{q}^T + 4\left(\mathbf{q}\mathbf{q}^T\right)\left(\mathbf{q}\mathbf{q}^T\right) = I - 4\mathbf{q}\mathbf{q}^T + 4\mathbf{q}\left(\mathbf{q}^T\mathbf{q}\right)\mathbf{q}^T = I - \cancel{4\mathbf{q}\mathbf{q}^T} + \cancel{4\mathbf{q}\mathbf{q}^T} = I$

2.4.3.19. As suggested by Corollary 2.5 in Section 2.4, begin by using Theorem 1.41 in Section 1.7 to construct a basis for $Col(A)$: Row reduce

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad \underset{-2R_1 + R_2 \to R_2}{\sim} \quad \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix},$$

so the two columns of A are its pivot columns.

Method 1: It follows that the two columns of A are a basis for $\mathbb{R}^2$ and thus that $Col(A) = \mathbb{R}^2$. Geometrically, the projection of $\mathbb{R}^2$ *onto* all of itself is the identity matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Method 2: Use the Gram-Schmidt process: Let

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{5},\ \mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \bullet \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{5}}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{3\sqrt{5}}{5}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Using Corollary 2.5 in Section 2.4, we have that the orthogonal projection onto $Col(A)$ is given by

$$P_A = \mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_2\mathbf{q}_2^T$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \end{bmatrix} + \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2.4.3.21. As suggested by Corollary 2.5 in Section 2.4, begin by using Theorem 1.41 in Section 1.7 to construct a basis for $Col(A)$: Row reduce

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \underset{-2R_1+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}, \underset{-2R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so the first two columns of A are its only pivot columns.

Use the Gram-Schmidt process: Let

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{5},\ \mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{5}} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4 \\ -5 \\ -2 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{\sqrt{45}}{5}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{\sqrt{45}}\begin{bmatrix} 4 \\ -5 \\ -2 \end{bmatrix}.$$

Using Corollary 2.5 in Section 2.4, we have that the orthogonal projection onto $Col(A)$ is given by

$$P_A = \mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_2\mathbf{q}_2^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \frac{1}{\sqrt{5}} [1 \quad 0 \quad 2] + \frac{1}{\sqrt{45}} \begin{bmatrix} 4 \\ -5 \\ -2 \end{bmatrix} \frac{1}{\sqrt{45}} [4 \quad -5 \quad -2]$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} + \frac{1}{45}\begin{bmatrix} 16 & -20 & -8 \\ -20 & 25 & 10 \\ -8 & 10 & 4 \end{bmatrix}. = \frac{1}{45}\begin{bmatrix} 25 & -20 & 10 \\ -20 & 25 & 10 \\ 10 & 10 & 40 \end{bmatrix}.$$

Section 2.5.2

2.5.2.1. The system is $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. The normal equations are $A^TA\mathbf{x} = A^T\mathbf{b}$, where

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

is invertible. There is only one l.s.s.:

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{11}\begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

2.5.2.3. The system is $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 4 & -1 \\ 4 & -3 \\ 2 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. The normal equations are $A^TA\mathbf{x} = A^T\mathbf{b}$, where

$$A^TA = \begin{bmatrix} 4 & 4 & 2 \\ -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 4 & -3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 36 & -18 \\ -18 & 11 \end{bmatrix}$$

is invertible. There is only one l.s.s.:

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{72}\begin{bmatrix} 11 & 18 \\ 18 & 36 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{72}\begin{bmatrix} 11 & 18 \\ 18 & 36 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 11 \\ 18 \end{bmatrix}.$$

2.5.2.5. The system is $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$. The normal equations are $A^TA\mathbf{x} = A^T\mathbf{b}$, where

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix}$$

is invertible. There is only one l.s.s.:

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{11}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 29 \\ 14 \end{bmatrix}.$$

2.5.2.7. Because Q and Q^T are both $n \times n$, by Theorem 1.21 in Section 1.5, $Q^TQ = I_n$ implies Q^{-1} exists and $Q^{-1} = Q^T$. It follows that $QQ^T = I_n$.

2.5.2.9. Take the hint and begin by defining $\mathbf{y} \triangleq [\frac{1}{m} \quad \frac{1}{m} \quad \ldots \quad \frac{1}{m}]^T$. Because we are assuming that at least two of the x_i's are distinct, we will be able to conclude that the set of vectors $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent: Without loss of generality, assume x_1 and x_2 are distinct (unequal). Then

$$0 = \alpha_1\mathbf{x} + \alpha_2\mathbf{y} = \alpha_1 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} + \alpha_2 \begin{bmatrix} 1/m \\ 1/m \\ \vdots \\ 1/m \end{bmatrix}$$

would imply

$$0 = \alpha_1 x_1 + \alpha_2\,\frac{1}{m} \quad \text{and} \quad 0 = \alpha_1 x_2 + \alpha_2\,\frac{1}{m}$$

hence

$$(\star)\quad \alpha_1 x_1 = -\alpha_2\,\frac{1}{m} = \alpha_1 x_2,$$

hence

$$\alpha_1(x_1 - x_2) = 0.$$

Because $x_1 \neq x_2$, it follows that $\alpha_1 = 0$. Further, $(\star)$ implies $-\alpha_2\,\frac{1}{m} = \alpha_1 x_1 = 0 \cdot x_1 = 0$, hence $\alpha_2 = 0$. This explains why $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

Taking the further hint to use the Cauchy-Schwarz inequality, because $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent,

$$|\overline{x}| = \left|\frac{1}{m}\,(x_1 + ... + x_m)\right| = |\langle \mathbf{x}, \mathbf{y}\rangle| < ||\mathbf{x}||\ \ ||\mathbf{y}|| = \sqrt{x_1^2 + ... + x_m^2}\ \left(\left(\frac{1}{m}\right)^2 + ... + \left(\frac{1}{m}\right)^2\right)^{1/2}$$

$$= \sqrt{x_1^2 + ... + x_m^2}\ \left(m \cdot \left(\frac{1}{m}\right)^2\right)^{1/2} = \frac{1}{\sqrt{m}}\ \sqrt{x_1^2 + ... + x_m^2},$$

hence

$$(\overline{x})^2 = |\overline{x}|^2 < \left(\frac{1}{\sqrt{m}}\right)^2\ \left(x_1^2 + ... + x_m^2\right),$$

that is,

$$(\overline{x})^2 < \frac{1}{m}\left(x_1^2 + ... + x_m^2\right) = \overline{x^2}\,.$$

It follows that $\overline{x^2} - \left(\overline{x}\right)^2$ is guaranteed to be positive.

2.5.2.11. We want to find a curve $y = f(x) \triangleq \mu x + \beta$ for data points $(x, y) = (x_i, y_i),\ i = 1, .., 3$, where at least two of the x_i's are distinct.

Method 1: Solving $y_i = \mu x_i + \beta x_i,\ i = 1, .., m$ can be restated as solving

$$A\begin{bmatrix}\mu\\ \beta\end{bmatrix} = \begin{bmatrix}0.9 & 1\\ 2.1 & 1\\ 2.8 & 1\end{bmatrix}\begin{bmatrix}\mu\\ \beta\end{bmatrix} = \begin{bmatrix}1.0\\ 2.0\\ 3.0\end{bmatrix}.$$

.

We calculate

$$A^T A = \begin{bmatrix}0.9 & 2.1 & 2.8\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}0.9 & 1\\ 2.1 & 1\\ 2.8 & 1\end{bmatrix} = \begin{bmatrix}13.06 & 5.8\\ 5.8 & 3\end{bmatrix}$$

and

$$A^T\mathbf{y} = \begin{bmatrix}0.9 & 2.1 & 2.8\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1.0\\ 2.0\\ 3.0\end{bmatrix} = \begin{bmatrix}13.5\\ 6.0\end{bmatrix}$$

The solution of the normal equations, $A^T A\begin{bmatrix}\mu\\ \beta\end{bmatrix} = A^T\mathbf{y}$, is

$$\begin{bmatrix}\mu\\ \beta\end{bmatrix} = \begin{bmatrix}13.06 & 5.8\\ 5.8 & 3\end{bmatrix}^{-1}\begin{bmatrix}13.5\\ 6.0\end{bmatrix} = \frac{1}{5.54}\begin{bmatrix}3 & -5.8\\ -5.8 & 13.06\end{bmatrix}\begin{bmatrix}13.5\\ 6.0\end{bmatrix} = \frac{1}{5.54}\begin{bmatrix}5.7\\ 0.06\end{bmatrix}.$$

The best such function is given by

$$f(x) = \mu x + \beta \approx 1.028880866\,x + 0.0108303249.$$

Method 2: We use formula (2.31) in Section 2.5 for the best line and start by calculating

$$\overline{x} = \frac{1}{3}(0.9 + 2.1 + 2.8) = \frac{5.8}{3},\quad \overline{x^2} = \frac{1}{3}(0.9^2 + 2.1^2 + 2.8^2) = \frac{13.06}{3},\quad \overline{y} = \frac{1}{3}(1.0 + 2.0 + 3.0) = 2,$$

and

$$\overline{x\,y} = \frac{1}{3}(0.9 \cdot 1.0 + 2.1 \cdot 2.0 + 2.8 \cdot 3.0) = \frac{13.5}{3}.$$

So, the best line is

$$y = \frac{1}{\overline{x^2} - \left(\overline{x}\right)^2}\left((\overline{xy} - \overline{x}\ \overline{y})x + (-\overline{x}\ \overline{xy} + \overline{x^2}\ \overline{y}) \right) = \frac{1}{\frac{13.06}{3} - \left(\frac{5.8}{3}\right)^2}\left(\left(\frac{13.5}{3} - \frac{5.8}{3} \cdot 2\right)x + \left(-\frac{5.8}{3} \cdot \frac{13.5}{3} + \frac{13.06}{3} \cdot 2\right) \right)$$

that is,

$$y \approx 1.028880866\,x + 0.01808303249.$$

2.5.2.13. Let x_i and y_i be the grades, on a 4 point scale, in Dynamics and Circuits, respectively, for the $i-$th student. We want to find a curve $y = f(x) \triangleq \mu x + \beta$ for data points $(x, y) = (x_i, y_i),\ i = 1, .., 6$, where at least two of the x_i's are distinct.

Method 1: Solving $y_i = \mu x_i + \beta x_i,\ i = 1, .., m$ can be restated as solving

$$A \begin{bmatrix} \mu \\ \beta \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3.7 & 1 \\ 2.3 & 1 \\ 2.3 & 1 \\ 1 & 1 \\ 2.7 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \beta \end{bmatrix} = \begin{bmatrix} 2.3 \\ 2.7 \\ 3 \\ 2.7 \\ 0 \\ 3.3 \end{bmatrix}.$$

.

We calculate

$$A^T A = \begin{bmatrix} 3 & 3.7 & 2.3 & 2.3 & 1 & 2.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 3.7 & 1 \\ 2.3 & 1 \\ 2.3 & 1 \\ 1 & 1 \\ 2.7 & 1 \end{bmatrix} = \begin{bmatrix} 41.56 & 15 \\ 15 & 6 \end{bmatrix}$$

and

$$A^T \mathbf{y} = \begin{bmatrix} 3 & 3.7 & 2.3 & 2.3 & 1 & 2.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2.3 \\ 2.7 \\ 3 \\ 2.7 \\ 0 \\ 3.3 \end{bmatrix} = \begin{bmatrix} 38.91 \\ 14.0 \end{bmatrix}$$

The solution of the normal equations, $A^T A \begin{bmatrix} \mu \\ \beta \end{bmatrix} = A^T \mathbf{y}$, is

$$\begin{bmatrix} \mu \\ \beta \end{bmatrix} = \begin{bmatrix} 41.56 & 15 \\ 15 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 38.91 \\ 14.0 \end{bmatrix} = \frac{1}{24.36} \begin{bmatrix} 6 & -15 \\ -15 & 41.56 \end{bmatrix} \begin{bmatrix} 38.91 \\ 14.0 \end{bmatrix} = \frac{1}{24.36} \begin{bmatrix} 23.46 \\ -1.81 \end{bmatrix}.$$

The best such function is given by

$$f(x) = \mu x + \beta \approx 0.9630541872\,x - 0.0743021346.$$

(a) Students who receive a B^-, that is, $x = 2.7$, in Dynamics, would get a grade of $y = f(2.7) \approx 0.9630541872 \cdot 2.7 - 0.0743021346 \approx 2.525944171$, that is, about a B^-, in Circuits.

(b) Students who receive a B^-, that is, $y = 2.7$, in Circuits, would get a grade of x in Dynamics, where $2.7 = y = f(x) \approx 0.9630541872 \cdot x - 0.0743021346$, hence $x = 0.9630541872^{-1}(2.7 + 0.0743021346) \approx 2.880733163$, that is, between a B^- and a B but closer to a B.

Method 2: We use formula (2.31) in Section 2.5 for the best line and start by calculating

$$\overline{x} = \frac{1}{6}(3+3.7+2.3+2.3+1+2.7) = 2.5, \quad \overline{x^2} = \frac{1}{6}(3^2+3.7^2+2.3^2+2.3^2+1^2+2.7^2) = \frac{41.56}{6},$$

$$\overline{y} = \frac{1}{6}(2.3+2.7+3+2.7+0+3.3) = \frac{14}{6},$$

and

$$\overline{x\,y} = \frac{1}{6}(3\cdot 2.3+3.7\cdot 2.7+2.3\cdot 3+2.3\cdot 2.7+1\cdot 0+2.7\cdot 3.3) = \frac{38.91}{6}.$$

So, the best line is

$$y = \frac{1}{\overline{x^2}-\left(\overline{x}\right)^2}\Big((\overline{xy}-\overline{x}\;\overline{y})x + (-\overline{x}\;\overline{xy}+\overline{x^2}\;\overline{y})\Big) = \frac{1}{\frac{41.56}{6}-(2.5)^2}\Big(\Big(\frac{38.91}{6}-2.5\cdot\frac{14}{6}\Big)x + \Big(-2.5\cdot\frac{38.91}{6}+\frac{41.56}{6}\cdot\frac{14}{6}\Big)\Big)$$

that is,

$$y \approx 0.9630541872\,x - 0.0743021346.$$

2.5.2.15. Yes, there can be infinitely many solutions, depending upon the matrix A^TA. For example, here's a modification of Example 2.20 in Section 2.5: For the system

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 3 & -3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

we calculate that

$$A^TA = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 14 & -14 \\ -14 & 14 \end{bmatrix}$$

is *not* invertible and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

We will use row reduction on the normal equations to find all of the l.s.s.:

$$[\,A^TA \mid A^T\mathbf{b}\,] = \left[\begin{array}{cc|c} 14 & -14 & 6 \\ -14 & 14 & -6 \end{array}\right] \quad \underset{\substack{R_1+R_2\to R_2 \\ \frac{1}{14}R_1\to R_1}}{\sim} \quad \left[\begin{array}{cc|c} 1 & -1 & \frac{3}{7} \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_2$ is the only free variable $\Rightarrow$ There are infinitely many l.s.s. given by

$$\mathbf{x} = \begin{bmatrix} -\frac{3}{7}+t \\ t \end{bmatrix}, \; -\infty < t < \infty.$$

2.5.2.17. $f(x) = Ae^{\alpha x}$, so $z = \ln y = \ln\big(f(x)\big) = \ln(Ae^{\alpha x}) = \alpha x + \ln A$. Define $\beta = \ln A$, so that we want to fit a line to the data in (x_i, z_i), where $z_i \triangleq \ln y_i)$, $i = 1, ..., 4$.
Method 1: To find the best fit line, the first method is

$$A\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1.00 & 1 \\ 2.00 & 1 \\ 3.00 & 1 \\ 4.00 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \ln 1.65 \\ \ln 2.70 \\ \ln 4.50 \\ \ln 7.35 \end{bmatrix}.$$

.

We calculate

$$A^T A = \begin{bmatrix} 1.00 & 2.00 & 3.00 & 4.00 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1.00 & 1 \\ 2.00 & 1 \\ 3.00 & 1 \\ 4.00 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$$

and

$$A^T \mathbf{z} \approx \begin{bmatrix} 1.00 & 2.00 & 3.00 & 4.00 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.5007752879 \\ 0.993251773 \\ 1.504077397 \\ 1.994700313 \end{bmatrix} = \begin{bmatrix} 14.97831228 \\ 4.992804771 \end{bmatrix}$$

The solution of the normal equations, $A^T A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A^T \mathbf{z}$, is

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \approx \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 14.97831228 \\ 4.992804771 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix} \begin{bmatrix} 14.97831228 \\ 4.992804771 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 9.985201399 \\ 0.0010203584 \end{bmatrix}.$$

The best linear function $\ln\big(f(x)\big)$ is given by

$$\ln f(x) = \alpha x + \beta \approx 0.4992600699\, x + 0.0000510179195.$$

Because $\beta = \ln A$, we have $A \approx 1.000051019$. The best fit is

$$y = Ae^{\alpha x} \approx 1.000051019\, e^{0.4992600699\, x}.$$

Method 2: We use formula (2.31) in Section 2.5 for the best line and start by calculating

$$\overline{x} = \frac{1}{4}(1.00 + 2.00 + 3.00 + 4.00) = 2.5, \quad \overline{x^2} = \frac{1}{4}(1.00^2 + 2.00^2 + 3.00^2 + 4.00^2) = 7.5,$$

$$\overline{z} = \frac{1}{4}(\ln 1.65 + \ln 2.70 + \ln 4.50 + \ln 7.35) \approx 1.248201193,$$

and

$$\overline{x\, z} = \frac{1}{4}(1.00 \cdot \ln 1.65 + 2.00 \cdot \ln 2.70 + 3.00 \cdot \ln 4.50 + 4.00 \cdot 7.35) \approx 3.744578069$$

So, the best line is

$$z = \ln f(x) = \frac{1}{\overline{x^2} - \big(\overline{x}\big)^2} \Big((\overline{xz} - \overline{x}\ \overline{z})x + (-\overline{x}\ \overline{xz} + \overline{x^2}\ \overline{z}) \Big)$$

$$= \frac{1}{7.5 - (2.5)^2} \Big((3.744578069 - 2.5 \cdot 1.248201193)x + (-2.5 \cdot 3.744578069 + 7.5 \cdot 1.248201193) \Big)$$

that is,

$$z \approx 0.4992600694\, x + 0.00005101941936.$$

Because $\beta = \ln A$, we have $A \approx 1.000051021$. The best fit is

$$y = Ae^{\alpha x} \approx 1.000051021\, e^{0.4992600694\, x}.$$

Section 2.6.3

2.6.3.1. $0 = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) \quad \Rightarrow$ eigenvalues $\lambda_1 = 0$, $\lambda_2 = 5$.

$[\, A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-2R_1 + R_2 \rightarrow R_2$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_1 = 0$.

$[\, A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-\frac{1}{4}R_1 \rightarrow R_1$, $-2R_1 + R_2 \rightarrow R_1$.

$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_2 = 5$.

The matrix $P = [\, \mathbf{p}^{(1)} \mid \mathbf{p}^{(2)} \,] = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ should diagonalize A.

To find a real, orthogonal matrix that diagonalizes A, all that we need to do is to normalize the columns because $\{\mathbf{p}^{(1)}, \mathbf{p}^{(2)}\}$ is already an orthogonal set. So, the real, orthogonal matrix

$$Q = \frac{1}{\sqrt{5}} [\, \mathbf{p}^{(1)} \mid \mathbf{p}^{(2)} \,] = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

should diagonalize A.

2.6.3.3. $0 = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 1) \Rightarrow$ eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$.

$[\, A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, after $-R_1 + R_2 \rightarrow R_2$, $\frac{1}{3}R_3 \rightarrow R_3$, $R_2 \leftrightarrow R_3$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_1 = -1$.

$[\, A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, after $R_1 + R_2 \rightarrow R_2$, $-R_1 \rightarrow R_1$, $R_2 \leftrightarrow R_3$.

$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_2 = 1$.

$[\, A - \lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, after $-\frac{1}{2}R_1 \rightarrow R_1$, $-R_1 + R_2 \rightarrow R_2$, $-\frac{2}{3}R_2 \rightarrow R_2$, $\frac{1}{2}R_2 + R_1 \rightarrow R_1$.

$\Rightarrow \mathbf{p}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_3 = 2$.

The matrix $P = [\, \mathbf{p}^{(1)} \mid \mathbf{p}^{(2)} \mid \mathbf{p}^{(3)} \,] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ should diagonalize A.

To find a real, orthogonal matrix that diagonalizes A, all that we need to do is to normalize the columns because $\{\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \mathbf{p}^{(3)}\}$ is already an orthogonal set. So, the real, orthogonal matrix

$$Q = \left[\frac{1}{\sqrt{2}}\mathbf{p}^{(1)} \;\vdots\; \frac{1}{\sqrt{2}}\mathbf{p}^{(2)} \;\vdots\; \mathbf{p}^{(3)} \right] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

should diagonalize A.

2.6.3.5. $0 = \begin{vmatrix} -2-\lambda & 4 & 4 \\ 4 & 7-\lambda & -5 \\ 4 & -5 & 7-\lambda \end{vmatrix} \underset{-R_2+R_3 \to R_3}{=} \begin{vmatrix} -2-\lambda & 4 & 4 \\ 4 & 7-\lambda & -5 \\ 0 & -12+\lambda & 12-\lambda \end{vmatrix}$

$\underset{R_3 \leftarrow (12-\lambda)R_3}{=} (12-\lambda) \begin{vmatrix} -2-\lambda & 4 & 4 \\ 4 & 7-\lambda & -5 \\ 0 & -1 & 1 \end{vmatrix} \underset{C_2+C_3 \to C_2}{=} (12-\lambda) \begin{vmatrix} -2-\lambda & 8 & 4 \\ 4 & 2-\lambda & -5 \\ 0 & 0 & 1 \end{vmatrix}$

$$= (12-\lambda) \begin{vmatrix} -2-\lambda & 8 \\ 4 & 2-\lambda \end{vmatrix} = (12-\lambda)\big((-2-\lambda)(2-\lambda) - 32\big) = (12-\lambda)(\lambda^2 - 36)$$

$\Rightarrow$ eigenvalues $\lambda_1 = -6$, $\lambda_2 = 6$, $\lambda_3 = 12$.

$[\, A - \lambda_1 I \mid \mathbf{0} \,] = \left[\begin{array}{rrr|r} 4 & 4 & 4 & 0 \\ 4 & 13 & -5 & 0 \\ 4 & -5 & 13 & 0 \end{array} \right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 9 & -9 & 0 \\ 0 & -9 & 9 & 0 \end{array} \right]$, after $-R_1 + R_2 \to R_2$, $-R_1 + R_3 \to R_3$, $\frac{1}{4}R_1 \to R_1$,

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

after $R_2 + R_3 \to R_3$, $\frac{1}{9}R_2 \to R_2$, $-R_2 + R_1 \to R_1$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_1 = -6$.

$[\, A - \lambda_2 I \mid \mathbf{0} \,] = \left[\begin{array}{rrr|r} -8 & 4 & 4 & 0 \\ 4 & 1 & -5 & 0 \\ 4 & -5 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right]$, after $\frac{1}{2}R_1 + R_2 \to R_2$, $\frac{1}{2}R_1 + R_3 \to R_3$, $-\frac{1}{8}R_1 \to R_1$,

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

after $R_2 + R_3 \to R_3$, $\frac{1}{3}R_2 \to R_2$, $\frac{1}{2}R_2 + R_1 \to R_1$.

$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_2 = 6$.

$[\, A - \lambda_3 I \mid \mathbf{0} \,] = \left[\begin{array}{rrr|r} -14 & 4 & 4 & 0 \\ 4 & -5 & -5 & 0 \\ 4 & -5 & -5 & 0 \end{array} \right] \sim \left[\begin{array}{rrr|r} 1 & -\frac{5}{4} & -\frac{5}{4} & 0 \\ -14 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$, after $-R_2 + R_3 \to R_3$, $R_1 \leftrightarrow R_2$, $\frac{1}{4}R_1 \to R_1$,

$$\sim \left[\begin{array}{rrr|r} 1 & -5/4 & -5/4 & 0 \\ 0 & -54/4 & -54/4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

after $14R_1+R_2\rightarrow R_2$,
$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right],$$
after $-\frac{4}{54}R_2\rightarrow R_2$, $\frac{5}{4}R_2+R_1\rightarrow R_1$.

$\Rightarrow \mathbf{p}^{(3)}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_3=12$.

The matrix $P=[\,\mathbf{p}^{(1)} \,\vdots\, \mathbf{p}^{(2)} \,\vdots\, \mathbf{p}^{(3)}\,]=\begin{bmatrix}-2 & 1 & 0\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}$ should diagonalize A.

To find a real, orthogonal matrix that diagonalizes A, all that we need to do is to normalize the columns because $\{\mathbf{p}^{(1)},\mathbf{p}^{(2)},\mathbf{p}^{(3)}\}$ is already an orthogonal set. So, the real, orthogonal matrix

$$Q=\left[\frac{1}{\sqrt{6}}\mathbf{p}^{(1)} \,\vdots\, \frac{1}{\sqrt{3}}\mathbf{p}^{(2)} \,\vdots\, \frac{1}{\sqrt{2}}\mathbf{p}^{(3)}\right]=\frac{1}{\sqrt{6}}\begin{bmatrix}-2 & \sqrt{2} & 0\\ 1 & \sqrt{2} & -\sqrt{3}\\ 1 & \sqrt{2} & \sqrt{3}\end{bmatrix}$$

should diagonalize A.

2.6.3.7. $0=\begin{vmatrix}-2-\lambda & 0 & 0\\ 0 & 3-\lambda & -1\\ 0 & -1 & 3-\lambda\end{vmatrix}=(-2-\lambda)\begin{vmatrix}3-\lambda & -1\\ -1 & 3-\lambda\end{vmatrix}=(-2-\lambda)\big((3-\lambda)^2-1\big)$

$$=(-2-\lambda)\big((3-\lambda)-1\big)\big((3-\lambda)+1\big)$$

$\Rightarrow$ eigenvalues $\lambda_1=-2$, $\lambda_2=2$, $\lambda_3=4$.

$[\,A-\lambda_1 I \mid \mathbf{0}\,]=\left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 0 & 5 & -1 & 0\\ 0 & -1 & 5 & 0\end{array}\right]\sim\left[\begin{array}{ccc|c}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$, after $R_1\leftrightarrow R_3$, $-R_1\rightarrow R_1$, $-5R_1+R_2\rightarrow R_2$, $\frac{1}{24}R_2\rightarrow R_2$, $5R_2+R_1\rightarrow R_1$.

$\Rightarrow \mathbf{p}^{(1)}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_1=-2$.

$[\,A-\lambda_2 I \mid \mathbf{0}\,]=\left[\begin{array}{ccc|c}-4 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\end{array}\right]\sim\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$, after $-\frac{1}{4}R_1\rightarrow R_1$, $R_2+R_3\rightarrow R_3$.

$\Rightarrow \mathbf{p}^{(2)}=\begin{bmatrix}0\\1\\1\end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_2=2$.

$[\,A-\lambda_3 I \mid \mathbf{0}\,]=\left[\begin{array}{ccc|c}-6 & 0 & 0 & 0\\ 0 & -1 & -1 & 0\\ 0 & -1 & -1 & 0\end{array}\right]\sim\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$, after $-\frac{1}{6}R_1\rightarrow R_1$, $-R_2+R_3\rightarrow R_3$, $-R_2\rightarrow R_2$.

$\Rightarrow \mathbf{p}^{(3)}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_3=4$.

The matrix $P=[\,\mathbf{p}^{(1)} \,\vdots\, \mathbf{p}^{(2)} \,\vdots\, \mathbf{p}^{(3)}\,]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{bmatrix}$ should diagonalize A.

To find a real, orthogonal matrix that diagonalizes A, all that we need to do is to normalize the columns because $\{\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \mathbf{p}^{(3)}\}$ is already an orthogonal set. So, the real, orthogonal matrix

$$Q = \left[\, \mathbf{p}^{(1)} \,\vdots\, \frac{1}{\sqrt{2}}\mathbf{p}^{(2)} \,\vdots\, \frac{1}{\sqrt{2}}\mathbf{p}^{(3)} \right] = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

should diagonalize A.

2.6.3.9. $0 = \begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} \underset{R_2+R_3\to R_3}{=} \begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ 0 & -9-\lambda & -9-\lambda \end{vmatrix}$

$$\underset{R_3 \leftarrow (-9-\lambda)R_3}{=} (-9-\lambda) \begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ 0 & 1 & 1 \end{vmatrix} \underset{C_2-C_3\to C_2}{=} (9-\lambda) \begin{vmatrix} 7-\lambda & 8 & 4 \\ 4 & -7-\lambda & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (-9-\lambda) \begin{vmatrix} 7-\lambda & 8 \\ 4 & -7-\lambda \end{vmatrix} = (-9-\lambda)\big((7-\lambda)(-7-\lambda) - 32\big) = (-9-\lambda)(\lambda^2 - 81)$$

$\Rightarrow$ eigenvalues $\lambda_1 = \lambda_2 = -9$, $\lambda_3 = 9$.

$[\, A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{rrr|r} 16 & 4 & -4 & 0 \\ 4 & 1 & -1 & 0 \\ -4 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1/4 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, after $\frac{1}{16}R_1 \to R_1$, $-4R_1 + R_2 \to R_2$, $4R_1 + R_3 \to R_3$.

$\Rightarrow$ free variables are $x_2 = c_1$ and $x_3 = c_2$

$$\Rightarrow \mathbf{v}_1 = \begin{bmatrix} -\frac{1}{4}c_1 + \frac{1}{4}c_2 \\ c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{p}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ are eigenvectors corr. to eigenvalue $\lambda_1 = \lambda_2 = -9$.

$[\, A - \lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{rrr|r} -2 & 4 & -4 & 0 \\ 4 & -17 & -1 & 0 \\ -4 & -1 & -17 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 2 & 0 \\ 0 & -9 & -9 & 0 \\ 0 & -9 & -9 & 0 \end{array}\right]$, after $2R_1 + R_2 \to R_2$, $-2R_1 + R_3 \to R_3$, $-\frac{1}{2}R_1 \to R_1$,

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after $-R_2 + R_3 \to R_2$, $-\frac{1}{9}R_2 \to R_2$, $2R_2 + R_1 \to R_1$.

$\Rightarrow \mathbf{p}^{(3)} = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector corr. to eigenvalue $\lambda_3 = 9$.

The matrix $P = [\, \mathbf{p}^{(1)} \,\vdots\, \mathbf{p}^{(2)} \,\vdots\, \mathbf{p}^{(3)} \,] = \begin{bmatrix} -1 & 1 & -4 \\ 4 & 0 & -1 \\ 0 & 4 & 1 \end{bmatrix}$ should diagonalize A.

Next, we will find a real, orthogonal matrix that diagonalizes A. We will need to use the Gram-Schmidt process on $\{\mathbf{p}^{(1)}, \mathbf{p}^{(2)}\}$ because it is not an orthogonal set of eigenvectors corr. to eigenvalue $\lambda_1 = \lambda_2 = -9$.

To start the Gram-Schmidt process, let

$$\mathbf{v}_1 \triangleq \mathbf{p}^{(1)},\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{17},$$

and

$$\mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{17}}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{p}^{(2)} - (\mathbf{p}^{(2)} \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \bullet \frac{1}{\sqrt{17}}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{17}}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \left(\frac{-1}{\sqrt{17}} \right) \frac{1}{\sqrt{17}}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$= \frac{4}{17}\begin{bmatrix} 4 \\ 1 \\ 17 \end{bmatrix}$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{4\sqrt{306}}{17}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{\sqrt{306}}\begin{bmatrix} 4 \\ 1 \\ 17 \end{bmatrix}.$$

So, the real, orthogonal matrix

$$Q = \left[\mathbf{q}_1 \ \vdots \ \mathbf{q}_2 \ \vdots \ \frac{1}{\sqrt{18}}\mathbf{p}^{(3)} \right] = \frac{1}{\sqrt{306}}\begin{bmatrix} -\sqrt{18} & 4 & -4\sqrt{17} \\ 4\sqrt{18} & 1 & -\sqrt{17} \\ 0 & 17 & \sqrt{17} \end{bmatrix}$$

should diagonalize A.

2.6.3.11. For every vector $\mathbf{x} = [\,x_1 \quad \dots \quad x_n\,]^T$ in $\mathbb{C}^n$,

$$\mathbf{x}^T\overline{\mathbf{x}} = x_1\overline{x_1} + \cdots + x_n\overline{x_n} = |x_1|^2 + \cdots + |x_n|^2$$

is real.

2.6.3.13. Suppose A is a real, $m \times n$ matrix, A^TA is invertible, $m > n$, and we define $B = A(A^TA)^{-1}A^T$. Then

$$B^2 = \big(A(A^TA)^{-1}A^T\big)^2 = \big(A(A^TA)^{-1}A^T\big)\big(A(A^TA)^{-1}A^T\big) = A\cancel{(A^TA)^{-1}}\cancel{(A^TA)}(A^TA)^{-1}A^T$$

$$= A\big(A^TA\big)^{-1}A^T = B,$$

hence $B^2 = B$.

2.6.3.15. The spectral decomposition (2.34) in Section 2.6 gives $A = \sum_{i=1}^n \lambda_i \mathbf{q}_i\mathbf{q}_i^T$ and we define

$$\sqrt{A} \triangleq \sum_{i=1}^n \sqrt{\lambda_i}\ \mathbf{q}_i\mathbf{q}_i^T.$$

We calculate that

$$\left(\sqrt{A}\right)^2 = \left(\sum_{i=1}^n \sqrt{\lambda_i}\ \mathbf{q}_i\mathbf{q}_i^T \right) \left(\sum_{j=1}^n \sqrt{\lambda_j}\ \mathbf{q}_j\mathbf{q}_j^T \right) = \sum_{i=1}^n \sum_{j=1}^n \left(\sqrt{\lambda_i}\ \mathbf{q}_i\mathbf{q}_i^T \right) \left(\sqrt{\lambda_j}\ \mathbf{q}_j\mathbf{q}_j^T \right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \sqrt{\lambda_i}\ \sqrt{\lambda_j}\ \mathbf{q}_i(\mathbf{q}_i^T\mathbf{q}_j)\mathbf{q}_j^T = \sum_{i=1}^{n}\sum_{j=1}^{n} \sqrt{\lambda_i}\ \sqrt{\lambda_j}\ \mathbf{q}_i \cdot \left\{ \begin{array}{ll} 1, & i=j \\ 0, & i\neq j \end{array} \right\} \cdot \mathbf{q}_j^T$$

$$= \sum_{i=1}^{n}(\sqrt{\lambda_i})^2\mathbf{q}_i\mathbf{q}_i^T, = \sum_{i=1}^{n}\lambda_i\mathbf{q}_i\mathbf{q}_i^T = A,$$

hence (2.38) in Section 2.6 is correct.

2.6.3.17. We define $\langle \mathbf{x}, \mathbf{y}\rangle_W \triangleq \langle W\mathbf{x}, \mathbf{y}\rangle \triangleq \mathbf{x}^T\, W^T\mathbf{y}$ and $||\,\mathbf{x}\,||_W \triangleq \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle_W}$, where W is a real, symmetric, positive definite $n \times n$ matrix.

The conclusions of Theorem 2.29 in Section 2.6 are the conclusions of Theorems 2.12 and 2.13 in Section 2.3 with $\langle \mathbf{x}, \mathbf{y}\rangle_W$ and $||\,\mathbf{x}\,||_W$ replacing $\langle \mathbf{x}, \mathbf{y}\rangle$ and $||\,\mathbf{x}\,||$, respectively.

Basically, the method of establishing these results is to apply Theorems 2.12 and 2.13's properties concerning $\langle \mathbf{x}, \mathbf{y}\rangle$ *and* $||\,\mathbf{x}\,||$ *to get properties for* $\langle \mathbf{x}, \mathbf{y}\rangle_W$ *and* $||\,\mathbf{x}\,||_W$.

Below, $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vectors in $\mathbb{R}^n$.

Regarding Theorem 2.12(a) in Section 2.3, using $W = W^T$ we calculate

$$\langle \mathbf{x}, \mathbf{y}\rangle_W = \mathbf{x}^T\, W^T\mathbf{y} = \big(W\mathbf{x}\big)^T\mathbf{y} = \mathbf{y}^T\big(W\mathbf{x}\big) = \mathbf{y}^T\big(W^T\mathbf{x}\big) = \mathbf{y}^T W^T\mathbf{x} = \langle \mathbf{y}, \mathbf{x}\rangle_W.$$

Regarding Theorem 2.12(b) in Section 2.3, we calculate

$$\langle \alpha\mathbf{x}, \mathbf{y}\rangle_W = (\alpha\mathbf{x})^T\, W^T\mathbf{y} = \alpha\mathbf{x}^T\, W^T\mathbf{y} = \alpha\langle \mathbf{x}, \mathbf{y}\rangle_W.$$

Regarding Theorem 2.12(c) in Section 2.3, we calculate

$$\langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}\rangle_W = (\mathbf{x}_1 + \mathbf{x}_2)^T\, W^T\mathbf{y} = \mathbf{x}_1^T\, W^T\mathbf{y} + \mathbf{x}_2^T\, W^T\mathbf{y} = \langle \mathbf{x}_1, \mathbf{y}\rangle_W + \langle \mathbf{x}_2, \mathbf{y}\rangle_W.$$

Regarding Theorem 2.12(d) in Section 2.3, we know that

$$\langle \mathbf{x}, \mathbf{x}\rangle_W = \mathbf{x}^T\, W^T\mathbf{x} = \mathbf{x}^T\, W\mathbf{x}$$

is positive as long as $\mathbf{x} \neq \mathbf{0}$, by the definition of W being positive definite; when $\mathbf{x} = \mathbf{0}$, then $\langle \mathbf{x}, \mathbf{x}\rangle_W = \mathbf{0}^T\, W^T\mathbf{0} = 0$. So, $\langle \mathbf{x}, \mathbf{x}\rangle \geq 0$, with equality only if $\mathbf{x} = \mathbf{0}$.

Regarding Theorem 2.13(a) in Section 2.3, $||\,\mathbf{x}\,||_W^2 = \langle \mathbf{x}, \mathbf{x}\rangle_W$ is true by our definitions of $\langle \mathbf{x}, \mathbf{y}\rangle_W$ and $||\,\mathbf{x}\,||_W$.

Regarding Theorem 2.13(b) in Section 2.3, we calculate

$$||\mathbf{x} + \mathbf{y}||_W^2 = \langle W(\mathbf{x} + \mathbf{y}), \mathbf{x} + \mathbf{y}\rangle = \langle W\mathbf{x}, \mathbf{x}\rangle + 2\langle W\mathbf{x}, \mathbf{y}\rangle + \langle W\mathbf{y}, \mathbf{y}\rangle = ||\,\mathbf{x}\,||_W^2 + ||\,\mathbf{y}\,||_W^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle_W.$$

Regarding Theorem 2.13(c) in Section 2.3, we know from Theorem 2.12(d) in Section 2.3 that $\mathbf{x}^T\, W^T\mathbf{x} \geq 0$, with equality only if $\mathbf{x} = \mathbf{0}$, so

$$||\mathbf{x}||_W = \sqrt{\mathbf{x}^T\, W^T\mathbf{x}} \geq \sqrt{0} = 0,$$

with equality only if $\mathbf{x} = \mathbf{0}$.

Regarding Theorem 2.13(d) in Section 2.3, we calculate

$$||\alpha\mathbf{x}||_W = \sqrt{(\alpha\mathbf{x})^T\, W^T(\alpha\mathbf{x})} = \sqrt{\alpha^2(\mathbf{x}^T\, W^T\mathbf{x})} = \sqrt{\alpha^2}\ \sqrt{\mathbf{x}^T\, W^T\mathbf{x}} = |\alpha|\ ||\,\mathbf{x}\,||_W.$$

Regarding Theorem 2.13(e) in Section 2.3, using the existence of $\sqrt{W}$ that satisfies

$$\big(\sqrt{W}\big)^2 = W = W^T = \Big(\big(\sqrt{W}\big)^T\Big)^2,$$

we calculate

$$|\langle \mathbf{x}, \mathbf{y}\rangle_W| = |\mathbf{x}^T W^T\mathbf{y}| = \left|\mathbf{x}^T\Big(\big(\sqrt{W}\big)^T\Big)^2\mathbf{y}\right| = \left|\mathbf{x}^T\big(\sqrt{W}\big)^T\big(\sqrt{W}\big)^T\mathbf{y}\right| = \left|\Big(\sqrt{W}\ \mathbf{x}\Big)^T\Big(\sqrt{W}\ \mathbf{y}\Big)\right|$$

$$\leq ||\sqrt{W}\ \mathbf{x}||\ ||\sqrt{W}\ \mathbf{y}||,$$

using the Cauchy-Schwarz inequality, that is, Theorem 2.13(e) in Section 2.3. Continuing, we have

$$|\langle \mathbf{x},\mathbf{y}\rangle_W| \leq ||\sqrt{W}\ \mathbf{x}||\ ||\sqrt{W}\ \mathbf{y}|| = \sqrt{\left(\sqrt{W}\mathbf{x}\right)^T\left(\sqrt{W}\mathbf{x}\right)} \cdot \sqrt{\left(\sqrt{W}\mathbf{y}\right)^T\left(\sqrt{W}\mathbf{y}\right)}$$

$$= \sqrt{\mathbf{x}^T\left(\sqrt{W}\right)^T\sqrt{W}\mathbf{x}} \cdot \sqrt{\mathbf{y}^T\left(\sqrt{W}\right)^T\sqrt{W}\mathbf{y}} = \sqrt{\mathbf{x}^T\left(\sqrt{W}\right)^T\left(\sqrt{W}\right)^T\mathbf{x}} \cdot \sqrt{\mathbf{y}^T\left(\sqrt{W}\right)^T\left(\sqrt{W}\right)^T\mathbf{y}}$$

$$= \sqrt{\mathbf{x}^T W^T \mathbf{x}} \cdot \sqrt{\mathbf{y}^T W^T \mathbf{y}} = ||\,\mathbf{x}\,||_W\ ||\,\mathbf{y}\,||_W.$$

Regarding Theorem 2.13(f) in Section 2.3, we calculate

$$||\mathbf{x}+\mathbf{y}||_W = \sqrt{\langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle_W} = \sqrt{(\mathbf{x}+\mathbf{y})W^T(\mathbf{x}+\mathbf{y})} = \sqrt{(\mathbf{x}+\mathbf{y})\left(\sqrt{W}\right)^T\left(\sqrt{W}\right)^T(\mathbf{x}+\mathbf{y})}$$

$$= \sqrt{\left(\sqrt{W}(\mathbf{x}+\mathbf{y})\right)^T\sqrt{W}(\mathbf{x}+\mathbf{y})} = \sqrt{\left(\sqrt{W}\mathbf{x}+\sqrt{W}\mathbf{y}\right)^T\left(\sqrt{W}\mathbf{x}+\sqrt{W}\mathbf{y}\right)}$$

$$= \left|\left|\sqrt{W}\mathbf{x}+\sqrt{W}\mathbf{y}\right|\right| \leq \left|\left|\sqrt{W}\mathbf{x}\right|\right| + \left|\left|\sqrt{W}\mathbf{y}\right|\right|,$$

using Theorem 2.13(f) in Section 2.3. Continuing, we have

$$||\mathbf{x}+\mathbf{y}||_W \leq \left|\left|\sqrt{W}\mathbf{x}\right|\right| + \left|\left|\sqrt{W}\mathbf{y}\right|\right| = \sqrt{\left(\sqrt{W}\mathbf{x}\right)^T\sqrt{W}\mathbf{x}} \cdot \sqrt{\left(\sqrt{W}\mathbf{y}\right)^T\sqrt{W}\mathbf{y}}$$

$$= \sqrt{\mathbf{x}^T\left(\sqrt{W}\right)^T\sqrt{W}\mathbf{x}} + \sqrt{\mathbf{y}^T\left(\sqrt{W}\right)^T\sqrt{W}\mathbf{y}} = \sqrt{\mathbf{x}^T\left(\sqrt{W}\right)^T\left(\sqrt{W}\right)^T\mathbf{x}} + \sqrt{\mathbf{y}^T\left(\sqrt{W}\right)^T\left(\sqrt{W}\right)^T\mathbf{y}}$$

$$= \sqrt{\mathbf{x}^T W^T \mathbf{x}} + \sqrt{\mathbf{y}^T W^T \mathbf{y}} = ||\,\mathbf{x}\,||_W + ||\,\mathbf{y}\,||_W.$$

2.6.3.19. Let $W = diag(b_1^{-2},...,b_m^{-2}) = W^T$. Then the **relative squared error** is

$$\sum_{i=1}^{m}\left(\frac{(A\mathbf{x})_i - b_i}{b_i}\right)^2 = \left\langle \begin{bmatrix} \frac{(A\mathbf{x})_1-b_1}{b_1} \\ \vdots \\ \frac{(A\mathbf{x})_m-b_m}{b_m} \end{bmatrix}, \begin{bmatrix} \frac{(A\mathbf{x})_1-b_1}{b_1} \\ \vdots \\ \frac{(A\mathbf{x})_m-b_m}{b_m} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \frac{(A\mathbf{x})_1-b_1}{b_1^2} \\ \vdots \\ \frac{(A\mathbf{x})_m-b_m}{b_m^2} \end{bmatrix}, \begin{bmatrix} (A\mathbf{x})_1 - b_1 \\ \vdots \\ (A\mathbf{x})_m - b_m \end{bmatrix} \right\rangle$$

$= \left\langle diag(b_1^{-2},...,b_m^{-2})(A\mathbf{x}-\mathbf{b}), (A\mathbf{x}-\mathbf{b})\right\rangle = \langle W(A\mathbf{x}-\mathbf{b}), (A\mathbf{x}-\mathbf{b})\rangle = (A\mathbf{x}-\mathbf{b})^T W^T (A\mathbf{x}-\mathbf{b}) = ||A\mathbf{x}-\mathbf{b}||_W^2.$

So, the problem of minimizing the relative squared error is a weighted least squares problem.

2.6.3.21. Exs.: $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Section 2.7.7

2.7.7.1. Use the Gram-Schmidt process on the columns of A: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{2},\ \mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \bullet \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{\sqrt{6}}{2}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

We have

$$Q = [\ \mathbf{q}_1 \ \vdots\ \mathbf{q}_2\ ] = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & -1 \\ \sqrt{3} & 1 \\ 0 & 2 \end{bmatrix}$$

and, using $r_{12} \triangleq \mathbf{a}_2 \bullet \mathbf{q}_1$,

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2}^{\,-1} \\ 0 & \frac{\sqrt{6}}{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}.$$

2.7.7.3. Use the Gram-Schmidt process on the columns of A: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. Then

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{3},\ \mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \bullet \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - \left(\frac{-1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix},$$

$r_{22} \triangleq ||\mathbf{v}_2|| = \frac{\sqrt{42}}{3}$, and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{\sqrt{42}}\begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}.$$

We have

$$Q = [\ \mathbf{q}_1 \ \vdots\ \mathbf{q}_2\ ] = \frac{1}{\sqrt{42}}\begin{bmatrix} \sqrt{14} & 1 \\ \sqrt{14} & -5 \\ \sqrt{14} & 4 \end{bmatrix}$$

and, using $r_{12} \triangleq \mathbf{a}_2 \bullet \mathbf{q}_1$,

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -\sqrt{3}^{\,-1} \\ 0 & \frac{\sqrt{42}}{3} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 3 & -1 \\ 0 & \sqrt{14} \end{bmatrix}.$$

2.7.7.5. We are given that $A = [\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n]$, where $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is an orthogonal set in $\mathbb{R}^m$. It follows that $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is linearly independent, so the Gram-Schmidt process can be used to find the QR factorization of A:

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1||,\ \mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{||\mathbf{a}_1||}\,\mathbf{a}_1,$$

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \mathbf{a}_2 - (\mathbf{a}_2 \bullet \frac{1}{||\mathbf{a}_1||}\,\mathbf{a}_1)\mathbf{q}_1 = \mathbf{a}_2 - 0 \cdot \mathbf{q}_1$$

hence

$$\mathbf{v}_2 = \mathbf{a}_2,\ r_{22} \triangleq ||\mathbf{v}_2||,\ \mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{||\mathbf{a}_2||}\,\mathbf{a}_2,$$

$$\vdots$$

$$\mathbf{v}_n \triangleq \mathbf{a}_n - (\mathbf{a}_n \bullet \mathbf{q}_1)\mathbf{q}_1 - \dots - (\mathbf{a}_n \bullet \mathbf{q}_{n-1})\mathbf{q}_{n-1} = \mathbf{a}_n - 0 \cdot \mathbf{q}_1 - \dots - -0 \cdot \mathbf{q}_{n-1} = \mathbf{a}_n,$$

$$r_{nn} \triangleq ||\mathbf{v}_n||,\ \mathbf{q}_n = r_{nn}^{-1}\mathbf{v}_n = \frac{1}{||\mathbf{a}_n||}\,\mathbf{a}_n.$$

So,

$$Q = \left[\ \frac{1}{||\mathbf{a}_1||}\,\mathbf{a}_1 \ \vdots\ \dots\ \vdots\ \frac{1}{||\mathbf{a}_n||}\,\mathbf{a}_n\ \right]$$

and

$$R = diag(\,||\mathbf{a}_1||, \dots, ||\mathbf{a}_n||\,).$$

2.7.7.7. Because A is invertible it must be square, say $n \times n$.

Because $A = [\,\mathbf{a}_1 \ \vdots\ \dots\ \vdots\ \mathbf{a}_n\,]$ is a real, invertible, upper triangular matrix, it follows that $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is linearly independent. The Gram-Schmidt process can be used to find the QR factorization of A:

$$\mathbf{v}_1 \triangleq \mathbf{a}_1 = a_{11}\mathbf{e}^{(1)},\ r_{11} \triangleq ||\mathbf{v}_1|| = |a_{11}|,$$

and

$$\mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{|a_{11}|}\,\mathbf{a}_1 = sgn(a_{11})\,\mathbf{e}^{(1)}.$$

Next, $\mathbf{a}_2 = a_{12}\mathbf{e}^{(1)} + a_{22}\mathbf{e}^{(2)}$, so

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \mathbf{a}_2 - \big(sgn(a_{11})a_{12}\big)\big(sgn(a_{11})\,\mathbf{e}^{(1)}\big) = \mathbf{a}_2 - \big(sgn(a_{11})\big)^2\,a_{12}\,\mathbf{e}^{(1)} = \mathbf{a}_2 - a_{12}\,\mathbf{e}^{(1)}$$

hence

$$\mathbf{v}_2 = a_{22}\mathbf{e}^{(2)},\ r_{22} \triangleq ||\mathbf{v}_2|| = |a_{22}|,$$

and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{1}{|a_{22}|}\,a_{22}\,\mathbf{e}^{(2)} = sgn(a_{22})\,\mathbf{e}^{(2)}.$$

$$\vdots$$

$\mathbf{a}_n = a_{1n}\mathbf{e}^{(1)} + \dots + a_{nn}\mathbf{e}^{(n)}$, so

$$\mathbf{v}_n \triangleq \mathbf{a}_n - (\mathbf{a}_n \bullet \mathbf{q}_1)\mathbf{q}_1 - \dots - (\mathbf{a}_n \bullet \mathbf{q}_{n-1})\mathbf{q}_{n-1}$$

$$= \mathbf{a}_n - \big(sgn(a_{11})\,a_{1n}\big)\big(sgn(a_{11})\,\mathbf{e}^{(1)}\big) - \dots - \big(sgn(a_{n-1,n-1})\,a_{12}\big)\big(sgn(a_{n-1,n-1})\,\mathbf{e}^{(n-1)}\big)$$

$$= ... = a_{nn}\mathbf{e}^{(n)},\ r_{nn} \triangleq ||\mathbf{v}_n|| = |a_{nn}|,$$

and

$$\mathbf{q}_n = r_{nn}^{-1}\mathbf{v}_n = \frac{1}{|a_{nn}|}\, a_{nn}\, \mathbf{e}^{(n)} = sgn(a_{nn})\, \mathbf{e}^{(n)}.$$

So,

$$Q = \left[\, sgn(a_{11})\, \mathbf{e}^{(1)} \ \vdots\ ...\ \vdots\ sgn(a_{nn})\, \mathbf{e}^{(n)} \,\right],$$

that is,

$$Q = diag\Big(sgn(a_{11}), ..., sgn(a_{nn})\Big),$$

and

$$R = \begin{bmatrix} |a_{11}| & a_{12}sgn(a_{11}) & . & . & . & a_{1n}sgn(a_{11}) \\ 0 & |a_{22}| & & & & a_{2n}sgn(a_{22}) \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ 0 & . & . & . & & |a_{nn}| \end{bmatrix}.$$

Note that the diagonal elements of R are $||\mathbf{v}_j|| > 0$, $j = 1, ..., n$.

2.7.7.9. Take the hint, and begin by using $R\mathbf{x} = Q^T\mathbf{b}$ to see that

$$A\mathbf{x} = Q(R\mathbf{x}) = Q(Q^T\mathbf{b}) = (QQ^T)\mathbf{b}.$$

Next, use $\mathbf{b} = Q\mathbf{c}$ to see that

$$A\mathbf{x} = (QQ^T)(Q\mathbf{c}) = Q(Q^TQ)\mathbf{c}) = Q(I)\mathbf{c} = Q\mathbf{c} = \mathbf{b}.$$

So, yes, $\mathbf{x} = R^{-1}Q^T\mathbf{b}$ is guaranteed to satisfy the original system, $A\mathbf{x} = \mathbf{b}$, as long as $\mathbf{b} = Q\mathbf{c}$ for some $\mathbf{c}$ and $A = QR$ is the QR factorization of A.

2.7.7.11. We see that $A = QR$ is the QR factorization with $Q = \begin{bmatrix} 2/3 & 1/\sqrt{5} \\ 2/3 & 0 \\ -1/3 & 2/\sqrt{5} \end{bmatrix}$ and $R = \begin{bmatrix} 3 & -3 \\ 0 & \sqrt{5} \end{bmatrix}$.

Then the unique solution (and thus the unique l.s.s.) of the system $A\mathbf{x} = \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is given by

$$\mathbf{x} = R^{-1}Q^T\mathbf{b} = \begin{bmatrix} 3 & -3 \\ 0 & \sqrt{5} \end{bmatrix}^{-1} \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix} \sqrt{5} & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 4/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{3\sqrt{5}} \begin{bmatrix} 41/(3\sqrt{5}) \\ 12/\sqrt{5} \end{bmatrix} = \frac{1}{45} \begin{bmatrix} 41 \\ 36 \end{bmatrix}.$$

2.7.7.13. We did not mention the case $m < n$ before Theorem 2.36 in Section 2.7 because it is impossible for the columns of an $m \times n$ matrix to be linearly independent if there are more columns than rows in the matrix. This is because of Corollary 1.3 in Section 1.7.

2.7.7.15. To start the Gram-Schmidt process, let

$$\mathbf{v}_1 \triangleq \mathbf{a}_1,\ r_{11} \triangleq ||\mathbf{v}_1|| = \sqrt{\mathbf{a}_1 \bullet \mathbf{a}_1} = \sqrt{2},$$

and

$$\mathbf{q}_1 = r_{11}^{-1}\mathbf{v}_1 = \frac{1}{\sqrt{2}}\,\mathbf{a}_1.$$

Next, let

$$\mathbf{v}_2 \triangleq \mathbf{a}_2 - (\mathbf{a}_2 \bullet \mathbf{q}_1)\mathbf{q}_1 = \mathbf{a}_2 - \Big(\mathbf{a}_2 \bullet \frac{1}{\sqrt{2}}\,\mathbf{a}_1\Big)\frac{1}{\sqrt{2}}\,\mathbf{a}_1 = \mathbf{a}_2 - \Big(\frac{1}{\sqrt{2}}\,\mathbf{a}_2 \bullet \mathbf{a}_1\Big)\frac{1}{\sqrt{2}}\,\mathbf{a}_1 = \mathbf{a}_2 - \Big(-\frac{1}{\sqrt{2}}\Big)\frac{1}{\sqrt{2}}\,\mathbf{a}_1$$

$$= \mathbf{a}_2 + \frac{1}{2}\,\mathbf{a}_1.$$

So

$$r_{22}^2 \triangleq ||\mathbf{v}_2||^2 = \Big|\Big|\mathbf{a}_2 + \frac{1}{2}\,\mathbf{a}_1\Big|\Big|^2 = \Big\langle \mathbf{a}_2 + \frac{1}{2}\,\mathbf{a}_1,\, \mathbf{a}_2 + \frac{1}{2}\,\mathbf{a}_1\Big\rangle = \langle\mathbf{a}_2,\,\mathbf{a}_2\rangle + \langle\mathbf{a}_2,\,\mathbf{a}_1\rangle + \frac{1}{4}\langle\mathbf{a}_1,\,\mathbf{a}_1\rangle = 2 + (-1) + \frac{1}{4}\cdot 2 = \frac{3}{2}.$$

We have

$$r_{22} = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}}$$

and

$$\mathbf{q}_2 = r_{22}^{-1}\mathbf{v}_2 = \frac{\sqrt{2}}{\sqrt{3}}\Big(\mathbf{a}_2 + \frac{1}{2}\,\mathbf{a}_1\Big) = \frac{2}{\sqrt{6}}\Big(\mathbf{a}_2 + \frac{1}{2}\,\mathbf{a}_1\Big).$$

The QR factorization is $A = QR$, where

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{a}_1 & \frac{1}{\sqrt{6}}\big(\mathbf{a}_1 + 2\mathbf{a}_2\big) \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix}.$$

2.7.7.17. We calculate

$$B = A^T A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

It is easy to see that the eigenvalues of $B = 2I_2$ are $\lambda_1 = \lambda_2 = 2$ and the corresponding eigenvectors are $\mathbf{v}_1 = \mathbf{e}^{(1)} = [1 \quad 0]$ and $\mathbf{v}_2 = \mathbf{e}^{(2)} = [0 \quad 1]$, respectively. It follows that $\sigma_1 = \sigma_2 = \sqrt{2}$, $\Sigma = S = diag(\sqrt{2}, \sqrt{2}) = \sqrt{2}I_2$, and $V = [\,\mathbf{e}^{(1)} \quad \mathbf{e}^{(2)}] = I_2$.

Using this we see that

$$U_1 = AV_1S^{-1} = AI_2\sqrt{2}^{-1}I_2 = \frac{1}{\sqrt{2}}A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Because A is 2×2 and U_1 is 2×2, $U = U_1$, that is, we do not need to find additional columns in the orthogonal matrix U.

To summarize, the SVD factorization is $A = U\Sigma V^T$, where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$, and $V^T = I_2$.

2.7.7.19. We calculate

$$B = A^T A = \begin{bmatrix} 2 & 4/\sqrt{5} & 2 \\ 1 & -8/\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4/\sqrt{5} & -8/\sqrt{5} \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{56}{5} & -\frac{12}{5} \\ -\frac{12}{5} & \frac{74}{5} \end{bmatrix}.$$

Next, find the eigenvalues of B:

$$0 = \begin{bmatrix} \frac{56}{5} - \mu & -\frac{12}{5} \\ -\frac{12}{5} & \frac{74}{5} - \mu \end{bmatrix} = \Big(\frac{56}{5} - \mu\Big)\Big(\frac{74}{5} - \mu\Big) - \frac{144}{25} = \mu^2 - \frac{130}{5}\,\mu + \frac{4000}{25} = \mu^2 - 26\mu + 160$$

$$= (\mu - 16)(\mu - 10).$$

The eigenvalues of B are $\mu_1 = 16$ and $\mu_2 = 10$. Correspondingly, $\sigma_1 = 4$ and $\sigma_2 = \sqrt{10}$.

Next, find the eigenvectors of B corresponding to its eigenvalues:

$$[\, B - \mu_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -\frac{24}{5} & -\frac{12}{5} & 0 \\ & & \\ -\frac{12}{5} & -\frac{6}{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ after } -\tfrac{1}{2}R_1 + R_2 \to R_2,\ -\tfrac{5}{24}R_1 \to R_1.$$

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector corr. to B's eigenvalue $\mu_1 = 16$.

$\Rightarrow \mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is a normalized eigenvector corr. to B's eigenvalue $\mu_1 = 16$.

$$[\, B - \mu_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} \frac{6}{5} & -\frac{12}{5} & 0 \\ & & \\ -\frac{12}{5} & \frac{24}{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ after } 2R_1 + R_2 \to R_2,\ \tfrac{5}{6}R_1 \to R_1.$$

$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corr. to B's eigenvalue $\mu_2 = 10$.

$\Rightarrow \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a normalized eigenvector corr. to eigenvalue $\mu_2 = 10$.

Because all of the σ's are positive, we have

$$V = V_1 = [\mathbf{v}_1 \ \ \mathbf{v}_2] = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Because all of the σ's are positive, we have that the 3×2 matrix $\Sigma = \begin{bmatrix} \multicolumn{2}{c}{S} \\ - & - \\ 0 & 0 \end{bmatrix}$, where $S = diag(4, \sqrt{10})$.

So, $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$. Using this we see that

$$U_1 = AV_1S^{-1} = \begin{bmatrix} 2 & 1 \\ 4/\sqrt{5} & -8/\sqrt{5} \\ 2 & 1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 4/\sqrt{5} & -8/\sqrt{5} \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{2}{\sqrt{10}} \\ & \\ \frac{2}{4} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & \frac{5}{\sqrt{10}} \\ -\sqrt{5} & 0 \\ 0 & \frac{5}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} \\ -1 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}.$$

The 3×3 real, orthogonal matrix $U = [\, U_1 \mid U_2 \,] = [\, \mathbf{u}_1 \quad \mathbf{u}_2 \mid \mathbf{u}_3 \,]$, where we can find $\mathbf{u}_3$ by the process in the Appendix to Section 2.4: First, calculate

$$\mathbf{w}_3 = \mathbf{e}^{(1)} - \langle \mathbf{e}^{(1)}, \mathbf{u}_1 \rangle \mathbf{u}_1 - \langle \mathbf{e}^{(1)}, \mathbf{u}_2 \rangle \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - (0) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$||\mathbf{w}_3|| = \frac{\sqrt{2}}{2}$, and finally

$$\mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

To summarize, the SVD factorization is

$A = U\Sigma V^T$, where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ -\sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$, and $V^T = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

2.7.7.21. Recall (2.60) in Section 2.7, that is, $A = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$, and also recall the thin SVD factorization $A = U_1 S V_1^T$. Essentially, our only use of (2.60) will be to make sure what are the correct notations of U_1, S, and V_1.

We have

$$\mathbf{x}^\star \triangleq V_1 S^{-1} U_1^T \mathbf{b} = V_1 \begin{bmatrix} \sigma_1^{-1} & 0 & . & . & . & 0 \\ 0 & \sigma_2^{-1} & & & & . \\ . & & & & . & . \\ . & & & & . & . \\ 0 & . & . & . & & \sigma_r^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_r^T \end{bmatrix} \mathbf{b}$$

$$= V_1 \begin{bmatrix} \sigma_1^{-1} & 0 & . & . & . & 0 \\ 0 & \sigma_2^{-1} & & & & . \\ . & & & & . & . \\ . & & & & . & . \\ 0 & . & . & . & & \sigma_r^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T\mathbf{b} \\ \vdots \\ \mathbf{u}_r^T\mathbf{b} \end{bmatrix} = V_1 \begin{bmatrix} \sigma_1^{-1}\mathbf{u}_1^T\mathbf{b} \\ \vdots \\ \sigma_r^{-1}\mathbf{u}_r^T\mathbf{b} \end{bmatrix} = [\, \mathbf{v}_1 \quad \dots \quad \mathbf{v}_r \,] \begin{bmatrix} \sigma_1^{-1}\mathbf{u}_1^T\mathbf{b} \\ \vdots \\ \sigma_r^{-1}\mathbf{u}_r^T\mathbf{b} \end{bmatrix}.$$

Using (1.42) in Section 1.7, we can rewrite this as

$$\mathbf{x}^\star = (\sigma_1^{-1}\mathbf{u}_1^T\mathbf{b})\mathbf{v}_1 + ... + (\sigma_r^{-1}\mathbf{u}_r^T\mathbf{b})\mathbf{v}_r,$$

that is, (2.61) in Section 2.7.

2.7.7.23. Define X to be the Moore-Penrose inverse A^+ we found in Example 2.34 in Section 2.7.

We will show that $X = A^+$ in Example 2.34 satisfies those properties.

First, we calculate that

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & -1 & -1 \\ \frac{3}{\sqrt{2}} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and, in Example 2.34 in Section 2.7, we calculated that

$$A^+ = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}.$$

(1) Regarding property (2.67)(a) in Section 2.7,

$$AXA = \begin{bmatrix} \frac{3}{\sqrt{2}} & -1 & -1 \\ \frac{3}{\sqrt{2}} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{2}} & -1 & -1 \\ \frac{3}{\sqrt{2}} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} & -1 & -1 \\ \frac{3}{\sqrt{2}} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & -1 & -1 \\ \frac{3}{\sqrt{2}} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A.$$

(2) Regarding property (2.67)(b) in Section 2.7,

$$XAX = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{2}} & -1 & -1 \\ \frac{3}{\sqrt{2}} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} = X.$$

(3) Finally, regarding property (2.67)(d) in Section 2.7,

$$AX = \begin{bmatrix} \frac{3}{\sqrt{2}} & -1 & -1 \\ \frac{3}{\sqrt{2}} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $(AX)^T = AX$.

By the way, the fact that in this example

$$AA^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

explains a little why A^+ is called a "pseudo-inverse."

2.7.7.25. Because A is a real, symmetric, positive definite matrix, it follows that $A = A^T$, all of its eigenvalues are positive, and there is an orthogonal matrix Q and diagonal matrix D such that $A = QDQ^T$. Without loss of generality suppose that $D = diag(\lambda_1, ..., \lambda_n)$, where $\lambda_1 \geq ... \geq \lambda_n > 0$.

To construct the SVD using the method following Theorem 2.37 in Section 2.7, we find

$$B = A^T A = A^2 = \big(QDQ^T\big)\big(QDQ^T\big) = QD\big(Q^T Q\big)DQ^T = Q^T D(I)DQ^T = QD^2Q^T$$

$$= Q\,diag(\lambda_1^2, ..., \lambda_n^2)\,Q^T.$$

It follows that the eigenvalues of B are $\lambda_1^2 \geq ... \geq \lambda_n^2 > 0$ and the corresponding eigenvectors can be chosen to be the columns of Q. It follows that $\sigma_j = \lambda_j$, for $j = 1, ..., n$.

So, $\Sigma = D$ and $V = Q$. It follows that

$$U = AV\Sigma^{-1} = (QDQ^T)QD^{-1} = QD(Q^TQ)D^{-1} = QD(I)D^{-1} = Q(DD^{-1}) = Q.$$

To summarize, if A is a real, symmetric, positive definite matrix, then its SVD factorization is $A = QDQ^T$, the factorization we found in Section 2.6.

2.7.7.27. Ex. (a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) We calculate

$$B = A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Next, find the eigenvalues of B:

$$0 = \begin{vmatrix} 2-\mu & 0 & 1 \\ 0 & 2-\mu & -1 \\ 1 & -1 & 2-\mu \end{vmatrix} \underset{R_1+R_2\to R_2}{=} \begin{vmatrix} 2-\mu & 0 & 1 \\ 2-\mu & 2-\mu & 0 \\ 1 & -1 & 2-\mu \end{vmatrix}$$

$$\underset{R_2\leftarrow(2-\mu)R_2}{=} (2-\mu)\begin{vmatrix} (2-\mu) & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 2-\mu \end{vmatrix}$$

and then expanding along the second column to get

$$= (2-\mu)\left(1\cdot\begin{vmatrix} 2-\mu & 1 \\ 1 & 2-\mu \end{vmatrix} - (-1)\cdot\begin{vmatrix} 2-\mu & 1 \\ 1 & 0 \end{vmatrix}\right) = (2-\mu)\big((2-\mu)^2-1-1\big) = (2-\mu)\big((2-\mu)^2-2\big).$$

The eigenvalues of B are $\mu_1 = 2+\sqrt{2}$, $\mu_2 = 2$, and $\mu_3 = 2-\sqrt{2}$. Correspondingly, $\sigma_1 = \sqrt{2+\sqrt{2}}$, $\sigma_2 = \sqrt{2}$, and $\sigma_3 = \sqrt{2-\sqrt{2}}$.

Next, find the eigenvectors of B corresponding to its eigenvalues:

$$[\,B-\mu_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -\sqrt{2} & 0 & 1 & 0 \\ 0 & -\sqrt{2} & -1 & 0 \\ 1 & -1 & -\sqrt{2} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -\sqrt{2} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\sqrt{2} & -1 & 0 \end{array}\right],$$

after $R_1 \leftrightarrow R_3$, $\sqrt{2}R_1 + R_3 \to R_3$, $-\frac{1}{\sqrt{2}}R_2 \to R_2$,

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after $\sqrt{2}R_2 + R_3 \to R_3$, $R_2 + R_1 \to R_1$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix}$ is an eigenvector corr. to B's eigenvalue $\mu_1 = 2+\sqrt{2}$.

$$\Rightarrow \mathbf{v}_1 = \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix} \text{ is a normalized eigenvector corr. to eigenvalue } \mu_1 = 2+\sqrt{2}.$$

$$[\,B-\mu_2 I \mid \mathbf{0}\,] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

after $R_1 \leftrightarrow R_3$, $R_2 + R_3 \rightarrow R_3$, $-R_2 \rightarrow R_2$.

$$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ is an eigenvector corr. to } B\text{'s eigenvalue } \mu_2 = 2.$$

$$\Rightarrow \mathbf{v}_2 = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ is a normalized eigenvector corr. to } B\text{'s eigenvalue } \mu_2 = 2.$$

$$[\,B-\mu_3 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} \sqrt{2} & 0 & 1 & 0 \\ 0 & \sqrt{2} & -1 & 0 \\ 1 & -1 & \sqrt{2} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & \sqrt{2} & 0 \\ 0 & 1 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} & -1 & 0 \end{array}\right],$$

after $R_1 \leftrightarrow R_3$, $-\sqrt{2}R_1 + R_3 \rightarrow R_3$, $\frac{1}{\sqrt{2}}R_2 \rightarrow R_2$,

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after $-\sqrt{2}R_2 + R_3 \rightarrow R_3$, $R_2 + R_1 \rightarrow R_1$.

$$\Rightarrow \mathbf{p}^{(3)} = \begin{bmatrix} -1 \\ 1 \\ \sqrt{2} \end{bmatrix} \text{ is an eigenvector corr. to } B\text{'s eigenvalue } \mu_3 = 2-\sqrt{2}.$$

$$\Rightarrow \mathbf{v}_3 = \tfrac{1}{2}\begin{bmatrix} -1 \\ 1 \\ \sqrt{2} \end{bmatrix} \text{ is a normalized eigenvector corr. to eigenvalue } \mu_3 = 2-\sqrt{2}.$$

Because all of the σ's are positive, we have

$$V = V_1 = [\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_3] = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & -1 \\ -1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & \sqrt{2} \end{bmatrix}$$

and

$$\Sigma = S = diag(\sqrt{2+\sqrt{2}},\ \sqrt{2},\ \sqrt{2-\sqrt{2}}) = \begin{bmatrix} \sqrt{2+\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2-\sqrt{2}} \end{bmatrix}.$$

Using this we see that

$$U = U_1 = AV\Sigma^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & -1 \\ -1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2+\sqrt{2}} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2-\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2\sqrt{2} & 0 \\ 2+\sqrt{2} & 0 & -2+\sqrt{2} \\ \sqrt{2} & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2+\sqrt{2}} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2-\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ \sqrt{2+\sqrt{2}} & 0 & -\sqrt{2-\sqrt{2}} \\ \sqrt{2/\sqrt{2+\sqrt{2}}} & 0 & \sqrt{2}/\sqrt{2-\sqrt{2}} \end{bmatrix}.$$

To summarize, the SVD factorization is $A = U\Sigma V^T$, where

$$U = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ \sqrt{2+\sqrt{2}} & 0 & -\sqrt{2-\sqrt{2}} \\ \sqrt{2}/\sqrt{2+\sqrt{2}} & 0 & \sqrt{2}/\sqrt{2-\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{2+\sqrt{2}} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{2-\sqrt{2}} \end{bmatrix},$$

and

$$V^T = \begin{bmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} \end{bmatrix}.$$

2.7.7.29. (a) Ex. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) We calculate

$$B = A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & 5 \end{bmatrix}$$

Next, find the eigenvalues of B:

$$0 = \begin{vmatrix} 1-\mu & 0 & 2 \\ 0 & 1-\mu & -1 \\ 2 & -1 & 5-\mu \end{vmatrix} = (1-\mu)\begin{vmatrix} 1-\mu & -1 \\ -1 & 5-\mu \end{vmatrix} + 2\begin{vmatrix} 0 & 1-\mu \\ 2 & -1 \end{vmatrix} = (1-\mu)\big((1-\mu)(5-\mu)-1\big) - 4(1-\mu)$$

$$= (1-\mu)\big((1-\mu)(5-\mu)-1-4\big) = (1-\mu)(\mu^2-6\mu) = \mu(1-\mu)(\mu-6).$$

The eigenvalues of B are $\mu_1 = 6$, $\mu_2 = 1$, and $\mu_3 = 0$. Correspondingly, $\sigma_1 = \sqrt{6}$, $\sigma_2 = 1$, and $\sigma_3 = 0$.

Next, find the eigenvectors of B corresponding to its eigenvalues:

$$[\,B - \mu_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -5 & 0 & 2 & 0 \\ 0 & -5 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{5} & 0 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & -1 & -\frac{1}{5} & 0 \end{array}\right],$$

after $-\frac{1}{5}R_1 \to R_1$, $-2R_1 + R_3 \to R_3$, $-\frac{1}{5}R_2 \to R_2$,

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{5} & 0 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after $R_2 + R_3 \to R_3$.

$\Rightarrow \mathbf{p}^{(1)} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ is an eigenvector corr. to B's eigenvalue $\mu_1 = 6$.

$\Rightarrow \mathbf{v}_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ is a normalized eigenvector corr. to eigenvalue $\mu_1 = 6$.

$$[\, B - \mu_2 I \mid \mathbf{0}\,] = \begin{bmatrix} 0 & 0 & 2 & | & 0 \\ 0 & 0 & -1 & | & 0 \\ 2 & -1 & 4 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix},$$

after $R_1 \leftrightarrow R_3$, $\frac{1}{2}R_1 \to R_1$, $2R_2 + R_3 \to R_3$, $-2R_2 + R_1 \to R_1$, $-R_2 \to R_2$.

$\Rightarrow \mathbf{p}^{(2)} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is an eigenvector corr. to B's eigenvalue $\mu_2 = 1$.

$\Rightarrow \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is a normalized eigenvector corr. to B's eigenvalue $\mu_2 = 1$.

$$[\, B - \mu_3 I \mid \mathbf{0}\,] = \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 2 & -1 & 5 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix},$$

after $-2R_1 + R_3 \to R_3$, $R_2 + R_3 \to R_3$.

$\Rightarrow \mathbf{p}^{(3)} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corr. to B's eigenvalue $\mu_3 = 0$.

$\Rightarrow \mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is a normalized eigenvector corr. to eigenvalue $\mu_3 = 0$.

$$V = [\, V_1 \mid \mathbf{v}_3 \,] = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 & \sqrt{6} & | & -2\sqrt{5} \\ -1 & 2\sqrt{6} & | & \sqrt{5} \\ 5 & 0 & | & \sqrt{5} \end{bmatrix}$$

and the 3×3 matrix Σ is

$$\Sigma = diag(\sigma_1, \sigma_2, \sigma_3) = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} S & | & O \\ -- & | & - \\ O & | & 0 \end{bmatrix}.$$

Using this we see that

$$[\, \mathbf{u}_1 \mid \mathbf{u}_2 \,] = U_1 = AV_1S^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{30}} \begin{bmatrix} 2 & \sqrt{6} \\ -1 & 2\sqrt{6} \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{30}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & \sqrt{6} \\ -1/\sqrt{6} & 2\sqrt{6} \\ 5/\sqrt{6} & 0 \end{bmatrix} = \frac{1}{6\sqrt{5}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -1 & 12 \\ 5 & 0 \end{bmatrix}$$

$$= \frac{1}{6\sqrt{5}} \begin{bmatrix} 12 & 6 \\ -6 & 12 \\ 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}$$

The 3×3 real, orthogonal matrix $U = [\, U_1 \mid U_2 \,] = [\, \mathbf{u}_1 \;\; \mathbf{u}_2 \mid \mathbf{u}_3 \,]$, where we can find $\mathbf{u}_3$ by the process in the Appendix to Section 2.4.

Because $Span\Big(\{\mathbf{u}_1, \mathbf{u}_2\}\Big) = Span\Big(\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}\}\Big)$, we should use $\mathbf{e}^{(3)}$ instead of $\mathbf{e}^{(1)}$.

First, calculate

$$\mathbf{w}_3 = \mathbf{e}^{(3)} - \langle \mathbf{e}^{(3)}, \mathbf{u}_1 \rangle \mathbf{u}_1 - \langle \mathbf{e}^{(3)}, \mathbf{u}_2 \rangle \mathbf{u}_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - (0)\, \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - (0)\, \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{e}^{(3)},$$

$||\mathbf{w}_3|| = 1$, and finally

$$\mathbf{u}_3 = \mathbf{e}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To summarize, the SVD factorization is $A = U \Sigma V^T$, where
$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $V^T = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 & -1 & 5 \\ \sqrt{6} & 2\sqrt{6} & 0 \\ -2\sqrt{5} & \sqrt{5} & \sqrt{5} \end{bmatrix}$.

Section 2.8.3

2.8.3.1. $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = E_1 A_3 = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \\ 0 & 1 & -2 \end{bmatrix}$, and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \\ 0 & 1 & -2 \end{bmatrix} = E_2(E_1 A_3) = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \\ 0 & 0 & -\frac{4}{3} \end{bmatrix} = U.$$

So, a LU factorization is

$$A_3 = LU,$$

where

$$L = (E_2E_1)^{-1} = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}.$$

To summarize, a LU factorization of A_3 is given by $L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}$, $\quad U = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \\ 0 & 0 & -\frac{4}{3} \end{bmatrix}$.

2.8.3.5. We want to find a lower triangular matrix $L = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}$ satisfying $A = LL^T$, that is,

$$\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} = LL^T = ... = \begin{bmatrix} \ell_{11}^2 & \ell_{11}\ell_{21} \\ \\ \ell_{11}\ell_{21} & \ell_{21}^2 + \ell_{22}^2 \end{bmatrix}.$$

The $(1,1)$ entry of A requires $\ell_{11}^2 = 3$, so one Cholesky factorization can use $\ell_{11} = \sqrt{3}$. After that, both the $(1,2)$ and $(2,1)$ entries of A require that $-2 = \ell_{11}\ell_{21}$, hence $\ell_{21} = -\frac{2}{\sqrt{3}}$. Finally, the $(2,2)$ entry of A requires $2 = \ell_{21}^2 + \ell_{22}^2 = \frac{4}{3} + \ell_{22}^2$. One Cholesky factorization of A is given by

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ -\frac{2}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ -\frac{2}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}^T.$$

2.8.3.7. Take the hint, and partition

$$L = \left[\begin{array}{cc|c} & L_{11} & \mathbf{0} \\ - & - & -\,- \\ \ell_{31} & \ell_{32} & \ell_{33} \end{array}\right], \quad \text{where} \quad L_{11} \text{ is } 2 \times 2,$$

specifically, $L_{11} = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}$ and correspondingly

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ -\,- & -\,- \\ A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{cc|c} & A_{11} & a_{12} \\ - & - & - \\ a_{31} & a_{32} & a_{33} \end{array}\right].$$

Then we calculate

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ -\,- & -\,- \\ A_{21} & A_{22} \end{array}\right] = A = LL^T == \left[\begin{array}{cc|c} & L_{11} & \mathbf{0} \\ - & - & -\,- \\ \ell_{31} & \ell_{32} & \ell_{33} \end{array}\right] \left[\begin{array}{cc|c} & & \ell_{31} \\ & L_{11}^T & \ell_{32} \\ - & - & -\,- \\ & \mathbf{0}^T & \ell_{33} \end{array}\right]$$

$$=\left[\begin{array}{cc|c} \multicolumn{2}{c|}{} & \ell_{11}\ell_{31}+\ell_{12}\ell_{32} \\ \multicolumn{2}{c|}{L_{11}L_{11}^T} & \ell_{21}\ell_{31}+\ell_{22}\ell_{32} \\ \hline \ell_{11}\ell_{31}+\ell_{12}\ell_{32} & \ell_{21}\ell_{31}+\ell_{22}\ell_{32} & \ell_{11}^2+\ell_{22}^2+\ell_{33}^2 \end{array}\right] = A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \quad (\star)$$

It follows that we need $A_{11} = L_{11}L_{11}^T$, which is a problem we have experience with from problems 2.8.3.4 and 2.8.3.5.

We want to find a lower triangular matrix $L_{11} = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}$ satisfying

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = A_{11} = L_{11}L_{11}^T = \begin{bmatrix} \ell_{11}^2 & \ell_{11}\ell_{21} \\ \\ \ell_{11}\ell_{21} & \ell_{21}^2 + \ell_{22}^2 \end{bmatrix}.$$

The $(1,1)$ entry of A requires $\ell_{11}^2 = 2$, so one Cholesky factorization can use $\ell_{11} = \sqrt{2}$. After that, both the $(1,2)$ and $(2,1)$ entries of A require that $-1 = \ell_{11}\ell_{21}$, hence $\ell_{21} = -\frac{1}{\sqrt{2}}$. Finally, the $(2,2)$ entry of A requires $2 = \ell_{21}^2 + \ell_{22}^2 = \frac{1}{2} + \ell_{22}^2$. One Cholesky factorization of A_{11} is given by

$$A_{11} = \begin{bmatrix} \sqrt{2} & 0 \\ \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix}^T.$$

$(\star)$ is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = A = LL^T = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} & \ell_{31} \\ \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & \ell_{32} \\ \\ 0 & 0 & \ell_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & \sqrt{2}\,\ell_{31} \\ \\ -1 & 2 & 0 \\ \\ \sqrt{2}\,\ell_{31} & -\frac{1}{\sqrt{2}}\,\ell_{31} + \frac{\sqrt{3}}{\sqrt{2}}\,\ell_{32} & \ell_{11}^2 + \ell_{22}^2 + \ell_{33}^2 \end{bmatrix}.$$

The $(3,1)$ entry of A requires

$$0 = \sqrt{2}\,\ell_{31} + 0 \cdot \ell_{32},$$

hence $\ell_{31} = 0$. The $(3,2)$ entry of A requires

$$-1 = -\frac{1}{\sqrt{2}}\,\ell_{31} + \frac{\sqrt{3}}{\sqrt{2}}\,\ell_{32} = -\frac{1}{\sqrt{2}} \cdot 0 + \frac{\sqrt{3}}{\sqrt{2}}\,\ell_{32},$$

hence $\ell_{32} = -\frac{\sqrt{2}}{\sqrt{3}}$. The $(3,3)$ entry of A requires

$$2 = \ell_{31}^2 + \ell_{22}^2 + \ell_{33}^2 = 0^2 + \Big(-\frac{\sqrt{2}}{\sqrt{3}}\Big)^2 + \ell_{33}^2 = \frac{2}{3} + \ell_{33}^2,$$

hence $\ell_{33} = \frac{2}{\sqrt{3}}$.

To summarize, the Cholesky factorization is $A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 \\ \\ 0 & -\sqrt{\frac{2}{3}} & \frac{2}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 \\ \\ 0 & -\sqrt{\frac{2}{3}} & \frac{2}{\sqrt{3}} \end{bmatrix}^T$

Section 2.9.2

2.9.2.1. Let $\mathbf{x} = [\,x_1 \quad x_2\,]^T$ be an unspecified unit vector. Let $x_1 = x$, hence $x_2^2 = 1 - x^2$. We calculate

$$\mathcal{R}_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = [\,x \quad x_2\,]\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} = 2x^2 + 2xx_2 - x_2^2 = 2x^2 + 2x(\pm\sqrt{1-x^2}) - (1-x^2)$$

$$= -1 + 3x^2 \pm 2x\sqrt{1-x^2} \triangleq f_\pm(x), \quad \text{for } -1 \le x \le 1.$$

Next, find the minimum and maximum values of $f_\pm(x)$, each of which is a function of a single variable:

$$f'_\pm(x) = 6x \pm 2\left(\sqrt{1-x^2} + x\cdot\frac{1}{2}\,\frac{1}{\sqrt{1-x^2}}\cdot(-2x)\right) = 6x \pm 2\left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}}\right)$$

$$= \frac{2}{\sqrt{1-x^2}}\left(3x\sqrt{1-x^2} \pm (1-2x^2)\right).$$

So, the critical points of $f_\pm(x)$ are where

$$3x\sqrt{1-x^2} \pm (1-2x^2) = 0,$$

that is,

$$3x\sqrt{1-x^2} = \mp(1-2x^2).$$

If x is a critical point, then

$$\left(3x\sqrt{1-x^2}\right)^2 = \left(\mp(1-2x^2)\right)^2,$$

that is,

$$9x^2(1-x^2) = (1-2x^2)^2,$$

that is,

$$0 = (1-4x^2+4x^4) - 9x^2(1-x^2) = 13x^4 - 13x^2 + 1.$$

The critical points for both $f_+(x)$ and $f_-(x)$ must satisfy

$$x^2 = \frac{13 \pm \sqrt{13^2 - 4\cdot 13\cdot 1}}{26} = \frac{1}{2}\left(1 \pm \frac{3}{\sqrt{13}}\right),$$

that is,

$$x = \frac{\epsilon_1}{\sqrt{2}}\sqrt{1 + \frac{3\epsilon_2}{\sqrt{13}}},$$

where independently $\epsilon_1 = \pm 1$ and $\epsilon_3 = \pm 1$.

Next, we evaluate the functions at any critical points that lie in the interval $-1 < x < 1$: Let $\epsilon_3 = \pm 1$. Note that

$$\left(\frac{\epsilon_1}{\sqrt{2}}\sqrt{1 + \frac{3\epsilon_2}{\sqrt{13}}}\right)^2 = \frac{1}{2}\left(1 + \frac{3\epsilon_2}{\sqrt{13}}\right).$$

We have

$$f_{\epsilon_3}\left(\frac{\epsilon_1}{\sqrt{2}}\sqrt{1 + \frac{3\epsilon_2}{\sqrt{13}}}\right) = -1 + 3\cdot\frac{1}{2}\left(1 + \frac{3\epsilon_2}{\sqrt{13}}\right) + 2\epsilon_3\,\frac{\epsilon_1}{\sqrt{2}}\sqrt{1 + \frac{3\epsilon_2}{\sqrt{13}}}\;\sqrt{1 - \frac{1}{2}\left(1 + \frac{3\epsilon_2}{\sqrt{13}}\right)}$$

$$= -1 + \frac{3}{2} + \frac{9\epsilon_2}{2\sqrt{13}} + \epsilon_1\epsilon_3\cancel{\sqrt{2}}\sqrt{1 + \frac{3\epsilon_2}{\sqrt{13}}}\;\sqrt{\frac{1}{\cancel{2}}\left(1 - \frac{3\epsilon_2}{\sqrt{13}}\right)} = -1 + \frac{3}{2} + \frac{9\epsilon_2}{2\sqrt{13}} + \epsilon_1\epsilon_3\sqrt{\left(1 + \frac{3\epsilon_2}{\sqrt{13}}\right)\left(1 - \frac{3\epsilon_2}{\sqrt{13}}\right)}$$

$$= -1 + \frac{3}{2} + \frac{9\epsilon_2}{2\sqrt{13}} + \epsilon_1\epsilon_3\sqrt{1 - \frac{9}{13}} = -1 + \frac{3}{2} + \frac{9\epsilon_2}{2\sqrt{13}} + \epsilon_1\epsilon_3\,\frac{2}{\sqrt{13}}.$$

Using all choices of ϵ_1, ϵ_2, and ϵ_3 gives eight possible values. The maximum of these is

$$M \triangleq -1 + \frac{3}{2} + \frac{9}{2\sqrt{13}} + \frac{2}{\sqrt{13}} = \frac{1}{2} + \frac{9+4}{2\sqrt{13}} = \frac{1+\sqrt{13}}{2},$$

and the minimum of these eight possible values is

$$m \triangleq -1 + \frac{3}{2} - \frac{9}{2\sqrt{13}} - \frac{2}{\sqrt{13}} = \frac{1}{2} - \frac{9+4}{2\sqrt{13}} = \frac{1-\sqrt{13}}{2}.$$

We also need to evaluate the functions at the endpoints of the interval $-1 \le x \le 1$:

$$f_+(\pm 1) = -1 + 3(\pm 1)^2 + 2(\pm 1)\sqrt{1-(\pm 1)^2} = 2$$

and similarly $f_-(\pm 1) = 2$.

So, the Rayleigh quotient for this 2×2 real, symmetric matrix gives maximum eigenvalue being $\frac{1+\sqrt{13}}{2}$ and minimum eigenvalue being $\frac{1-\sqrt{13}}{2}$.

By the way, Section 2.1's usual way of finding the eigenvalues using the characteristic equation gives the result that the exact eigenvalues of A are $\frac{1\pm\sqrt{13}}{2}$.

2.9.2.3. Let $||\mathbf{x}|| = [\,x \quad y \quad z\,]^T$ be an unspecified nonzero vector. We calculate

$$\mathcal{R}_A(\mathbf{x}) = \frac{\mathbf{x}^T A\mathbf{x}}{||\mathbf{x}||^2} = \frac{\mathbf{x}^T A\mathbf{x}}{x^2+y^2+z^2} = \frac{1}{x^2+y^2+z^2}\cdot[\,x \quad y \quad z\,]\begin{bmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \frac{2x^2 + 2\sqrt{3}xy - z^2}{x^2+y^2+z^2} \triangleq f(x,y,z)$$

Mathematica™on $f(x,y,z)$ over the unit cube, as in Example 2.37 in Section 2.9, gives maximum value of $+3$ and minimum value of -1.

2.9.2.5. $\mathbf{x}_+ = [1 \quad 1 \quad 1 \quad 1 \quad 1]^T$ and $\mathbf{x}_- = [1 \quad -1 \quad 1 \quad -1 \quad 1]^T$ give estimates $\mathcal{R}_A(\mathbf{x}_+) = 4 \approx \lambda_1$, the maximum eigenvalue of A, and for the minimum eigenvalue of A gives the estimate $\lambda_5 \approx \mathcal{R}_A(\mathbf{x}_-) = -\frac{4}{5}$, after calculating

$$\mathcal{R}_A(\mathbf{x}_+) = \frac{1}{||\mathbf{x}_+||^2}\,[1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{5}\,[1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]\begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = 4$$

and

$$\mathcal{R}_A(\mathbf{x}_-) = \frac{1}{||\mathbf{x}_-||^2}\,[1 \;\; -1 \;\; 1 \;\; -1 \;\; 1]\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{5}\,[1 \;\; -1 \;\; 1 \;\; -1 \;\; 1]\begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = -\frac{4}{5}.$$

2.9.2.7. Denoting, as usual in this section, by λ_n (respectively, λ_1) the minimum (respectively, maximum) eigenvalue of the real, symmetric matrix A, we have that $\lambda_n \le \mathcal{R}_A(\mathbf{x}) \le \lambda_1$, for all nonzero vectors $\mathbf{x}$.

For each index $i = 1, ..., n$, using in the Rayleigh quotient the trial unit vector $\mathbf{e}^{(i)}$ gives

$$\mathcal{R}_A(\mathbf{e}^{(i)}) = (\mathbf{e}^{(i)})^T A\mathbf{e}^{(i)} = (\mathbf{e}^{(i)})^T\,[\,a_{ij}\,]\,\mathbf{e}^{(i)} = (\mathbf{e}^{(i)})^T\begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} = a_{ii}.$$

So, $\lambda_n \le a_{ii} \le \lambda_1$.

2.9.2.9. (a) Because $\{t\mathbf{q}_1, (1-t)\mathbf{q}_n\}$ is also an orthogonal set of vectors, the Pythagorean theorem implies

$$||\mathbf{x}(t)||^2 = ||t\mathbf{q}_1 + (1-t)\mathbf{q}_n||^2 = ||t\mathbf{q}_1||^2 + ||(1-t)\mathbf{q}_n||^2 = (|t|\ ||\mathbf{q}_1||)^2 + (|1-t|\ ||\mathbf{q}_n||)^2 = (|t|\cdot 1)^2 + (|1-t|\cdot 1)^2$$

$$= |t|^2 + |1-t|^2 = t^2 + (1-t)^2,$$

using the assumption that t is real.

Further, because

$$t^2 + (1-t)^2 = t^2 + t^2 - 2t + 1 = 2t^2 - 2t + 1 = 2(t^2 - t) + 1$$

$$= 2\Big(t^2 - t + \frac{1}{4} - \frac{1}{4}\Big) + 1 = 2\Big(t^2 - t + \frac{1}{4}\Big) - \frac{1}{2} + 1 = 2\Big(t - \frac{1}{2}\Big)^2 + \frac{1}{2} \ge \frac{1}{2} > 0,$$

we know that $\mathbf{x}(t) \ne \mathbf{0}$.

(b) Define a function of a single variable by

$$f(t) \triangleq \mathcal{R}_A\big(\mathbf{x}(t)\big) = \frac{\mathbf{x}(t)^T A\mathbf{x}(t)}{||\mathbf{x}(t)||^2}.$$

$\mathbf{x}(t) = t\mathbf{q}_1 + (1-t)\mathbf{q}_n$ is differentiable in t, and has derivative $\mathbf{x}'(t) = \mathbf{q}_1 - \mathbf{q}_n$, hence $\mathbf{x}(t)$ is a continuous function of t. It follows that $\mathbf{x}(t)^T A\mathbf{x}(t)$ is continuous in t. We also know that the denominator, $||\mathbf{x}(t)||^2 = t^2 + (1-t)^2$ is differentiable in t, hence is a continuous function of t. Finally, we know that the denominator is never zero. Putting all of these results together imply that $f(t)$ is a continuous function.

(c) Because $f(t)$ is continuous on the interval $-1 \le t \le 1$, the Intermediate Value Theorem applies. Because the range $f(\,[-1,\,1]\,)$ contains both $\lambda_n = f(0)$ and $\lambda_1 = f(1)$, every w between λ_n and λ_1 must be in $f(\,[-1,\,1]\,)$, that is, there is a t_w such that $f(t_w) = w$.

Section 2.10.8

2.10.8.1. Define the function $q(x) \equiv 1$ and suppose p is a polynomial. On the space $\mathcal{P}_n$ consisting of all polynomials of degree less than or equal to n with real coefficients, define the inner product

$$\langle p, q \rangle \triangleq \int_{-1}^{1} p(x)q(x)dx. \tag{2.78}$$

The Cauchy-Schwarz inequality implies

$$\left| \int_{-1}^{1} p(x) \cdot\, dx \right| = |\langle p, q \rangle| \le ||p|| \cdot ||q|| = \left(\int_{-1}^{1} |p(x)|^2 \, dx \right)^{1/2} \left(\int_{-1}^{1} |1|^2 \, dx \right)^{1/2} = \left(\int_{-1}^{1} |p(x)|^2 \, dx \right)^{1/2} \cdot \sqrt{2},$$

hence

$$\left| \frac{1}{2} \int_{-1}^{1} p(x) \, dx \right| \le \frac{\sqrt{2}}{2} \left(\int_{-1}^{1} |p(x)|^2 \, dx \right)^{1/2},$$

hence

$$|\overline{p}| \triangleq \left| \frac{1}{2} \int_{-1}^{1} p(x) dx \right| \le \left(\frac{1}{2} \int_{-1}^{1} |p(x)|^2 dx \right)^{1/2} \triangleq p_{rms}.$$

2.10.8.3. We are given that both A and B are bounded linear operators on an i.p. space $\mathcal{V}$, so the result of problem 2.10.8.2 implies $||BA|| \le ||B|| \, ||A||$. Because B is the algebraic inverse of A, $BA\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathcal{V}$. By definition of the norm of a bounded linear operator,

$$||\mathbf{x}|| = ||BA\mathbf{x}|| \le ||BA|| \, ||\mathbf{x}|| \le ||B|| \, ||A|| \, ||\mathbf{x}||.$$

Choosing any unit vector $\mathbf{x}$ implies that

$$1 \le ||B|| \, ||A||,$$

hence

$$||B|| \ge (\, ||A||\,)^{-1}.$$

2.10.8.5. For *all* $\mathbf{x}, \mathbf{y}$ in $\mathbb{C}^n$,

$$\langle \mathbf{x}, A^* \mathbf{y} \rangle = \langle A\mathbf{x}, \mathbf{y} \rangle \triangleq (A\mathbf{x})^T \overline{\mathbf{y}} = \mathbf{x}^T A^T \overline{\mathbf{y}} = \mathbf{x}^T \overline{(\overline{A^T} \mathbf{y})} = \langle \mathbf{x}, \overline{A^T} \mathbf{y} \rangle.$$

So, we need $A^* = \overline{A^T}$.

2.10.8.7. We are given that $\mathbf{x}_k \rightharpoonup \mathbf{x}_\infty$ and $\mathbf{y}_k \to \mathbf{y}_\infty$. We calculate that the sequence of scalars $\langle \mathbf{x}_k, \mathbf{y}_k \rangle$ satisfy

$$|\langle \mathbf{x}_k, \mathbf{y}_k \rangle - \langle \mathbf{x}_\infty, \mathbf{y}_\infty \rangle| = |\langle \mathbf{x}_k, \mathbf{y}_k \rangle - \langle \mathbf{x}_k, \mathbf{y}_\infty \rangle + \langle \mathbf{x}_k, \mathbf{y}_\infty \rangle - \langle \mathbf{x}_\infty, \mathbf{y}_\infty \rangle|$$

$$\le |\langle \mathbf{x}_k, \mathbf{y}_k \rangle - \langle \mathbf{x}_k, \mathbf{y}_\infty \rangle| + |\langle \mathbf{x}_k, \mathbf{y}_\infty \rangle - \langle \mathbf{x}_\infty, \mathbf{y}_\infty \rangle| = |\langle \mathbf{x}_k, \mathbf{y}_k - \mathbf{y}_\infty \rangle| + |\langle \mathbf{x}_k - \mathbf{x}_\infty, \mathbf{y}_\infty \rangle|$$

$$\le ||\mathbf{x}_k|| \cdot ||\mathbf{y}_k - \mathbf{y}_\infty|| + |\langle \mathbf{x}_k - \mathbf{x}_\infty, \mathbf{y}_\infty \rangle|$$

Both of these terms converge to 0 as $k \to \infty$, but for two different reasons: Concerning the first term, because $\mathbf{x}_k \rightharpoonup \mathbf{x}_\infty$ we know that the sequence of norms $\{\, ||\mathbf{x}_k||\,\}_{k=1}^{\infty}$ is bounded, that is, there is an $M \ge 0$ for which $||\mathbf{x}_k|| \le M$ for all k. So,

$$||\mathbf{x}_k|| \cdot ||\mathbf{y}_k - \mathbf{y}_\infty|| \le M ||\mathbf{y}_k - \mathbf{y}_\infty|| \to 0 \text{ as } k \to \infty.$$

Concerning the second term, we were given that $\mathbf{x}_k \rightharpoonup \mathbf{x}_\infty$. By definition of $\rightharpoonup$, this implies that for the fixed vector $\mathbf{y}_\infty$ we have $\langle \mathbf{x}_k - \mathbf{x}_\infty, \mathbf{y}_\infty \rangle \to 0$ as $k \to \infty$.

2.10.8.9. For *all* $\mathbf{x}$ in $\mathcal{V}$, using the triangle inequality we have

$$||A\mathbf{x}|| \triangleq \left|\left| \lambda_1 \langle \mathbf{x}, \mathbf{u}^{(1)} \rangle \mathbf{u}^{(1)} + \cdots + \lambda_n \langle \mathbf{x}, \mathbf{u}^{(n)} \rangle \mathbf{u}^{(n)} \right|\right| \le \left|\left| \lambda_1 \langle \mathbf{x}, \mathbf{u}^{(1)} \rangle \mathbf{u}^{(1)} \right|\right| + \cdots + \left|\left| \lambda_n \langle \mathbf{x}, \mathbf{u}^{(n)} \rangle \mathbf{u}^{(n)} \right|\right|,$$

and then using (2.79)(a) in Section 2.10,

$$||A\mathbf{x}|| \le |\lambda_1 \langle \mathbf{x}, \mathbf{u}^{(1)} \rangle| \cdot ||\mathbf{u}^{(1)}|| + \cdots + |\lambda_n \langle \mathbf{x}, \mathbf{u}^{(n)} \rangle| \cdot ||\mathbf{u}^{(n)}||,$$

and using the information that the $\mathbf{u}^{(j)}$ are unit vectors,

$$||A\mathbf{x}|| \le |\lambda_1 \langle \mathbf{x}, \mathbf{u}^{(1)} \rangle| + \cdots + |\lambda_n \langle \mathbf{x}, \mathbf{u}^{(n)} \rangle| = |\lambda_1| \cdot |\langle \mathbf{x}, \mathbf{u}^{(1)} \rangle| + \cdots + |\lambda_n| \cdot |\langle \mathbf{x}, \mathbf{u}^{(n)} \rangle|,$$

and then, using the Cauchy-Schwarz inequality,

$$||A\mathbf{x}|| \le |\lambda_1| \cdot ||\mathbf{x}||\ ||\mathbf{u}^{(1)}|| + \cdots + |\lambda_n| \cdot ||\mathbf{x}||\ ||\mathbf{u}^{(n)}||,$$

and again using the information that $||\mathbf{u}^{(j)}|| = 1$,

$$||A\mathbf{x}|| \le |\lambda_1| \cdot ||\mathbf{x}|| + \cdots + |\lambda_n| \cdot ||\mathbf{x}||.$$

This being true for *all* $\mathbf{x}$ in $\mathcal{V}$, it follows that A is bounded and $||\,A\,|| \le |\lambda_1| + \cdots + |\lambda_n|$.

To go further, we want to find the value of $||A||$, not just an upper bound on $||A||$. How?

Add more unit vectors to the set $\{\mathbf{u}^1, ..., \mathbf{u}^n\}$ to get an o.n. basis, $\{\mathbf{u}^1, ..., \mathbf{u}^n, ...\}$, for $\mathcal{V}$. [We use the ellipsis ... to indicate that there may be finitely many or infinitely many more vectors in the basis.]

Then, no matter how many more vectors there are in the o.n. basis for $\mathcal{V}$, we have

$$\mathbf{x} = \sum_{i=1} x_i \mathbf{u}^{(i)},$$

where the notation $\sum_{i=1}$ indicates that the sum may have a finite or infinite number of terms, depending on the dimension of $\mathcal{V}$.

We calculate using Parseval's identity that

$$||\mathbf{x}||^2 = \left|\left| \sum_{i=1} x_i \mathbf{u}^{(i)} \right|\right|^2 = \sum_{i=1} |x_i|^2$$

and

$$A\mathbf{x} \triangleq \sum_{j=1}^{n} \lambda_j \langle \mathbf{x}, \mathbf{u}^{(j)} \rangle \mathbf{u}^{(j)} = \sum_{j=1}^{n} \lambda_j \left\langle \sum_{i=1} x_i \mathbf{u}^{(i)}, \mathbf{u}^{(j)} \right\rangle \mathbf{u}^{(j)}.$$

Using orthogonality,

$$(\star) \quad A\mathbf{x} = \sum_{j=1}^{n} \lambda_j x_j \mathbf{u}^{(j)}.$$

Note that, even though $\mathbf{x}$ and $||\mathbf{x}||^2$ may involve infinitely many terms, orthogonality made $A\mathbf{x}$ be a finite sum.

Again using Parseval's identity,

$$||A\mathbf{x}||^2 = \left|\left| \sum_{j=1}^{n} \lambda_j x_j \mathbf{u}^{(j)} \right|\right|^2 = \sum_{j=1}^{n} |\lambda_j x_j|^2 = \sum_{j=1}^{n} |\lambda_j|^2 |x_j|^2.$$

But, we were given that $|\lambda_1| \ge |\lambda_2| \ge ... \ge |\lambda_n|$, so

$$||A\mathbf{x}||^2 \le \sum_{j=1}^{n} |\lambda_1|^2 |x_j|^2 = |\lambda_1|^2 \sum_{j=1}^{n} |x_j|^2,$$

so,

$$||A\mathbf{x}||^2 \le |\lambda_1|^2 \sum_{i=1}^{n} |x_i|^2 = |\lambda_1|^2 ||\mathbf{x}||^2.$$

It follows that $||A|| \le |\lambda_1|$.

But, in fact, choosing $\mathbf{x} = \mathbf{u}^{(1)} = 1 \cdot \mathbf{u}^{(1)} + 0 \cdot \mathbf{u}^{(2)} + ...$, we have, from $(\star)$,

$$A\mathbf{x} = \lambda_1 \mathbf{u}^{(1)},$$

so

$$||A\mathbf{x}||^2 = |\lambda_1|^2 \cdot 1 = |\lambda_1|^2 \cdot ||\mathbf{u}^{(1)}||^2 = |\lambda_1|^2 \cdot ||\mathbf{x}||^2.$$

It follows that $||A|| \ge |\lambda_1$. Combined with the previous result, we get

$$||A|| = |\lambda_1|.$$

2.10.8.11. For *all* $\mathbf{x}, \mathbf{y}$ in $\mathcal{H}$,

$$\left\langle \mathbf{x}, \left(A^{-1}\right)^* \mathbf{y} \right\rangle = \left\langle A^{-1}\mathbf{x}, \mathbf{y} \right\rangle.$$

Let $\mathbf{w} = A^{-1}\mathbf{x}$ and $\mathbf{v} = (A^*)^{-1}\mathbf{y}$, hence $A\mathbf{w} = \mathbf{x}$ and $A^*\mathbf{v} = \mathbf{y}$. It follows that

$$\left\langle \mathbf{x}, \left(A^*\right)^{-1} \mathbf{y} \right\rangle = \langle \mathbf{x}, \mathbf{v} \rangle = \langle A\mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{w}, A^*\mathbf{v} \rangle = \left\langle \mathbf{w}, A^*\big((A^*)^{-1}\mathbf{y}\big) \right\rangle = \left\langle \mathbf{w}, \big(A^*(A^*)^{-1}\big)\mathbf{y} \right\rangle$$

$$= \langle \mathbf{w}, \mathbf{y} \rangle = \left\langle A^{-1}\mathbf{x}, \mathbf{y} \right\rangle = \left\langle \mathbf{x}, \left(A^{-1}\right)^* \mathbf{y} \right\rangle.$$

To summarize, for *all* $\mathbf{x}, \mathbf{y}$ in $\mathcal{H}$,

$$\left\langle \mathbf{x}, \left(A^*\right)^{-1} \mathbf{y} \right\rangle = \left\langle \mathbf{x}, \left(A^{-1}\right)^* \mathbf{y} \right\rangle.$$

It follows that

$$\left(A^*\right)^{-1} = \left(A^{-1}\right)^*.$$

2.10.8.13. For each $\mathbf{x} = [\, x_1 \quad \dots \quad x_n \,]$ in $\mathbb{C}^n$,

$$||\, A\mathbf{x}\, ||^2 = \left|\left| \begin{bmatrix} \sum_{k=1}^{n} a_{1k}x_k \\ \vdots \\ \sum_{k=1}^{n} a_{nk}x_k \end{bmatrix} \right|\right|^2 = \sum_{j=1} \left| \sum_{k=1}^{n} a_{jk}x_k \right|^2.$$

For each index $j = 1, ..., n$, let $\mathbf{A}_{*j}$ denote the $j-$th column of A. Then

$$\sum_{k=1}^{n} a_{jk}x_k = \mathbf{A}_{*j} \bullet \mathbf{x},$$

so the Cauchy-Schwarz inequality implies

$$\left| \sum_{k=1}^{n} a_{jk}x_k \right| \le ||\mathbf{A}_{*j}||\ ||\mathbf{x}||,$$

hence

$$\left| \sum_{k=1}^{n} a_{jk}x_k \right|^2 \le |\mathbf{A}_{*j}||^2\ ||\mathbf{x}||^2 = \left(\sum_{k=1}^{n} |a_{jk}|^2 \right) ||\mathbf{x}||^2.$$

It follows that

$$|| A\mathbf{x} ||^2 = \sum_{j=1} \left| \sum_{k=1}^{n} a_{jk} x_k \right|^2 \le \sum_{j=1} \left(\sum_{k=1}^{n} |a_{jk}|^2 \right) ||\mathbf{x}||^2,$$

hence

$$|| A\mathbf{x} ||^2 \le ||\mathbf{x}||^2 \cdot \left(\sum_{j=1} \sum_{k=1}^{n} |a_{jk}|^2 \right).$$

This implies

$$|| A\mathbf{x} || \le ||\mathbf{x}|| \cdot \sqrt{\sum_{j=1} \sum_{k=1}^{n} |a_{jk}|^2}.$$

Because this is true for *all* $\mathbf{x}$ in $\mathbb{C}^n$, A is a bounded linear operator and

$$|| A || \le \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} |a_{jk}|^2} \triangleq || A ||_F.$$

Chapter Three

Section 3.1.4

3.1.4.1. $\mu(t) = \exp\left(\int 1\ dt\right) = e^t \quad \Rightarrow \dfrac{d}{dt}\left[\, e^t y \,\right] = e^t\dot{y} + e^t y = e^t e^{-2t} = e^{-t} \quad \Rightarrow e^t y = \displaystyle\int e^{-t}\ dt = -e^{-t} + c$

$\Rightarrow y = e^{-t}\Big(- e^{-t} + c\Big) \quad \Rightarrow$ General solution of the ODE is $\quad y = -e^{-2t} + c\,e^{-t}$, where c =arb. const.

3.1.4.3. ODE in standard form is $\dot{y} - 3t^{-1}y = t^3$:

$\mu(t) = \exp\left(\displaystyle\int -3t^{-1}\ dt\right) = \exp(-3\ln t) = \exp(\ln t^{-3}) = t^{-3} \quad \Rightarrow \dfrac{d}{dt}\left[\, t^{-3}y \,\right] = t^{-3}\dot{y} - 3t^{-4}y = 1$

$\Rightarrow t^{-3}y = \displaystyle\int 1\ dt = t + c \quad \Rightarrow y = t^3(t+c)$

$\Rightarrow$ General solution of the ODE is $y = t^4 + c\,t^3$, where c =arb. const.

3.1.4.5. ODE has a particular solution $y_{_P}(t) = t^2\ln(t)$. The corresponding homogeneous ODE, $t\dot{y} - 2y = 0$, that is, $\dot{y} - 2t^{-1}y = 0$, has general solution $\quad y = c\mu(t) = c\exp\left(\displaystyle\int 2t^{-1}dt\right) = c\exp\Big(2\ln\, t\Big)$ $= c\exp\Big(\ln\, t^2\Big) = c\,t^2$,

so the general solution of the original ODE is $y{=}y_{_P}(t) + y_{_h}(t){=}c\,t^2{+}t^2\ln(t)$, where c=arb. const.

Satisfying the IC: $3 = y(1) + c\,1^2 + 1^2\ln(1) = c + 0 = c$.

Solution of the IVP is

$$y = 3\cdot t^2 + t^2\ln\, t = t^2(3 + \ln\, t).$$

3.1.4.7. ODE in standard form is $\dot{y} + \dfrac{3}{t}y = \dfrac{e^{-2t}}{t^3} \quad \Rightarrow \mu(t){=}\exp\left(\displaystyle\int 3t^{-1}\ dt\right){=}\exp(3\ln t){=}\exp(\ln t^3){=}\ t^3$

$\Rightarrow \dfrac{d}{dt}\left[\, t^3 y \,\right] = t^3\dot{y} + 3t^2 y = e^{-2t} \quad \Rightarrow t^3 y = \displaystyle\int e^{-2t}\ dt = -\dfrac{1}{2}\,e^{-2t} + c$

$\Rightarrow$ General solution of the ODE is $y = -\frac{1}{2}\,t^{-3}e^{-2t} + c\,t^{-3}$, where c =arb. const.

Satisfying the IC: $-1 = y(1) = -\frac{1}{2}\,e^{-2} + c \quad \Rightarrow c = -1 + \frac{1}{2}\,e^{-2}. \quad \Rightarrow$ Solution of the IVP is

$$y = -\frac{1}{2}\,t^{-3}e^{-2t} + \big(-1 + \frac{1}{2}\,e^{-2}\big)\,t^{-3}.$$

3.1.4.9. ODE in standard form is $\dot{y} + \dfrac{t-1}{t}\,y = -t$

$\Rightarrow \mu(t) = \exp\left(\displaystyle\int \dfrac{t-1}{t}\ dt\right) = \exp\left(\displaystyle\int (1 - t^{-1})\ dt\right) = \exp\left(t - \ln t\right) = e^t\cdot\exp\Big(\ln(t^{-1})\Big) = t^{-1}e^t$

$\Rightarrow \dfrac{d}{dt}\left[t^{-1}e^t y\right]{=}\ t^{-1}e^t\dot{y} + t^{-1}e^t\cdot\dfrac{t-1}{t}\,y{=}\ -t^{-1}e^t\cdot t \Rightarrow t^{-1}e^t y = -\int e^t\ dt = -e^t + c$

$\Rightarrow$ General solution of the ODE is $y = -t + c\,t\,e^{-t}$, where c =arb. const.

Satisfying the IC: $-2 = y(1) = -1 + c\,e^{-1} \Rightarrow c = -e$.

Solution of the IVP is $y = -t - e\cdot te^{-t} = -t(1 + e^{-(t-1)})$.

3.1.4.11. ODE in standard form is $\dot{A} + \dfrac{6}{100-2t}\,A = 4$

$\Rightarrow \mu(t) = \exp\left(\displaystyle\int \dfrac{6}{100-2t}\ dt\right) = \exp\Big(-3\ln(100-2t)\Big) = \exp\Big(\ln\Big((100-2t)^{-3}\Big)\Big) = (100-2t)^{-3}$

$$\Rightarrow \frac{d}{dt}\big[\,(100-2t)^{-3}A\,\big]=(100-2t)^{-3}\dot{A}+\frac{6}{(100-2t)^4}\,A=4(100-2t)^{-3}$$

$$\Rightarrow (100-2t)^{-3}A=\int 4(100-2t)^{-3}dt=(100-2t)^{-2}+c$$

$\Rightarrow$ General solution of the ODE is $A=(100-2t)+c(100-2t)^3$, where $c=$arb. const.
Satisfying the IC: $10=A(0)=100+c\cdot 100^3 \Rightarrow c=-90\cdot 100^{-3}$.
Solution of the IVP is $A=(100-2t)-90\cdot 100^{-3}(100-2t)^3$, that is, $A=100(1-0.02t)-90\cdot(1-0.02t)^3$.

3.1.4.13. ODE in standard form is $\dot{y}+3y=2t \quad \Rightarrow \mu(t)=\exp\left(\int 3\;dt\right)=e^{3t}$

$\Rightarrow \dfrac{d}{dt}\big[\,e^{3t}y\,\big]=e^{3t}\dot{y}+3e^{3t}y=e^{3t}\cdot 2t=2te^{3t}$

$\Rightarrow$ Using integration by parts, $e^{3t}y=\int 2te^{3t}\;dt=2t\cdot\frac{1}{3}\,e^{3t}-\frac{2}{3}\int e^{3t}\;dt=\frac{2}{3}\,t\,e^{3t}-\frac{2}{9}e^{3t}+c$

$\Rightarrow$ General solution of the ODE is $y=\frac{2}{3}\,t-\frac{2}{9}+c\,e^{-3t}$, where $c=$arb. const.

Satisfying the IC: $0=y(1)=\frac{2}{3}-\frac{2}{9}+ce^{-3}\Rightarrow c=-\frac{4}{9}e^3. \quad \Rightarrow$ Solution of the IVP is

$$y=\frac{2}{3}\,t-\frac{2}{9}-\frac{4}{9}\,e^{-3(t-1)}.$$

3.1.4.15. $\mu(t)=\exp\left(\int\frac{2t}{1+t^2}\;dt\right)=\exp\big(\ln(1+t^2)\big)=(1+t^2)$

$\Rightarrow \dfrac{d}{dt}\big[\,(1+t^2)y\,\big]=(1+t^2)\dot{y}+2t\,y=(1+t^2)\cdot\dfrac{1}{t}=(t^{-1}+t) \quad \Rightarrow (1+t^2)y=\displaystyle\int(t^{-1}+t)\;dt=\ln t+\frac{1}{2}\,t^2+c$

$\Rightarrow$ General solution of the ODE is $y=(1+t^2)^{-1}\Big(\ln t+\frac{1}{2}\,t^2+c\Big)$, where $c=$arb. const.

Satisfying the IC: $-1=y(2)=\frac{1}{5}(\ln 2+2+c) \quad \Rightarrow c=-7-\ln 2$.

$\Rightarrow$ Solution of the IVP is $y=(1+t^2)^{-1}\Big(-7-\ln 2+\ln t+\frac{1}{2}\,t^2\Big)$.

3.1.4.17. $\mu(t)=\exp\left(\int\frac{t+1}{t}\;dt\right)=\exp\left(\int(1+t^{-1})\,dt\right)=\exp\left(t+\ln t\right)=e^t\cdot\exp\left(\ln t\right)=t\,e^t$

$\Rightarrow \frac{d}{dt}\big[t\,e^ty\big]=t\,e^t\dot{y}+(t+1)e^t\,y=t\,e^t\cdot\frac{1}{t}\,e^{-2t}=e^{-t} \quad \Rightarrow t\,e^ty=\int e^{-t}\;dt=-e^{-t}+c$

$\Rightarrow$ General solution of the ODE is $y=-t^{-1}e^{-2t}+c\,t^{-1}e^{-t}$, where $c=$arb. const.

Satisfying the IC: $0=y(1)=-e^{-2}+c\,e^{-1}\Rightarrow c=e^{-1}. \quad \Rightarrow$ Solution of the IVP is

$$y=-t^{-1}e^{-2t}+e^{-1}t^{-1}e^{-t}=-t^{-1}e^{-2t}+t^{-1}e^{-(t+1)}.$$

3.1.4.19. ODE in standard form is $\dot{y}+y=\sin t$: $\Rightarrow \mu(t)=\exp\left(\int 1\,dt\right)=e^t \Rightarrow \dfrac{d}{dt}\big[\,e^t\,y\,\big]=e^t\,\dot{y}+e^t\,y=$ $e^t\,\sin t$

$\Rightarrow e^t\;y=\displaystyle\int e^t\,\sin t\;dt=\frac{e^t}{2}\,(-\cos t+\sin t)+c$

$\Rightarrow$ General solution of the ODE is $y=\frac{1}{2}\,(-\cos t+\sin t)+c\,e^{-t}$, where $c=$arb. const. Satisfying the IC:

$$1=y(0)=\frac{1}{2}\,(-1+0)+c \quad \Rightarrow c=\frac{3}{2}.$$

Solution of the IVP is

$$y=\frac{1}{2}\Big(-\cos t+\sin t\Big)+\frac{3}{2}\,e^{-t}.$$

The term $\frac{3}{2}\,e^{-t} \to 0$ as $t \to \infty$, hence is transient. The steady state solution is

$$y_S(t) = \frac{1}{2}\big(-\cos t + \sin t\big).$$

3.1.4.21. $\mu(t) = \exp\left(\int 1\ dt\right) = e^t \quad \Rightarrow \dfrac{d}{dt}\big[\,e^t y\,\big] = e^t\dot{y} + e^t y = e^t\cos 2t$

$\Rightarrow e^t y = \displaystyle\int e^t \cos 2t\ dt = \frac{e^t}{5}\Big(\cos 2t + 2\sin 2t\Big) + c$

$\Rightarrow$ General solution of the ODE is $y = \frac{1}{5}\big(\cos 2t + 2\sin 2t\big) + c\,e^{-t}$, where c =arb. const. Satisfying the IC:

$$0 = y(0) = \frac{1}{5} + c \quad \Rightarrow c = -\frac{1}{5}.$$

Solution of the IVP is

$$y = \frac{1}{5}\big(\cos 2t + 2\sin 2t\big) - \frac{1}{5}\,e^{-t}.$$

The term $-\frac{1}{5}\,e^{-t} \to 0$ as $t \to \infty$, hence is transient. The steady state solution is

$$y_S(t) = \frac{1}{5}\big(\cos 2t + 2\sin 2t\big).$$

3.1.4.23. (a) $\mu(t) = \exp\left(\displaystyle\int \alpha\ dt\right) = e^{\alpha t} \quad \Rightarrow \dfrac{d}{dt}\big[\,e^{\alpha t} y\,\big] = e^{\alpha t}\dot{y} + \alpha e^{\alpha t} y = 2\,e^{\alpha t}$

$\Rightarrow e^{\alpha t} y = \displaystyle\int 2\,e^{\alpha t}\ dt = \frac{2}{\alpha}\,e^{\alpha t} + c \Rightarrow$ General solution of the ODE is $y = \frac{2}{\alpha} + c\,e^{-\alpha t}$, where c =arb. const. Satisfying the IC:

$$1 = y(0) = \frac{2}{\alpha} + c \quad \Rightarrow c = 1 - \frac{2}{\alpha}.$$

Solution of the IVP is

$$y = \frac{2}{\alpha} + \left(1 - \frac{2}{\alpha}\right)e^{-\alpha t}.$$

(b) Solve $2 = y(1) = \frac{2}{\alpha} + \left(1 - \frac{2}{\alpha}\right)e^{-\alpha}$, that is, after multiplying through by $\alpha > 0$ and rearranging terms,

$$0 = f(\alpha) \triangleq 2 - 2\alpha + (\alpha - 2)e^{-\alpha}.$$

A graphing calculator and successive zooming in gives estimates

0.64893617
0.64361702
0.64361702
0.64378324
0.64378324
0.64379363
0.64379883
0.64379753
0.64379769
0.64379777
0.64379775
0.64379776

so we guess that $\alpha \approx 0.64379776$.

3.1.4.25. ODE in standard form is $\dot{T}+\alpha\, T = \alpha\, M$, where $\alpha > 0$ and M are constants $\Rightarrow \mu(t) = \exp\left(\int \alpha\, dt\right) = e^{\alpha t}$

$$\Rightarrow \frac{d}{dt}\big[\, e^{\alpha t} T\,\big] = e^{\alpha t}\dot{T} + \alpha\, e^{\alpha t} T = \alpha\, M\, e^{\alpha t} \quad \Rightarrow e^{\alpha t} T = \int \alpha\, M\, e^{\alpha t}\; dt = M\, e^{\alpha t} + c$$

$\Rightarrow$ General solution of the ODE is $T = M + c\, e^{-\alpha t}$, where c =arb. const. Satisfying the IC:

$$T_0 = T(0) = M + c \quad \Rightarrow c = T_0 - M.$$

Solution of the IVP is the object's temperature as a function of time:

$$T = M + (T_0 - M)e^{-\alpha t}.$$

We are given that the temperature of the medium is $M = 20^\circ\, C$. Other given information is found in the table below, assuming t is measured in minutes after 1:00 pm. and T, the temperature of the object, is measured in $^\circ\, C$.

	Time	*Object's Temperature*
	0	T_0
$(\star)$	3	250
	4	200

Here T_0 is the unknown temperature at 1:00 pm. that we are to solve for.

Substituting $M = 20$ and the known data into the solution of the ODE, we get

$$\left\{\begin{array}{l} 250 = 20 + (T_0 - 20)e^{-3\alpha} \\ 200 = 20 + (T_0 - 20)e^{-4\alpha} \end{array}\right\},$$

which implies $230e^{3\alpha} = (T_0 - 20) = 180e^{4\alpha}$, hence

$$\frac{230}{180} = \frac{e^{4\alpha}}{e^{3\alpha}} = e^{\alpha}.$$

So, $\alpha = \ln\dfrac{230}{180} = \ln\dfrac{23}{18} \approx 0.245122458$. Substitute
this into either of the two equations in $(\star)$, for example, the first, to get

$$(T_0 - 20) = 230\big(e^{\alpha}\big)^3 = 230 \cdot \left(\frac{23}{18}\right)^3 \approx 479.8371056$$

so the temperature at 1:00 pm. was $T_0 \approx 499.8371056 \approx 500^\circ\, C$.

3.1.4.27. ODE in standard form is $\dot{T}+\alpha\, T = \alpha\, M$, where $\alpha > 0$ and M re constants $\Rightarrow \mu(t) = \exp\left(\int \alpha\, dt\right) = e^{\alpha t}$

$$\Rightarrow \frac{d}{dt}\big[\, e^{\alpha t} T\,\big] = e^{\alpha t}\dot{T} + \alpha\, e^{\alpha t} T = \alpha\, M\, e^{\alpha t} \quad \Rightarrow e^{\alpha t} T = \int \alpha\, M\, e^{\alpha t}\; dt = M\, e^{\alpha t} + c$$

$\Rightarrow$ General solution of the ODE is $T = M + c\, e^{-\alpha t}$, where c =arb. const. Satisfying the IC:

$$T_0 = T(0) = M + c \quad \Rightarrow c = T_0 - M.$$

Solution of the IVP is the person's temperature as a function of time:

$$T = M + (T_0 - M)e^{-\alpha t}.$$

We are given that the temperature of the medium is $M = 21.1^\circ\, C$. Other given information is found in the table below, assuming t is measured in hours after 11 am. and T, the temperature of the person, is measured in $^\circ\, C$.

Time	*Person's Temperature*
$t^\star$	36.95
0	34.8
0.5	34.3

Substituting $M = 21.1$ and the known data into the solution of the ODE, we get

$$\left\{ \begin{array}{l} 34.8 = T_0 \\ 34.3 = 21.1 + (T_0 - 21.1)e^{-0.5\alpha} \end{array} \right\},$$

which implies $34.3 = 21.1 + (34.8 - 21.1)e^{-0.5\alpha}$, $\Rightarrow \frac{13.2}{13.7} = e^{-0.5\alpha}$ $\Rightarrow \ln\left(\frac{13.2}{13.7}\right) = \ln e^{-0.5\alpha} = -0.5\alpha$.

So, $\alpha = -\dfrac{1}{0.5}\ln\left(\dfrac{13.2}{13.7}\right) \approx 0.0743580065$.

Substitute this into the solution of the IVP to get

$$36.95 = T(t^\star) = 21.1 + 13.7e^{-\alpha t^\star} \quad \Rightarrow \frac{15.85}{13.7} = e^{-\alpha t^\star}$$

hence

$$t^\star = -\frac{1}{0.0743580065}\ln\left(\frac{15.85}{13.7}\right) \approx -1.960430011\, hours.$$

The person died about 1.960430011 *hours* before 11 am., that is, at about 9:02 am.

3.1.4.29. Assume that y measures the vertical displacement of the object *down* from the point from which it was released. Newton's second law of motion says that $\dfrac{d}{dt}[\, mv\,] = \Sigma Forces$. Here, the forces are (1) the force of gravity, $F = mg$, and (2) the air resistance force, $F = -4v$, where $v = \dot{y}$ is the velocity of the object. The air resistance force opposes the motion downward. So,

$$\frac{d}{dt}[\, mv\,] = mg - 4v$$

We are also given that $m = 0.5\,kg$, g is approximately $9.81\,m/s^2$, and that the object is released from rest, that is, $v(0) = 0$.

Assuming instead that g is exactly $9.81\,m/s^2$, v satisfies the IVP $\dot{v} = 9.81 - 8v$, $v(0) = 0$. In standard form this is $\dot{v} + 8v = 9.81$, so $\mu(t) = \exp\left(\displaystyle\int 8\; dt\right) = e^{8t}$. Multiply through by the integrating factor to get

$$\frac{d}{dt}[\, e^{8t}v\,] = e^{8t}\dot{v} + 8e^{8t}v = 9.81e^{8t},$$

and then take the indefinite integral of both sides with respect to t to get $e^{8t}v = \displaystyle\int 9.81e^{8t}\; dt = \frac{9.81}{8}\, e^{8t} + c$. The general solution of the ODE is $v = \frac{9.81}{8} + c\,e^{-8t}$. The IC requires $0 = v(0) = \frac{9.81}{8} + c$, so $c = -\frac{9.81}{8}$ and the solution of the IVP is

$$v = \frac{9.81}{8}\left(1 - e^{-8t}\right).$$

The term $-\frac{9.81}{8}\, e^{-8t}$ is transient, so the steady state velocity is

$$v_S = \frac{9.81}{8} \approx 1.22625\, m/s.$$

This is also known as the *terminal velocity*.

3.1.4.31. (a) Let A be the number of acres occupied by the plant and t be measured in years. The two given assumptions give two terms in $\dot{A}$: (i) -10, because goats are consuming the plant, and (ii) kA, the increase of the plant at a rate proportional to the current acreage in the absence of goats. So, $\dot{A} = -10 + kA$, where k is a positive constant.

(b) In standard form the ODE is $\dot{A} - kA = -10$, so the integrating factor $\mu(t) = e^{-kt}$ gives

$$\frac{d}{dt}[\, e^{-kt}A\,] = e^{-kt}\dot{A} - ke^{-kt}A = -10e^{-kt},$$

and then take the indefinite integral of both sides with respect to t to get $e^{-kt}A = \int -10e^{-kt}\,dt = \frac{10}{k}\,e^{-kt} + c$. The general solution of the ODE is

$$A = \frac{10}{k} + c\,e^{kt}.$$

3.1.4.33. The ODE is $-mg = m\dot{v} + u\dot{m}$, where $v(0) = v_0$, $g = 32ft/s^2$, $m = m_0(1 - \frac{t}{200})$, and m_0 is a constant. The ODE can be rewritten as $\dot{v} = -g - u\,\frac{\dot{m}}{m}$. In standard form this is

$$\dot{v} = -32 - u\,\frac{-\frac{m_0}{200}}{m_0\big(1 - \frac{t}{200}\big)} = -32 + \frac{u\,m_0}{200} \cdot \frac{1}{m_0\big(1 - \frac{t}{200}\big)}\ ,$$

so

$$v = \int \left(-32 + \frac{u}{200} \cdot \frac{1}{\big(1 - \frac{t}{200}\big)}\right) dt = -32t - u\ln\big(1 - \frac{t}{200}\big) + c.$$

The IC requires $v_0 = 0 - u \cdot 0 + c$, so the velocity as a function of time is

$$v = -32t - u\ln\big(1 - \frac{t}{200}\big) + v_0.$$

The velocity when the rocket stops burning, that is, when $t = 190s$, is

$$v(190) = -32 \cdot 190 - u\ln(0.05) + v_0 = -6080 + u\ln 20 + v_0.$$

By the way, to leave the earth's effective gravitational control, for example to go to the Moon, we need to reach the escape velocity of approximately $25,000\,m/h$, that is, $36,667\,ft/s$, so we need this hypothetical single stage rocket's u to satisfy $36,667 = -6080 + u\ln 20 + v_0$. We can assume $v_0 = 0$ is the initial velocity, so we need $u \approx 14,270 ft/s$. I don't know if this hypothetical rocket's nozzle gas speed is realistic.

The Apollo rocket configuration used a three stage rocket in order to lower the constant $(1 - \alpha_f)$ which is the fraction of the rocket mass after firing divided by its initial mass when fully fueled. When in orbit around the Earth, the vehicle's configuration was re-arranged and then the third stage rocket was re-fired to start the rest of the journey to the Moon.

3.1.4.35. $\mu(t) = \exp\left(\int 2t\,dt\right) = e^{t^2} \quad \Rightarrow \frac{d}{dt}[\,e^{t^2}y\,] = e^{t^2}\dot{y} + 2t\,e^{t^2}y = e^{t^2}$

$\Rightarrow e^{t^2}y(t) - e^0 y(0) = \int_0^t \frac{d}{ds}[\,e^{s^2}y(s)\,]\,ds = \int_0^t e^{s^2}ds \quad \Rightarrow$ Using $y(0) = 3$,

$y(t) = 3e^{-t^2} + \int_0^t e^{-(t^2 - s^2)}ds$ solves the IVP.

3.1.4.37. $\dot{y} + e^{t^2}y = 1 \quad \Rightarrow \mu(t) = \exp\left(\int_0^t e^{s^2}\,ds\right) \Rightarrow \frac{d}{dt}[\,\mu(t)y\,] = \mu(t)\dot{y} + \mu(t)e^{t^2}y = \mu(t) \cdot 1$

Note that $\mu(0) = \exp\left(\int_0^0 e^{s^2}\,ds\right) = 1$. We have $\mu(t)y(t) - y(0) = \int_0^t \frac{d}{ds}[\,\mu(s)y(s)\,]\,ds = \int_0^t \mu(s)ds$

$\Rightarrow y(t) = \frac{1}{\mu(t)}\left(y(0) + \int_0^t \mu(s)ds\right)$, that is,

$$y(t) = \exp\left(-\int_0^t e^{s^2}ds\right) \cdot \left(y(0) + \int_0^t \exp\left(\int_0^s e^{u^2}du\right)ds\right)$$

solves the IVP.

Section 3.2.4

3.2.4.1. Rewrite the ODE in the form $M + N\dot{y} = 0$: $-\Big(e^{2t} - y + t\ln y\Big) + \Big(t - \dfrac{t^2}{2y} + e^{-3y}\Big)\dot{y} = 0$, so $M(t,y) = -e^{2t} + y - t\ln y$ and $N(t,y) = t - \dfrac{t^2}{2y} + e^{-3y}$. Next, check the exactness criterion:

$$0 + 1 - \frac{t}{y} = \frac{\partial}{\partial y}\big[-e^{2t} + y - t\ln y\big] = \frac{\partial}{\partial y}\big[M(t,y)\big] =^{?} \frac{\partial}{\partial t}\big[N(t,y)\big] = \frac{\partial}{\partial t}\big[t - \frac{t^2}{2y} + e^{-3y}\big] = 1 - \frac{2t}{2y} + 0,$$

so, yes, ODE is exact.

A "potential function" $\phi(t,y)$ would have

$$-e^{2t} + y - t\ln(y) = M(t,y) = \frac{\partial}{\partial t}\big[\phi(t,y)\big],$$

hence

$$\phi(t,y) = \int \big(-e^{2t} + y - t\ln y\big)\,\partial t = -\frac{1}{2}\,e^{2t} + ty - \frac{1}{2}t^2\ln y + f(y),$$

where $f(y)$ is an arbitrary function of *only* y. Our symbol $\int \ldots\, \partial t$ is shorthand for the operation of anti-partial-differentiation with respect to t.

The reason we have an arbitrary function $f(y)$ instead of an arbitrary constant is because $\dfrac{\partial}{\partial t}[f(y)] \equiv 0$. Note also that because $f(y)$ is a function of y alone, $\dfrac{\partial}{\partial y}[f(y)] = \dfrac{df}{dy}$.

$\phi(t,y)$ must also satisfy

$$t - \frac{t^2}{2y} + e^{-3y} = N(t,y) = \frac{\partial}{\partial y}\big[\phi(t,y)\big] = \frac{\partial}{\partial y}\Big[-\frac{1}{2}\,e^{2t} + ty - \frac{1}{2}t^2\ln y + f(y)\Big] = t - \frac{t^2}{2y} + \frac{df}{dy},$$

so

$$e^{-3y} = \frac{df}{dy}.$$

We have $f(y) = -\frac{1}{3}\,e^{-3y}$; we could add an arbitrary constant but it would turn out to be redundant because our solutions are the curves $\phi(t,y) = C$.

Putting everything together, we have the solutions of the ODE are the curves

$$C = \phi(t,y) = -\frac{1}{2}\,e^{2t} + ty - \frac{1}{2}t^2\ln y - \frac{1}{3}\,e^{-3y},$$

where C is an arbitrary constant.

3.2.4.3. The ODE is separable. Multiply through by $\dfrac{1}{y-1}\,dt$ to get $\dfrac{dy}{y-1} = \dfrac{t\,dt}{(1+t^2)}$ and then integrate both sides to get

$$\ln|y-1| = \int \frac{dy}{y-1}\,dy = \int \frac{t\,dt}{(1+t^2)} = \frac{1}{2}\,\ln(1+t^2) + c.$$

Raise e to both sides to get

$$(\pm 1)(y-1) = |y-1| = e^{\ln|y-1|} = e^{\frac{1}{2}\,\ln(1+t^2)+c}$$
$$= e^{\ln(1+t^2)^{1/2}}e^c = e^c\sqrt{1+t^2}.$$

Multiply through by (± 1) and define $K = (\pm 1)e^c$ to get solutions $y - 1 = K(1+t^2)$, that is,

$$y = 1 + K\sqrt{1+t^2},$$

where K is an arbitrary non-zero constant.

In addition, there is an equilibrium solution $y(t) \equiv 1$, because $f(t,y) = \dfrac{t(y-1)}{t^2+1}$ has $f(t,1) \equiv 0$.

(a) The IC requires

$$3 = y(1) = 1 + K\sqrt{1+1^2} = 1 + K\sqrt{2} \quad \Rightarrow K = \sqrt{2},$$

so the solution is

$$y_1(t) = 1 + \sqrt{2}\,\sqrt{1+t^2} = 1 + \sqrt{2(1+t^2)}.$$

(b) The IC requires $1 = y(1)$, which is satisfied by the equilibrium solution. So, the solution for the second IC is $y_2(t) \equiv 1$.

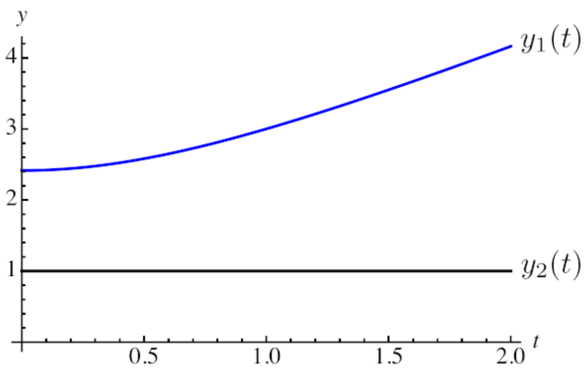


Figure 1: Problem 3.2.4.3: Solutions for two different ICs

3.2.4.5. $\dfrac{dA}{dt} = -\alpha A \quad \Rightarrow \displaystyle\int \frac{dA}{A} = -\int \alpha dt \;\Rightarrow \ln|A| = -\alpha t + c \quad \Rightarrow (\pm 1)A = |A| = e^{\ln|A|} = e^{-\alpha t + c} =$ $e^{-\alpha t}e^{c}$

$$\Rightarrow A = (\pm 1)e^{c}e^{-\alpha t} = Ke^{-\alpha t},$$

for arbitrary non-zero constant K. IC $A_0 = A(0) = K$, so the solution of the IVP is $A(t) = A_0 e^{-\alpha t}$.

The half-life, $t^\star$, satisfies

$$A_0 e^{-\alpha t^\star} = A(t^\star) = \frac{1}{2}A_0 \;\Rightarrow e^{-\alpha t^\star} = \frac{1}{2} \quad \Rightarrow -\alpha t^\star = \ln\frac{1}{2} = \ln(2^{-1}) = -\ln 2,$$

so

$$\alpha = \frac{1}{t^\star}\ln 2 = \frac{1}{5730}\ln 2,$$

using the half-life for C^{14}, assuming time is measured in years.

The sarcophagus has 63% of what would be in a present day sample. Assume that this measurement 63% was done at time $t = 0$ and the wood was cut at time $\tau < 0$.

The table below summarizes data we will use.

Time	*Amount of* C^{14}
τ	$A(\tau)$
0	$0.63\,A(\tau)$

We have

$$A_0 = A(0) = 0.63\,A(\tau) = 0.63\,A_0 e^{-\alpha\tau} \quad \Rightarrow e^{-\alpha\tau} = \frac{1}{0.63} \quad \Rightarrow -\alpha\tau = \ln e^{-\alpha\tau} = \ln\frac{1}{0.63} = \ln 0.63^{-1} = -\ln 0.63$$

$$\Rightarrow \tau = \frac{1}{\alpha}\ln 0.63 = 5730\cdot\frac{\ln 0.63}{\ln 2} \approx -3819.482006.$$

Measured with the three significant digits implicit in the half-life being $t^\star = 5730$, the sarcophagus was buried about 3820 years ago.

3.2.4.7. $\frac{dP}{dt} = kP \quad \Rightarrow \int \frac{dP}{P} = \int k\,dt \ \Rightarrow \ln|P| = kt + c \quad \Rightarrow (\pm 1)P = |P| = e^{\ln|P|} = e^{kt+c} = e^{kt}e^{c}$

$$\Rightarrow P = (\pm 1)e^{c}e^{kt} = Ce^{-\alpha t},$$

for arbitrary non-zero constant C. IC $P_0 = P(0) = C$, so the solution of the IVP is $P(t) = P_0 e^{kt}$.

The table below summarizes data we will use. We will measure time in years after July 1, 1980.

Time	*Population*
0	4.473
7	5.055
T	10.000

We have $P_0 = 4.473$, so

$$P(t) = 4.473e^{kt}$$

and so

$$5.055 = P(7) = 4.473e^{7k}.$$

This implies $\ln \frac{5.055}{4.473} = 7k$, hence

$$k = \frac{1}{7}\ln\left(\frac{5.055}{4.473}\right) \approx 0.0174740754.$$

We want to solve for T the equation for the third entry in the data table:

$$10.000 = P(T) = 4.473e^{kT}$$

implies

$$T = \frac{1}{k}\ln\left(\frac{10.000}{4.473}\right) \approx 46.04110653.$$

So, this model predicts that the Earth's population of human beings will reach 10 billion on about July 16, 2026.

By the way, $k \approx 0.0174740754$ says that in this model the Earth's population of human beings has a growth rate of about 1.74740754 % per year.

Also, by the way, the Earth's population on September 9, 2012 was about billion.

3.2.4.9. The uncertainty in the data for both P_0 and $P(7)$ means that eventually we will need two study four cases:

(1) $P_0 = 4.423$ and $P(7) = 5.105$,
(2) $P_0 = 4.523$ and $P(7) = 5.005$,
(3) $P_0 = 4.423$ and $P(7) = 5.005$,
(4) $P_0 = 4.523$ and $P(7) = 5.105$.

$$\frac{dP}{dt} = kP \quad \Rightarrow \int \frac{dP}{P} = \int k\,dt \ \Rightarrow \ln|P| = kt + c \quad \Rightarrow (\pm 1)P = |P| = e^{\ln|P|} = e^{kt+c} = e^{kt}e^{c}$$

$$\Rightarrow P = (\pm 1)e^{c}e^{kt} = Ce^{-\alpha t},$$

for arbitrary non-zero constant C. IC $P_0 = P(0) = C$, so the solution of the IVP is $P(t) = P_0 e^{kt}$.

The uncertainty in the initial datum was that P_0 ranged between 4.423 and 4.523.

Case 1: The table below summarizes some of the data we will use. We will measure time in years after July 1, 1980.

Time	*Population*
0	4.423
7	5.105
T	10.000

Of the four cases, this case has the greatest increase of population between July 1, 1980 and July 1, 1987, thus producing the greatest k. Intuitively this should give the smallest value of T of the four cases.

We have $P_0 = 4.423$, so

$$P(t) = 4.423e^{kt}$$

and so

$$5.105 = P(7) = 4.423e^{7k}.$$

This implies $\ln \frac{5.105}{4.423} = 7k$, hence

$$k = \frac{1}{7}\ln\left(\frac{5.105}{4.423}\right) \approx 0.0204860361.$$

We want to solve for T the equation for the third entry in the data table:

$$10.000 = P(T) = 4.423e^{kT}$$

implies

$$T = \frac{1}{k}\ln\left(\frac{10.000}{4.423}\right) \approx 39.82063148.$$

So, this case of the model predicts that the Earth's population of human beings will reach 10 billion on about April 26, 2020.

Case 2: The table below summarizes some of the data we will use. We will measure time in years after July 1, 1980.

Time	*Population*
0	4.523
7	5.005
T	10.000

We have $P_0 = 4.523$, so

$$P(t) = 4.523e^{kt}$$

and so

$$5.005 = P(7) = 4.523e^{7k}.$$

This implies $\ln \frac{5.005}{4.523} = 7k$, hence

$$k = \frac{1}{7}\ln\left(\frac{5.005}{4.523}\right) \approx 0.0144659889.$$

We want to solve for T the equation for the third entry in the data table:

$$10.000 = P(T) = 4.523e^{kT}$$

implies

$$T = \frac{1}{k}\ln\left(\frac{10.000}{4.523}\right) \approx 54.84655133.$$

So, this case of the model predicts that the Earth's population of human beings will reach 10 billion on about May 5, 2035.

Case 3: The table below summarizes some of the data we will use. We will measure time in years after July 1, 1980.

Time	*Population*
0	4.423
7	5.005
T	10.000

We have $P_0 = 4.423$, so

$$P(t) = 4.423e^{kt}$$

and so

$$5.005 = P(7) = 4.423e^{7k}.$$

This implies $\ln \frac{5.005}{4.423} = 7k$, hence

$$k = \frac{1}{7} \ln\left(\frac{5.005}{4.423}\right) \approx 0.0176598877.$$

We want to solve for T the equation for the third entry in the data table:

$$10.000 = P(T) = 4.423e^{kT}$$

implies

$$T = \frac{1}{k} \ln\left(\frac{10.000}{4.423}\right) \approx 46.19320961.$$

So, this case of the model predicts that the Earth's population of human beings will reach 10 billion on about September 10, 2026.

Case 4: The table below summarizes some of the data we will use. We will measure time in years after July 1, 1980.

Time	*Population*
0	4.523
7	5.105
T	10.000

We have $P_0 = 4.523$, so

$$P(t) = 4.523e^{kt}$$

and so

$$5.105 = P(7) = 4.523e^{7k}.$$

This implies $\ln \frac{5.105}{4.523} = 7k$, hence

$$k = \frac{1}{7} \ln\left(\frac{5.105}{4.523}\right) \approx 0.0172921373.$$

We want to solve for T the equation for the third entry in the data table:

$$10.000 = P(T) = 4.523e^{kT}$$

implies

$$T = \frac{1}{k} \ln\left(\frac{10.000}{4.523}\right) \approx 45.88268001.$$

So, this case of the model predicts that the Earth's population of human beings will reach 10 billion on about May 18, 2026.

The model, including uncertainties, predicts that the Earth's human population will reach 10 billion approximately between April 26, 2020 and May 5, 2035. That is an uncertainty of only about 15 years in the final conclusion, which is a lot of uncertainty!

The results for the four cases agree with the intuition about the comparisons among the four values of T: Case 1 did give the smallest value of T and Case 2 did give the largest value of T.

3.2.4.11. (a) Let $y = y(t)$ be the position of the particle. We are given that $\dot{y}(t) = k\,|\,y(t)\,|^2 = ky^2$, for some constant k. The table below summarizes the data we will use.

Time	*Position*
0	2
3	4

Separation of variables gives $\quad -y^{-1} = \int \frac{dy}{y^2} = \int k\,dt = kt + c, \quad$ so

$$y = -\frac{1}{kt + c}.$$

This function exists for $t \neq -\frac{c}{k}$.

The IC requires $2 = y(0) = -\frac{1}{0+c}$, so $c = -\frac{1}{2}$. The solution of the IVP is unique on some open interval containing $t = 0$ and is

$$y(t) = -\frac{1}{kt - \frac{1}{2}} \cdot \frac{2}{2} = \frac{2}{1 - 2kt}.$$

The second data point gives

$$4 = y(3) = \frac{2}{1 - 6k},$$

hence $4(1 - 6k) = 2$, hence $k = \frac{1}{12}$. The solution satisfying all of the given data is

$$y(t) = \frac{2}{1 - 2\frac{1}{12}\,t} = \frac{2}{1 - \frac{1}{6}\,t}\,.$$

(b) The particle reaches position $y = 8$ when $8 = \dfrac{2}{1 - \frac{1}{6}\,t}$, that is, $8\big(1 - \frac{1}{6}\,t\big) = 2$. The particle reaches position $y = 8$ at time $t = 4.5$.

(c) The particle reaches position $y = 8$ when $1000 = \dfrac{2}{1 - \frac{1}{6}\,t}$, that is, $1000\big(1 - \frac{1}{6}\,t\big) = 2$. The particle reaches position $y = 1000$ at time $t = 5.988$.

(d) The particle lives during the time interval the solution exists. The solution stops existing when the denominator is zero, that is, when $t = 6$. So, the particle lives for $0 \leq t < 6$.

3.2.4.13. Multiply both sides by $(3 + 3y^2 - x)$ to rewrite the ODE in the form $\quad (3 + 3y^2 - x)\,\dfrac{dy}{dx} = (2x + y)$, that is,

$$-(2x + y) + (3 + 3y^2 - x)\,\frac{dy}{dx} = 0,$$

so $M(t, y) = -2x - y$ and $N = 3 + 3y^2 - x$. Next, check the exactness criterion:

$$-1 = \frac{\partial}{\partial y}\big[-2x - y\big] = \frac{\partial}{\partial y}\big[M(x, y)\big] =^{?} \frac{\partial}{\partial x}\big[N(x, y)\big] = \frac{\partial}{\partial x}\big[3 + 3y^2 - x\big] = -1,$$

so, yes, ODE is exact.

A "potential function" $\phi(x, y)$ would have

$$-2x - y = M(x, y) = \frac{\partial}{\partial x}\big[\phi(x, y)\big],$$

hence

$$\phi(x, y) = \int (-2x - y)\,\partial x = -x^2 - xy + f(y),$$

where $f(y)$ is an arbitrary function of *only* y. Our symbol $\int \ldots\, \partial x$ is shorthand for the operation of anti-partial-differentiation with respect to x.

The reason we have an arbitrary function $f(y)$ instead of an arbitrary constant is because $\dfrac{\partial}{\partial x}[f(y)] \equiv 0$. Note also that because $f(y)$ is a function of y alone, $\dfrac{\partial}{\partial y}[f(y)] = \dfrac{df}{dy}$.

$\phi(t, y)$ must also satisfy

$$3 + 3y^2 - x = N(x, y) = \frac{\partial}{\partial y}\big[\phi(x, y)\big] = \frac{\partial}{\partial y}\Big[-x^2 - xy + f(y)\Big] = 0 - x + \frac{df}{dy},$$

so

$$\frac{df}{dy} = 3 + 3y^2.$$

We have $f(y) = 3y + y^3$; we could add an arbitrary constant but it would turn out to be redundant because our solutions are the curves $\phi(x, y) = C$.

Putting everything together, we have the solutions of the ODE are the curves

$$C = \phi(x, y) = -x^2 - xy + 3y + y^3,$$

where C is an arbitrary constant.

The IC requires $y(0) = 1$, that is,

$$C = \phi(0, 1) = -0^2 - 0 \cdot 1 + 3 \cdot 1 + 1^3 = 4,$$

so the implicit solution is the curve

$$4 = -x^2 - xy + 3y + y^3.$$

3.2.4.15. Multiply both sides by $(\sin t + t\cos y + y)$ to rewrite the ODE in the form

$$(\sin t + t\cos y + y)\,\dot{y} = -\,(\sin y + y\cos t - 4)$$

that is,

$$(\sin y + y\cos t - 4) + (\sin t + t\cos y + y)\,\frac{dy}{dt} = 0,$$

so $M(t, y) = \sin y + y\cos t - 4$ and $N(t, y) = \sin t + t\cos y + y$. Next, check the exactness criterion:

$$\cos y + \cos t = \frac{\partial}{\partial y}\big[\sin y + y\cos t - 4\big] = \frac{\partial}{\partial y}\big[M(t, y)\big] \overset{?}{=} \frac{\partial}{\partial t}\big[N(t, y)\big] = \frac{\partial}{\partial t}\big[\sin t + t\cos y + y\big] = \cos t + \cos y,$$

so, yes, ODE is exact.

A "potential function" $\phi(t, y)$ would have

$$\sin y + y\cos t - 4 = M(t, y) = \frac{\partial}{\partial t}\big[\,\phi(t, y)\,\big],$$

hence

$$\phi(t, y) = \int (\sin y + y\cos t - 4)\,\partial t = t\sin y + y\sin t - 4t + f(y),$$

where $f(y)$ is an arbitrary function of *only* y. Our symbol $\int ... \,\partial x$ is shorthand for the operation of anti-partial-differentiation with respect to x.

The reason we have an arbitrary function $f(y)$ instead of an arbitrary constant is because $\frac{\partial}{\partial t}[f(y)] \equiv 0$. Note also that because $f(y)$ is a function of y alone, $\frac{\partial}{\partial y}[f(y)] = \frac{df}{dy}$.

$\phi(t, y)$ must also satisfy

$$\sin t + t\cos y + y = N(t, y) = \frac{\partial}{\partial y}[\,\phi(t, y)\,] = \frac{\partial}{\partial y}\Big[\,t\sin y + y\sin t - 4t + f(y)\,\Big] = t\cos y + \sin t - 0 + \frac{df}{dy},$$

so

$$\frac{df}{dy} = y.$$

We have $f(y) = \frac{1}{2}\,y^2$; we could add an arbitrary constant but it would turn out to be redundant because our solutions are the curves $\phi(t, y) = C$.

Putting everything together, we have the solutions of the ODE are the curves

$$C = \phi(t, y) = t\sin y + y\sin t - 4t + \frac{1}{2}\,y^2,$$

where C is an arbitrary constant.

3.2.4.17. Separation of variables gives $-k\,t+c=\int -k\,dt=\int \dfrac{(A+x)dx}{x}=\int\Big(\dfrac{A}{x}+1\Big)dx=A\ln x+x.$
[Note that biologically $x\geq 0$ so we don't need the absolute sign in $\ln|x|$.]

This gives implicit solutions relating x and t. Note that the goal of the problem is to find a certain time, so it might help to solve for t in terms of x, instead of the usual method of solving for x in terms of t!

We get

$$t=-\frac{1}{k}\Big(-c+A\ln x+x\Big)$$

The table below summarizes the data we will use.

Time	*Alcohol concentration*
0	0.024
$t^\star$	0.008

(a) Assuming $A=0.005$ and $k=0.01$, the first data point requires

$$0=-100\Big(-c+0.005\,\ln 0.024+0.024\Big),$$

hence

$$c=0.005\,\ln 0.024+0.024\approx 0.0053514928.$$

So, we can solve for

$$t^\star\approx -100\Big(-0.0053514928+0.005\ln 0.008+0.008\Big)\approx 2.149306144\,hours.$$

So, it would take about 2 hours and 9 minutes for the person's blood alcohol concentration to be within the legal limit.

(b) Ex: 1: Assuming $A=0.005$ and $k=0.007$, the first data point requires

$$0=-\frac{1}{.007}\Big(-c+0.005\,\ln 0.024+0.024\Big),$$

hence, as in part (a),

$$c=0.005\,\ln 0.024+0.024\approx 0.0053514928.$$

So, we can solve for

$$t^\star\approx -.007^{-1}\cdot\Big(-0.0053514928+0.005\ln 0.008+0.008\Big)\approx 3.070437349\,hours.$$

So, it would take about 3 hours and 4 minutes for the person's blood alcohol concentration to be within the legal limit. This is much longer than when we assumed $A=0.005$ and $k=0.01$ in part (a).

The results in part (b)'s Ex: 1 were for the same A as in part (a) but for a smaller value of k. The positive proportionality constant k partly measures how fast alcohol is "cleared" out of the bloodstream, because

$$f(x,k,A)\triangleq -\frac{kx}{A+x}$$

has

$$\frac{\partial f}{\partial k}=-\frac{x}{A+x}<0,\ \text{ for } x>0 \text{ and constant } A>0\,.$$

So, it makes sense that the smaller the value of k the longer it takes for the person's blood alcohol concentration to be within the legal limit.

Ex: 2: Assuming $A=0.01$ and $k=0.01$, the first data point requires

$$0=-\frac{1}{.01}\Big(-c+0.01\,\ln 0.024+0.024\Big),$$

hence

$$c = 0.01\ \ln 0.024 + 0.024 \approx -0.0132970145.$$

So, we can solve for

$$t^\star \approx -100\Big(0.0132970145 + 0.01 \ln 0.008 + 0.008\Big)$$
$$\approx 2.698612289\, hours.$$

So, it would take about 2 hours and 42 minutes for the person's blood alcohol concentration to be within the legal limit. When $A = 0.01$, it takes longer for alcohol concentration to go down compared to the situation in part (a), where $A = 0.005$.

The results in part (b)'s Ex: 2 were for the same k as in part (a) but for a larger value of A. The positive constant A partly measures how slowly alcohol is "cleared" out of the bloodstream, because $f(x, k, A) = -\frac{kx}{A+x}$ has

$$\frac{\partial f}{\partial A} = \frac{kx}{(A+x)^2} > 0, \text{ for } x > 0 \text{ and constants } A, k > 0\,.$$

So, it makes sense that the larger the value of A the longer it takes for the person's blood alcohol concentration to be within the legal limit.

Here are several physiological factors that could affect the time it takes for alcohol to be metabolized:

- As an adult person ages usually their metabolism decreases, so their ability to clear alcohol from their bloodstream slows down.
- I believe it is physiologically true that, on average women metabolize alcohol more slowly than men. This would lead to a smaller k or A for women than for men.
- I suspect that a person with a higher BMI (body mass index) would tend to metabolize alcohol more slowly than a person with a lower BMI, although this may simply be correlated with having a lower metabolism in general.

So, factors that would tend to increase A and/or decrease k would be to have a higher BMI, be female, or be older.

(c) Add a positive constant, b, to the right hand side of the ODE, to get $\dot{x} = -\frac{kx}{A+x} + b$.

3.2.4.19. Separation of variables, when $y \neq 1$, gives $\quad 2(y-1)^{1/2} = \displaystyle\int \frac{dy}{(y-1)^{1/2}} = \int dt = t + c,\quad$ so

$$y = 1 + \Big(\frac{t+c}{2}\Big)^2.$$

The IC requires $1 = y(2) = 1 + \left(\frac{2+c}{2}\right)^2$, so $c = -2$. Separation of variables gives

$$y_1(t) = 1 + \Big(\frac{t-2}{2}\Big)^2.$$

as an example of a solution of the IVP. The ODE also has an equilibrium solution, $y_2(t) \equiv 1$, that happens to satisfy the IC.

3.2.4.21. Separation of variables, when $y \neq 0$, gives $\quad 3\,y^{1/3} = \displaystyle\int \frac{dy}{y^{2/3}} = \int dt = t + c,\quad$ so

$$y = \Big(\frac{t+c}{3}\Big)^3.$$

This function exists for all t.

The IC requires $0 = y(0) = \left(\frac{4(0+c)}{5}\right)^{5/4}$, so $c = 0$. Separation of variables gives

$$y_1(t) = \Big(\frac{t}{3}\Big)^3.$$

as an example of a solution of the IVP. The ODE also has an equilibrium solution, $y_2(t) \equiv 0$, that happens to satisfy the IC.

3.2.4.23. (a) Separation of variables, when $y \neq 0$, gives $\quad -y^{-1} = \int \dfrac{dy}{y^2} = \int dt = t + c, \quad$ so

$$y = -\frac{1}{t+c}.$$

This function exists for $t \neq -c$.

The IC requires $3 = y(0) = -\frac{1}{0+c}$, so $c = -\frac{1}{3}$. The solution of the IVP is unique on some open interval containing $t = 0$, by Picard's Theorem, and is

$$y(t) = -\frac{1}{t-\frac{1}{3}} \cdot \frac{3}{3} = \frac{3}{1-3t},$$

This exists on any open interval not containing $t = \frac{1}{3}$; but, to be a solution of the IVP it has to exist on an open interval containing $t = 0$. So, the solution $y(t)$ exists only on the open interval $(-\infty, \frac{1}{3})$.

So, $\delta = \frac{1}{3}$ in the notation of this problem.

(b) In this problem $t_0 = 0$ and $y_0 = 3$. The hypotheses of Picard's Theorem 3.6 are satisfied on *any* rectangle

$$\mathcal{R}_{\alpha,\beta} \triangleq \{(t,y){:}\, -\alpha \leq t \leq \alpha,\ 3-\beta \leq y \leq 3+\beta\}$$

because $f(t,y) \triangleq y^2$ is continuous and has continuous partial derivative with respect to y everywhere. So, we can take α and β to be any positive numbers.

Picard's Theorem 3.6 establishes that there is an open time interval containing $t_0 = 0$ on which IVP has exactly one solution. Being an open interval containing $t = 0$ implies that it contains some open interval of the form $-\bar{\alpha} \leq t \leq \bar{\alpha}$ on which IVP has exactly one solution.

(c) Picard's Theorem 3.7 gives more information: The hypotheses ask for $\bar{\alpha}$ and $\bar{\beta}$ sufficiently small that

$$(\star)\ 0 < \bar{\alpha} \leq \alpha, 0 < \bar{\beta} \leq \beta, M\bar{\alpha} \leq \bar{\beta}, K\bar{\alpha} < 1,$$

where

$$|f(t,y)| \leq M \quad \text{and} \quad \left|\frac{\partial f}{\partial y}(t,y)\right| \leq K$$

for all (t,y) in $\mathcal{R}_{\alpha,\beta}$.

Whatever positive numbers α and β we take. we will have $M = (3+\beta)^2$ because $|f(t,y)| \triangleq |y|^2$ and $|3+\beta| > |3-\beta|$ for any $\beta > 0$. Similarly, $\left|\frac{\partial f}{\partial y}(t,y)\right| = 2|y|$, so $K = 2(3+\beta)$.

So, $(\star)$ requires

$$(\star) 0 < \bar{\alpha} \leq \alpha, 0 < \bar{\beta} \leq \beta, (3+\beta)^2\bar{\alpha} \leq \bar{\beta}, 2(3+\beta)\bar{\alpha} < 1.$$

The last of these requirements and the requirement that $\beta > 0$ together imply that $2(3+0)\bar{\alpha} < 1$, that is,

$$\bar{\alpha} < \frac{1}{6}.$$

In fact, using all of the requirements of $(\star)$ may make the guaranteed, theoretical time open interval of existence even smaller.

The theoretically guaranteed time open interval of existence is *at most*$-\frac{1}{6} < t < \frac{1}{6}$ but the actual time interval of existence is $\big(-\infty, \frac{1}{3}\big)$.

In fact, when we found $\bar{\alpha} < \frac{1}{6}$ it did not satisfy all of the conditions in $(\star)$. The third condition, $(3+\beta)^2\bar{\alpha} \leq \bar{\beta}$, along with the second condition, $0 < \bar{\beta} \leq \beta$, imply that

$$(3+\bar{\beta})^2\bar{\alpha} < (3+\beta)^2\bar{\alpha} \leq \bar{\beta},$$

hence

$$(3+\bar{\beta})^2\bar{\alpha} < \bar{\beta}.$$

This implies

$$\bar{\alpha} < \frac{\bar{\beta}}{(3+\bar{\beta})^2} \triangleq g(\bar{\beta}).$$

Calculus I techniques and the fact that $g'(\bar{\beta}) = \frac{3-\bar{\beta}}{(3+\bar{\beta})^3}$, can be used to explain why the global maximum value of $g(\bar{\beta})$ on the open interval $0<\bar{\beta}<\infty$ is achieved at its only critical point, $\bar{\beta} = 3$. So, in fact, we need

$$\bar{\alpha} < g(3) = \frac{1}{12},$$

which is more restrictive than $\bar{\alpha} < \frac{1}{6}$.

Section 3.3.8

3.3.8.1. Characteristic equation $0 = s^2 + 8s + 15 = (s + 5)(s + 3) \quad \Rightarrow$ roots are $s = -5, -3$
$\Rightarrow$ General solution of the ODE is $y(t) = c_1 e^{-5t} + c_2 e^{-3t}$, c_1, c_2 = arbitrary consts. All solutions have $\lim_{t\to\infty} y(t) = 0$, and the time constant is

$$\tau \triangleq \frac{1}{r_{min}} = \frac{1}{\min\{3, 5\}} = \frac{1}{3}.$$

3.3.8.3. Characteristic equation $0 = s^2 + s - \dfrac{15}{4} = \left(s + \dfrac{5}{2}\right)\left(s - \dfrac{3}{2}\right) \quad \Rightarrow$ roots are $s = -\frac{5}{2}, \frac{3}{2}$
$\Rightarrow$ General solution of the ODE is $y(t)=c_1 e^{-5t/2} + c_2 e^{3t/2}$, c_1, c_2 = arbitrary consts. Not all solutions have $\lim_{t\to\infty} y(t) = 0$, so there is no time constant.

3.3.8.5. Characteristic equation $0 = s^2 + 8s + 18 = (s + 4)^2 + 2 \quad \Rightarrow$ roots are $s = -4 \pm i\sqrt{2}$
$\Rightarrow$ General solution of the ODE is $y(t)=c_1 e^{-4t} \cos(\sqrt{2}\,t) + c_2 e^{-4t} \sin(\sqrt{2}\,t)$, where c_1, c_2 =arb. consts. All solutions have $\lim_{t\to\infty} y(t) = 0$, and the time constant is $\tau = \dfrac{1}{4}$.

3.3.8.7. Characteristic equation $0 = s^2 - 2s - 3 = (s + 1)(s - 3) \quad \Rightarrow$ roots are $s = -1, 3$
$\Rightarrow$ General solution of the ODE is $y(t)=c_1 e^{-t} + c_2 e^{3t}$, c_1, c_2 = arbitrary consts. The ICs require

$$\left\{\begin{array}{rl} 5 = y(0) = & c_1 \ \ + c_2 \\ 7 = \dot{y}(0) = & -c_1 + 3c_2 \end{array}\right\},$$

which implies

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so the solution of the IVP is

$$y(t) = 2e^{-t} + 3e^{3t}.$$

3.3.8.9. Characteristic equation $0 = s^2 + s + \dfrac{9}{4} = \left(s + \dfrac{1}{2}\right)^2 + 2 \quad \Rightarrow$ roots are $s = -\dfrac{1}{2} \pm i\sqrt{2}$
$\Rightarrow$ General solution of the ODE is $y(t)=c_1 e^{-t/2} \cos(\sqrt{2}\,t) + c_2 e^{-t/2} \sin(\sqrt{2}\,t)$, where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -\frac{1}{2}\, c_1 e^{-t/2} \cos(\sqrt{2}\,t) - \sqrt{2}\, c_1 e^{-t/2} \sin(\sqrt{2}\,t) - \frac{1}{2}\, c_2 e^{-t/2} \sin(\sqrt{2}\,t) + \sqrt{2}\, c_2 e^{-t/2} \cos(\sqrt{2}\,t).$$

The ICs require

$$\left\{\begin{array}{rl} 7 = y(0) = & c_1 \\ 0 = \dot{y}(0) = & -\frac{1}{2}\, c_1 + \sqrt{2} c_2 \end{array}\right\},$$

which implies $c_1 = 7$ and $c_2 = \dfrac{7}{2\sqrt{2}}$. The solution of the IVP is

$$y(t) = 7e^{-t/2} \Big(\cos(\sqrt{2}\,t) + \frac{1}{2\sqrt{2}} \sin(\sqrt{2}\,t) \Big).$$

3.3.8.11. Characteristic equation $0 = s^2 + 3s - 10 = (s + 5)(s - 2) \quad \Rightarrow$ roots are $s = -5, 2$
$\Rightarrow$ General solution of the ODE is $y(t)=c_1 e^{-5t} + c_2 e^{2t}$, c_1, c_2 = arbitrary consts. The ICs require

$$\left\{\begin{array}{rl} 1 = y(0) = & c_1 \ \ + c_2 \\ -3 = \dot{y}(0) = & -5c_1 + 2c_2 \end{array}\right\},$$

which implies

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

so the solution of the IVP is

$$y(t) = \frac{1}{7}\big(5e^{-5t} + 2e^{2t}\big).$$

3.3.8.13. Characteristic equation $0 = s^2 + s + \dfrac{1}{4} = \Big(s + \dfrac{1}{2}\Big)^2 \quad \Rightarrow$ roots are $s = -\frac{1}{2}, -\frac{1}{2}$

$\Rightarrow$ General solution of the ODE is $y(t) = c_1 e^{-t/2} + c_2 t e^{-t/2}$, $c_1, c_2 =$ arbitrary consts. It follows that

$$\dot{y}(t) = -\frac{1}{2} c_1 e^{-t/2} + c_2 \Big(1 - \frac{1}{2} t\Big) e^{-t/2},$$

where c_1, c_2 =arbitrary constants.

The ICs require

$$\left\{ \begin{array}{rl} -1 = y(0) = & c_1 \\ 2 = \dot{y}(0) = & -\frac{1}{2} c_1 + c_2 \end{array} \right\},$$

which implies $c_1 = -1$ and $c_2 = \frac{3}{2}$. The solution of the IVP is

$$y(t) = \Big(-1 + \frac{3}{2}\, t\Big) e^{-t/2}.$$

3.3.8.15. Characteristic equation $0 = s^2 + s + 1 = \Big(s + \dfrac{1}{2}\Big)^2 + \dfrac{3}{4} \quad \Rightarrow$ roots are $s = -\dfrac{1}{2} \pm i\, \dfrac{\sqrt{3}}{2}$

$\Rightarrow$ General solution of the ODE is $y(t) = c_1 e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2} t\Big) + c_2 e^{-t/2} \sin\Big(\frac{\sqrt{3}}{2} t\Big)$, where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -\tfrac{1}{2} c_1 e^{-t/2} \cos\Big(\tfrac{\sqrt{3}}{2} t\Big) - \tfrac{\sqrt{3}}{2} c_1 e^{-t/2} \sin\Big(\tfrac{\sqrt{3}}{2} t\Big) - \tfrac{1}{2} c_2 e^{-t/2}\, \sin\Big(\tfrac{\sqrt{3}}{2}\Big) + \tfrac{\sqrt{3}}{2} c_2 e^{-t/2} \cos\Big(\tfrac{\sqrt{3}}{2}\Big).$$

The ICs require

$$\left\{ \begin{array}{rl} -2 = y(0) = & c_1 \\ -2 = \dot{y}(0) = & -\frac{1}{2}\, c_1 + \frac{\sqrt{3}}{2} c_2 \end{array} \right\},$$

which implies $c_1 = -2$ and $c_2 = -2\sqrt{3}$. The solution of the IVP is

$$y(t) = e^{-t/2}\Big(-2\cos\Big(\frac{\sqrt{3}}{2}\, t\Big) - 2\sqrt{3}\, \sin\Big(\frac{\sqrt{3}}{2}\, t\Big)\Big).$$

Here, the amplitude phase form (3.39) is

$$y(t) = A e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \delta\Big),$$

where

$$\left\{ \begin{array}{rl} -2 = & A\cos\delta \\ -2\sqrt{3} = & A\sin\delta \end{array} \right\},$$

hence $A = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4$ and $\tan\delta = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$. Because $(c_1, c_2) = (-2, -2\sqrt{3})$ is in the third quadrant,

$$\delta = \pi + \arctan(\sqrt{3}) = \pi + \Big(\frac{\pi}{3}\Big) = \frac{4\pi}{3}.$$

The IVP's solution in amplitude phase form is

$$y(t) = 4e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \frac{4\pi}{3}\Big).$$

3.3.8.17. Characteristic equation $0 = s^2 + 2s + 5 = (s+1)^2 + 4 \quad \Rightarrow$ roots are $s = -1 \pm i\,2$
$\Rightarrow$ General solution of the ODE is $y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$, where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -c_1 e^{-t}\cos 2t - 2\,c_1 e^{-t}\sin 2t - c_2 e^{-t}\sin 2t + 2\,c_2 e^{-t}\cos 2t.$$

The ICs require

$$\left\{ \begin{array}{rcl} -2 = y(0) = & c_1 \\ 6 = \dot{y}(0) = & -c_1 + 2c_2 \end{array} \right\},$$

which implies $c_1 = -2$ and $c_2 = 2$. The solution of the IVP is

$$y(t) = e^{-t}\Big(-2\cos 2t + 2\sin 2t \Big).$$

Here, the amplitude phase form (3.39) is

$$y(t) = Ae^{-t}\cos(2t - \delta),$$

where

$$\left\{ \begin{array}{r} -2 = A\cos\delta \\ 2 = A\sin\delta \end{array} \right\},$$

hence $A = \sqrt{(-2)^2 + (2)^2} = 2\sqrt{2}$ and $\tan\delta = \frac{2}{-2} = -1$. Because $(c_1, c_2) = (-2, 2)$ is in the second quadrant,

$$\delta = \pi + \arctan(-1) = \pi + \Big(-\frac{\pi}{4} \Big).$$

The IVP's solution in amplitude phase form is

$$y(t) = 2\sqrt{2}\,e^{-t}\cos\Big(2t - \frac{3\pi}{4} \Big).$$

3.3.8.19. Characteristic equation $0 = s^2 + 4s + 5 = (s+2)^2 + 1 \quad \Rightarrow$ roots are $s = -2 \pm i$
$\Rightarrow$ General solution of the ODE is $y(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$, where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -2c_1 e^{-2t}\cos t - c_1 e^{-2t}\sin t - 2c_2 e^{-2t}\sin t + c_2 e^{-2t}\cos t.$$

The ICs require

$$\left\{ \begin{array}{rcl} -1 = y(0) = & c_1 \\ 2 + \sqrt{3} = \dot{y}(0) = & -2c_1 + c_2 \end{array} \right\},$$

which implies $c_1 = -1$ and $c_2 = \sqrt{3}$. The solution of the IVP is

$$y(t) = e^{-2t}\Big(-\cos t + \sqrt{3}\,\sin t \Big).$$

Here, the amplitude phase form (3.39) is

$$y(t) = Ae^{-2t}\cos(t - \delta),$$

where

$$\left\{ \begin{array}{r} -1 = A\cos\delta \\ \sqrt{3} = A\sin\delta \end{array} \right\},$$

hence $A = \sqrt{(-1)^2 + \big(\sqrt{3}\big)^2} = 2$ and $\tan\delta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$. Because $(c_1, c_2) = (-1, \sqrt{3})$ is in the second quadrant, $\delta = \pi + \arctan\big(-\sqrt{3}\big) = \pi + \Big(-\frac{\pi}{3} \Big) = \frac{2\pi}{3}$. The IVP's solution in amplitude phase form is

$$y(t) = 2\,e^{-2t}\cos\Big(t - \frac{2\pi}{3} \Big).$$

3.3.8.21. Characteristic equation $0 = 5s^2 + 20s + 60 = 5\big((s+2)^2 + 8\big)$ $\Rightarrow$ roots are $s = -2 \pm i\sqrt{8}$
$\Rightarrow$ General solution of the ODE is $y(t) = c_1 e^{-2t}\cos\left(\sqrt{8}\,t\right) + c_2 e^{-2t}\sin\left(\sqrt{8}\,t\right)$, where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -2c_1 e^{-2t}\cos\left(\sqrt{8}\,t\right) - \sqrt{8}\,c_1 e^{-2t}\sin\left(\sqrt{8}\,t\right) - 2c_2 e^{-2t}\sin\left(\sqrt{8}\,t\right) + \sqrt{8}\,c_2 e^{-2t}\cos\left(\sqrt{8}\,t\right).$$

The ICs require

$$\left\{\begin{array}{rl} -2 = y(0) = & c_1 \\ 0 = \dot{y}(0) = & -2c_1 + \sqrt{8}\,c_2 \end{array}\right\},$$

which implies $c_1 = -2$ and $c_2 = -\sqrt{2}$. The solution of the IVP is

$$y(t) = e^{-2t}\left(-2\cos\left(\sqrt{8}\,t\right) - \sqrt{2}\sin\left(\sqrt{8}\,t\right)\right).$$

Here, the amplitude phase form (3.39) is

$$y(t) = Ae^{-2t}\cos(\sqrt{8}\,t - \delta),$$

where

$$\left\{\begin{array}{r} -2 = A\cos\delta \\ -\sqrt{2} = A\sin\delta \end{array}\right\},$$

hence $A = \sqrt{(-2)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{6}$ and $\tan\delta = \frac{-\sqrt{2}}{-2} = \frac{1}{\sqrt{2}}$. Because $(c_1, c_2) = (-2, -\sqrt{2})$ is in the third quadrant, $\delta = \pi + \arctan\dfrac{1}{\sqrt{2}}$. The IVP's solution in amplitude phase form is

$$y(t) = \sqrt{6}\,e^{-2t}\cos\left(\sqrt{8}\,t - \pi - \arctan\frac{1}{\sqrt{2}}\right).$$

3.3.8.23. Note that $m = 1$.
(a) Ex. $b = 3, k = 2$ would be in the overdamped case, because then $b^2 - 4mk = 3^2 - 4\cdot 1\cdot 2 = 1 > 0$
(b) Ex. $b = 3, k = 3$ would be in the underdamped case because then $b^2 - 4mk = 3^2 - 4\cdot 1\cdot 3 = -3 < 0$
(c) Ex. $b = 2, k = 1$ would be in the critically damped case because then $b^2 - 4mk = 2^2 - 4\cdot 1\cdot 1 = 0$

3.3.8.25. Both ODEs have $m = 1$ and $b = 2$. Figure 3.15's first graph shows positive damping but has at least two zeros, which is impossible in the overdamped or critically damped cases. So, Figure 3.15's first graph is in the underdamped case. So $4 = b^2 < 4mk_1 = 4\cdot 1\cdot k_1$, hence $k_1 > 1$. Only choice (e) satisfies this criterion.

Figure 3.15's second graph also shows positive damping and is in the underdamped case, so $4 = b^2 < 4mk = 4\cdot 1\cdot k_2$, hence $k_2 > 1$. All but choice (c) satisfy this criterion.

Final conclusion: only pair (e) could conceivably give the graphs shown in Figure 3.15.

3.3.8.27. We are given that $m = 2\,kg$ and that is would stretch a (implicitly, vertical) spring $\ell = 0.392\,m$. Because $|F_{spring}| = k\ell = weight = mg$ and $g \approx 9.81\,m/s^2$, we have approximately

$$k(0.392) = (2)(9.81)$$

hence approximately $k = 50\,N/m$. Concerning the coordinate system, use displacement downward from the equilibrium position to give $y > 0$ and the equilibrium position to be $y = 0$. So, the ICs are $y(0) = 0\,m$ and $\dot{y}(0) = -5\,m/s$. We are given that the damping force is $F = -20v$, where the velocity is $v = \dot{y}$.

The IVP is

$$\left\{\begin{array}{r} 2\ddot{y} + 20\dot{y} + 50y = 0 \\ y(0) = 0 \\ \dot{y}(0) = -5 \end{array}\right\}.$$

The ODE has characteristic equation $0 = 2s^2 + 20s + 50 = 2\big(s^2 + 10s + 25\big) = 2(s+5)^2$, so the roots are $s = -5, -5$. The solutions of the ODE are

$$y(t) = c_1 e^{-5t} + c_2\, t\, e^{-5t},$$

where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -5c_1 e^{-5t} + c_2(1-5t)\, e^{-5t}$$

The ICs require

$$\left\{ \begin{array}{rcl} 0 = y(0) = & c_1 \\ -5 = \dot{y}(0) = & -5c_1 + c_2 \end{array} \right\},$$

which implies $c_1 = 0$ and $c_2 = -5$. The solution of the IVP is the position of the mass as a function of time, namely

$$y(t) = -5\, t\, e^{-5t}$$

and

$$\dot{y}(t) = -5(1-5t)\, e^{-5t}.$$

The furthest from the equilibrium position that the object goes happens either at $t = 0$ or at a time when $y(t)$ has a local maximum or a local minimum, because $\lim_{t\to\infty} y(t) = 0$. But, because $y(0) = 0$, that is, the object was released form the equilibrium position, this will not happen at $t = 0$.

We have that $0 = \dot{y}(t) = -5(1-5t)\, e^{-5t}$ only at $t = 0.2$, and $y(0.2) = -5\,(0.2)\, e^{-5(0.2)} = e^{-1}$. The furthest from the equilibrium position the object can go is $\frac{1}{e}$ meters.

3.3.8.29. The solution corresponds to a second order LCCHODE whose characteristic equation has roots $s = -3 \pm i$, hence the characteristic equation is, or is equivalent to after dividing by a constant,

$$0 = (s+3)^2 + 1^2 = s^2 + 6s + 10.$$

We were given that the ODE has the form $\ddot{y} + b\dot{y} + ky = 0$, whose characteristic equation is $0 = s^2 + bs + k$. So, we need $b = 6$ and $k = 10$.

3.3.8.31. I does not oscillate infinitely often when the ODE is not in the underdamped case. The characteristic equation is $\frac{1}{2}\, s^2 + 10s + \frac{1}{C} = 0$, so the roots are

$$s = \frac{-10 \pm \sqrt{10^2 - 4 \cdot \frac{1}{2} \cdot \frac{1}{C}}}{2 \cdot \frac{1}{2}} = -10 \pm \sqrt{100 - \frac{2}{C}}\,.$$

So, I does not oscillate infinitely often $\iff 100 \ge \frac{2}{C} \iff C \ge \frac{1}{50}$.

So, the current I does not oscillate infinitely often $\iff C \ge \frac{1}{50}$.

3.3.8.33. The time between successive local maxima on the graph is the quasi-period

$$\mathsf{T} = \frac{2\pi}{\nu} = 4.000 - 2.000 = 2.000,$$

so $\nu = \pi$. The roots of the characteristic equation for $\ddot{y} + p\dot{y} + qy = 0$ are

$$\alpha \pm i\nu = s = -\frac{p}{2} \pm i\,\frac{\sqrt{4q - p^2}}{2}.$$

The logarithmic decrement is

$$\frac{2\pi\alpha}{\nu} = D \triangleq \ln\big(y(t_2)/y(t_1)\big) = \ln\frac{3.9876}{5.1234},$$

so

$$\alpha = \frac{\nu D}{2\pi} = \frac{\pi \ln \frac{3.9876}{5.1234}}{2\pi} \approx -0.1253143675.$$

So,

$$-\frac{p}{2} = \alpha \approx -0.1253143675 \Rightarrow p \approx 0.2506287350$$

and

$$\pi = \nu = \frac{\sqrt{4q-p^2}}{2} \Rightarrow (2\pi)^2 = 4q - p^2 \quad \Rightarrow q = \frac{1}{4}\left(4\pi^2 + p^2\right) \approx \frac{1}{4}\left(4\pi^2 + 0.2506287350^2\right)$$

so

$$q \approx 9.885308092.$$

So, the ODE is approximately

$$\ddot{y} + 0.2506287350\,\dot{y} + 9.885308092\,y = 0.$$

3.3.8.35. (a) a1 is the only correct choice because of the Principle of Linear Superposition for the homogeneous ODE $\ddot{y} + p(t)\dot{y} + q(t)y = 0$. Note that neither a2 nor a3 can be true if a1 is true.

(b) b3 is the only correct choice because without knowing if $W(y_1, y_2)$ is non-zero at at least one value of t in $\mathcal{I}$ we cannot establish that $c_1y_1(t) + c_2y_2(t)$ is a general solution of $(\star)$ on I

3.3.8.39. A critically damped ODE has general solution $y(t) = c_1e^{-\alpha t} + c_2\,t\,e^{-\alpha t}$ for some positive constant α. It follows that $\dot{y}(t) = -\alpha c_1 e^{-\alpha t} + c_2(1-\alpha t)e^{-\alpha t}$.

Because the system is released from rest, $\dot{y}(0) = 0$. The ICs require

$$\left\{\begin{array}{c} y_0 = y(0) = c_1 \\ 0 = \dot{y}(0) = -\alpha c_1 + c_2 \end{array}\right\},$$

for some constant y_0, which implies $c_1 = y_0$ and $c_2 = \alpha y_0$. the solution of the IVP is

$$y(t) = y_0 e^{-\alpha t}\left(1 + \alpha t\right)$$

and

$$\dot{y}(t) = -\alpha y_0 e^{-\alpha t} + \alpha y_0(1-\alpha t)e^{-\alpha t} = -\alpha^2\,t\,y_0 e^{-\alpha t}.$$

If $y_0 = 0$ then the system stays at rest for all $t \geq 0$. In this case, the maximum deviation from equilibrium occurs at $t = 0$ and is $|y_0| = 0$.

If $y_0 \neq 0$ it follows that $\dot{y}(t) \neq 0$ for all $t \geq 0$. Because $\lim_{t\to\infty} y(t) = 0$ and there is no critical point for $t \geq 0$, it follows that the maximum deviation from equilibrium occurs at $t = 0$ and is $|y_0|$.

3.3.8.41. (a) Define $y_1(t) = e^t$ and $y_2(t) = e^{t^2}$. To verify that they solve the ODE, we calculate

$$(1-2t)\ddot{y}_1 + (1+4t^2)\dot{y}_1 + (-2+2t-4t^2)y_1 = (1-2t)e^t + (1+4t^2)e^t + (-2+2t-4t^2)e^t$$

$$= \Big((1-2t) + (1+4t^2) + (-2+2t-4t^2)\Big)e^t \equiv 0 \cdot e^t \equiv 0$$

and $\dot{y}_2 = 2te^{t^2}$ and $\ddot{y}_2 = \left(2+4t^2\right)e^{t^2}$ imply

$$(1-2t)\ddot{y}_2 + (1+4t^2)\dot{y}_2 + (-2+2t-4t^2)y_2 = (1-2t)\left(2+4t^2\right)e^{t^2} + (1+4t^2)2te^{t^2} + (-2+2t-4t^2)e^{t^2}$$

$$= \left((1-2t)(2+4t^2) + (1+4t^2)2t + (-2+2t-4t^2)\right)e^{t^2}$$

$$= \left(-8t^3 + 4t^2 - \cancel{4t} + 2 + \cancel{2t} + 8t^3 - 2 + \cancel{2t} - 4t^2\right)e^{t^2} \cdot e^{t^2} \equiv 0.$$

(b) Because

$$W\big(y_1(t), y_2(t)\big) = \begin{vmatrix} e^t & e^{t^2} \\ e^t & 2te^{t^2} \end{vmatrix} = (2t-1)e^{t+t^2} \neq 0,$$

except at $t = \frac{1}{2}$, it follows from Theorem 3.13 that $\left\{e^t, e^{t^2}\right\}$ is a complete set of basic solutions on any open interval $\mathcal{I}$ that does *not* include $t = \frac{1}{2}$.

(c) Let $y(t) = c_1 e^t + c_2 e^{t^2}$. It follows that $\dot{y}(t) = c_1 e^t + c_2\, 2\,t\,e^{t^2}$. The ICs require

$$\left\{\begin{array}{c} 5 = y(\frac{1}{2}) = c_1 e^{1/2} + c_2 e^{1/4} \\ \\ -3 = \dot{y}(\frac{1}{2}) = c_1 e^{1/2} + c_2 e^{1/4} \end{array}\right\},$$

which would require $5 = -3$. The system of equations for c_1, c_2 is inconsistent, that is, has no solution. So, there is no solution of this IVP.

(d) Our difficulty in part (c) of not being able to solve the IVP does not contradict the Existence and Uniqueness conclusion of Theorem 3.8 because the ODE, when written in standard form,

$$\ddot{y} + \frac{1+4t^2}{1-2t}\,\dot{y} + \frac{-2+2t-4t^2}{1-2t}\,y = 0,$$

does not satisfy the hypothesis that both of the coefficients $p(t)$ and $q(t)$ must be continuous on some open interval containing the initial time $t = \frac{1}{2}$.

3.3.8.43. $\ddot{y} - \omega^2 y = 0$ has characteristic equation $s^2 - \omega^2 = 0$, which has roots $s = \pm\omega$. The general solution is

$$y(t) = c_1 e^{\omega t} + c_2 e^{-\omega t},$$

where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = \omega\big(c_1 e^{\omega t} - c_2 e^{-\omega t}\big),$$

The boundary condition (BC) requires

$$0 = \dot{y}(L) = \omega\big(c_1 e^{\omega L} - c_2 e^{-\omega L}\big).$$

Recall that the problem assumed that ω is a positive constant.

The BC is satisfied if $c_1 = c_2 = 0$. If, instead $c_1 \neq 0$ or $c_2 \neq 0$, then we need $c_1 e^{\omega L} = c_2 e^{-\omega L}$. Multiply both sides by $e^{\omega L}$ to get $c_1 e^{2\omega L} = c_2$. The solutions that satisfy the BC in this case are

$$y(t) = c_1\big(e^{\omega t} + e^{2\omega L} e^{-\omega t}\big),$$

where c_1 is an arbitrary constant.

This is enough to finish the problem, but we can also go on to get a nicer looking formula: This can be rewritten as

$$y(t) = 2c_1 e^{\omega L} \cdot \frac{1}{2}\left(e^{-\omega L} e^{\omega t} + e^{\omega L} e^{-\omega t}\right),$$

that is, solutions can be written in the form

$$y(t) = c \cdot \frac{e^{\omega(L-t)} + e^{-\omega(L-t)}}{2} = c\,\cosh\big(\omega(L-t)\big),$$

where $c \triangleq 2c_1 e^{\omega L}$ is an arbitrary constant.

3.3.8.45. We calculate the Wronskian

$$W\big(\alpha y_1(t) - 3y_2(t), y_1(t) - \alpha y_2(t)\big) = \begin{vmatrix} \alpha y_1 - 3y_2 & y_1 - \alpha y_2 \\ \alpha y_1{}' - 3y_2{}' & y_1{}' - \alpha y_2{}' \end{vmatrix} = (\alpha y_1 - 3y_2)(y_1{}' - \alpha y_2{}') - (y_1 - \alpha y_2)(\alpha y_1{}' - 3y_2{}')$$

$$= \cancel{\alpha y_1 y_1{}'} - 3y_2 y_1{}' - \alpha^2 y_1 y_2{}' + \cancel{3\alpha y_2 y_2{}'} - \cancel{\alpha y_1 y_1{}'} + \alpha^2 y_2 y_1{}' + 3y_1 y_2{}' - \cancel{3\alpha y_2 y_2{}'} = (-\alpha^2 + 3)(y_1 y_2{}' - y_2 y_1{}')$$
$$= (-\alpha^2 + 3)W\big(y_1, y_2\big),$$

so $\{\alpha y_1(t) - 3y_2(t), y_1(t) - \alpha y_2(t)\}$ is also a compete set of basic solutions on I exactly when $-\alpha^2 + 3 \neq 0$, that is, for all α *except* $\alpha = \pm\sqrt{3}$.

3.3.8.47. Calculus I results say that the relative maxima of $y(t) = Ae^{\alpha t}\cos(\nu t - \delta)$ occur where $\dot{y}(t) = 0$ and alternate with relative minima, unless $\dot{y}(t) = 0$ does not change sign as t passes through some critical point.

We calculate

$$\dot{y}(t) = A\big(\alpha e^{\alpha t}\cos(\nu t-\delta) - \nu e^{\alpha t}\sin(\nu t-\delta)\big) = Ae^{\alpha t}\big(\alpha\cos(\nu t-\delta) - \nu\sin(\nu t-\delta)\big) = Ae^{\alpha t}\rho\cos\big((\nu t-\delta)-\eta\big)$$

where we use polar coordinates, that is, the amplitude phase form, to express

$$\alpha\cos(\nu t-\delta) - \nu\sin(\nu t-\delta) = \rho\cos\big((\nu t-\delta)-\eta\big).$$

This requires

$$\left\{\begin{array}{l} \alpha = \rho\cos\eta \\ \nu = -\rho\sin\eta \end{array}\right\},$$

ao $\rho = \sqrt{\alpha^2+\nu^2}$ and $\tan\eta = -\frac{\nu}{\alpha}$. Note that both $\alpha \neq 0$ and $\nu \neq 0$ are assumed.

Because $\dot{y}(t) = A\rho e^{\alpha t}\cos(\nu t - \delta - \eta)$, the critical points are at t_n satisfying

$$\nu t_n - \delta - \eta = \Big(n - \frac{1}{2}\Big)\pi,$$

that is,

$$t_n = \frac{1}{\nu}\left(\delta + \eta + \Big(n - \frac{1}{2}\Big)\pi\right),$$

where n are integers. From this, it follows that successive critical points are at a distance apart equal to

$$t_{n+1} - t_n = \frac{1}{\nu}\bigg(\delta+\eta+\Big((n+1)-\frac{1}{2}\Big)\pi\bigg) - \frac{1}{\nu}\bigg(\delta+\eta+\Big(n-\frac{1}{2}\Big)\pi\bigg) = \frac{\pi}{\nu}.$$

Also, the trigonometric function $\cos\theta$ changes sign as θ passes through its successive zeros, so $y'(t)$ changes sign as t passes through each critical point.

Because of the changes of sign, it follows that relative maxima and relative minima alternate successively. So, consecutive relative maxima differ by

$$t_{n+2} - t_n = 2\cdot\frac{\pi}{\nu} = \frac{2\pi}{\nu},$$

as we were asked to show.

Section 3.4.4

3.4.4.1. The characteristic polynomial, $\mathcal{P}(s) \triangleq s^3 + s^2 - 2$, has the easy to find root $s = 1$ because $\mathcal{P}(1) = 0$. We factor

$$0 = \mathcal{P}(s) = s^3 + s^2 - 2 = (s-1)(s^2 + 2s + 2).$$

Because $s^2 + 2s + 2 = (s+1)^2 + 1$ the three roots of the characteristic polynomial are

$$s = 1, -1 \pm i.$$

The general solution of the ODE is

$$y(t) = c_1 e^t + c_2 e^{-t}\cos t + c_3 e^{-t}\sin t,$$

where c_1, c_2, c_3 =arbitrary constants.

3.4.4.3. The characteristic equation is $(s+1)^3 = 0$, so the roots are $s = -1, -1, -1$. The general solution of the ODE is

$$y(t) = c_1 e^{-t} + c_2\, t\, e^{-t} + c_3\, t^2\, e^{-t},$$

where c_1, c_2, c_3 =arbitrary constants.

3.4.4.5. The characteristic equation is $0 = s^3 - 2s^2 - 15s = s(s^2 - 2s - 15) = s(s+3)(s-5)$, so the roots are $s = -3, 0, 5$. The general solution of the ODE is

$$y(t) = c_1 e^{-3t} + c_2 + c_3 e^{5t},$$

where c_1, c_2, c_3 =arbitrary constants.

The ICs require

$$\left\{ \begin{array}{rl} 0 = y(0) = & c_1 + c_2 + c_3 \\ 0 = \dot{y}(0) = & -3c_1 \qquad + 5c_3 \\ 1 = \ddot{y}(0) = & 9c_1 \qquad + 25c_3 \end{array} \right\},$$

which, using a calculator, implies that

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 5 \\ 9 & 0 & 25 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{120}\begin{bmatrix} 5 \\ -8 \\ 3 \end{bmatrix},$$

so the solution of the IVP is

$$y(t) = \frac{1}{120}\left(5e^{-3t} - 8 + 3e^{5t}\right).$$

There is no time constant, because not all solutions have $\lim_{t\to\infty} y(t) = 0$.

3.4.4.7. The characteristic equation is

$$0 = s^4 - 2s^2 - 3 = (s^2+1)(s^2-3)$$

so the roots are $s = -\sqrt{3}, \sqrt{3}, \pm i$. The general solution of the ODE is

$$y(t) = c_1 e^{-\sqrt{3}\,t} + c_2 e^{\sqrt{3}\,t} + c_3\cos t + c_4 \sin t,$$

where c_1, c_2, c_3, c_4 =arbitrary constants. It follows that

$$\dot{y}(t) = -\sqrt{3}\, c_1 e^{-\sqrt{3}\,t} + \sqrt{3}\, c_2 e^{\sqrt{3}\,t} - c_3 \sin t + c_4 \cos t,$$

$$\ddot{y}(t) = 3c_1 e^{-\sqrt{3}\,t} + 3c_2 e^{\sqrt{3}\,t} - c_3 \cos t - c_4 \sin t,$$

and
$$\dddot{y}(t) = -3^{3/2}c_1 e^{-\sqrt{3}\,t} + 3^{3/2}\sqrt{3}\,c_2 e^{\sqrt{3}\,t} + c_3 \sin t - c_4 \cos t.$$
The ICs require
$$\left\{\begin{array}{rcrrrr} 0 = y(0) = & & c_1 & +\, c_2 & +\, c_3 & \\ 0 = \dot{y}(0) = & & -\sqrt{3}\,c_1 & +\, \sqrt{3}\,c_2 & & +\, c_4 \\ 0 = \ddot{y}(0) = & & 3c_1 & +\, 3c_2 & -\, c_3 & \\ -2 = \dddot{y}(0) = & & -3\sqrt{3}\,c_1 & +\, 3\sqrt{3}\,c_2 & & -\, c_4 \end{array}\right\},$$
which, using a calculator, implies that
$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -\sqrt{3} & \sqrt{3} & 0 & 1 \\ 3 & 3 & -1 & 0 \\ -3\sqrt{3} & 3\sqrt{3} & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1/(4\sqrt{3}) \\ 1/(4\sqrt{3}) \\ 0 \\ 1/2 \end{bmatrix},$$
so the solution of the IVP is
$$y(t) = \frac{1}{4\sqrt{3}}\,e^{-\sqrt{3}\,t} - \frac{1}{4\sqrt{3}}\,e^{\sqrt{3}\,t} + \frac{1}{2}\sin t.$$

3.4.4.9. The characteristic equation is $0 = s^4 - 8s^3 + 17s^2 - 8s + 16$. We are given that it has roots $s = \pm i$ because we were given that $y = \sin t$ is one of the solutions of the ODE.

It follows that $(s-i)(s+i) = (s^2+1)$ is a factor of the characteristic polynomial. We calculate that the characteristic equation is
$$0 = s^4 - 8s^3 + 17s^2 - 8s + 16 = (s^2+1)(s^2 - 8s + 16) = (s^2+1)(s-4)^2$$
so the roots are $s = 4, 4, \pm i$. The general solution of the ODE is
$$y(t) = c_1 e^{4t} + c_2\, t\, e^{4t} + c_3 \cos t + c_4 \sin t,$$
where c_1, c_2, c_3, c_4 =arbitrary constants.

3.4.4.11. (a) The characteristic equation is
$$0{=}s^3 + 8{=}(s+2)(s^2 - 2s + 4) = (s+2)\big((s-1)^2 + 3\big)$$
so the roots are $s = -2,\ 1 \pm i\,\sqrt{3}$. The general solution of the ODE is
$$y(t) = c_1 e^{-2t} + c_2\, e^t \cos(\sqrt{3}\,t) + c_3\, e^{-t} \sin(\sqrt{3}\,t),$$
where c_1, c_2, c_3 =arbitrary constants.

(b) The characteristic equation is
$$0 = s^3 - 2 = (s - \sqrt[3]{2})\big(s^2 + \sqrt[3]{2}\,s + \sqrt[3]{4}\big) = (s - \sqrt[3]{2})\left(\left(s + \frac{\sqrt[3]{2}}{2}\right)^2 + \sqrt[3]{4} - \left(\frac{\sqrt[3]{2}}{2}\right)^2\right)$$
$$= (s - \sqrt[3]{2})\left(\left(s + \frac{\sqrt[3]{2}}{2}\right)^2 + \sqrt[3]{4} - \frac{\sqrt[3]{4}}{4}\right)$$
so the roots are
$$s = \sqrt[3]{2},\ -\frac{\sqrt[3]{2}}{2} \pm i\,\frac{\sqrt[3]{2}\cdot\sqrt{3}}{2}.$$
The general solution of the ODE is
$$y(t) = c_1\, e^{\sqrt[3]{2}\,t} + c_2\, e^{-\sqrt[3]{2}\,t/2} \cos\left(\frac{\sqrt[3]{2}\cdot\sqrt{3}t}{2}\right) + c_3\, e^{-\sqrt[3]{2}t/2} \sin\left(\frac{\sqrt[3]{2}\cdot\sqrt{3}t}{2}\right),$$

where c_1, c_2, c_3 =arbitrary constants.

3.4.4.13. For $n = 3$, expanding the determinant along the first row gives

$$W\big(y_1(t), y_2(t), y_3(t)\big) = \begin{vmatrix} y_1 & y_2 & y_3 \\ \dot y_1 & \dot y_2 & \dot y_3 \\ \ddot y_1 & \ddot y_2 & \ddot y_3 \end{vmatrix} = y_1 \begin{vmatrix} \dot y_2 & \dot y_3 \\ \ddot y_2 & \ddot y_3 \end{vmatrix} - y_2 \begin{vmatrix} \dot y_1 & \dot y_3 \\ \ddot y_1 & \ddot y_3 \end{vmatrix} + y_3 \begin{vmatrix} \dot y_1 & \dot y_2 \\ \ddot y_1 & \ddot y_2 \end{vmatrix}$$

$$= y_1(\dot y_2\ddot y_3 - \dot y_3\ddot y_2) - y_2(\dot y_1\ddot y_3 - \dot y_3\ddot y_1) + y_3(\dot y_1\ddot y_2 - \dot y_2\ddot y_1).$$

Take the time derivative to get

$$\begin{aligned}
\dot W(t) =& \dot y_1(\dot y_2\ddot y_3 - \dot y_3\ddot y_2) - \dot y_2(\dot y_1\ddot y_3 - \dot y_3\ddot y_1) + \dot y_3(\dot y_1\ddot y_2 - \dot y_2\ddot y_1) + y_1(\cancel{\ddot y_2\ddot y_3} + \dot y_2\dddot y_3 - \cancel{\ddot y_3\ddot y_2} - \dot y_3\dddot y_2) \\
& - y_2(\cancel{\ddot y_1\ddot y_3} + \dot y_1\dddot y_3 - \cancel{\ddot y_3\ddot y_1} - \dot y_3\dddot y_1) + y_3(\cancel{\ddot y_1\ddot y_2} + \dot y_1\dddot y_2 - \cancel{\ddot y_2\ddot y_1} - \dot y_2\dddot y_1) \\
=& \begin{vmatrix} \dot y_1 & \dot y_2 & \dot y_3 \\ \dot y_1 & \dot y_2 & \dot y_3 \\ \ddot y_1 & \ddot y_2 & \ddot y_3 \end{vmatrix} + y_1\big(\dot y_2(-p_1(t)\ddot y_3 - \cancel{p_2(t)\dot y_3} - \cancel{p_3(t)y_3}) - \dot y_3(-p_1(t)\ddot y_2 - \cancel{p_2(t)\dot y_2} - \cancel{p_3(t)y_2})\big) \\
& + y_2\big(\dot y_1(-p_1(t)\ddot y_3 - \cancel{p_2(t)\dot y_3} - \cancel{p_3(t)y_3}) - \dot y_3(-p_1(t)\ddot y_1 - \cancel{p_2(t)\dot y_1} - \cancel{p_3(t)y_1})\big) \\
& + y_3\big(\dot y_1(-p_1(t)\ddot y_2 - \cancel{p_2(t)\dot y_2} - \cancel{p_3(t)y_2}) - \dot y_2(-p_1(t)\ddot y_1 - \cancel{p_2(t)\dot y_1} - \cancel{p_3(t)y_1})\big) \\
=& \, 0 + \begin{vmatrix} \dot y_1 & \dot y_2 & \dot y_3 \\ \dot y_1 & \dot y_2 & \dot y_3 \\ -p_1(t)\ddot y_1 & -p_1(t)\ddot y_2 & -p_1(t)\ddot y_3 \end{vmatrix} = -p_1(t) \begin{vmatrix} \dot y_1 & \dot y_2 & \dot y_3 \\ \dot y_1 & \dot y_2 & \dot y_3 \\ \ddot y_1 & \ddot y_2 & \ddot y_3 \end{vmatrix} = -p_1(t)W(t).
\end{aligned}$$

Section 3.5.1

3.5.1.1. The characteristic equation is $0 = n(n-1) + 5n - 2 = n^2 + 4n - 2$, so the roots are

$$n = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-2)}}{2} = -2 \pm \sqrt{6}.$$

The general solution of the ODE is

$$y(r) = c_1 r^{-2+\sqrt{6}} + c_2 r^{-2-\sqrt{6}}\,,$$

where c_1, c_2 =arbitrary constants.

3.5.1.3. The characteristic equation is $0 = n(n-1) + n + 4 = n^2 + 4$, so the roots are $n = \pm\, i\, 2$. The general solution of the ODE is

$$y(r) = c_1 \cos(2 \ln r) + c_2 \sin(2 \ln r),$$

where c_1, c_2 =arbitrary constants.

3.5.1.5. The characteristic equation is $0 = n(n-1) + 5n + 4 = n^2 + 4n + 4 = (n+2)^2$, so the roots are $n = -2, -2$. The general solution of the ODE is

$$y(r) = c_1 r^{-2} + c_2 r^{-2} \ln r\,,$$

where c_1, c_2 =arbitrary constants.

3.5.1.7. The characteristic equation is $0 = n(n-1) - 2 = n^2 - n - 2 = (n+1)(n-2)$, so the roots are $n = -1, 2$. The general solution of the ODE is

$$y(r) = c_1 r^{-1} + c_2 r^2\,,$$

where c_1, c_2 =arbitrary constants. It follows that $y'(r) = -c_1 r^{-2} + 2c_2 r$. The ICs require

$$\left\{ \begin{array}{rcl} 0 = y(e) &=& e^{-1} c_1 + e^2 c_2 \\ 11 = y'(e) &=& -e^{-2} c_1 + 2\, e\, c_2 \end{array} \right\},$$

which implies

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{-1} & e^2 \\ -e^{-2} & 2e \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 11 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e & -e^2 \\ e^{-2} & e^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 11 \end{bmatrix} = \frac{11}{3} \begin{bmatrix} -e^2 \\ e^{-1} \end{bmatrix},$$

so the solution of the IVP is

$$y(r) = \frac{11}{3}(-e^2 r^{-1} + e^{-1} r^2).$$

3.5.1.9. The characteristic equation is $0 = n(n-1) - n + 5 = n^2 - 2n + 5 = (n-1)^2 + 4$, so the roots are $n = 1 \pm\, i\, 2$. The general solution of the ODE is

$$y(r) = c_1 r \cos(2 \ln r) + c_2 r \sin(2 \ln r),$$

where c_1, c_2 =arbitrary constants. It follows that

$$y'(r) = c_1 \cos(2 \ln r) - c_1 r \sin(2 \ln r) \cdot \frac{2}{r} + c_2 \sin(2 \ln r) + c_2 r \cos(2 \ln r) \cdot \frac{2}{r}$$

that is,

$$y'(r) = (c_1 + 2c_2) \cos(2 \ln r) + (c_2 - 2c_1) \sin(2 \ln r).$$

Using the fact that $\ln 1 = 0$, the ICs require

$$\left\{ \begin{array}{l} -2 = y(1) \ = c_1 \\ \quad 0 = y'(1) = c_1 + 2c_2 \end{array} \right\},$$

which implies $c_1 = -2$ and $c_2 = 1$. So, the solution of the IVP is

$$y(r) = -2r\cos(2\ln r) + r\sin(2\ln r).$$

3.5.1.11. The characteristic equation is $0 = n(n-1) + n - (2m)^2 = n^2 - (2m)^2 = (n-2m)(n+2m)$, so the roots are $n = \pm\, 2m$.
(a) *Case 1*: If $m = 0$, the root $n = 0$ is repeated. Using $r^0 \equiv 1$, the general solution of the ODE is

$$y(r) = c_1 + c_2 \ln r,$$

where c_1, c_2 =arbitrary constants.
(b) *Case 2*: If the integer $m \geq 1$, the general solution of the ODE is

$$y(r) = c_1 r^{2m} + c_2 r^{-2m},$$

where c_1, c_2 =arbitrary constants.

Chapter Four

Section 4.1.5

4.1.5.1. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 5s + 6 = (s + 3)(s + 2)$
$\Rightarrow \mathsf{L}_1 = -3, -2$
$f(t) = 3e^{-t} \Rightarrow \mathsf{L}_2 = -1 \Rightarrow$ Superlist is $\mathsf{L} = -3, -2, -1$
$\Rightarrow y(t) = c_1e^{-3t} + c_2e^{-2t} + c_3e^{-t} \Rightarrow y_p(t) = Ae^{-t}$, where A is a constant to be determined:

$$3e^{-t} = \ddot{y}_p + 5\dot{y}_p + 6y_p = Ae^{-t} - 5Ae^{-t} + 6Ae^{-t} = 2Ae^{-t}$$

$\Rightarrow A = \frac{3}{2} \Rightarrow \quad y_p(t) = \dfrac{3}{2}\, e^{-t}$.
The general solution of the ODE is

$$y(t) = c_1e^{-3t} + c_2e^{-2t} + \frac{3}{2}\, e^{-t},$$

where c_1, c_2 =arbitrary constants.

4.1.5.3. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 5s + 6 = (s + 3)(s + 2)$
$\Rightarrow \mathsf{L}_1 = -3, -2$
$f(t) = 2e^{-3t} \Rightarrow \mathsf{L}_2 = -3 \Rightarrow$ Superlist is $\mathsf{L} = -3, -2, -3$
$\Rightarrow y(t) = c_1e^{-3t} + c_2e^{-2t} + c_3\, t\, e^{-3t} \Rightarrow y_p(t) = A\, t\, e^{-3t}$, where A is a constant to be determined:
$\Rightarrow \dot{y}_p(t) = A(1 - 3t)e^{-3t} \Rightarrow \ddot{y}_p(t) = A(-6 + 9t)e^{-3t}$

$$2e^{-3t} = \ddot{y}_p + 5\dot{y}_p + 6y_p = A(-6 + 9t)e^{-3t} + 5A(1 - 3t)e^{-3t} + 6A\, t\, e^{-3t} = -A\, e^{-3t}$$

$\Rightarrow A = -2 \Rightarrow y_p(t) = -2\, t\, e^{-3t}$.
Alternatively, we could find A using the shift theorem:

$$2e^{-3t} = \ddot{y}_p + 5\dot{y}_p + 6y_p = (D+2)(D+3)[\, A\, t\, e^{-3t}\,] = (D+2)\big[\,(D+3)[\, A\, t\, e^{-3t}\,]\,\big] = (D+2)[\, Ae^{-3t}\,] = -A\, e^{-3t},$$

hence $A = -2$.

The general solution of the ODE is

$$y(t) = -2\, t\, e^{-3t} + c_1e^{-3t} + c_2e^{-2t},$$

where c_1, c_2 =arbitrary constants.

4.1.5.5. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 - 1 \Rightarrow \mathsf{L}_1 = -1, 1$
$f(t) = e^{-t} + 5e^{-2t} \Rightarrow \mathsf{L}_2 = -2, -1 \Rightarrow$ Superlist is $\mathsf{L} = -1, 1, -2, -1$
$\Rightarrow y(t) = c_1e^{-t} + c_2e^{t} + c_3e^{-2t} + c_4\, t\, e^{-t}$ t $\Rightarrow y_p(t) = A\, e^{-2t} + B\, t\, e^{-t}$, where A, B are constants to be determined:

$\Rightarrow \dot{y}_p(t) = -2A\, e^{-2t} + B(1 - t)e^{-t} \quad \Rightarrow \ddot{y}_p(t) = 4A\, e^{-2t} + B(-2 + t)e^{-t}$

$$e^{-t} + 5e^{-2t} = \ddot{y}_p - y_p = 4A\, e^{-2t} + B(-2 + t)e^{-t} - A\, e^{-2t} - B\, t\, e^{-t} = -2Be^{-t} + 3Ae^{-2t}$$

$\Rightarrow B = -\frac{1}{2}$ and $A = \frac{5}{3} \Rightarrow y_p(t) = \dfrac{5}{3}\, e^{-2t} - \dfrac{1}{2}\, t\, e^{-t}$.
The general solution of the ODE is

$$y(t) = \frac{5}{3}\, e^{-2t} - \frac{1}{2}\, t\, e^{-t} + c_1e^{-t} + c_2e^{t},$$

where c_1, c_2 =arbitrary constants.

4.1.5.7. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + s - 12 = (s+4)(s-3)$ $\Rightarrow \mathsf{L}_1 = -4, 3$

$f(t) = 5e^{-4t} \Rightarrow \mathsf{L}_2 = -4 \Rightarrow$ Superlist is $\mathsf{L} = -4, 3, -4$

$\Rightarrow y(t) = c_1 e^{-4t} + c_2 e^{3t} + c_3\, t\, e^{-4t} \quad \Rightarrow y_p(t) = A\, t\, e^{-4t}$, where A is a constant to be determined:

$\Rightarrow \dot{y}_p(t) = A(1-4t)e^{-4t} \quad \Rightarrow \ddot{y}_p(t) = A(-8+16t)e^{-4t}$

$$5e^{-4t} = \ddot{y}_p + \dot{y}_p - 12y_p = A(-8+16t)e^{-4t} + A(1-4t)e^{-4t} - 12A\, t\, e^{-4t} = -7Ae^{-4t}$$

$\Rightarrow A = -\frac{5}{7} \Rightarrow y_p(t) = -\dfrac{5}{7}\, t\, e^{-4t}$.

The general solution of the ODE is

$$y(t) = -\frac{5}{7}\, t\, e^{-4t} + c_1 e^{-4t} + c_2 e^{3t},$$

where c_1, c_2 =arbitrary constants.

4.1.5.9. We are given a particular solution $y_p(t) = -e^{-2t}\cos(e^t)$. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 3s + 2 = (s+2)(s+1) \Rightarrow y_h(t) = c_1 e^{-2t} + c_2 e^{-t}$

$\Rightarrow$ The general solution of the ODE is

$$y(t) = y_p(t) + y_h(t) = -e^{-2t}\cos(e^t) + c_1 e^{-2t} + c_2 e^{-t},$$

where c_1, c_2 =arbitrary constants. It follows from the chain rule that

$$\dot{y}(t) = 2e^{-2t}\cos(e^t) + e^{-2t}\sin(e^t)\cdot e^t - 2c_1 e^{-2t} - c_2 e^{-t}.$$

The ICs require

$$\left\{ \begin{array}{l} 0 = y(0) = -\cos(1) \qquad\qquad + \quad c_1 + c_2 \\ 0 = \dot{y}(0) = 2\cos(1) + \sin(1) - 2c_1 - c_2 \end{array} \right\},$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \cos(1) \\ -2\cos(1) - \sin(1) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \cos(1) \\ -2\cos(1) - \sin(1) \end{bmatrix} = \begin{bmatrix} \cos(1) + \sin(1) \\ -\sin(1) \end{bmatrix}$$

The solution of the IVP is

$$y(t) = -e^{-2t}\cos(e^t) + \big(\cos(1) + \sin(1)\big)e^{-2t} - \sin(1)e^{-t}.$$

4.1.5.11. We are given a particular solution $y_p(t) = -\dfrac{1}{9}\, te^t + \dfrac{1}{6}\, t^2 e^t$. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + s - 2 = (s+2)(s-1) \Rightarrow y_h(t) = c_1 e^{-2t} + c_2 e^t$

$\Rightarrow$ The general solution of the ODE is

$$y(t) = y_p(t) + y_h(t) = -\frac{1}{9}\, te^t + \frac{1}{6}\, t^2 e^t + c_1 e^{-2t} + c_2 e^t,$$

where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -\frac{1}{9}e^t - \frac{1}{9}\, t\, e^t + \frac{1}{3}\, te^t + \frac{1}{6}t^2 e^t - 2c_1 e^{-2t} + c_2 e^t.$$

The ICs require

$$\left\{ \begin{array}{l} \;\;\,0 = y(0) = \;\;0 \;+\; c_1 \;+ c_2 \\ -2 = \dot{y}(0) = -\frac{1}{9} - 2c_1 + c_2 \end{array} \right\},$$

so
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -17/9 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -17/9 \end{bmatrix} = \frac{17}{27}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
The solution of the IVP is
$$y(t) = -\frac{1}{9}\,te^t + \frac{1}{6}\,t^2e^t + \frac{17}{27}\big(e^{-2t} - e^t\big).$$

4.1.5.13. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 - 1 \Rightarrow \mathsf{L}_1 = -1, 1$
$f(t) = e^{-2t} \Rightarrow \mathsf{L}_2 = -2 \Rightarrow$ Superlist is $\mathsf{L} = -1, 1, -2$
$\Rightarrow y(t) = c_1e^{-t} + c_2e^t + c_3e^{-2t} \Rightarrow y_p(t) = Ae^{-2t}$, where A is a constant to be determined:
$$e^{-2t} = \ddot{y}_p(t) - y_p(t) = 4Ae^{-2t} - Ae^{-2t} = 3Ae^{-2t} \quad\Rightarrow\quad A = \frac{1}{3} \quad\Rightarrow\quad y_p(t) = \frac{1}{3}\,e^{-2t}.$$
The general solution of the ODE is
$$y(t) = \frac{1}{3}\,e^{-2t} + c_1e^{-t} + c_2e^t,$$
where c_1, c_2 =arbitrary constants. It follows that
$$\dot{y}(t) = -\frac{2}{3}e^{-2t} - c_1e^{-t} + c_2e^t.$$

The ICs require
$$\left\{\begin{array}{rl} 0 = y(0) = & \frac{1}{3} + c_1 + c_2 \\ \\ 0 = \dot{y}(0) = & -\frac{2}{3} - c_1 + c_2 \end{array}\right\},$$
so
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \frac{1}{3}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$
The solution of the IVP is
$$y(t) = \frac{1}{3}\,e^{-2t} - \frac{1}{2}\,e^{-t} + \frac{1}{6}\,e^t.$$

4.1.5.15. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + s + 5 = \big(s + \frac{1}{2}\big)^2 + \frac{19}{4}$
$\Rightarrow \mathsf{L}_1 = -\frac{1}{2} \pm i\,\frac{\sqrt{19}}{2}$
$f(t) = 10 \Rightarrow \mathsf{L}_2 = 0 \Rightarrow$ Superlist is $\mathsf{L} = -\frac{1}{2} \pm i\,\frac{\sqrt{19}}{2}, 0$
$\Rightarrow y(t) = c_1e^{-t/2}\cos\big(\frac{\sqrt{19}}{2}\,t\big) + c_2e^{-t/2}\sin\big(\frac{\sqrt{19}}{2}\,t\big) + c_3 \Rightarrow y_p(t) = A$, where A is a constant to be determined:
$$10 = \ddot{y}_p(t) + \dot{y}_p(t) + 5y_p(t) = 0 + 0 + 5A \quad\Rightarrow\quad A = 2 \quad \Rightarrow y_p(t) = 2.$$

The general solution of the ODE is $y(t) = 2 + c_1e^{-t/2}\cos\Big(\frac{\sqrt{19}}{2}\,t\Big) + c_2e^{-t/2}\sin\Big(\frac{\sqrt{19}}{2}\,t\Big)$, where c_1, c_2 =arbitrary constants. It follows that
$$\dot{y}(t) = \Big(-\frac{1}{2}\,c_1 + \frac{\sqrt{19}}{2}\,c_2\Big)e^{-t/2}\cos\Big(\frac{\sqrt{19}}{2}\,t\Big) + \Big(-\frac{\sqrt{19}}{2}\,c_1 - \frac{1}{2}\,c_2\Big)e^{-t/2}\sin\Big(\frac{\sqrt{19}}{2}\,t\Big).$$

The ICs require
$$\left\{\begin{array}{l} 0 = y(0) = 2 + c_1 \\ \\ 0 = \dot{y}(0) = \ -\frac{1}{2}\,c_1 + \frac{\sqrt{19}}{2}\,c_2 \end{array}\right\},$$
so $c_1 = -2$ and $c_2 = -\frac{2}{\sqrt{19}}$. The solution of the IVP is
$$y(t) = 2 - 2e^{-t/2}\Big(\cos\Big(\frac{\sqrt{19}}{2}\,t\Big) + \frac{1}{\sqrt{19}}\,\cos\Big(\frac{\sqrt{19}}{2}\,t\Big)\Big).$$

4.1.5.17. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 2s + 5 = (s+1)^2 + 4$
$\Rightarrow \mathsf{L}_1 = -1 \pm 2i$

$f(t) = \sin 2t \Rightarrow \mathsf{L}_2 = \pm 2\,i \Rightarrow$ Superlist is $\mathsf{L} = -1 \pm 2i,\ \pm 2\,i$

$y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + c_3\cos 2t + c_4\sin 2t$

$\Rightarrow y_p(t) = A\cos 2t + B\sin 2t$, where A, B are constants to be determined:

$$\sin 2t = \ddot{y}_p + 2\dot{y}_p + 5y_p = -4A\cos 2t - 4B\sin 2t + 2(-2A\sin 2t + 2B\cos 2t) + 5A\cos 2t + 5B\sin 2t$$

$$= (A+4B)\cos 2t + (B-4A)\sin 2t$$

$$\Rightarrow \left\{\begin{array}{r} A + 4B = 0 \\ -4A + \ B = 1 \end{array}\right\}$$

so

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$\Rightarrow$ The steady state solution is

$$y_S(t) = y_p(t) = \frac{1}{17}(-4\cos 2t + \sin 2t),$$

because the terms $c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$ are transient no matter what are the values of the constants c_1, c_2.

The amplitude of the steady state solution is

$$Amplitude = \frac{1}{17}\sqrt{(-4)^2 + 1^2} = \frac{1}{\sqrt{17}}\,.$$

4.1.5.19. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 4s + 5 = (s+2)^2 + 1$
$\Rightarrow \mathsf{L}_1 = -2 \pm i$

$f(t) = \cos t \Rightarrow \mathsf{L}_2 = \pm i \Rightarrow$ Superlist is $\mathsf{L} = -2 \pm i,\ \pm i$

$\Rightarrow y(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t + c_3\cos t + c_4\sin t \quad \Rightarrow y_p(t) = A\cos t + B\sin t$, where A, B are constants to be determined:

$$\cos t = \ddot{y}_p + 4\dot{y}_p + 5y_p = -A\cos t - B\sin t + 4(-A\sin t + B\cos 2t) + 5A\cos t + 5B\sin t$$

$$= (4A+4B)\cos t + (4B-4A)\sin t$$

$$\Rightarrow \left\{\begin{array}{r} 4A + 4B = 1 \\ -4A + 4B = 0 \end{array}\right\}$$

so

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -4 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ The steady state solution is

$$y_S(t) = y_p(t) = \frac{1}{8}(\cos t + \sin t),$$

because the terms $c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$ are transient no matter what are the values of the constants c_1, c_2.

The amplitude of the steady state solution is

$$Amplitude = \frac{1}{8}\sqrt{1^2 + 1^2} = \frac{\sqrt{2}}{8} = \frac{1}{\sqrt{32}}\,.$$

4.1.5.21. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 2s + 5 = (s+1)^2 + 4$ $\Rightarrow \mathsf{L}_1 = -1 \pm 2i$

$f(t) = 3 \Rightarrow \mathsf{L}_2 = 0 \Rightarrow$ Superlist is $\mathsf{L} = -1 \pm 2i, 0$

$\Rightarrow y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + c_3 \quad \Rightarrow y_p(t) = A$, where A is a constant to be determined:

$$3 = \ddot{y}_p + 2\dot{y}_p + 5y_p = 0 + 0 + 5A$$

$\Rightarrow$ The steady state solution is

$$y_S(t) = y_p(t) = \frac{3}{5},$$

because the terms $c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$ are transient no matter what are the values of the constants c_1, c_2. So, even though we were given initial conditions we didn't need to satisfy them in order to answer the question that was asked!

The amplitude of the steady state solution is $Amplitude = \frac{3}{5}$.

4.1.5.23. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = 2s^2 + 4 = 2(s^2 + 2) \Rightarrow \mathsf{L}_1 = \pm\sqrt{2}\, i$

$f(t) = f_0 e^{-2t} \Rightarrow \mathsf{L}_2 = -2 \Rightarrow$ Superlist is $\mathsf{L} = \pm\sqrt{2}\, i,\, -2$

$\Rightarrow y(t) = c_1 \cos(\sqrt{2}\, t) + c_2 \sin(\sqrt{2}\, t) + c_3 e^{-2t} \quad \Rightarrow y_p(t) = Ae^{-2t}$, where A is a constant to be determined:

$$f_0 e^{-2t} = 2\ddot{y}_p + 4y_p = 8e^{-2t} + 4Ae^{-2t}$$

$\Rightarrow y_p(t) = \frac{f_0}{12}e^{-2t}$. Note that in this problem the particular solution is transient, not part of the steady state solution.

The general solution of the ODE is $y(t) = y_p(t) + y_h(t) = \dfrac{f_0}{12}e^{-2t} + c_1\cos(\sqrt{2}\, t) + c_2\sin(\sqrt{2}\, t)$. It follows that

$$\dot{y}(t) = -\frac{f_0}{6}e^{-2t} - \sqrt{2}\, c_1 \cos(\sqrt{2}\, t) + \sqrt{2}\, c_2 \sin(\sqrt{2}\, t).$$

The ICs require

$$\left\{ \begin{array}{l} 0 = y(0) = \ \ \frac{f_0}{12} + c_1 \\ \\ 0 = \dot{y}(0) = -\frac{f_0}{6} + \sqrt{2}\, c_2 \end{array} \right\},$$

so $c_1 = -\frac{f_0}{12}$ and $c_2 = \frac{f_0}{6\sqrt{2}}$. The solution of the IVP is

$$y(t) = \frac{f_0}{12}e^{-2t} - \frac{f_0}{12}\cos(\sqrt{2}\, t) + \frac{f_0}{6\sqrt{2}}\sin(\sqrt{2}\, t)$$

$\Rightarrow$ The steady state solution is

$$y_S(t) = \frac{f_0}{12}\big(-\cos(\sqrt{2}\, t) + \sqrt{2}\,\sin(\sqrt{2}\, t)\big),$$

because the term $\frac{f_0}{12}e^{-2t}$ is transient. Note that the "steady state solution" is not a solution of the nonhomogeneous ODE.

The amplitude of the steady state solution is

$$Amplitude = \frac{|f_0|}{12}\sqrt{(-1)^2 + (\sqrt{2})^2} = \frac{|f_0|}{4\sqrt{3}}\,.$$

4.1.5.25. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 - 2s + 2 = (s-1)^2 + 1$ $\Rightarrow \mathsf{L}_1 = 1 \pm i$

$f(t) = t \Rightarrow \mathsf{L}_2 = 0, 0 \Rightarrow$ Superlist is $\mathsf{L} = 1 \pm i,\, 0, 0$

$\Rightarrow y(t) = c_1 e^t \cos t + c_2 e^t \sin t + c_3 + c_4\, t \quad \Rightarrow y_p(t) = A + B\, t$, where A, B are constants to be determined:

$$t = \ddot{y}_p - 2\dot{y}_p + 2y_p = 0 - 2B + 2A + 2B\, t$$

$\Rightarrow 1 = 2B$ and $0 = 2A - 2B \Rightarrow B = \frac{1}{2},\ A = \frac{1}{2} \quad \Rightarrow y_p(t) = \dfrac{1}{2}(1+t)$.

The general solution of the ODE is

$$y(t) = y_p(t) + y_h(t) = \frac{1}{2}(1+t) + c_1 e^t \cos t + c_2 e^t \sin t.$$

It follows that

$$\dot{y}(t) = \frac{1}{2} + (c_1 + c_2)e^t \cos t + (-c_1 + c_2)e^t \sin t.$$

Because $\cos \pi = -1$ and $\sin \pi = 0$, the ICs require

$$\left\{ \begin{array}{l} 0 = y(\pi) = \frac{1}{2}(1+\pi) - c_1 e^{\pi} \\ \\ 0 = \dot{y}(\pi) = \frac{1}{2} - (c_1 + c_2)e^{\pi} \end{array} \right\}.$$

It follows that $c_1 = \frac{1}{2}(1+\pi)e^{-\pi}$ and $c_2 = -\frac{\pi}{2}e^{-\pi}$.
The solution of the IVP is

$$y(t) = \frac{1}{2}(1+t) + \frac{1}{2}(1+\pi)e^{t-\pi}\cos t - \frac{\pi}{2}e^{t-\pi}\sin t.$$

4.1.5.27. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^3 + 2s^2 + s + 2$. We are given that $\cos t$ is a solution of the corresponding LCCHODE so (s^2+1) is a factor of $\mathcal{P}(s)$. We factor

$$0 = \mathcal{P}(s) = (s^2+1)(s+2)$$

$\Rightarrow \mathsf{L}_1 = -2, \pm i$
$f(t) = 4 - 3e^{-2t} \Rightarrow \mathsf{L}_2 = -2, 0 \Rightarrow$ Superlist is $\mathsf{L} = -2, \pm i,\ -2, 0$
$\Rightarrow y(t) = c_1 e^{-2t} + c_2 \cos t + c_3 \sin t + c_4 + c_5\, t\, e^{-2t} \quad \Rightarrow y_p(t) = A + B\, t\, e^{-2t}$, where A, B are constants to be determined.

It follows that $\dot{y}_p(t) = B(1-2t)e^{-2t}$,
$\ddot{y}_p(t) = B(-4+4t)e^{-2t}$, and $\dddot{y}_p(t) = B(12-8t)e^{-2t}$.

$$\begin{aligned} 4 - 3e^{-2t} = \dddot{y}_p + 2\ddot{y}_p + \dot{y}_p + 2y_p &= B(12-8t)e^{-2t} + 2B(-4+4t)e^{-2t} + B(1-2t)e^{-2t} + 2A + 2B\, t\, e^{-2t} \\ &= 2A + 5Be^{-2t} \end{aligned}$$

$\Rightarrow A = 2,\ B = -\frac{3}{5} \Rightarrow y_p(t) = 2 - \dfrac{3}{5}\, t\, e^{-2t}$.

The general solution of the ODE is

$$y(t) = y_p(t) + y_h(t)$$

that is,

$$y(t) = 2 - \frac{3}{5}\, t\, e^{-2t} + c_1 e^{-2t} + c_2 \cos t + c_3 \sin t,$$

where $c_1, c_2, c_3 =$arbitrary constants.

4.1.5.29. We are given that $\ddot{y}_1 + 2\dot{y}_1 + 5y_1 = -10t + 11$ and that $\ddot{y}_2 + 2\dot{y}_2 + 5y_2 = -17\cos 2t$. It follows that

$$t - \frac{11}{10} + 2\cos 2t = -\frac{1}{10}\big(-10t+11\big) - \frac{2}{17}\big(-17\cos 2t\big) = -\frac{1}{10}\big(\ddot{y}_1 + 2\dot{y}_1 + 5y_1\big) - \frac{2}{17}\big(\ddot{y}_2 + 2\dot{y}_2 + 5y_2\big) = \ddot{y} + 2\dot{y} + 5y,$$

where $y(t) = -\frac{1}{10}\, y_1(t) - \frac{2}{17}\, y_2$.

So, a particular solution of $\ddot{y}+2\dot{y}+5y=t-\frac{11}{10}+2\cos 2t$ is given by $y(t)=-\frac{1}{10}\,y_1(t)-\frac{2}{17}\,y_2$.

The corresponding LCCHODE is $\ddot{y}+2\dot{y}+5y=0$, whose characteristic equation is $0=s^2+2s+5=(s+1)^2+4$, whose roots are $s=-1\pm i2$. The general solution of the ODE $\ddot{y}+2\dot{y}+5y=t-\frac{11}{10}+2\cos 2t$ is

$$y(t)=-\frac{1}{10}\,y_1(t)-\frac{2}{17}\,y_2+c_1e^{-t}\cos 2t+c_2e^{-t}\sin 2t,$$

that is, the solution is

$$y(t)=-\frac{1}{10}(-2t+3)-\frac{2}{17}(-\cos 2t-4\sin 2t)+c_1e^{-t}\cos 2t+c_2e^{-t}\sin 2t,$$

where c_1,c_2 =arbitrary constants.

4.1.5.31. This problem asks us to "reverse engineer" a solution of an ODE to find the ODE it satisfies. The solution, $y(t)=e^{-t}+e^{-2t}-e^{-3t}$, could come from a superlist $\mathsf{L}=-3,-2,-1$ for the method of undetermined coefficients for a non-homogeneous second order ODE. There are three ways that this can happen: $\mathsf{L}_1=-3,-2$, or $\mathsf{L}_1=-3,-1$, or $\mathsf{L}_1=-2,-1$.

Ex. 1: With $\mathsf{L}_1=-3,-2$ and $\mathsf{L}_2=-1$ we would have an ODE of the form

$$f_0e^{-t}=(D+3)(D+2)[\,y\,]=(D^2+5D+6)[\,y\,],$$

for some choice of constant f_0. In this case, we would have $y_h(t)=e^{-2t}-e^{-3t}$. Substitute in $y_p(t)=e^{-t}$ to get

$$f_0e^{-t}=(D^2+5D+6)[\,y_p\,]=(D^2+5D+6)[\,e^{-t}\,]=\big((-1)^2+5(-1)+6\big)e^{-t}=2e^{-t},$$

so $f_0=2$. So, an example of an ODE that could have $y(t)=e^{-t}+e^{-2t}-e^{-3t}$ as one of its solutions is

$$(\star)\quad \ddot{y}+5\dot{y}+6y=2e^{-t}.$$

Ex. 2: With $\mathsf{L}_1=-3,-1$ and $\mathsf{L}_2=-2$ we would have an ODE of the form

$$f_0e^{-2t}=(D+3)(D+1)[\,y\,]=(D^2+4D+3)[\,y\,],$$

for some choice of constant f_0. In this case, we would have $y_h(t)=e^{-t}-e^{-3t}$. Substitute in $y_p(t)=e^{-2t}$ to get

$$f_0e^{-2t}=(D^2+4D+3)[\,y_p\,]=(D^2+4D+3)[\,e^{-2t}\,]=\big((-2)^2+4(-2)+3\big)e^{-2t}=-e^{-2t},$$

so $f_0=-1$. So, an example of an ODE that could have $y(t)=e^{-t}+e^{-2t}-e^{-3t}$ as one of its solutions is

$$(\star\star)\quad \ddot{y}+4\dot{y}+3y=-e^{-2t}.$$

In fact, there are infinitely many examples based on each of $(\star)$ and $(\star\star)$, for example, $5\ddot{y}+25\dot{y}+30y=10e^{-t}$.

Section 4.2.5

4.2.5.1. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 4 \Rightarrow \mathsf{L}_1 = \pm\, i\, 2$
$f(t) = -3\cos 2t \Rightarrow \mathsf{L}_2 = \pm\, i\, 2 \Rightarrow$ Superlist is $\mathsf{L} = \pm\, i\, 2,\, \pm\, i\, 2$
$\Rightarrow y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t\cos 2t + c_4 t\sin 2t \quad \Rightarrow y_p(t) = At\cos 2t + Bt\sin 2t$, where A, B are constants to be determined.

It follows that $\dot{y}_p(t) = A\cos 2t - 2A\, t\sin 2t + B\sin 2t + 2B\, t\cos 2t$ and $\ddot{y}_p(t) = -4A\sin 2t - 4A\, t\cos 2t + 4B\cos 2t - 4B\, t\sin 2t$.

Substitute into the original, non-homogeneous ODE to get

$$-3\cos 2t = \ddot{y}_p + 4y_p = -4A\sin 2t - 4A\, t\cos 2t + 4B\cos 2t - 4B\, t\sin 2t + 4A\, t\cos 2t + 4B\, t\sin 2t$$

$$= -4A\sin 2t + 4B\cos 2t$$

$\Rightarrow A = 0$ and $B = -\frac{3}{4} \Rightarrow y_p(t) = -\dfrac{3}{4}\, t\sin 2t.$

The general solution of the ODE is $y(t) = -\frac{3}{4}\, t\sin 2t + c_1\cos 2t + c_2 \sin 2t$, where c_1, c_2 =arbitrary constants.

4.2.5.3. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 16 \Rightarrow \mathsf{L}_1 = \pm\, i\, 4$
$f(t) = 4\cos 4t \Rightarrow \mathsf{L}_2 = \pm\, i\, 4 \Rightarrow$ Superlist is $\mathsf{L} = \pm\, i\, 4,\, \pm\, i\, 4$
$y(t) = c_1\cos 4t + c_2\sin 4t + c_3 t\cos 4t + c_4 t\sin 4t \quad \Rightarrow y_p(t) = At\cos 4t + Bt\sin 4t$, where A, B are constants to be determined.

It follows that $\dot{y}_p(t) = A\cos 4t - 4A\, t\sin 4t + B\sin 4t + 4B\, t\cos 4t$ and $\ddot{y}_p(t) = -8A\sin 4t - 16At\cos 4t + 8B\cos 4t - 16Bt\sin 4t$.

Substitute into the original, non-homogeneous ODE to get

$$4\cos 4t = \ddot{y}_p + 16y_p = -8A\sin 4t - 16A\, t\cos 4t + 8B\cos 4t - 16B\, t\sin 4t + 16A\, t\cos 4t + 16B\, t\sin 4t$$

$$= -8A\sin 4t + 8B\cos 4t$$

$\Rightarrow A = 0$ and $B = \frac{1}{2} \Rightarrow y_p(t) = \dfrac{1}{2}\, t\sin 4t.$

The general solution of the ODE is $y(t) = \frac{1}{2}\, t\sin 4t + c_1\cos 4t + c_2\sin 4t$, where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = \frac{1}{2}\,\sin 4t + 2t\cos 4t - 4c_1\sin 4t + 4c_2\cos 4t.$$

The ICs require

$$\left\{ \begin{array}{rl} -1 = y(0) = & c_1 \\ 3 = \dot{y}(0) = & \qquad 4c_2 \end{array} \right\},$$

so $c_1 = -1$ and $c_2 = \frac{3}{4}$. The solution of the IVP is

$$y(t) = \frac{1}{2}\, t\sin 4t - \cos 4t + \frac{3}{4}\,\sin 4t.$$

4.2.5.5. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 2s + 2 = (s+1)^2 + 1$
$\Rightarrow \mathsf{L}_1 = -1 \pm i$
$f(t) = \sin t \Rightarrow \mathsf{L}_2 = \pm\, i \Rightarrow$ Superlist is $\mathsf{L} = -1 \pm i,\, \pm i$
$\Rightarrow y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + c_3\cos t + c_4\sin t \quad \Rightarrow y_p(t) = A\cos t + B\sin t$, where A, B are constants to be determined:

$$\sin t = \ddot{y}_p + 2\dot{y}_p + 2y_p = -A\cos t - B\sin t - 2A\sin t + 2B\cos t + 2A\cos t + 2B\sin t = (A+2B)\cos t + (B-2A)\sin t$$

$$\Rightarrow \left\{ \begin{array}{rl} A + 2B = 0 \\ -2A + \ B = 1 \end{array} \right\}$$

so

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ The steady state solution is

$$y_S(t) = y_p(t) = \frac{1}{5}(-2\cos t + \sin t),$$

because the terms $c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$ are transient no matter what are the values of the constants c_1, c_2.

The amplitude phase form (3.39) for the steady state solution is $y(t) = \alpha\cos(t-\delta)$, where

$$\left\{ \begin{array}{c} -\frac{2}{5} = A = \alpha\cos\delta \\ \\ \frac{1}{5} = B = \alpha\sin\delta \end{array} \right\},$$

hence $\alpha = \frac{1}{5}\sqrt{(-2)^2+1^2} = \frac{1}{\sqrt{5}}$ and $\tan\delta = \frac{1/5}{-2/5} = -\frac{1}{2}$. Because $(A,B) = \big(-\frac{2}{5}, \frac{1}{5}\big)$ is in the second quadrant, $\delta = \pi + \arctan\big(-\dfrac{1}{2}\big) = \pi - \arctan\dfrac{1}{2}$.

The steady state solution in amplitude phase form is

$$y_S(t) = \frac{1}{\sqrt{5}}\cos\Big(t - \pi + \arctan\frac{1}{2}\Big),$$

whose amplitude is $Amplitude = \dfrac{1}{\sqrt{5}}$.

4.2.5.7. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 2s + 3 = (s+1)^2 + 2$
$\Rightarrow \mathsf{L}_1 = -1 \pm i\sqrt{2}$

$f(t) = \sin 2t \Rightarrow \mathsf{L}_2 = \pm 2i \Rightarrow$ Superlist is $\mathsf{L} = -1 \pm i\sqrt{2}, \pm 2i$ implies

$$y(t) = c_1 e^{-t}\cos(\sqrt{2}\,t) + c_2 e^{-t}\sin(\sqrt{2}\,t) + c_3\cos 2t + c_4\sin 2t$$

$\Rightarrow y_p(t) = A\cos 2t + B\sin 2t$, where A, B are constants to be determined:

$$\sin 2t = \ddot{y}_p + 2\dot{y}_p + 3y_p = -4A\cos 2t - 4B\sin t - 4A\sin 2t + 4B\cos 2t + 3A\cos 2t + 3B\sin 2t$$

$$= (-A+4B)\cos 2t + (-B-4A)\sin 2t$$

$$\Rightarrow \left\{ \begin{array}{c} -A+4B = 0 \\ -4A - B = 1 \end{array} \right\}$$

so

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

$\Rightarrow$ The steady state solution is

$$y_S(t) = y_p(t) = \frac{1}{17}(-4\cos 2t - \sin 2t),$$

because the terms $c_1 e^{-t}\cos(\sqrt{2}\,t) + c_2 e^{-t}\sin(\sqrt{2}\,t)$ are transient no matter what are the values of the constants c_1, c_2.

The amplitude phase form (3.39) for the steady state solution is $y(t) = \alpha\cos(2t-\delta)$, where

$$\left\{ \begin{array}{c} -\frac{4}{17} = A = \alpha\cos\delta \\ \\ -\frac{1}{17} = B = \alpha\sin\delta \end{array} \right\},$$

hence $\alpha = \frac{1}{17}\sqrt{(-4)^2+(-1)^2} = \frac{1}{\sqrt{17}}$ and $\tan\delta = \frac{-1/17}{-4/17} = \frac{1}{4}$. Because $(A,B) = \big(-\frac{4}{17}, -\frac{1}{17}\big)$ is in the third quadrant, $\delta = \pi + \arctan\dfrac{1}{4}$.

The steady state solution in amplitude phase form is

$$y_S(t) = \frac{1}{\sqrt{17}}\,\cos\left(2t - \pi - \arctan\frac{1}{4}\right),$$

whose amplitude is $Amplitude = \dfrac{1}{\sqrt{17}}$.

4.2.5.9. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 2s + 10 = (s+1)^2 + 9$
$\Rightarrow \mathsf{L}_1 = -1 \pm 3i$
$f(t) = 74\cos 3t \Rightarrow \mathsf{L}_2 = \pm 3i \Rightarrow$ Superlist is $\mathsf{L} = -1 \pm 3i,\ \pm 3i$ implies

$$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t + c_3\cos 3t + c_4 \sin 3t$$

$\Rightarrow y_p(t) = A\cos 3t + B\sin 3t$, where A, B are constants to be determined:

$$\begin{aligned} 74\cos 3t = \ddot{y}_p + 2\dot{y}_p + 10y_p &= -9A\cos 3t - 9B\sin t - 6A\sin 3t + 6B\cos 3t + 10A\cos 3t + 10B\sin 3t \\ &= (A + 6B)\cos 3t + (B - 6A)\sin 3t \\ &\Rightarrow \left\{\begin{array}{r} A + 6B = 74 \\ -6A + B = 0 \end{array}\right\} \end{aligned}$$

so

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ -6 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 74 \\ 0 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 1 & -6 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 74 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}.$$

The general solution of the ODE is

$$y(t) = 2\cos 3t + 12\sin 3t + c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t,$$

where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -6\sin 3t + 36\cos 3t + (-c_1 + 3c_2)e^{-t}\cos 3t + (-c_2 + 3c_2)e^{-t}\sin 3t.$$

The ICs require

$$\left\{\begin{array}{l} -1 = y(0) = \ \ 2 + c_1 \\ \ \ \ 2 = \dot{y}(0) = 36 - c_1 + 3c_2 \end{array}\right\},$$

so $c_1 = -3$ and $c_2 = -\frac{37}{3}$. The solution of the IVP is

$$y(t) = 2\cos 3t + 12\sin 3t - 3e^{-t}\cos 3t - \frac{37}{3}\,e^{-t}\sin 3t.$$

$\Rightarrow$ The steady state solution is

$$y_S(t) = y_p(t) = 2\cos 3t + 12\sin 3t,$$

because the terms $-3e^{-t}\cos 3t - \frac{37}{3}\,e^{-t}\sin 3t$ go to zero as $t \to \infty$.

The amplitude phase form (3.39) for the steady state solution is $y(t) = \alpha\cos(3t - \delta)$, where

$$\left\{\begin{array}{r} 2 = A = \alpha\cos\delta \\ \\ 12 = B = \alpha\sin\delta \end{array}\right\},$$

hence $\alpha = \sqrt{2^2 + 12^2} = \sqrt{148} = 2\sqrt{37}$ and $\tan\delta = \frac{-12}{2} = -6$. Because $(A, B) = (2, 12)$ is in the first quadrant, $\delta = \arctan(6) = \arctan 6$.

The steady state solution in amplitude phase form is

$$y_S(t) = 2\sqrt{37}\,\cos\left(3t - \arctan 6\right),$$

whose amplitude is $Amplitude = 2\sqrt{37}$.

4.2.5.11. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 2s + 6 = (s+1)^2 + 5$
$\Rightarrow \mathsf{L}_1 = -1 \pm \sqrt{5}\, i$
$f(t) = \sin 2t \Rightarrow \mathsf{L}_2 = \pm 2i \Rightarrow$ Superlist is $\mathsf{L} = -1 \pm \sqrt{5}\, i,\ \pm 2i$ implies

$$y(t) = c_1 e^{-t}\cos(\sqrt{5}\, t) + c_2 e^{-t}\sin(\sqrt{5}\, t) + c_3\cos 2t + c_4 \sin 2t$$

$\Rightarrow y_p(t) = A\cos 2t + B\sin 2t$, where A, B are constants to be determined:

$$\sin 2t = \ddot{y}_p + 2\dot{y}_p + 6y_p = -4A\cos 2t - 4B\sin 2t - 4A\sin 2t + 4B\cos 2t + 6A\cos 2t + 6B\sin 2t$$

$$= (2A + 4B)\cos 2t + (2B - 4A)\sin 2t$$

$$\Rightarrow \left\{ \begin{array}{c} 2A + 4B = 0 \\ -4A + 2B = 1 \end{array} \right\}$$

so

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The general solution of the ODE is $y(t) = \frac{1}{10}(-2\cos 2t + \sin 2t) + c_1 e^{-t}\cos(\sqrt{5}\, t) + c_2 e^{-t}\sin(\sqrt{5}\, t)$, where $c_1, c_2 =$arbitrary constants. It follows that

$$\dot{y}(t) = \frac{1}{10}(4\sin 2t + 2\cos 2t) + (-c_1 + \sqrt{5}\, c_2)e^{-t}\cos\sqrt{5}\, t + (-c_2 + \sqrt{5}\, c_2)e^{-t}\sin\sqrt{5}\, t.$$

The ICs require

$$\left\{ \begin{array}{c} -3 = y(0) = -\frac{2}{10} + c_1 \\ \\ 3 - 3\sqrt{5} = \dot{y}(0) = \ \frac{2}{10} - c_1 + \sqrt{5}\, c_2 \end{array} \right\},$$

so $c_1 = -\frac{14}{5}$ and $c_2 = -3$. The solution of the IVP is

$$y(t) = \frac{1}{10}(-2\cos 2t + \sin 2t) - \frac{14}{5}\, e^{-t}\cos(\sqrt{5}\, t) - 3e^{-t}\sin(\sqrt{5}\, t),$$

$\Rightarrow$ The steady state solution is

$$y_S(t) = y_p(t) = \frac{1}{10}(-2\cos 2t + \sin 2t),$$

because the other terms in $y(t)$ go to zero as $t \to \infty$.

The amplitude phase form (3.39) for the steady state solution is $y(t) = \alpha\cos(2t - \delta)$, where

$$\left\{ \begin{array}{c} 2 = A = \alpha\cos\delta \\ -12 = B = \alpha\sin\delta \end{array} \right\},$$

hence $\alpha = \frac{1}{10}\sqrt{(-2)^2 + (1)^2} = \frac{1}{2\sqrt{5}}$ and $\tan\delta = \frac{1/10}{-2/10} = -\frac{1}{2}$. Because $(A, B) = \left(-\frac{2}{10}, \frac{1}{10}\right)$ is in the second quadrant, $\delta = \pi + \arctan\left(-\dfrac{1}{2}\right) = \pi - \arctan\dfrac{1}{2}$.

The steady state solution in amplitude phase form is

$$y_S(t) = \frac{1}{2\sqrt{5}}\, \cos\left(2t - \pi + \arctan\frac{1}{2}\right),$$

whose amplitude is $Amplitude = \dfrac{1}{2\sqrt{5}}$.

4.2.5.13. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 9 \quad \Rightarrow \mathsf{L}_1 = \pm\, 3i$
$f(t) = 5\cos(\sqrt{8}\,t) \Rightarrow \mathsf{L}_2 = \pm\sqrt{8}\,i \Rightarrow$ Superlist is $\mathsf{L} = \pm\, 3i,\ -1 \pm\, 2\sqrt{2}\,i$ implies

$$y(t) = c_1\cos 3t + c_2\sin 3t + c_3\cos(2\sqrt{2}\,t) + c_4\sin(2\sqrt{2}\,t)$$

$\Rightarrow y_p(t) = A\cos(2\sqrt{2}\,t) + B\sin(2\sqrt{2}\,t)$, where A, B are constants to be determined:

$$\begin{aligned} 5\cos(2\sqrt{2}\,t) = \ddot{y}_p + 9y_p &= -8A\cos(2\sqrt{2}\,t) - 8B\sin(2\sqrt{2}\,t) + 9A\cos(2\sqrt{2}\,t) + 9B\sin(2\sqrt{2}\,t) \\ &= A\cos(2\sqrt{2}\,t) + B\sin(2\sqrt{2}\,t) \end{aligned}$$

$\Rightarrow A = 5$ and $B = 0$.
The general solution of the ODE is $y(t) = 5\cos(2\sqrt{2}\,t) + c_1\cos 3t + c_2\sin 3t$, where $c_1, c_2 =$arbitrary constants. It follows that

$$\dot{y}(t) = -10\sqrt{2}\,\sin(2\sqrt{2}\,t) - 3c_1\sin 3t + 3c_2\cos 3t.$$

The ICs require

$$\left\{\begin{array}{l} 0 = y(0) = 5 + c_1 \\ \\ 0 = \dot{y}(0) = 3c_2 \end{array}\right\},$$

so $c_1 = -5$ and $c_2 = 0$. The solution of the IVP is

$$y(t) = 5\big(\cos(2\sqrt{2}\,t) - \cos 3t\big).$$

(a) The natural frequency is $\omega_0 = 3$ and the forcing frequency of this undamped system is $\omega = 2\sqrt{2}$. The frequency of the beats is

$$\zeta \triangleq \frac{|2\sqrt{2}-3|}{2} = \frac{3-2\sqrt{2}}{2}.$$

(b) The maximum amplitude of the motion given by $y(t) = 5\big(\cos(2\sqrt{2}\,t) - \cos 3t\big)$ is 10. Intuitively, there are times when both $\cos(2\sqrt{2}\,t) \approx 1$ and $\cos 3t \approx -1$, because the motion is quasi-periodic, not periodic.

4.2.5.15. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = ms^2 + bs + k$
$\Rightarrow \mathsf{L}_1 = -\frac{b}{2m} \pm \frac{\sqrt{b^2-4mk}}{2m}$
$f(t) = f_0\sin\omega t \Rightarrow \mathsf{L}_2 = \pm\, i\,\omega \Rightarrow$ Superlist is $\mathsf{L} = -\frac{b}{2m} \pm \frac{\sqrt{b^2-4mk}}{2m},\ \pm i\omega$
$y(t) = y_h(t) + c_3\cos\omega t + c_4\sin\omega t \quad \Rightarrow y_p(t) = A\cos\omega t + B\sin\omega t$, where A, B are constants to be determined:

$$\begin{aligned} f_0\sin\omega t = m\ddot{y}_p + b\dot{y}_p + ky_p &= -m\omega^2 A\cos\omega t - m\omega^2 B\sin\omega t + b(-\omega A\sin\omega t - \omega B\cos\omega t) + k(A\cos\omega t + B\sin\omega t) \\ &= \big((k - m\omega^2)\,A + b\omega B\big)\cos\omega t + \big((k - m\omega^2)\,B - b\omega A\big)\sin\omega t, \end{aligned}$$

so $0 = (k - m\omega^2)A + b\omega B$ and $f_0 = (k - m\omega^2)B - b\omega A$. This can be written as a 2×2 system

$$\left\{\begin{array}{rcrl} (k - m\omega^2)A & + & b\omega B & = 0 \\ \\ -b\omega A & + & (k - m\omega^2)B & = f_0 \end{array}\right\},$$

whose solution is

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} k - m\omega^2 & b\omega \\ -b\omega & k - m\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f_0 \end{bmatrix} = \frac{1}{(k - m\omega^2)^2 + (b\omega)^2} \begin{bmatrix} k - m\omega^2 & -b\omega \\ b\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} 0 \\ f_0 \end{bmatrix}.$$

So, a particular solution of the IVP is given by

$$y_p(t) = \frac{f_0}{(k - m\omega^2)^2 + (b\omega)^2}\left(-b\omega\cos\omega t + (k - m\omega^2)\sin\omega t\right).$$

$b > 0$ implies that $y_h(t) \to 0$ as $t \to \infty$, that is, $y_h(t)$ is a transient solution, and the initial data $y_0, \dot{y}_0$ don't affect this fact! The steady state solution is

$$y_S(t) = y_p(t) = \frac{f_0}{(k - m\omega^2)^2 + (b\omega)^2} \left(-b\omega \cos \omega t + (k - m\omega^2) \sin \omega t\right).$$

4.2.5.17. (a) From the graph in the textbook's Figure 4.11, the steady state solution appears to have period 4. But the period of the steady state solution is $2\pi/\omega$, so $\omega = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$.
(b) From the graph in the textbook's Figure 4.11, the steady state solution appears to have *Amplitude* ≈ 1.7.
(c) According to formula (4.11), the amplitude of the steady state solution of an ODE $m\ddot{y}+b\dot{y}+ky = f_0 \cos \omega t$ is

$$(\star) \quad Amplitude = \frac{|f_0|}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}.$$

In this problem, we were given $f_0 = 11$, $m = 1$, and $b = 4$, in part (a) we derived that $\omega = \frac{\pi}{2}$, and in part (b) we found *amplitude* ≈ 1.7. Using these values and equation $(\star)$ we can find an approximate value of k: $(\star)$ implies

$$(1.7)^2 \approx Amplitude^2 = \frac{f_0^2}{(k - m\omega^2)^2 + (b\omega)^2} = \frac{(11)^2}{\left(k - 1 \cdot \left(\frac{\pi}{2}\right)^2\right)^2 + (4 \cdot \frac{\pi}{2})^2},$$

which implies

$$\left(k - 1 \cdot \left(\frac{\pi}{2}\right)^2\right)^2 + (4 \cdot \frac{\pi}{2})^2 \approx \frac{(11)^2}{(1.7)^2}.$$

This implies

$$\left(k - \left(\frac{\pi}{2}\right)^2\right)^2 \approx \frac{(11)^2}{(1.7)^2} - (2\pi)^2,$$

hence

$$k \approx \left(\frac{\pi}{2}\right)^2 \pm \sqrt{\frac{(11)^2}{(1.7)^2} - (2\pi)^2} \approx 4.013394149.$$

So, there are two possibilities:

$$k^+ \approx 4.013394149 \quad \text{and} \quad k^- \approx 0.9214080517$$

Because $m = 1$, the natural frequency of the system is, rounded off to the two significant digits of accuracy of data we read from the graph:

$$\omega_0^+ = \sqrt{k^+} \approx \sqrt{4.013394149} \approx 2.0$$

and

$$\omega_0^- = \sqrt{k^-} \approx \sqrt{0.9214080517} \approx 0.96.$$

4.2.5.19. As $d \to 0^+$, eventually d will be a very small but still positive number, hence the steady state solution is a steady state oscillation of $\ddot{y} + 2d\dot{y} + y = \cos t$. From formula (4.11) and noting that $b = 2d$, $m = 1$, $k = 1$, and $\omega = 1$, the amplitude of the steady state oscillation is

$$(\star) \quad Amplitude = \frac{|f_0|}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}} = \frac{1}{\sqrt{(1-1)^2 + (2d)^2}} = \frac{1}{2d} > 0.$$

As $d \to 0^+$, the *Amplitude* $\to +\infty$.

We can think of pure resonance as an "oscillation" with amplitude that becomes unbounded as $t \to \infty$. This is not quite the same thing as a steady state oscillation depending on a parameter d whose *Amplitude* $\to +\infty$ as $d \to 0^+$.

From the practical point of view, having a sufficiently small damping coefficient can be almost as powerful as having no damping and pure resonance.

4.2.5.21. By Hooke's Law, $mg = k\ell$, where ℓ is the stretch of the spring in equilibrium and m is the mass of the object compressing or stretching the spring. We are given that $\ell = 0.1$ in $= \frac{1}{120}$ ft, so the mass of the machine, m, and the spring constant, k, satisfy

$$\omega_0^2 = \frac{k}{m} = \frac{g}{\ell} = \frac{32\,ft/s^2}{(1/120)ft} = 32 \cdot 120\,s^{-2} = 2^8 \cdot 15\,s^{-2},$$

where ω_0 is the natural frequency of the undamped system. Pure resonance vibrations will occur if

$$\omega = \omega_0 = \sqrt{2^8 \cdot 15\,s^{-2}} = 16\sqrt{15}\,s^{-1} \approx 61.96773354 s^{-1}.$$

I.e., pure resonance vibrations occur if the machine spins at

$$\frac{\omega}{2\pi} \text{ revolutions per second} \approx 9.862471105\,Hz.$$

4.2.5.23. This problem asks us to "reverse engineer" a solution of an ODE to find the ODE it satisfies. The solution, $y(t) = 2t\sin 2t + 5\cos(2t - \delta)$, could come from a superlist $\mathsf{L} = \pm 2i, \pm 2i$. Some of the roots in L come from L_1 and others from L_2.

Because the right hand side of the ODE is $f_0 \cos\omega t$, $\mathsf{L}_2 = \pm i\omega$. To be part of the superlist L, we must have $\underline{\omega = 2}$.

It follows that the list L_1 is the rest of L, hence $\mathsf{L}_1 = \pm 2i$.

We were given that $k = 6$, in N/m. So, the corresponding homogeneous ODE has characteristic polynomial

$$0 = ms^2 + bs + 6 = m\big(s - (-i2)\big)\big(s - (-i2)\big) = m(s - i2)(s + i2) = m(s^2 + 4),$$

hence $\underline{b = 0}$ and $4m = 6$. The latter implies $\underline{m = \frac{3}{2}}$ and the ODE is

$$\frac{3}{2}\ddot{y} + 6y = f_0 \cos 2t.$$

To find f_0 we can substitute into the ODE either the whole given solution, $y(t) = 2t\sin 2t + 5\cos(2t - \delta)$, or the particular solution, which is $y_p(t) = 2t\sin 2t$ because $\mathsf{L}_2 = \pm 2i$. We calculate $\dot{y}_p(t) = 2\sin 2t + 4t\cos 2t$ and $\ddot{y}_p(t) = 8\cos 2t - 8t\sin 2t$.

We substitute all of this into the ODE to get

$$f_0 \cos 2t = \frac{3}{2}\,\ddot{y}_p + 6y_p = \frac{3}{2}(8\cos 2t - 8t\sin 2t) + 12t\sin 2t = 12\cos 2t,$$

so $\underline{f_0 = 12}$.

4.2.5.25. The ODE is $\ddot{y} + b\dot{y} + 9y = f_0 \cos\omega t$.

(a) In order to have a solution of the form $y(t) = (\textit{steady state solution}) + (\textit{transient solution})$, in particular to have a transient solution, we must have $b > 0$. Other than that, there are no restrictions on the parameters. Here's an Ex: $\underline{b = 1}$, $\underline{f_0 = 1}$, and $\underline{\omega = 1}$.

(b) In order to have a solution $y(t)$ that exhibits pure resonance, there must be no damping, that is, $b = 0$, and matching frequencies $\omega = \omega_0$. Then, the ODE becomes $\ddot{y} + 9y = f_0 \cos\omega t$. The natural frequency is $\omega_0 = 3$, so $\underline{\omega = 3}$. Here's an Ex: $\underline{b = 0}$, $\underline{f_0 = 1}$, and $\underline{\omega = 3}$.

(c) In order to have a solution $y(t)$ that exhibits the beats phenomenon, we must have no damping, that is, $b = 0$, and different frequencies $\omega \neq \omega_0$. Then, the ODE becomes $\ddot{y} + 9y = f_0 \cos\omega t$. The natural frequency is $\omega_0 = 3$, so $\omega \neq 3$. Here's an Ex: $\underline{b = 0}$, $\underline{f_0 = 1}$, and $\underline{\omega = 2}$.

4.2.5.27. Substitute
$$\omega^* \triangleq \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}$$
into $G_{\max}$ to get
$$G_{\max} = G(\omega^*) = \frac{1}{\sqrt{(k-m(\omega^*)^2)^2 + (b\omega^*)^2}} = \frac{1}{\sqrt{\Big(k - m\big(\omega_0^2 - \frac{b^2}{2m^2}\big)\Big)^2 + b^2\big(\omega_0^2 - \frac{b^2}{2m^2}\big)}}\,.$$

But, $\omega_0^2 = \dfrac{k}{m}$, so $k - m\omega_0^2 = 0$, and thus
$$G_{\max} = \frac{1}{\sqrt{\Big(m\cdot\dfrac{b^2}{2m^2}\Big)^2 + b^2\big(\dfrac{k}{m} - \dfrac{b^2}{2m^2}\big)}} = \frac{1}{\sqrt{\Big(\dfrac{b^2}{2m}\Big)^2 + \dfrac{kb^2}{m} - \dfrac{b^4}{2m^2}}} = \frac{1}{\sqrt{\dfrac{b^4}{4m^2} + \dfrac{kb^2}{m} - \dfrac{b^4}{2m^2}}}$$
$$= \frac{1}{\sqrt{\dfrac{kb^2}{m} - \dfrac{b^4}{4m^2}}} = \frac{1}{\sqrt{\dfrac{b^2}{4m^2}\,(4km - b^2)}}.$$

Because $b > 0$ and $m > 0$, $\sqrt{\dfrac{b^2}{4m^2}} = \dfrac{b}{2m}$, so
$$G_{\max} = G(\omega^*) = \frac{1}{\dfrac{b}{2m}\,\sqrt{4km - b^2}} = \frac{1}{b}\cdot\frac{1}{\frac{\sqrt{4km-b^2}}{2m}} = \frac{1}{\nu b}\,,$$
as we were asked to show.

4.2.5.29. This problem asks us to "reverse engineer" a solution of an ODE to find an ODE it satisfies. The solution, $y(t) = 1.5 + 0.75\cos(t - \frac{\pi}{3}) - 0.5e^{-t/5}$, could come from a superlist $\mathsf{L} = 0, -\frac{1}{5}, \pm i$. Some of the roots in L come from L_1 and others from L_2.

(a) Here, we assume that the ODE has the form $\ddot{y} + \omega_0^2 y = \delta + \gamma e^{-\alpha t}$. The right hand side of the ODE, $\delta + \gamma e^{-\alpha t}$, where δ, γ, and α are constants, implies that $\mathsf{L}_2 = 0, -\frac{1}{5}$. It follows from L and L_2 that $\mathsf{L}_1 = \pm i$, hence $\omega_0 = 1$. So, the ODE is of the form
$$(\star)\quad \ddot{y} + y = \delta + \gamma e^{-t/5}$$

The solutions of this ODE come from the superlist and are
$$y(t) = c_1\cos t + c_2\sin t + c_3 + c_4 e^{-t/5}$$
$\Rightarrow$ The form of the particular solution is $y_p(t) = A + Be^{-t/5}$, where A, B are constants.

To find γ and δ, we can substitute into the ODE $(\star)$ either the whole given solution, $y(t) = 1.5 + 0.75\cos(t - \frac{\pi}{3}) - 0.5e^{-t/5}$, or the part of that solution that could be the particular solution, namely $y_p(t) = 1.5 - 0.5e^{-t/5}$. We calculate
$$\delta + \gamma e^{-t/5} = \ddot{y}_p + y_p = (0 - 0.02e^{-t/5}) + 1.5 - 0.5e^{-t/5} = 1.5 - 0.52e^{-t/5}.$$
$\Rightarrow$ $\delta = 1.5$ and $\gamma = -0.52$. The final conclusion is that the ODE
$$\ddot{y} + y = 1.5 - 0.52e^{-t/5}$$
fits the desired form and has $y(t) = 1.5 + 0.75\cos(t - \frac{\pi}{3}) - 0.5e^{-t/5}$ as one of its solutions.

(b) Again, the superlist is $\mathsf{L} = 0, -\frac{1}{5}, \pm i$. Here, we assume that the ODE has the form $(D^2+\omega_0^2)(D+\varepsilon)[\,y\,] = \eta$. The right hand side of the ODE, η, where η is a constant, implies that $\mathsf{L}_2 = 0$. It follows from L and L_2 that $\mathsf{L}_1 = -\varepsilon, \pm i$, hence $\omega_0 = 1$ and $\varepsilon = -\frac{1}{5}$. So, the ODE is

$$(\star\star)\quad (D^2+1)\Big(D+\frac{1}{5}\Big)[\,y\,] = \eta$$

The solutions of this ODE come from the superlist and are

$$y(t) = c_1\cos t + c_2\sin t + c_3 e^{-t/5} + c_4$$

$\Rightarrow$ The form of the particular solution is $y_p(t) = A$, where A is a constant.

To find η, we can substitute into the ODE $(\star\star)$ either the whole given solution, $y(t) = 1.5 + 0.75\cos(t - \frac{\pi}{3}) - 0.5e^{-t/5}$, or the part of that solution that could be the particular solution, namely $y_p(t) = 1.5$. We calculate

$$\eta = (D^2+1)\Big(D+\frac{1}{5}\Big)[\,y_p\,] = \frac{1.5}{5}$$

$\Rightarrow \eta = 0.3$. The final conclusion is that the ODE

$$0.3 = (D^2+1)\Big(D+\frac{1}{5}\Big)[\,y\,] = \dddot{y} + \frac{1}{5}\,\ddot{y} + \dot{y} + \frac{1}{5}\,y$$

fits the desired from and has $y(t) = 1.5 + 0.75\cos(t - \frac{\pi}{3}) - 0.5e^{-t/5}$ as one of its solutions.

(c) Here, we assume that the ODE is homogeneous. This implies that the corresponding homogeneous ODE has characteristic equation

$$0 = (s^2+1)\left(\Big(s+\frac{1}{5}\Big)s\right).$$

So, the ODE is

$$0 = (D^2+1)\Big(D+\frac{1}{5}\Big)D[\,y\,] = y^{(iv)} + \frac{1}{5}\,\dddot{y} + \ddot{y} + \frac{1}{5}\,\dot{y}.$$

4.2.5.31. (a) ODE $\dot{y} + \delta y = f_0\cos\omega t$ has lists $\mathsf{L}_1 = -\delta$ and $\mathsf{L}_2 = \pm\, i\omega$, so the super list is $\mathsf{L} = -\delta, \pm\, i\omega$. This gives

$$y(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + c_3 e^{-\delta t}$$

$\Rightarrow y_p(t) = A\cos(\omega t) + B\sin(\omega t)$, where A, B are constants to be determined. Substitute $y_p(t)$ into the ODE:

$$f_0\cos\omega t = \dot{y}_p + \delta y_p = -A\omega\sin(\omega t) + \omega B\cos(\omega t) + \delta A\cos(\omega t) + \delta B\sin(\omega t) = (\omega B + \delta A)\cos(\omega t) + (-\omega A + \delta B)\sin(\omega t)$$

so $0 = -A\omega + \delta B$ and $f_0 = \omega B + \delta A$. This can be written as a 2×2 system

$$\left\{\begin{array}{rl} -\omega A + \delta B & = 0 \\ \delta A + \omega B & = f_0 \end{array}\right\},$$

whose solution is

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -\omega & \delta \\ \delta & \omega \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f_0 \end{bmatrix} = -\frac{1}{\omega^2+\delta^2}\begin{bmatrix} \omega & -\delta \\ -\delta & -\omega \end{bmatrix}\begin{bmatrix} 0 \\ f_0 \end{bmatrix}.$$

So, a particular solution of the IVP is given by

$$y_p(t) = \frac{f_0}{\omega^2+\delta^2}\left(\delta\cos\omega t + \omega\sin\omega t\right).$$

$\delta > 0$ implies that $y_h(t) = c_3 e^{-\delta t} \to 0$ as $t \to \infty$, that is, $y_h(t)$ is a transient solution, and the initial datum y_0 doesn't affect this fact! The steady state solution is

$$y_S(t) = y_p(t) = \frac{f_0}{\omega^2+\delta^2}\left(-\delta\cos\omega t + \omega\sin\omega t\right)$$

(b) ODE $\dot{y} + \delta y = f_0 \cos \omega t$ has lists $\mathsf{L}_1 = -\delta$ and $\mathsf{L}_2 = \pm i\omega$, so the super list is $\mathsf{L} = -\delta, \pm i\omega$. This gives

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + c_3 e^{-\delta t}$$

$\Rightarrow y_p(t) = A\cos(\omega t) + B\sin(\omega t)$, where A, B are constants to be determined. Substitute $y_p(t)$ into the ODE:

$$f_0 \sin \omega t = \dot{y}_p + \delta y_p = -A\omega \sin(\omega t) + \omega B \cos(\omega t) + \delta A \cos(\omega t) + \delta B \sin(\omega t)$$

$$= (\omega B + \delta A)\cos(\omega t) + (-\omega A + \delta B)\sin(\omega t)$$

so $0 = -A\omega + \delta B$ and $f_0 = \omega B + \delta A$. This can be written as a 2×2 system

$$\left\{ \begin{array}{ll} -\omega A + \delta B & = f_0 \\ \delta A + \omega B & = 0 \end{array} \right\},$$

whose solution is

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -\omega & \delta \\ \delta & \omega \end{bmatrix}^{-1} \begin{bmatrix} f_0 \\ 0 \end{bmatrix} = -\frac{1}{\omega^2 + \delta^2} \begin{bmatrix} \omega & -\delta \\ -\delta & -\omega \end{bmatrix} \begin{bmatrix} f_0 \\ 0 \end{bmatrix}.$$

So, a particular solution of the IVP is given by

$$y_p(t) = \frac{f_0}{\omega^2 + \delta^2} \left(-\omega \cos \omega t + \delta \sin \omega t\right).$$

$\delta > 0$ implies that $y_{_h}(t) = c_3 e^{-\delta t} \to 0$ as $t \to \infty$, that is, $y_h(t)$ is a transient solution, and the initial datum y_0 doesn't affect this fact! The steady state solution is

$$y_{_S}(t) = y_p(t) = \frac{f_0}{\omega^2 + \delta^2} \left(-\omega \cos \omega t + \delta \sin \omega t\right)$$

(c) Using the results of parts (a) and (b), along with the non-homogeneous superposition principle in Theorem 4.2, we get that the steady state solution of $\dot{y} + \delta y = f_0 \cdot (a \cos \omega t + b \sin \omega t)$ is

$$y_{_S}(t) = \frac{f_0}{\omega^2 + \delta^2} \cdot \Big(a\left(\delta \cos \omega t + \omega \sin \omega t\right) + b\left(-\omega \cos \omega t + \delta \sin \omega t\right)\Big),$$

that is,

$$y_{_S}(t) = \frac{f_0}{\omega^2 + \delta^2}\Big((a\delta - b\omega)\cos \omega t + (a\omega + b\delta)\sin \omega t\Big).$$

Section 4.3.2

4.3.2.1. The corresponding homogeneous ODE, $y''+4y=0$, where $' = \frac{d}{dx}$, has characteristic polynomial s^2+4, so the homogeneous solution is

$$y_h(x) = c_1\cos 2x + c_2\sin 2x.$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(x) = v_1(x)\cos 2x + v_2(x)\sin 2x,$$

where $v_1(x), v_2(x)$ are functions to be determined later.

We assume that

$$v_1'(x)\cos 2x + v_2'(x)\sin 2x \equiv 0,$$

so

$$y'(x) = (-2\sin 2x)v_1(x) + (2\cos 2x)v_2(x)$$

and thus

$$y''(x) = (-2\sin 2x)v_1'(x) + (2\cos 2x)v_2'(x) - (4\cos 2x)v_1(x) - (4\sin 2x)v_2(x).$$

Substitute all of that into the original, non-homogeneous ODE $y''+4y = \frac{1}{\sin 2x}$, which is in standard form. We get

$$(-2\sin 2x)v_1'(x) + (2\cos 2x)v_2'(x) = \frac{1}{\sin 2x}.$$

So, $v_1'(x), v_2'(x)$ should satisfy the system of linear equations

$$\left\{\begin{array}{rl} (\cos 2x)v_1'(x) + (\sin 2x)v_2'(x) & = 0 \\ (-2\sin 2x)v_1'(x) + (2\cos 2x)v_2'(x) & = \frac{1}{\sin 2x} \end{array}\right\}.$$

Using the inverse of a 2×2 matrix, we get

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{\sin 2x} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2\cos 2x & -\sin 2x \\ 2\sin 2x & \cos 2x \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{\sin 2x} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ \frac{\cos 2x}{\sin 2x} \end{bmatrix}.$$

We obtain

$$v_1(x) = \int v_1'(x)dx = \int -\frac{1}{2}\,dx = \ldots = -\frac{1}{2}\,x + c_1,$$

where c_1 =arbitrary constant.

Substituting $w=\sin 2x$, we also get

$$v_2(x) = \frac{1}{2}\int \frac{\cos 2x}{\sin 2x}\,dx = \frac{1}{4}\int \frac{1}{w}\,dw = \frac{1}{4}\ln|\sin 2x| + c_2,$$

where c_2 is an arbitrary constant.

Putting everything together, the general solution of the ODE is

$$y(x) = -\frac{1}{2}\,x\cos(2x) + \frac{1}{4}\,\sin(2x)\ln|\sin 2x| + c_1\cos 2x + c_2\sin 2x,$$

where c_1, c_2 =arbitrary constants.

4.3.2.3. The corresponding homogeneous ODE, $\ddot{y}+y=0$ has characteristic polynomial s^2+1, so the homogeneous solution is

$$y_h(t) = c_1\cos t + c_2\sin t.$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(t) = v_1(t)\cos t + v_2(t)\sin t,$$

where $v_1(t), v_2(t)$ are functions to be determined later.

We assume that

$$\dot v_1(t)\cos t + \dot v_2(t)\sin t \equiv 0,$$

so

$$\dot y(t) = (-\sin t)v_1(t) + (\cos t)v_2(t)$$

and thus

$$\ddot y(t) = (-\sin t)\dot v_1(t) + (\cos t)\dot v_2(t) - (\cos t)v_1(t) - (\sin t)v_2(t).$$

Substitute all of that into the original, non-homogeneous ODE $\ddot y + y = \sec^2(t)$, which is in standard form. We get

$$(-\sin t)\dot v_1(t) + (\cos t)\dot v_2(t) = \sec^2(t).$$

So, $\dot v_1(t), \dot v_2(t)$ should satisfy the system of linear equations

$$\left\{ \begin{array}{cl} (\cos t)\dot v_1(t) + (\sin t)\dot v_2(t) & = 0 \\ \\ (-\sin t)\dot v_1(t) + (\cos t)\dot v_2(t) & = \sec^2(t) \end{array} \right\}.$$

Using the inverse of a 2×2 matrix, we get

$$\begin{bmatrix} \dot v_1 \\ \\ \dot v_2 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ \\ -\sin t & \cos t \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \\ \sec^2(t) \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ \\ \sec^2(t) \end{bmatrix} = \begin{bmatrix} -\frac{\sin t}{\cos^2 t} \\ \\ \sec(t) \end{bmatrix}.$$

We obtain, using the substitution $w = \cos t$,

$$v_1(t) = \int \dot v_1(t)dt = \frac{-\sin t}{\cos^2 t}\,dt = \int \frac{1}{w^2}\,dw = -\frac{1}{w} + c_1 = -\frac{1}{\cos t} + c_1,$$

where c_1 =arbitrary constant. We also get

$$v_2(t) = \int \dot v_2(t)dt = \int \sec t\,dt = \ln|\tan t + \sec t| + c_2,$$

where c_2 is an arbitrary constant.

Putting everything together, the general solution of the ODE is

$$y(t) = -\cos(t)\cdot\frac{1}{\cos t} + \sin(t)\ln|\tan t + \sec t| + c_1\cos t + c_2\sin t,$$

hence

$$y(t) = -1 + \sin(t)\ln|\tan t + \sec t| + c_1\cos t + c_2\sin t,$$

where c_1, c_2 =arbitrary constants.

4.3.2.5. The corresponding homogeneous ODE is the Cauchy-Euler ODE $x^2y'' - 2xy' + 2y = 0$, where $' = \frac{d}{dx}$. Substituting $y(x) = x^m$ into that ODE gives characteristic equation $0 = m(m-1) - 2m + 2 = m^2 - 3m + 2 = (m-1)(m-2)$. The homogeneous solution is

$$y_h(x) = c_1x^1 + c_2x^2.$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(x) = v_1(x)\cdot x \ + \ v_2(x)\cdot x^2,$$

where $v_1(x), v_2(x)$ are functions to be determined later.

We assume that

$$v_1'(x)\cdot x \;+\; v_2'(x)\cdot x^2 \equiv 0,$$

so

$$y'(x) = 1\cdot v_1(x) \;+\; 2x\cdot v_2(x)$$

and thus

$$y''(x) = v_1'(x) + 2x\cdot v_2'(x) + 2v_2(x).$$

Put the non-homogeneous ODE into standard form by dividing through by x^2 to get $y''-2x^{-1}y'+2x^{-2}y = x$. Substitute into that ODE to get

$$1\cdot v_1'(x) + 2x\cdot v_2'(x) = x.$$

So, $v_1'(x), v_2'(x)$ should satisfy the system of linear equations

$$\left\{\begin{array}{ll} x\,v_1'(x) + x^2\,v_2'(x) & = 0 \\ \\ v_1'(x) + 2x\,v_2'(x) & = x \end{array}\right\}.$$

Using the inverse of a 2×2 matrix, we get

$$\begin{bmatrix} v_1' \\ \\ v_2' \end{bmatrix} = \begin{bmatrix} x & x^2 \\ \\ 1 & 2x \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \\ x \end{bmatrix} = \frac{1}{x^2}\begin{bmatrix} 2x & -x^2 \\ \\ -1 & x \end{bmatrix}\begin{bmatrix} 0 \\ \\ x \end{bmatrix} = \begin{bmatrix} -x \\ \\ 1 \end{bmatrix}.$$

We obtain

$$v_1(x) = \int v_1'(x)dx = \int -x\,dx = -\frac{1}{2}\,x^2 + c_1$$

and

$$v_2(x) = \int v_2'(x)dx = \int 1\;dx = x + c_2,$$

where c_1, c_2 are arbitrary constants.

Putting everything together, the general solution of the ODE is

$$y(x) = \left(-\frac{1}{2}\,x^2 + c_1\right)x + (x + c_2)\,x^2,$$

hence

$$y(x) = \frac{1}{2}\,x^3 + c_1 x + c_2 x^2,$$

where c_1, c_2 =arbitrary constants.

4.3.2.7. The corresponding homogeneous ODE is the Cauchy-Euler ODE $x^2y''-5xy'+8y=0$, where $' = \frac{d}{dx}$. Substituting $y(x) = x^m$ into that ODE gives characteristic equation $0 = m(m-1)-5m+8 = m^2-6m+8 = (m-2)(m-4)$. The homogeneous solution is

$$y_h(x) = c_1x^2 + c_2x^4.$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(x) = v_1(x)\cdot x^2 + v_2(x)\cdot x^4,$$

where $v_1(x), v_2(x)$ are functions to be determined later.

We assume that

$$v_1'(x)\cdot x + v_2'(x)\cdot x^4 \equiv 0,$$

so

$$y'(x) = 2x\cdot v_1(x) + 4x^3\cdot v_2(x)$$

and thus
$$y''(x) = 2x\, v_1'(x) + 4x^3\, v_2'(x) + 2v_1(x) + 12x^2 v_2(x).$$

Put the non-homogeneous ODE into standard form by dividing through by x^2 to get $y''-5x^{-1}y'+8x^{-2}y = xe^{-x}$. Substitute into that ODE to get
$$2x \cdot v_1'(x) + 4x^3 \cdot v_2'(x) = xe^{-x}.$$

So, $v_1'(x), v_2'(x)$ should satisfy the system of linear equations
$$\left\{ \begin{array}{ll} x^2\, v_1'(x) + x^4\, v_2'(x) & = 0 \\ \\ 2x\, v_1'(x) + 4x^3\, v_2'(x) & = xe^{-x} \end{array} \right\}.$$

Using the inverse of a 2×2 matrix, we get
$$\begin{bmatrix} v_1' \\ \\ v_2' \end{bmatrix} = \begin{bmatrix} x^2 & x^4 \\ \\ 2x & 4x^3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \\ xe^{-x} \end{bmatrix} = \frac{1}{2x^5} \begin{bmatrix} 4x^3 & -x^4 \\ \\ -2x & x^2 \end{bmatrix} \begin{bmatrix} 0 \\ \\ xe^{-x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\, e^{-x} \\ \\ \frac{1}{2}\, x^{-2}e^{-x} \end{bmatrix}.$$

We obtain
$$v_1(x) = \int v_1'(x)dx = \int -\frac{1}{2}\, e^{-x}dx = \frac{1}{2}\, e^{-x} + c_1.$$

We also get, because we cannot find the indefinite integral, that
$$v_2(x) = c_2 + \frac{1}{2} \int_0^x s^{-2}e^{-s}ds,$$

where c_1, c_2 are arbitrary constants.

Putting everything together, the general solution of the ODE is
$$y(x) = \left(\frac{1}{2}\, e^{-x} + c_1\right) x^2 + \left(c_2 + \frac{1}{2} \int_1^x s^{-2}e^{-s}ds\right) x^4,$$

hence
$$y(x) = c_1x^2 + c_2x^4 + \frac{1}{2}\, x^2e^{-x} + \frac{1}{2}\, x^4 \int_1^x s^{-2}e^{-s}ds,$$

where c_1, c_2 =arbitrary constants.

4.3.2.9. The corresponding homogeneous ODE is the Cauchy-Euler ODE $r^2y'' - 6ry' + 12y = 0$, where $' = \frac{d}{dr}$. Substituting $y(r) = r^m$ into that ODE gives characteristic equation $0 = m(m-1) - 6m + 8 = m^2 - 7m + 12 = (m-3)(m-4)$. The homogeneous solution is
$$y_h(r) = c_1r^3 + c_2r^4.$$

Let's try to find a solution of the non-homogeneous ODE in the form
$$y(r) = v_1(r) \cdot r^3 + v_2(r) \cdot r^4,$$

where $v_1(r), v_2(r)$ are functions to be determined later.

We assume that
$$v_1'(r) \cdot r^3 + v_2'(r) \cdot r^3 \equiv 0,$$

so
$$y'(r) = 3r^2 \cdot v_1(r) + 4r^4 \cdot v_2(r)$$

and thus
$$y''(r) = 3r^2v_1'(r) + 4r^3 \cdot v_2'(r) + 6rv_1(r) + 12r^2v_2(r).$$

Put the non-homogeneous ODE into standard form by dividing through by r^2 to get $y'' - 6r^{-1}y' + 12r^{-2}y = r^3 \sin 2r$. Substitute into that ODE to get

$$3r^2 \cdot v_1'(r) + 4r^3 \cdot v_2'(r) = r^3 \sin 2r.$$

So, $v_1'(r), v_2'(r)$ should satisfy the system of linear equations

$$\left\{ \begin{array}{ll} r^3 v_1'(r) + r^4\, v_2'(r) & = 0 \\ 3r^2 v_1'(r) + 4r^3\, v_2'(r) & = r^3 \sin 2r \end{array} \right\}.$$

Using the inverse of a 2×2 matrix, we get

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} r^3 & r^4 \\ 3r^2 & 4r^3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ r^3 \sin 2r \end{bmatrix} = \frac{1}{r^6} \begin{bmatrix} 4r^3 & -r^4 \\ -3r^2 & r^3 \end{bmatrix} \begin{bmatrix} 0 \\ r^3 \sin 2r \end{bmatrix} = \begin{bmatrix} -r\,\sin 2r \\ \sin 2r \end{bmatrix}.$$

We obtain, using integration by parts, that

$$v_1(r) = \int v_1'(r)dr = \int r(-\sin 2r)dr = \left(r\left(\frac{1}{2}\,\cos 2r\right) - \int 1 \cdot \left(\frac{1}{2}\,\cos 2r\right)dr \right) = \frac{1}{2}\,r\,\cos 2r - \frac{1}{4}\,\sin 2r + c_1,$$

where c_1 is an arbitrary constant. We also get

$$v_2(r) = \int \sin 2r\,dr = -\frac{1}{2}\cos 2r + c_2,$$

where c_2 is an arbitrary constant.

Putting everything together, the general solution of the ODE is

$$y(r) = \left(\cancel{\frac{1}{2}\,r\,\cos 2r} - \frac{1}{4}\,\sin 2r + c_1 \right) r^3 + \left(\cancel{-\frac{1}{2}\cos 2r} + c_2 \right) r^4,$$

hence

$$y(r) = c_1 r^3 + c_2 r^4 - \frac{1}{4} r^3 \sin 2r,$$

where c_1, c_2 =arbitrary constants.

4.3.2.11. The corresponding homogeneous ODE is the Cauchy-Euler ODE $r^2y'' - 4ry' + 6y = 0$, where $' = \frac{d}{dr}$. Substituting $y(r) = r^m$ into that ODE gives characteristic equation $0 = m(m-1) - 4m + 6 = m^2 - 5m + 6 = (m-2)(m-3)$. The homogeneous solution is

$$y_h(r) = c_1\,r^2 + c_2\,r^3.$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(r) = v_1(r) \cdot r^2 + v_2(r) \cdot r^3,$$

where $v_1(r), v_2(r)$ are functions to be determined later.

We assume that

$$v_1'(r) \cdot r^2 + v_2'(r) \cdot r^3 \equiv 0,$$

so

$$y'(r) = 2r \cdot v_1(r) + 3r^2 \cdot v_2(r)$$

and thus

$$y''(r) = 2r \cdot v_1'(r) + 3r^2 \cdot v_2'(r) + 2\,v_1(r) + 6r\,v_2(r).$$

Put the non-homogeneous ODE into standard form by dividing through by r^2 to get $y'' - 4r^{-1}y' + 6r^{-2}y = 1$. Substitute into that ODE to get

$$2r \cdot v_1'(r) + 3r^2 \cdot v_2'(r) = 1.$$

So, $v_1'(r), v_2'(r)$ should satisfy the system of linear equations

$$\left\{\begin{array}{rl} r^2\, v_1'(r) + r^3\, v_2'(r) & = 0 \\ \\ 2r\, v_1'(r) + 3r^2\, v_2'(r) & = 1 \end{array}\right\}.$$

Using the inverse of a 2×2 matrix, we get

$$\begin{bmatrix} v_1' \\ \\ v_2' \end{bmatrix} = \begin{bmatrix} r^2 & r^3 \\ \\ 2r & 3r^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} = \frac{1}{r^4} \begin{bmatrix} 3r^2 & -r^3 \\ \\ -2r & r^2 \end{bmatrix} \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{r} \\ \\ \frac{1}{r^2} \end{bmatrix}.$$

We obtain that

$$v_1(r) = \int v_1'(r) dr = \int -\frac{1}{r}\, dr = -\ln|r| + c_1,$$

where c_1 =arbitrary constant.

We also get

$$v_2(r) = \int \frac{1}{r^2}\, dr = -\frac{1}{r} + c_2,$$

where c_2 is an arbitrary constant.

Putting everything together, the general solution of the ODE is

$$y(r) = (-\ln|r| + c_1)\, r^2 + \left(-\frac{1}{r} + c_2\right) r^3,$$

hence

$$y(r) = \widetilde{c}_1\, r^2 + \widetilde{c}_2\, r^3 - r^2 \ln|r|,$$

where $\widetilde{c}_1, \widetilde{c}_2$ =arbitrary constants.

4.3.2.13. The corresponding homogeneous ODE, $\ddot{y} + \dot{y} = 0$ has characteristic polynomial $s^2 + s = s(s+1)$, so the homogeneous solution is

$$y_h(t) = c_1 + c_2 e^{-t}.$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(t) = v_1(t) + v_2(t)e^{-t},$$

where $v_1(t), v_2(t)$ are functions to be determined later.

We assume that

$$(\star) \quad \dot{v}_1(t) + \dot{v}_2(t)e^{-t} \equiv 0,$$

so

$$\dot{y}(t) = -e^{-t} v_2(t)$$

and thus

$$\ddot{y}(t) = -e^{-t}\dot{v}_2(t) + e^{-t} v_2(t).$$

Substitute all of that into the original, non-homogeneous ODE $\ddot{y} + \dot{y} = e^{-t}$, which is in standard form. We get

$$-e^{-t}\dot{v}_2(t) = e^{-t}.$$

So, $\dot{v}_2(t) \equiv -1$, whose solution is $v_2(t) = -t + c_2$, where c_2=arbitrary constant.

From $(\star)$ we get $\dot{v}_1(t) = -\dot{v}_2(t)e^{-t} = e^{-t}$, whose solution is $v_2(t) = -e^{-t} + c_1$.

Putting everything together, the general solution of the ODE is

$$y(t) = (-e^{-t} + c_1) + (-t + c_2)e^{-t}$$

hence

$$y(t) = -t\, e^{-t} + \widetilde{c}_1 + \widetilde{c}_2\, e^{-t},$$

where $\widetilde{c}_1, \widetilde{c}_2$ =arbitrary constants. [Note that the e^{-t} term that came from $v_1(t)$ was added to c_2 to get the arbitrary constant $\widetilde{c}_2$.]

4.3.2.15. The corresponding homogeneous ODE, $\ddot{y}+4\dot{y}+4y=0$ has characteristic polynomial $s^2+4s+4=(s+2)^2$, so the homogeneous solution is

$$y_h(t) = c_1 e^{-2t} + c_2\, t\, e^{-2t}.$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(t) = v_1(t)e^{-2t} + v_2(t)\, t\, e^{-2t},$$

where $v_1(t), v_2(t)$ are functions to be determined later.

We assume that

$$\dot{v}_1(t)e^{-2t} + \dot{v}_2(t)\, t\, e^{-2t} \equiv 0,$$

so

$$\dot{y}(t) = (-2e^{-2t})v_1(t) + (1-2t)e^{-2t}v_2(t)$$

and thus

$$\ddot{y}(t) = (-2e^{-2t})\dot{v}_1(t) + (1-2t)e^{-2t}\dot{v}_2(t) + 4e^{-2t}v_1(t) + (-4+4t)e^{-2t}v_2(t).$$

Substitute all of that into the original, non-homogeneous ODE $\ddot{y}+4\dot{y}+4y=\sqrt{t}\,e^{-2t}$, which is in standard form. We get

$$(-2e^{-2t})\dot{v}_1(t) + (1-2t)e^{-2t}\dot{v}_2(t) = \sqrt{t}\,e^{-2t}.$$

So, $\dot{v}_1(t), \dot{v}_2(t)$ should satisfy the system of linear equations

$$\left\{ \begin{array}{ll} e^{-2t}\dot{v}_1(t) + te^{-2t}\dot{v}_2(t) & = 0 \\ \\ -2e^{-2t}\dot{v}_1(t) + (1-2t)e^{-2t}\dot{v}_2(t) & = \sqrt{t}\,e^{-2t} \end{array} \right\}.$$

Dividing each equation by e^{-2t} gives an equivalent system,

$$\left\{ \begin{array}{ll} \dot{v}_1(t) + t\,\dot{v}_2(t) & = 0 \\ \\ -2\dot{v}_1(t) + (1-2t)\dot{v}_2(t) & = \sqrt{t} \end{array} \right\}.$$

Using the inverse of a 2×2 matrix, we get

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & t \\ -2 & 1-2t \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \sqrt{t} \end{bmatrix} = \begin{bmatrix} 1-2t & -t \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{t} \end{bmatrix} = \begin{bmatrix} -t^{3/2} \\ t^{1/2} \end{bmatrix}.$$

We obtain, using the substitution $w=\cos t$,

$$v_1(t) = \int \dot{v}_1(t)dt = -\int -t^{3/2}\,dt = -\frac{2}{5}\,t^{5/2} + c_1,$$

where c_1 =arbitrary constant. We also get

$$v_2(t) = \int \dot{v}_2(t)dt = \int t^{1/2}\,dt = \frac{2}{3}\,t^{3/2} + c_2,$$

where c_2 =arbitrary constant.

Putting everything together, the general solution of the ODE is

$$y(t) = \Big(-\frac{2}{5}\,t^{5/2} + c_1\Big)e^{-2t} + \Big(\frac{2}{3}\,t^{3/2} + c_2\Big)t\,e^{-2t}$$

hence

$$y(t) = \frac{4}{15}\,t^{5/2}\,e^{-2t} + c_1 e^{-2t} + c_2\,t\,e^{-2t},$$

where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -\frac{8}{15}\, t^{5/2}\, e^{-2t} + \frac{2}{3}\, t^{3/2}\, e^{-2t} - 2c_1 e^{-2t} + c_2(1-2t)e^{-2t},$$

where c_1, c_2 =arbitrary constants.

The ICs require

$$\left\{ \begin{array}{c} -1 = y(1) = \left(\frac{4}{15} + c_1 \ + c_2\right) e^{-2} \\ \\ 0 = \dot{y}(1) = \left(\frac{2}{15} - 2c_1 - c_2\right) e^{-2} \end{array} \right\},$$

so, after some calculations, $c_1 = e^2 + \frac{2}{5}$, hence $c_2 = -2\Big(e^2 + \frac{1}{3}\Big)$.

The solution of the IVP is

$$y(t) = \frac{4}{15}\, t^{5/2}\, e^{-2t} - e^{-2t} - 2\, t\, e^{-2t},$$

that is,

$$y(t) = \left(e^2 + \frac{2}{5} - 2\Big(e^2 + \frac{1}{3}\Big)t + \frac{4}{15}\, t^{5/2}\right) e^{-2t}.$$

4.3.2.17. The corresponding homogeneous ODE is the Cauchy-Euler ODE $r^2y'' + ry' + y = 0$, where $' = \frac{d}{dr}$. Substituting $y(r) = r^m$ into that ODE gives characteristic equation $0 = m(m-1) + m + 1 = m^2 + 1$. The roots are $m = \pm i$, so the homogeneous solution is

$$y_h(r) = c_1 \cos(\ln r) + c_2 \sin(\ln r).$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(r) = v_1(r) \cdot \cos(\ln r) + v_2(r) \cdot \sin(\ln r),$$

where $v_1(r), v_2(r)$ are functions to be determined later.

We assume that

$$v_1'(r) \cdot \cos(\ln r) + v_2'(r) \cdot \sin(\ln r) \equiv 0,$$

so

$$y'(r) = -r^{-1} \sin(\ln r) \cdot v_1(r) + r^{-1} \cos(\ln r) \cdot v_2(r)$$

and thus

$$y''(r) = -r^{-1} \sin(\ln r) v_1'(r) + r^{-1} \cos(\ln r) \cdot v_2'(r) + \big(r^{-2} \sin(\ln r) - r^{-2} \cos(\ln r)\big) v_1(r)$$
$$+ \big(-r^{-2} \cos(\ln r) - r^{-2} \sin(\ln r)\big) v_2(r).$$

Put the non-homogeneous ODE into standard form by dividing through by r^2 to get $y''+r^{-1}y'+r^{-2}y = 1$. Substitute into that ODE to get

$$-r^{-1} \sin(\ln r) \cdot v_1'(r) + r^{-1} \cos(\ln r) \cdot v_2'(r) = 1.$$

So, $v_1'(r), v_2'(r)$ should satisfy the system of linear equations

$$\left\{ \begin{array}{rl} \cos(\ln r) \cdot v_1'(r) + \sin(\ln r) \cdot v_2'(r) & = 0 \\ \\ -r^{-1} \sin(\ln r) \cdot v_1'(r) + r^{-1} \cos(\ln r) \cdot v_2'(r) & = 1 \end{array} \right\}.$$

Using the inverse of a 2×2 matrix, we get

$$\begin{bmatrix} v_1' \\ \\ v_2' \end{bmatrix} = \begin{bmatrix} \cos(\ln r) & \sin(\ln r) \\ \\ -r^{-1} \sin(\ln r) & r^{-1} \cos(\ln r) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} = \frac{1}{r^{-1}} \begin{bmatrix} r^{-1} \cos(\ln r) & -\sin(\ln r) \\ \\ r^{-1} \sin(\ln r) & \cos(\ln r) \end{bmatrix} \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} = \begin{bmatrix} -r \sin(\ln r) \\ \\ r \cos(\ln r) \end{bmatrix}.$$

The substitution $w = \ln r$, hence $dr = r\,dw = e^w dw$, along with the integration formula (3.10), gives

$$v_1(r) = \int v_1'(r)dr = \int -r\sin(\ln r)\,dr = -\int e^w \sin(w)\,e^w dw = -\int e^{2w}\sin(w)\,dw$$

$$= -\frac{e^{2w}}{2^2+1^2}\cdot(2\cdot\sin w - \cos w) + c_1 = \frac{1}{5}\,r^2\big(-2\sin(\ln r) + \cos(\ln r)\big) + c_1,$$

where c_1 =arbitrary constant.

Similarly, using the integration formula (3.9) we get

$$v_2(r) = \int v_2'(r)dr = \int r\cos(\ln r)\,dr = \int e^w\cos(w)\,e^w dw = \int e^{2w}\cos(w)\,dw = \frac{e^{2w}}{2^2+1^2}\cdot(\sin w + 2\cos w) + c_2$$

$$= \frac{1}{5}\,r^2\big(\sin(\ln r) + 2\cos(\ln r)\big) + c_2,$$

where c_2 =arbitrary constant.

Putting everything together, the general solution of the ODE is

$$y(r) = \frac{1}{5}\,r^2\Big(\big(\cancel{-2\sin(\ln r)} + \cos(\ln r)\big)\cos(\ln r) + \big(\sin(\ln r) + \cancel{2\cos(\ln r)}\big)\sin(\ln r)\Big) + c_1\cos(\ln r) + c_2\sin(\ln r),$$

hence

$$y(r) = \frac{1}{5}\,r^2\Big(\cos^2(\ln r) + \sin^2(\ln r)\Big) + c_1\cos(\ln r) + c_2\sin(\ln r).$$

So, the general solution of the ODE is

$$y(r) = \frac{1}{5}\,r^2 + c_1\cos(\ln r) + c_2\sin(\ln r).$$

where c_1, c_2 =arbitrary constants. It follows that

$$y'(r) = \frac{4}{5}\,r - c_1\,r^{-1}\sin(\ln r) + c_2\,r^{-1}\cos(\ln r).$$

The ICs require

$$\left\{\begin{array}{rl} 3 = y(1) = \frac{1}{5} + & c_1 \\ -1 = y'(1) = \frac{2}{5} & \qquad + c_2 \end{array}\right\},$$

so $c_1 = \frac{14}{5}$ and $c_2 = -\frac{7}{5}$.

The solution of the IVP is

$$y(r) = \frac{1}{5}\,r^2 + \frac{14}{5}\,\cos(\ln r) - \frac{7}{5}\,\sin(\ln r).$$

4.3.2.19. We are given that the corresponding homogeneous ODE has solutions $y_1(x) = e^x$ and $y_2(x) = 1 + x + \frac{1}{2}x^2$. We calculate that their Wronskian is

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & 1 + x + \frac{1}{2}x^2 \\ e^x & 1 + x \end{vmatrix} = -\frac{1}{2}\,x^2\,e^x \neq 0,$$

so this set of functions $\{y_1(x), y_2(x)\}$ is linearly independent on any interval that does not contain $x = 0$.

So, the general solution of the corresponding homogeneous ODE is

$$y_h(x) = c_1\,e^x + c_2\big(1 + x + \frac{1}{2}x^2\big).$$

Let's try to find a solution of the non-homogeneous ODE in the form

$$y(x) = v_1(x)\cdot e^x \;+\; v_2(x)\cdot(1 + x + \frac{1}{2}x^2),$$

where $v_1(x), v_2(x)$ are functions to be determined later.

We assume that

$$v_1'(x) \cdot e^x \; + \; v_2'(x) \cdot \big(1 + x + \frac{1}{2}x^2\big) \equiv 0,$$

so

$$y'(x) = e^x \cdot v_1(x) \; + \; (1+x) \cdot v_2(x)$$

and thus

$$y''(x) = e^x\, v_1'(x) + (1+x)\, v_2'(x) + e^x\, v_1(x) + v_2(x).$$

Put the non-homogeneous ODE into standard form by dividing through by x to get $y'' - \frac{x+2}{x}\, y' + \frac{2}{x}\, y = x^2$. Substitute into that ODE to get

$$e^x \; \cdot v_1'(x) + (1+x) \cdot v_2'(x) = x^2.$$

So, $v_1'(x), v_2'(x)$ should satisfy the system of linear equations

$$\left\{ \begin{array}{ll} e^x\, v_1'(x) + \big(1 + x + \frac{1}{2}x^2\big)\, v_2'(x) & = 0 \\ \\ e^x\, v_1'(x) + (1+x) v_2'(x) & = x^2 \end{array} \right\}.$$

Using the inverse of a 2×2 matrix, we get

$$\begin{bmatrix} v_1' \\ \\ v_2' \end{bmatrix} = \begin{bmatrix} e^x & 1 + x + \frac{1}{2}x^2 \\ \\ e^x & 1+x \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \\ x^2 \end{bmatrix} = \frac{1}{-\frac{1}{2}\, x^2\, e^x} \begin{bmatrix} 1+x & -\big(1 + x + \frac{1}{2}x^2\big) \\ \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 0 \\ \\ x^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\big(1 + x + \frac{1}{2}x^2\big)e^{-x} \\ \\ -2 \end{bmatrix}.$$

We obtain, using integration by parts twice, that

$$v_1(x) = \int v_1'(x) dx = \int 2\big(1 + x + \frac{1}{2}x^2\big)e^{-x} dx = 2\left(\big(1 + x + \frac{1}{2}x^2\big)(-e^{-x}) - \int (1+x)(-e^{-x}) dx \right)$$

$$= -2\big(1 + x + \frac{1}{2}x^2\big)e^{-x} + \int 2(1+x)e^{-x} dx = -2\big(1 + x + \frac{1}{2}x^2\big)e^{-x} + \left(2(1+x)(-e^{-x}) - \int 2(-e^{-x}) dx \right)$$

$$= e^{-x}\left(-2\big(1 + x + \frac{1}{2}x^2\big) - 2(1+x) - 2 + c_1 \right) = e^{-x}\left(-6 - 4x - x^2\right) + c_1,$$

where c_1 =arbitrary constant, and

$$v_2(x) = \int v_2'(x) dx = \int -2 \; dx = -2x + c_2,$$

where c_1, c_2 are arbitrary constants.

Putting everything together, the general solution of the ODE is

$$y(x) = -6 - 4x - x^2 - 2x\big(1 + x + \frac{1}{2}x^2\big) + c_1 e^x + c_2\big(1 + x + \frac{1}{2}x^2\big),$$

hence

$$y(x) = -6 - 4x - x^2 - 2x - 2x^2 - x^3 + c_1 e^x + c_2\big(1 + x + \frac{1}{2}x^2\big),$$

that is,

$$y(x) = -6 - 6x - 3x^2 - x^3 + c_1 e^x + c_2\big(1 + x + \frac{1}{2}x^2\big)$$

where c_1, c_2 =arbitrary constants.

4.3.2.21. The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 5s + 6 = (s+3)(s+2)$ $\Rightarrow \mathsf{L}_1 = -3, -2$

$f(t) = t^2 e^{-t} \Rightarrow \mathsf{L}_2 = -1, -1, -1 \Rightarrow$ Superlist is $\mathsf{L} = -3, -2, -1, -1, -1$

$\Rightarrow y(t) = c_1 e^{-3t} + c_2 e^{-2t} + c_3 e^{-t} + c_4 t\, e^{-t} + t^2\, c_5 e^{-t} \Rightarrow y_p(t) = Ae^{-t} + Bt\, e^{-t} + Ct^2\, e^{-t}$, where A, B, C are constants to be determined.

It follow that $\dot{y}_p(t) = -Ae^{-t} + B(1-t)e^{-t} + C(2t - t^2)e^{-t}$ and $\ddot{y}_p(t) = Ae^{-t} + B(-2+t)e^{-t} + C(2 - 4t + t^2)e^{-t}$.

Substitute all of this into the original, non-homogenous ODE to get

$$t^2 e^{-t} = \ddot{y}_p + 5\dot{y}_p + 6y_p$$

$$= Ae^{-t} + B(-2+t)e^{-t} + C(2-4t+t^2)e^{-t} - 5Ae^{-t} + 5B(1-t)e^{-t} + 5C(2t-t^2)e^{-t} + 6Ae^{-t} + 6Bt\, e^{-t} + 6Ct^2\, e^{-t}$$

$$= (2A + 3B + 2C)e^{-t} + (2B + 6C)t\, e^{-t} + (2C)t^2\, e^{-t}$$

$t^2\, e^{-t}$ terms $\Rightarrow C = \frac{1}{2}$.

$t\, e^{-t}$ terms $\Rightarrow 2B + 6C = 0 \Rightarrow B = -\frac{3}{2}$,

e^{-t} terms $\Rightarrow 2A + 3B + 2C = 0 \Rightarrow A = \frac{7}{4} \quad \Rightarrow y_p(t) = \dfrac{7}{4}\, e^{-t} - \dfrac{3}{2}\, t\, e^{-t} + \dfrac{1}{2}\, t^2\, e^{-t}$.

The general solution of the ODE is

$$y(t) = c_1 e^{-3t} + c_2 e^{-2t} + \frac{7}{4}\, e^{-t} - \frac{3}{2}\, t\, e^{-t} + \frac{1}{2}\, t^2\, e^{-t},$$

where c_1, c_2 =arbitrary constants. This agrees with the conclusion of Example 4.16.

I think variation of parameters, as we did in Example 4.16, was easier for this problem.

Section 4.4.1

4.4.1.1. $\mathcal{L}\left[\,-5e^{3t}+\sin 2t\,\right]=-5\mathcal{L}\left[\,e^{3t}\,\right]+\mathcal{L}\left[\,\sin 2t\,\right]=-\dfrac{5}{s-3}+\dfrac{2}{s^2+4}$

4.4.1.3. $\mathcal{L}\left[\,1+at+\dfrac{1}{2!}(at)^2+\dfrac{1}{3!}(at)^3\,\right]=\mathcal{L}\left[\,1\,\right]+a\mathcal{L}\left[\,t\,\right]+\dfrac{a^2}{2!}\mathcal{L}\left[\,t^2\,\right]+\dfrac{a^3}{3!}\mathcal{L}\left[\,t^3\,\right]=\dfrac{1}{s}+\dfrac{a}{s^2}+\dfrac{a^2}{s^3}+\dfrac{a^3}{s^4}$
$=\dfrac{1}{s}\left(1+\dfrac{a}{s}+\left(\dfrac{a}{s}\right)^2+\left(\dfrac{a}{s}\right)^3\right)$

4.4.1.5. $e^{t/2}\cos\left(\dfrac{\sqrt{3}}{2}t\right)=\mathcal{L}\left[\,e^{t/2}f(t)\,\right]$, where $f(t)=\cos\left(\frac{\sqrt{3}}{2}t\right)$. Using Table entry L1.8, we have

$$\mathcal{L}\left[\,e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right]=\mathcal{L}\left[\,\cos\left(\frac{\sqrt{3}}{2}t\right)\right]\Big|_{s\mapsto(s-\frac{1}{2})}=\frac{s}{s^2+(\sqrt{3}/2)^2}\Big|_{s\mapsto(s-\frac{1}{2})}=\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{s-\frac{1}{2}}{s^2+s+1}\,.$$

4.4.1.7. Partial fractions: $\dfrac{s-2}{(s+2)(s^2+1)}=\dfrac{A}{s-2}+\dfrac{Bs+C}{s^2+1}\quad\Rightarrow\quad(\star)\;\; s-2=A(s^2+1)+(Bs+C)(s+2)$. Substitute $s=-2$ into $(\star)$ to get

$$-2-2=A((-2)^2+1)+(Bs+C)(-2+2)=5A$$

so $A=-\frac{4}{5}$. Substitute this into $(\star)$ to get

$$s-2=-\frac{4}{5}(s^2+1)+(Bs+C)(s+2),$$

hence

$$Bs+C=\frac{s-2+\frac{4}{5}\,s^2+\frac{4}{5}}{(s+2)}=\frac{\frac{4}{5}\,s^2+s-\frac{6}{5}}{(s+2)}=\frac{1}{5}\cdot\frac{4s^2+5s-6}{(s+2)}=\frac{1}{5}(4s-3).$$

So,

$$\mathcal{L}^{-1}\left[\,\frac{s-2}{(s+2)(s^2+1)}\,\right]=-\frac{4}{5}\,\mathcal{L}\left[\,\frac{1}{s+2}\,\right]+\frac{1}{5}\,\mathcal{L}\left[\,\frac{4s-3}{s^2+1}\,\right]=-\frac{4}{5}\,e^{-2t}+\frac{1}{5}(4\cos t-3\sin t).$$

4.4.1.9. $\mathcal{L}^{-1}\left[\,\dfrac{-s+4}{s^2+4s+7}\,\right]=\mathcal{L}^{-1}\left[\,\dfrac{-s+4}{(s+2)^2+3}\,\right]=\mathcal{L}^{-1}\left[\,\dfrac{-(s+2-2)+4}{(s+2)^2+3}\,\right]=\mathcal{L}^{-1}\left[\,\dfrac{-(s+2)+6}{(s+2)^2+3}\,\right]$

$$=\mathcal{L}^{-1}\left[\,F(s+2)\,\right]=e^{-2t}\big(-\cos(\sqrt{3}\,t)+\frac{6}{\sqrt{3}}\sin(\sqrt{3}\,t)\big)=e^{-2t}\big(-\cos(\sqrt{3}\,t)+2\sqrt{3}\,\sin(\sqrt{3}\,t)\big).$$

4.4.1.11. Partial fractions: $\dfrac{s+6}{(s+3)^2(s^2+2s+2)}=\dfrac{A}{s+3}+\dfrac{B}{(s+3)^2}+\dfrac{Cs+E}{(s^2+2s+2)}$

$\Rightarrow\quad(\star)\;\; s+6=A(s+3)(s^2+2s+2)+B(s^2+2s+2)+(Cs+E)(s+3)^2$. Substitute $s=-3$ into $(\star)$ to get

$$3=A\cdot 0+5B+C\cdot 0$$

so $B=\frac{3}{5}$. Substitute this into $(\star)$ to get

$$(\star)\;\; s+6=A(s+3)(s^2+2s+2)+\frac{3}{5}(s^2+2s+2)+(Cs+E)(s+3)^2,$$

hence

$$(\star\star)\;\; A(s^2+2s+2)+(Cs+E)(s+3)=\frac{s+6-\frac{3}{5}(s^2+2s+2)}{s+3}=\frac{1}{5}\cdot\frac{-3s^2-s+24}{s+3}=\frac{1}{5}(-3s+8).$$

Substitute $s=-3$ into ($\star\star$) to get
$$5A = \frac{17}{5}$$
so $A=\frac{17}{25}$. Substitute this into ($\star\star$) to get
$$(\star\star\star)\quad \frac{1}{5}(-3s+8) = \frac{17}{25}(s^2+2s+2) + (Cs+E)(s+3),$$
hence
$$Cs+E = \frac{\frac{1}{5}(-3s+8) - \frac{17}{25}(s^2+2s+2)}{(s+3)} = \frac{1}{25}\cdot\frac{-17s^2-49s+6}{(s+3)} = \frac{1}{25}(-17s+2).$$
So,
$$\mathcal{L}^{-1}\left[\frac{s+6}{(s+3)^2(s^2+2s+2)}\right] = \frac{17}{5}\,\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] + \frac{3}{5}\,\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right] + \frac{1}{25}\,\mathcal{L}^{-1}\left[\frac{-17s+2}{s^2+2s+2}\right]$$
$$= \frac{17}{25}\,e^{-3t} + \frac{3}{5}\,t\,e^{-3t} + \frac{1}{25}\,\mathcal{L}^{-1}\left[\frac{-17(s+1-1)+2}{(s+1)^2+1}\right]$$
$$= \frac{17}{25}\,e^{-3t} + \frac{3}{5}\,t\,e^{-3t} + e^{-t}\left(-\frac{17}{25}\cos t + \frac{19}{25}\sin t\right).$$

4.4.1.13. Define $Y(s) \triangleq \mathcal{L}[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\dot{y}-2y=3e^{4t}$, and use the IC $y(0)=-1$ to get $sY-(-1)-2Y=\frac{3}{s-4}$. This gives
$$Y(s) = -\frac{1}{s-2} + \frac{3}{(s-2)(s-4)}$$
and then we use a partial fractions expansion:
$$\frac{3}{(s-2)(s-4)} = \frac{A}{s-2} + \frac{B}{s-4},$$
where A, B are constants to be determined. After multiplying both sides by $(s-2)(s-4)$ we get ($\star$) $\;3 = A(s-4)+B(s-2)$. Substituting $s=2$ and $s=4$ into ($\star$) we get
$$\left\{\begin{array}{ll} @s=2: & 3=-2A \\ @s=4: & 3=\ \ 2B \end{array}\right\},$$
so $A=-\frac{3}{2}, B=\frac{3}{2}$. So,
$$y(t)=\mathcal{L}^{-1}[\,Y(s)\,] = \mathcal{L}^{-1}\left[-\frac{1}{s-2} - \frac{3}{2}\cdot\frac{1}{s-2} + \frac{3}{2}\frac{1}{s-4}\right] = \mathcal{L}^{-1}\left[-\frac{5}{2}\cdot\frac{1}{s-2} + \frac{3}{2}\frac{1}{s-4}\right] = -\frac{5}{2}\,e^{2t} + \frac{3}{2}\,e^{4t}.$$
There is no steady state solution.

4.4.1.15. Define $Y(s) \triangleq \mathcal{L}[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\ddot{y}+3\dot{y}-10y=0$, and use the ICs $y(0)=1, \dot{y}(0)=-3$ to get $s^2Y-s+3+3(sY-1)-10Y=0$. This gives
$$Y(s) = \frac{s}{s^2+3s-10} = \frac{s}{(s+5)(s-2)}.$$
Partial fractions: $\dfrac{s}{s^2+3s-10} = \dfrac{A}{s+5} + \dfrac{B}{s-2} \quad\Rightarrow\quad (\star)\;\; s = A(s-2)+B(s+5)$. We get
$$\left\{\begin{array}{ll} @s=-5: & -5=-7A \\ @s=2: & \ \ 2=\ \ 7B \end{array}\right\},$$

so $A = \frac{5}{7}, B = \frac{2}{7}$. Substitute this into $(\star)$ to get that the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{5}{7}\cdot\frac{1}{s+5} + \frac{2}{7}\cdot\frac{B}{s-2}\right] = \frac{5}{7}\,e^{-5t} + \frac{2}{7}\,e^{2t}.$$

There is no steady state solution.

4.4.1.17. (a) The corresponding LCCHODE's characteristic polynomial is $\mathcal{P}(s) = s^2 + 9 \quad \Rightarrow \mathsf{L}_1 = \pm\, i\, 3$
$f(t) = 10te^{-t} \Rightarrow \mathsf{L}_2 = -1, -1 \quad \Rightarrow$ Superlist is $\mathsf{L} = \pm\, i\, 3, -1, -1$ implies

$$y(t) = c_1\cos(3t) + c_2\sin(3t) + c_3e^{-t} + c_4te^{-t}$$

$\Rightarrow y_p(t) = Ae^{-t} + Bte^{-t}$, where A, B are constants to be determined. First calculate $\dot{y}_p(t) = -Ae^{-t} + B(-t+1)e^{-t}$ and then $\ddot{y}_p(t) = Ae^{-t} + B(t-2)e^{-t}$. We have

$$10te^{-t} = \ddot{y}_p + 9y_p = Ae^{-t} + B(t-2)e^{-t} + 9(Ae^{-t} + Bte^{-t}) = (10A - 2B)e^{-t} + 10Bte^{-t}$$

hence $B = 1$ and thus $A = \frac{1}{5}$.

The general solution of the ODE is $y(t) = c_1\cos(3t) + c_2\sin(3t) + \dfrac{1}{5}\,e^{-t} + te^{-t}$, where c_1, c_2 =arbitrary constants. It follows that

$$\dot{y}(t) = -3c_1\sin(3t) + 3c_2\cos(3t) - \frac{1}{5}\,e^{-t} + (-t+1)e^{-t}.$$

The ICs require

$$\left\{\begin{array}{ll} 0 = y(0) = c_1 + \frac{1}{5} & \\ & \\ 0 = \dot{y}(0) = & 3c_2 - \frac{1}{5} + 1 \end{array}\right\},$$

so $c_1 = -\frac{1}{5}$ and $c_2 = -\frac{4}{15}$. The solution of the IVP is

$$y(t) = -\frac{1}{5}\cos(3t) - \frac{4}{15}\sin(3t) + \frac{1}{5}\,e^{-t} + te^{-t},$$

[By the way, the steady state solution is $y_S(t) = -\dfrac{1}{5}\cos(3t) - \dfrac{4}{15}\sin(3t)$.]

(b) Define $Y(s) \triangleq \mathcal{L}[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\ddot{y} + 9y = 10te^{-t}$, and use the ICs $y(0) = 0, \dot{y}(0) = 0$ to get $s^2Y + 9Y = \frac{10}{(s+1)^2}$. This gives

$$Y(s) = \frac{10}{(s^2+9)(s+1)^2}\,.$$

Partial fractions: $\dfrac{10}{(s^2+9)(s+1)^2} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{Cs+E}{(s^2+9)}$

$\Rightarrow \quad (\star)\;\; 10 = A(s+1)(s^2+9) + B(s^2+9) + (Cs+E)(s+1)^2$. Substitute $s = -1$ into $(\star)$ to get

$$10 = A\cdot 0 + 10B + C\cdot 0$$

so $B = 1$. Substitute this into $(\star)$ to get

$$(\star)\;\; 10 = A(s+1)(s^2+9) + 1\cdot(s^2+9) + (Cs+E)(s+1)^2,$$

hence

$$(\star\star)\;\; A(s^2+9) + (Cs+E)(s+1) = \frac{10 - s^2 - 9}{s+1} = \frac{1-s^2}{s+1} = -s+1$$

Substitute $s=-1$ into $(\star\star)$ to get $10A = 2$ so $A = -\frac{1}{5}$. Substitute this into $(\star\star)$ to get

$$(\star\star\star)\quad -s+1 = \frac{1}{5}(s^2+9) + (Cs+E)(s+1),$$

hence

$$Cs+E = \frac{-s+1-\frac{1}{5}(s^2+9)}{(s+1)} = \frac{1}{5}\cdot\frac{-s^2-5s-4}{(s+1)} = \frac{1}{5}(-s-4).$$

So,

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{5}\cdot\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{5}\cdot\frac{-s-4}{(s^2+9)}\right] = \frac{1}{5}\,\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{5}\,\mathcal{L}^{-1}\left[\frac{-s-4}{s^2+9}\right]$$

$$= \frac{1}{5}\,e^{-t} + t\,e^{-t} - \frac{1}{5}\big(\cos 3t + \frac{4}{3}\,\sin 3t\big).$$

[By the way, the steady state solution is $y_S(t) = -\dfrac{1}{5}\cos(3t) - \dfrac{4}{15}\sin(3t)$.]

4.4.1.19. Define $Y(s) \triangleq \mathcal{L}[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\ddot{y}+2\dot{y}+2y = \sin t$, and use the ICs $y(0)=a, \dot{y}(0)=b$ to get $s^2Y - as - b + 2(sY-a) + 2Y = \frac{1}{s^2+1}$. This gives

$$Y(s) = \frac{as+b+2a}{s^2+2s+2} + \frac{1}{(s^2+1)\big(s^2+2s+1\big)}\,.$$

Note that $\mathcal{L}^{-1}\left[\dfrac{as+b+2a}{s^2+2s+2}\right] = \mathcal{L}^{-1}\left[\dfrac{as+b+2a}{(s+1)^2+1}\right] = e^{-t}(...)$ will be transient and thus will not be part of the desired steady state solution.

Partial fractions: $\dfrac{1}{(s^2+1)\big(s^2+2s+1\big)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+E}{s^2+2s+2}$

$\Rightarrow\quad (\star)\ \ s = (As+B)(s^2+2s+2)+(Cs+E)(s^2+1) = (A+C)s^3+(2A+B+E)s^2+(2A+2B+C)s+(2B+E)$. Sorting by powers of s we get

$$\left\{\begin{array}{ll} s^3: & 0 = A \qquad\quad +C \\ s^2: & 0 = 2A + \ B \qquad + E \\ s^1: & 0 = 2A + 2B + C \\ s^0: & 1 = \qquad 2B \qquad + E \end{array}\right\},$$

so

$$\begin{bmatrix} A \\ B \\ C \\ E \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & 2 & 1 & -2 \\ -2 & -1 & 2 & 1 \\ 6 & -2 & -1 & 2 \\ 4 & 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Substitute these results into $(\star)$ to get that

$$Y(s)) = \frac{as+b+2a}{s^2+2s+2} + \frac{1}{5}\left(\frac{-2s+1}{s^2+1} + \frac{2s+3}{s^2+2s+2}\right).$$

As noted earlier, all terms whose denominator is $(s+1)^2+1$ are transient. The steady state solution is

$$y_S(t) = \frac{1}{5}\,\mathcal{L}^{-1}\left[\frac{-2s+1}{s^2+1}\right] = \frac{1}{5}(-2\cos t + \sin t).$$

4.4.1.21. $\mathcal{L}^{-1}[\,y(t)\,]$ has terms that come from the Laplace transform of $\cos(\sqrt{3}\,t)$ and/or $\sin(\sqrt{3}\,t)$, as well as from $\cos(t)$ and/or $\sin(t)$. For a second order ODE that models a sinusoidally forced oscillator problem, this can only happen if there is no damping and two unequal frequencies 1 and $\sqrt{3}$. This can happen only in the beats phenomenon case.

4.4.1.23. $\mathcal{L}\left[\,p_n(t)\,\right] = \mathcal{L}\left[\,1 + t + \frac{1}{2!}(t)^2 + ... + \frac{1}{n!}(t)^n\,\right] = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2!}\cdot\frac{2!}{s^3} + ... + \frac{1}{n!}\cdot\frac{n!}{s^{n+1}}$. Using a finite geometric series we get

$$\mathcal{L}\left[\,p_n(t)\,\right] = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + ... + \frac{1}{s^{n+1}} = \frac{1}{s}\left(1 + \frac{1}{s} + \frac{1}{s^2} + ... + \frac{1}{s^n}\right) = \frac{1}{s}\cdot\frac{1-\left(\frac{1}{s}\right)^{n+1}}{1-\frac{1}{s}}$$

$$= \frac{1-\left(\frac{1}{s}\right)^{n+1}}{s-1} \;\to\; \frac{1}{s-1} = \mathcal{L}\left[\,e^t\,\right], \quad \text{as} \quad n\to\infty, \text{ for } s>1.$$

Section 4.5.8

4.5.8.1. $\mathcal{L}\left[\,7+3e^{-2t}-t\,step(t-4)\,\right]=7\mathcal{L}\left[\,1\,\right]+3\mathcal{L}\left[\,e^{-2t}\,\right]-\mathcal{L}\left[\,t\,step(t-4)\,\right]=7\cdot\dfrac{1}{s}+3\cdot\dfrac{1}{s+2}-e^{-4s}\mathcal{L}\left[\,(t+4)\,\right]$

$$=\frac{7}{s}+\frac{3}{s+2}-\left(\frac{4}{s}+\frac{1}{s^2}\right)e^{-4s}.$$

4.5.8.3. $f(t)=\left\{\begin{array}{cc}-t+4, & 0\le t<2\\ t, & t\ge 2\end{array}\right\}=(-t+4)+\big(t-(-t+4)\big)step(t-2)=-t+4+2(t-2)step(t-2)$, so

$$\mathcal{L}\left[\,f(t)\,\right]=\mathcal{L}\left[\,-t+4+2(t-2)step(t-2)\,\right]=-\frac{1}{s^2}+\frac{4}{s}+\frac{2}{s^2}\,e^{-2s}.$$

4.5.8.5. $\mathcal{L}\left[\,h(t)\,\right]=\mathcal{L}\left[\displaystyle\int_0^t e^{-(t-u)}\cos(2u)du\right]=\mathcal{L}\left[\,e^{-t}*\cos(2t)\,\right]=\dfrac{1}{s+1}\cdot\dfrac{s}{s^2+4}=\dfrac{s}{(s+1)(s^2+4)}$

4.5.8.7. $\mathcal{L}\left[\dfrac{1}{\sqrt{3}}e^{-t/2}\sin\left(\dfrac{\sqrt{3}t}{2}\right)\right]=\dfrac{1}{\sqrt{3}}\mathcal{L}\left[\sin\left(\dfrac{\sqrt{3}t}{2}\right)\right]\Big|_{s\mapsto(s+\frac{1}{2})}=\dfrac{1}{\sqrt{3}}\cdot\dfrac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\dfrac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$

$$=\frac{\frac{1}{2}}{s^2+s+1}$$

so only (a) is correct.

4.5.8.9. Partial fractions: $\dfrac{1}{s(s-2)}=\dfrac{A}{s}+\dfrac{B}{s-2}\iff 1=A(s-2)+Bs$, where A,B are constants to be determined: Substitute in $s=0$ to find $1=-2A$, and substitute in $s=2$ to find that $1=2B$. So

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2-2s}\right]=\mathcal{L}^{-1}\left[e^{-s}\cdot\frac{1}{2}\left(-\frac{1}{s}+\frac{1}{s-2}\right)\right]=\frac{1}{2}\mathcal{L}^{-1}\left[-\frac{1}{s}+\frac{1}{s-2}\right]\Big|_{t\mapsto t-1}step(t-1)$$

$$=\frac{1}{2}\big(-1+e^{2(t-1)}\big)step(t-1).$$

4.5.8.11. <u>Method 1</u>: Using Table entry L1.13 with $\omega=2$, we get

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right]=\frac{1}{4}\mathcal{L}^{-1}\left[\frac{2\cdot 2\cdot s}{(s^2+2^2)^2}\right]=\frac{1}{4}\,t\sin(2t).$$

<u>Method 2</u>: $\mathcal{L}^{-1}\left[\dfrac{s}{(s^2+4)^2}\right]=\mathcal{L}^{-1}\left[\dfrac{1}{(s^2+4)}\cdot\dfrac{s}{(s^2+4)}\right]=\dfrac{1}{2}\sin(2t)*\cos(2t)=\dfrac{1}{2}\displaystyle\int_0^t\sin\big(2(t-u)\big)\cos(2u)du$

$$=\frac{1}{2}\int_0^t(\sin(2t)\cos(2u)-\cos(2t)\sin(2u))\cos(2u)du$$

$$=\frac{1}{2}\left(\sin(2t)\int_0^t\cos^2(2u)du-\cos(2t)\int_0^t\sin(2u)\cos(2u)du\right)$$

$$=\frac{1}{2}\left(\sin(2t)\int_0^t\frac{1}{2}\big(1+\cos(4u)\big)du-\cos(2t)\int_0^t\frac{1}{2}\sin(4u)du\right)$$

$$=\frac{1}{4}\left(\sin(2t)\Big[u+\frac{1}{4}\sin(4u)\Big]_0^t-\cos(2t)\Big[-\frac{1}{4}\cos(4u)\Big]_0^t\right)$$

$$=\frac{1}{4}\left(\sin(2t)\Big(t+\frac{1}{4}\sin(4t)\Big)-\cos(2t)\Big(\frac{1}{4}-\frac{1}{4}\cos(4t)\Big)\right)$$

$$= \frac{1}{4}\left(t\sin(2t) - \frac{1}{4}\cos(2t) + \frac{1}{4}\Big(\sin(2t)\sin(4t) + \cos(2t)\cos(4t)\Big)\right)$$

$$= \frac{1}{4}\left(t\sin(2t) - \frac{1}{4}\cos(2t) + \frac{1}{4}\cos(2t-4t)\right) = \frac{1}{4}\left(t\sin(2t) - \frac{1}{4}\cos(2t) + \frac{1}{4}\cos(-2t)\right) = \frac{1}{4}\,t\sin(2t).$$

By the way, using a convolution, as we did in Method 2, is probably how the Table entry L1.13 was first found.

4.5.8.13. Method 1: Using Table entries L1.13 and L1.14 with $\omega = 2$, we get

$$\mathcal{L}^{-1}\left[\frac{4}{(s^2+4)^2}\right] = \mathcal{L}^{-1}\left[\frac{\frac{1}{2}(s^2+4) - \frac{1}{2}(s^2-4)}{(s^2+4)^2}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s^2-4}{(s^2+4)^2}\right] = \frac{1}{4}\sin 2t - \frac{1}{2}t\cos 2t.$$

Method 2: $\mathcal{L}^{-1}\left[\frac{4}{(s^2+4)^2}\right] = \mathcal{L}^{-1}\left[\frac{2}{(s^2+4)}\cdot\frac{2}{(s^2+4)}\right] = \sin(2t) * \sin(2t) = \int_0^t \sin\big(2(t-u)\big)\sin(2u)du$

$$= \int_0^t (\sin(2t)\cos(2u) - \cos(2t)\sin(2u))\sin(2u)du = \left(\sin(2t)\int_0^t \sin(2u)\cos(2u)du - \cos(2t)\int_0^t \sin^2(2u)du\right)$$

$$= \left(\sin(2t)\int_0^t \frac{1}{2}\,\sin(4u)du - \cos(2t)\int_0^t \frac{1}{2}\big(1-\cos(4u)\big)du\right)$$

$$= \frac{1}{2}\left(\sin(2t)\Big[-\frac{1}{4}\,\cos(4u)\Big]_0^t - \cos(2t)\Big[u - \frac{1}{4}\,\sin(4u)\Big]_0^t\right)$$

$$= \frac{1}{2}\left(\sin(2t)\Big(\frac{1}{4} - \frac{1}{4}\,\cos(4t)\Big) - \cos(2t)\Big(t - \frac{1}{4}\,\sin(4t)\Big)\right)$$

$$= \frac{1}{2}\left(-t\cos(2t) + \frac{1}{4}\sin(2t) + \frac{1}{4}\Big(-\sin(2t)\cos(4t) + \cos(2t)\sin(4t)\Big)\right)$$

$$= \frac{1}{2}\left(-t\cos(2t) + \frac{1}{4}\sin(2t) - \frac{1}{4}\sin(2t-4t)\right) = \frac{1}{2}\left(-t\cos(2t) + \frac{1}{4}\sin(2t) + \frac{1}{4}\sin(2t)\right)$$

$$= -\frac{1}{2}\,t\cos(2t) + \frac{1}{4}\sin(2t).$$

By the way, using a convolution, as we did in Method 2, is probably how the Table entry L1.14 was first found.

4.5.8.15. Define $Y(s) \triangleq \mathcal{L}[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\dot{y} + 3y = 2 - step(t-1)$, and use the IC $y(0) = 4$ to get $sY - 4 + 3Y = \frac{2}{s} - \frac{1}{s}e^{-s}$. This gives

$$Y(s) = \frac{4}{s+3} + \frac{2}{s(s+3)} - \frac{1}{s(s+3)}e^{-s}$$

and then we use a partial fractions expansion:

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3},$$

where A, B are constants to be determined. After multiplying both sides by $s(s+3)$ we get $(\star)\quad 1 = A(s+3) + Bs$. Substituting $s = 0$ and $s = -3$ into $(\star)$ we get

$$\left\{\begin{array}{rl} @s = 0: & 1 = 3A \\ @s = -3: & 1 = -3B \end{array}\right\},$$

so $A = \frac{1}{3}, B = -\frac{1}{3}$. So,

$$y(t) = \mathcal{L}^{-1}\left[\,Y(s)\,\right] = \mathcal{L}^{-1}\left[\frac{4}{s+3} + \frac{2}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right) - \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right)e^{-s}\right]$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{10}{s+3} + \frac{2}{s} - \left(\frac{1}{s} - \frac{1}{s+3}\right)e^{-s}\right] = \frac{1}{3}\left(10e^{-3t} + 2 + \left(-1 + e^{-3(t-1)}\right) step(t-1)\right).$$

The graph of the solution is given in Figure 4.1.

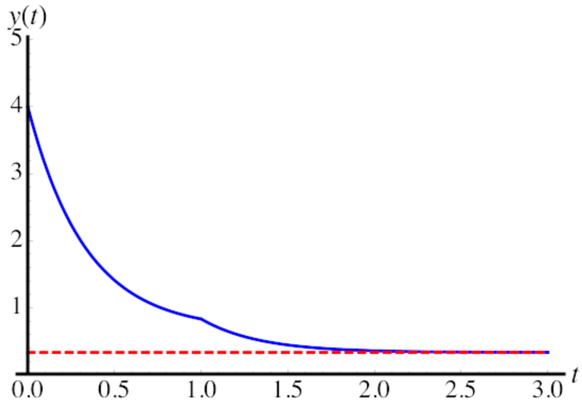


Figure 1: Answer for problem 4.5.8.15

4.5.8.17. Define $Y(s) \triangleq \mathcal{L}[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\ddot{y}+4y = 2-step(t-2)$, and use the ICs $y(0) = 0,\ \dot{y}(0) = -5$ to get $s^2Y - (0)s - (-5) + 4Y = \frac{1}{s}e^{-cs}$. This gives

$$Y(s) = \frac{-5}{s^2+4} + \frac{1}{s(s^2+4)}e^{-cs}$$

and then we use a partial fractions expansion:

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4},$$

where A, B, C are constants to be determined. After multiplying both sides by $s(s^2+4)$ we get $(\star)\quad 1 = A(s^2+4) + (Bs+C)s$. Substituting $s = 0$ into $(\star)$ we get $1 = 4A$. Substituting in the value of A gives

$$Bs + C = \frac{1 - \frac{1}{4}(s^2+4)}{s} = -\frac{1}{4}\,s.$$

So, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\,Y(s)\,\right] = \mathcal{L}^{-1}\left[\frac{-5}{s^2+4} + \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)e^{-cs}\right] = -\frac{5}{2}\,\sin(2t) + \frac{1}{4}\Big(1 - \cos\big(2(t-c)\big)\Big)step(t-c).$$

For the case when $c = \sqrt{2}$, the graph of the solution is given in Figure 4.3. Note that in this case the steady state solution is

$$y_S(t) = -\frac{5}{2}\sin 2t + \frac{1}{4}\Big(1 - \cos\big(2(t-\sqrt{2})\big)\Big) = -\frac{5}{2}\sin 2t + \frac{1}{4} - \frac{1}{4}\,\cos(2\sqrt{2})\cos 2t - \frac{1}{4}\,\sin(2\sqrt{2})\sin 2t$$

$$= \frac{1}{4} - \frac{1}{4}\,\cos(2\sqrt{2})\cos(2t) + \big(-\frac{5}{2} - frac14\,\sin(2\sqrt{2})\big)\sin(2t),$$

which has amplitude $\frac{1}{4} + \sqrt{\big(-\frac{1}{4}\,\cos(2\sqrt{2})\big)^2 + \big(-\frac{5}{2} - \frac{1}{4}\,\sin(2\sqrt{2})\big)^2}$, as indicated by the dashed horizontal lines in Figure 4.3.

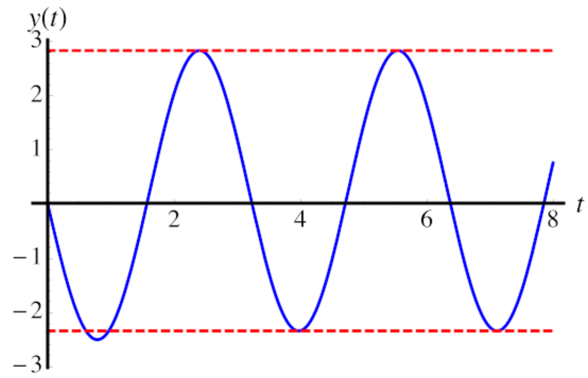


Figure 2: Answer for problem 4.5.8.17

4.5.8.19. Define $Y(s) \triangleq \mathcal{L}\,[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\dot{y} + 3y = \delta(t-2)$, and use the IC $y(0) = -1$ to get $sY - (-1) + 3Y = e^{-2s}$. This gives

$$Y(s) = -\frac{1}{s+3} + \frac{1}{s+3}e^{-2s}$$

So,

$$y(t) = \mathcal{L}^{-1}\,[\,Y(s)\,] = \mathcal{L}^{-1}\left[\frac{1}{s+3} + \frac{1}{s+3}e^{-2s}\right] = e^{-3t} + e^{-3(t-2)}step(t-2).$$

The graph of the solution is given in Figure 4.5.

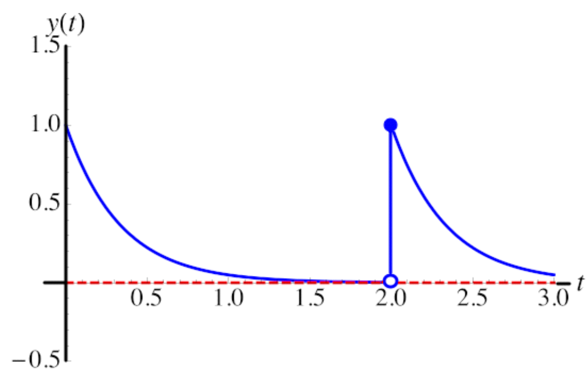


Figure 3: Answer for problem 4.5.8.19

4.5.8.21. Define $Y(s) \triangleq \mathcal{L}\,[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\ddot{y} + 1y = \frac{s}{s^2+1}$, and use the unspecified ICs Cs $y(0) = a$, $\dot{y}(0) = b$ to get $s^2Y - (a)s - b + Y = \frac{s}{s^2+1}$. This gives

$$Y(s) = \frac{as+b}{s^2+1} + \frac{s}{(s^2+1)^2}.$$

Using Table entries L1.13 and L1.14 with $\omega = 1$, we get

$$y(t) = \mathcal{L}^{-1}\left[\frac{as+b}{s^2+1} + \frac{s}{(s^2+1)^2}\right] = a\cos(t) + b\sin(t) + \mathcal{L}^{-1}\left[\frac{\frac{1}{2}(s^2+1) - \frac{1}{2}(s^2-1)}{(s^2+1)^2}\right]$$

$$= a\cos(t) + b\sin(t) + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = a\cos(t) + b\sin(t) + \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t).$$

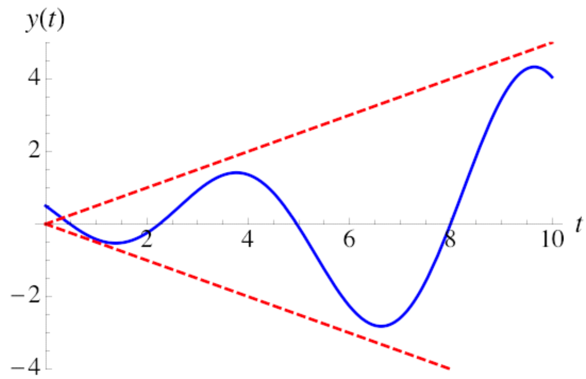


Figure 4: Answer for problem 4.5.8.21

The graph of the solution, when $a = \frac{1}{2}$ and $b = -1$, is given in Figure 4.7. The red dashed line show the "envelope" $y = \pm\frac{1}{2}\,t$.

4.5.8.23. Define $Y(s) \triangleq \mathcal{L}[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\ddot{y} + 4y = f(t)$, and use the ICs $y(0) = 0,\ \dot{y}(0) = 0$ to get $s^2Y - (0)s - 0 + 4Y = F(s)$, where $F(s) \triangleq \mathcal{L}[\,f(t)\,]$. So, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\cdot F(s)\right] = \frac{1}{2}\,\sin(2t) * f(t) = \int_0^t \frac{1}{2}\,\sin\big(2(t-u)\big)f(u)\,du,$$

as was desired.

4.5.8.25. Define $Y(s) \triangleq \mathcal{L}[\,y(t)\,]$. Take the Laplace transform of both sides of the ODE $\dot{y} + 5y = f(t)$, and use the IC $y(0) = 0$ to get $sY + 5Y = G(s)$, where $G(s) \triangleq \mathcal{L}[\,g(t)\,]$. So, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{s+5}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+5}\cdot G(s)\right] = e^{-5t} * g(t) = \int_0^t e^{-5(t-u)}g(u)\,du = e^{-5t}\int_0^t e^{5u}g(u)\,du.$$

Continuing as in Example 4.34, we analyze the solution: First, if $0 \le t \le \pi$, then

$$I(t) \triangleq \int_0^t e^{5u}g(u)du = \int_0^t e^{5u}\cdot u\,du = \left[\frac{u\,e^{5u}}{5} - \frac{e^{5u}}{25}\right]_0^t = \left(\frac{t\,e^{5t}-0}{5} - \frac{e^{5t}-1}{25}\right) = \frac{1+(5t-1)e^{5t}}{25}.$$

Next, if $\pi \le t \le 2\pi$, then a property of definite integration yields

$$\int_0^t e^{5u}g(u)du = \int_0^\pi e^{5u}\cdot u\,du + \int_\pi^t e^{5u}\cdot(u-\pi)du = I(\pi) + \left[\frac{(u-\pi)\,e^{5u}}{5} - \frac{e^{5u}}{25}\right]_\pi^t$$

$$= \frac{1+(5\pi-1)e^{5\pi}}{25} + \left(\frac{(t-\pi)e^{5t}-0}{5} - \frac{e^{5t}-1}{25}\right) = \frac{1+(5\pi-1)e^{5\pi}}{25} + \frac{1+(5(t-\pi)-1)e^{5t}}{25}$$

If $2\pi \le t \le 3\pi$, then

$$\int_0^t e^{5u}g(u)du = \int_0^{2\pi} e^{5u}g(u)du + \int_{2\pi}^t e^{5u}\cdot(u-2\pi)\,du = I(2\pi) + \left[\frac{(u-2\pi)\,e^{5u}}{5} - \frac{e^{5u}}{25}\right]_{2\pi}^t$$

$$= \frac{1+(5\pi-1)e^{5\pi}}{25} + \frac{1+(5\pi-1)e^{10\pi}}{25} + \left(\frac{(t-2\pi)e^{5t}-0}{5} - \frac{e^{5t}-1}{25}\right)$$

$$= \frac{1+(5\pi-1)e^{5\pi}}{25} + \frac{1+(5\pi-1)e^{10\pi}}{25} + \left(\frac{1+(5(t-2\pi)-1)e^{5t}}{25}\right)$$

If $3\pi \le t \le 4\pi$, then

$$\int_0^t e^{5u} g(u)du = \int_0^{3\pi} e^{2u} g(u)du + \int_{3\pi}^t e^{2u} \cdot (u-3\pi)du = I(3\pi) + \left[\frac{(u-3\pi)\,e^{5u}}{5} - \frac{e^{5u}}{25} \right]_{3\pi}^t$$

$$= \frac{1+(5\pi-1)e^{5\pi}}{25} + \frac{1+(5\pi-1)e^{10\pi}}{25} + \frac{1+(5\pi-1)e^{15\pi}}{25} + \left(\frac{(t-3\pi)e^{5t}-0}{5} - \frac{e^{5t}-1}{25} \right)$$

$$= \frac{1+(5\pi-1)e^{5\pi}}{25} + \frac{1+(5\pi-1)e^{10\pi}}{25} + \frac{1+(5\pi-1)e^{15\pi}}{25} + \left(\frac{1+(5(t-3\pi)-1)e^{5t}}{25} \right)$$

The solution of the IVP is

$$y(t) = \frac{1}{25}\, e^{-5t} \cdot \left\{ \begin{array}{ll} 1+(5t-1)e^{5t}, & 0 \le t \le \pi \\ 2+(5\pi-1)e^{5\pi}+(5(t-\pi)-1)e^{5t}, & \pi \le t \le 2\pi \\ 3+(5\pi-1)e^{5\pi}+(5\pi-1)e^{10\pi}+(5(t-2\pi)-1)e^{5t}, & 2\pi \le t \le 3\pi \\ 4+(5\pi-1)e^{5\pi}+(5\pi-1)e^{10\pi}+(5\pi-1)e^{15\pi}+(5(t-3\pi)-1)e^{5t}, & 3\pi \le t \le 4\pi \\ \vdots & \vdots \end{array} \right\}.$$

In general, for $k\pi \le t \le (k+1)\pi$,

$$y(t) = \frac{1}{25}e^{-5t} \left(k+1+(5(t-k\pi)-1)e^{5t} + (5\pi-1)\sum_{j=1}^{k} e^{5k\pi} \right).$$

The solution is graphed in Figure 4.8. By the way, despite appearances, the solution graphed in Figure 4.8 is *not* periodic. For example, $y\left(\frac{\pi}{4}\right) \approx 0.117868$ versus $y\left(\frac{13\pi}{4}\right) \approx 0.128671$. Nevertheless, there is a steady state solution hiding in Figure 4.8.

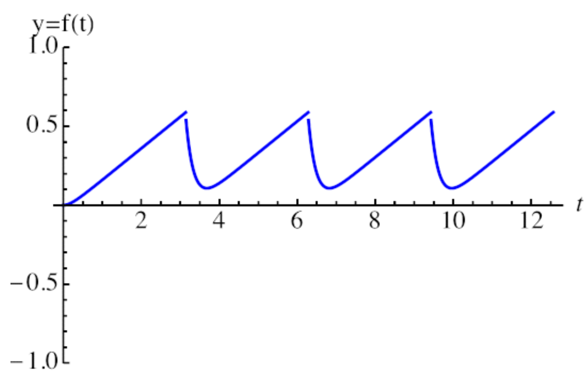


Figure 5: Answer for problem 4.5.8.25

Section 4.6.4

4.6.4.1. Substitute $y_k = r^k$ into $0 = y_{k+2} - y_k$ to get the characteristic equation $0 = r^2 - 1 = (r-1)(r+1)$. The general solution of the difference equation is $y_k = c_1\, 1^k + c_2\,(-1)^k$, that is,

$$y_k = c_1 + c_2\,(-1)^k,$$

where c_1, c_2 are arbitrary constants.

4.6.4.3. Substitute $y_k = r^k$ into $0 = y_{k+2} + y_k$ to get the characteristic equation $0 = r^2 + 1$, so the roots are $r = \pm i$. The root $r = i$ can be rewritten in polar exponential form as

$$i = r = \rho\, e^{i\omega} = \rho\cos\omega + \, i\,\rho\sin\omega = 1 \cdot e^{i(\pi/2)},$$

hence $\rho = 1$ and $\omega = \dfrac{\pi}{2}$. The general solution of the difference equation is

$$y_k = \rho^k\big(c_1\, \cos(\omega k) + c_2\, \sin(\omega k)\big),$$

that is,

$$y_k = c_1\, \cos\big(\frac{\pi}{2}\,k\big) + c_2\, \sin\big(\frac{\pi}{2}\,k\big),$$

where c_1, c_2 are arbitrary constants.

4.6.4.5. Substitute $y_k = r^k$ into $0 = y_{k+3} - y_k$ to get the characteristic equation $0 = r^3 - 1 = (r-1)(r^2+r+1) = (r-1)\Big(\big(r+\frac{1}{2}\big)^2 + \frac{3}{4}\Big)$. The roots are $r = 1$ and $r = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$.

The root $r = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$ can be rewritten in polar exponential form as

$$-\frac{1}{2} + i\,\frac{\sqrt{3}}{2} = r = \rho\, e^{i\omega}, = \rho\cos\omega + \, i\,\rho\sin\omega = e^{i\,(2\pi/3)},$$

hence $\rho = 1$ and $\omega = \dfrac{2\pi}{3}$. The general solution of the difference equation is

$$y_k = c_1\, 1^k + c_2\, 1^k\, \cos\big(\omega k\big) + c_2\, 1^k\, \sin\big(\omega k\big),$$

that is,

$$y_k = c_1 + c_2\, \cos\big(\frac{2\pi}{3}\,k\big) + c_3\, \sin\big(\frac{2\pi}{3}\,k\big),$$

where c_1, c_2, c_3 are arbitrary constants.

4.6.4.7. The corresponding homogeneous difference equation is $y_{k+2} - y_k = 0$, which has characteristic equation $0 = r^2 - 1 = (r-1)(r+1)$. $\Rightarrow \mathsf{L}_1 : -1,\, 1.$

The inhomogeneity $f_k = k \Rightarrow \mathsf{L}_2 : 1, 1 \Rightarrow$ The superlist is $\mathsf{L} : -1,\, 1,\, 1,\, 1$

$$\Rightarrow y_k = c_1\, 1^k + c_2(-1)^k + c_3\, k\, 1^k + c_4\, k^2\, 1^k$$

$\Rightarrow$ The correct form of a particular solution is $y_k^{(p)} = A\,k \cdot 1^k + B\,k^2 \cdot 1^k$, that is

$$y_k^{(p)} = A\,k + B\,k^2,$$

where A, B are constants to be determined. Substitute that into the original non-homogeneous difference equation:

$$0 \cdot 1 - 1 \cdot k = -k = y_{k+2}^{(p)} - y_k^{(p)} = A(k+2) + B(k+2)^2 - Ak - Bk^2 = 2A + 4B + 4Bk$$

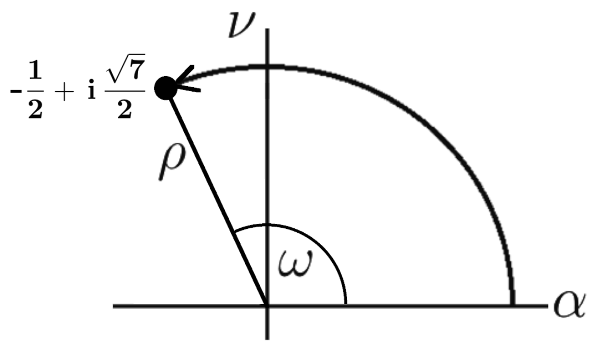


Figure 6: Characteristic root for problem 4.6.4.9

$\Rightarrow -1 = 4B \Rightarrow B = -\frac{1}{4}$, and $0 = 2A + 4B \Rightarrow A = -2B = \frac{1}{2}$

$$\Rightarrow y_k^{(p)} = \frac{1}{2}\,k - \frac{1}{4}\,k^2.$$

The general solution of the non-homogeneous difference equation is

$$y_k = y_k^{(h)} + y_k^{(p)} = c_1 + c_2(-1)^k + \frac{1}{2}\,k - \frac{1}{4}\,k^2,$$

where c_1, c_2 =arbitrary constants.

4.6.4.9. Substitute $y_k = r^k$ into $0 = y_{k+2} + y_{k+1} + 2y_k$ to get the characteristic equation $0 = r^2 + 2r + 1 = \big(r + \frac{1}{2}\big)^2 + \frac{7}{4}$. The root in the second quadrant is shown in Figure 4.10:

$$r = -\frac{1}{2} + i\,\frac{\sqrt{7}}{2}.$$

It can be rewritten in polar exponential form as

$$-\frac{1}{2} + i\,\frac{\sqrt{7}}{2} = r = \rho\,e^{i\omega} = \rho\cos\omega + \,i\,\rho\sin\omega = \sqrt{\big(-\frac{1}{2}\big)^2 + \big(\frac{\sqrt{7}}{2}\big)^2}\;\; e^{i\ \arctan(-\sqrt{7})},$$

hence $\rho = \sqrt{2}$ and $\omega = \pi - \arctan(\sqrt{7})$.

The general solution of the difference equation is

$$y_k = \rho^k\big(c_1\,\cos(\omega k) + c_2\,\sin(\omega k)\big),$$

that is,

$$y_k = 2^{k/2}\big(c_1\,\cos(\omega k) + c_2\,\sin(\omega k)\big),$$

where c_1, c_2 are arbitrary constants.

The ICs require

$$\left\{\begin{array}{l} 2 = y_0 = c_1 \\ 1 = y_1 = \sqrt{2}\,(\cos\omega)c_1 + \sqrt{2}\,(\sin\omega)c_2 \end{array}\right\},$$

hence $c_1 = 2$ and

$$c_2 = (\sin\omega)^{-1}\left(\frac{1}{\sqrt{2}} - c_1\cos\omega\right).$$

As we can see in Figure 4.10,

$$\cos\omega = \frac{-1/2}{\sqrt{2}} \quad \text{and} \quad \sin\omega = \frac{\sqrt{7}/2}{\sqrt{2}},$$

hence

$$c_2 = (\sin\omega)^{-1}\Big(\frac{1}{\sqrt{2}} - c_1\cos\omega\Big) = \frac{\sqrt{2}}{\sqrt{7}/2}\Big(\frac{1}{\sqrt{2}} - 2\cdot\frac{-1/2}{\sqrt{2}}\Big) = \frac{2\sqrt{2}}{\sqrt{7}}\Big(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\Big) = \frac{4}{\sqrt{7}}.$$

The solution of the IVP is

$$y_k = 2^{k/2}\left(2\,\cos(\omega k) + \frac{4}{\sqrt{7}}\,\sin(\omega k)\right),$$

where $\omega = \pi - \arctan(\sqrt{7})$.

4.6.4.11. Substitute $y_k = r^k$ into $0 = y_{k+2} + 4y_{k+1} + 13y_k =$ to get the characteristic equation $0 = r^2 + 4r + 13 = (r+2)^2 + 9$, so the roots are $r = -2 \pm 3i$. The root $r = -2 + 3i$ can be rewritten in polar exponential form as

$$r = -2 + 3i = \rho\cos\omega + i\,\rho\sin\omega$$

It can be rewritten in polar exponential form as

$$-2 + 3i = r = \rho\, e^{i\omega} = \sqrt{(-2)^2 + (3)^2}e^{i\,\omega},$$

hence $\rho = \sqrt{13}$. Because ω is in the second quadrant, $\omega = \pi + \arctan\big(3/(-2)\big) = \pi - \arctan(3/2)$.

The general solution of the difference equation is

$$y_k = \rho^k\big(c_1\,\cos(\omega k) + c_2\,\sin(\omega k)\big),$$

that is,

$$y_k = 13^{k/2}\big(c_1\,\cos(\omega k) + c_2\,\sin(\omega k)\big),$$

where c_1, c_2 are arbitrary constants.

The ICs require

$$-2 = y_0 = c_1$$
$$1 = y_1 = \sqrt{13}(\cos\omega)c_1 + \sqrt{13}(\sin\omega)c_2,$$

hence $c_1 = -2$ and $c_2 = (\sin\omega)^{-1}\Big(\frac{1}{\sqrt{13}} + 2\cos\omega\Big)\Big)$.

Because

$$\cos\omega = \frac{-2}{\sqrt{13}} \quad\text{and}\quad \sin\omega = \frac{3}{\sqrt{13}},$$

it follows that

$$c_2 = \frac{\sqrt{13}}{3}\Big(\frac{1}{\sqrt{13}} + 2\cdot\frac{-2}{\sqrt{13}}\Big) = \frac{\sqrt{13}}{3}\cdot\frac{-3}{\sqrt{13}} = -1.$$

The solution of the IVP is

$$y_k = 13^{k/2}\left(-2\,\cos(\omega k) - \sin(\omega k)\right),$$

where $\omega = \pi - \arctan(3/2)$.

4.6.4.13. As in Example 4.38, we will find a LCCHΔE satisfied by $y_k \triangleq |A_k|$. First, look at some small sized examples of the matrix A_k and their determinants.

For $k = 3$,

$$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

By expanding along the first row

$$y_3 = |A_3| = (1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + (0)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1.$$

For $k=4$, by expanding along the first row we have

$$(\star)\quad y_4 = |A_4| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = (1)\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} + (0) + (0) = 1\cdot y_3 - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}.$$

This suggests that it would be good to also define

$$A_2 \triangleq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad y_2 \triangleq |A_2| = 0.$$

Then we see from $(\star)$ that

$$y_4 = y_3 - y_2.$$

In fact, in general, by expanding the determinant along the first row, we have

$$y_k = \begin{vmatrix} 1 & 1 & 0 & 0 & . & . & . & & & & 0 \\ 1 & 1 & 1 & 0 & & & & & & & . \\ 0 & 1 & 1 & 1 & & & & & & & . \\ . & & & . & . & . & & & & & . \\ . & & & & . & . & . & & & & \\ . & & & & & . & . & 1 & 1 & 0 & \\ & & & & & & & 1 & 1 & 1 & \\ 0 & . & . & . & & & & 0 & 1 & 1 & \end{vmatrix} = (1)\begin{vmatrix} 1 & 1 & 0 & . & . & . & & & 0 \\ 1 & 1 & 1 & & & & & & . \\ 0 & 1 & 1 & & & & & & . \\ . & & . & . & . & & & & . \\ . & & & . & . & . & & & \\ . & & & & & . & 1 & 1 & 1 \\ 0 & . & . & . & & & 0 & 1 & 1 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 & 0 & . & . & . & & 0 \\ 0 & 1 & 1 & & & & & . \\ & 1 & 1 & & & & & . \\ . & & . & . & . & & & . \\ . & & & . & . & . & & \\ . & & & & & . & 1 & 1 \\ 0 & . & . & . & & & 1 & 1 \end{vmatrix}.$$

Now, expand the second term along its first column to get

$$y_k = 1\cdot y_{k-1} \; - \; (1)\begin{vmatrix} 1 & 1 & 0 & . & . & . & 0 \\ 1 & 1 & 1 & & & & . \\ & . & . & . & & & . \\ . & & . & . & . & & \\ . & & & & 1 & 1 & 1 \\ 0 & . & . & . & 0 & 1 & 1 \end{vmatrix} = 1\cdot y_{k-1} - (1)(1)y_{k-2}.$$

So, y_k satisfies the second order difference equation $y_k = y_{k-1} - y_{k-2}$, for $k \ge 3$, that is,

$$y_{k+2} = y_{k+1} - y_k, \text{ for } k \ge 1,$$

whose characteristic polynomial, $r^2 - r + 1$ has a complex conjugate pair of roots

$$r = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \rho\,\cos\omega \,+\, i\,\rho\,\sin\omega.$$

As in equation (4.64), $\rho = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$. Because $\tan\omega = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$ and ω is in the first quadrant of the $(\alpha,\nu)-$plane, we have $\omega = \dfrac{\pi}{3}$ and

$$y_k = c_1\rho^k\,\cos(\omega k) + c_2\rho^2\,\sin(\omega k) = c_1\,\cos\left(\frac{\pi}{3}\,k\right) + c_2\,\sin\left(\frac{\pi}{3}\,k\right),$$

where c_1, c_2 are arbitrary constants.

To satisfy the initial conditions,

$$\left\{ \begin{array}{c} 0 = y_2 = c_1\cos 2\omega + c_2\sin 2\omega \\ \\ -1 = y_3 = c_1\cos 3\omega + c_2\sin 3\omega \end{array} \right\},$$

we get

$$\begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \begin{bmatrix} \cos 2\omega & \sin 2\omega \\ \\ \cos 3\omega & \sin 3\omega \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \\ -1 \end{bmatrix} = \frac{1}{\cos 2\omega \sin 3\omega - \sin 2\omega \cos 3\omega} \begin{bmatrix} \sin 3\omega & -\sin 2\omega \\ \\ -\cos 3\omega & \cos 2\omega \end{bmatrix} \begin{bmatrix} 0 \\ \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sin(3\omega - 2\omega)} \begin{bmatrix} \sin 2\omega \\ \\ -\cos 2\omega \end{bmatrix} = \frac{1}{\sin \omega} \begin{bmatrix} \sin 2\omega \\ \\ -\cos 2\omega \end{bmatrix}$$

Because

$$\sin \omega = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \sin 2\omega = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \quad \text{and} \quad \cos 2\omega = \cos \frac{2\pi}{3} = -\frac{1}{2},$$

we have

$$\begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{3}/2} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

So,

$$\det(A_k) = y_k = \cos\left(\frac{\pi}{3}\, k\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\pi}{3}\, k\right), k \ge 3.$$

This can be rewritten as

$$\det(A_k) \;=\; \frac{2}{\sqrt{3}}\Big(\frac{\sqrt{3}}{2} \cos\left(\frac{\pi}{3}\, k\right) - \left(-\frac{1}{2}\right) \sin\left(\frac{\pi}{3}\, k\right)\Big) \;=\; \frac{2}{\sqrt{3}}\Big(\sin \frac{2\pi}{3} \cos\left(\frac{\pi}{3}\, k\right) - \cos \frac{2\pi}{3} \sin\left(\frac{\pi}{3}\, k\right)\Big) \;=\; \frac{2}{\sqrt{3}} \sin\left(\frac{\pi}{3}(2-k)\right),$$

for $k \ge 3$.

By the way, we can conclude that for all $k \ge 3$,

$$\Big|\det(A_k)\Big| = \left|\cos\left(\frac{\pi}{3}\, k\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\pi}{3}\, k\right)\right| \le \left|\cos\left(\frac{\pi}{3}\, k\right)\right| + \left|\frac{1}{\sqrt{3}} \sin\left(\frac{\pi}{3}\, k\right)\right| \le 1 + \frac{1}{\sqrt{3}}.$$

4.6.4.15. As in Example 4.38, we will find a LCCHΔE satisfied by $y_k \triangleq |A_k|$. First, look at some small sized examples of the matrix A_k and their determinants.

For $k = 3$,

$$A_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

By expanding along the first row

$$y_3 = |A_3| = (-2) \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} - (1) \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + (0) \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = ... = -4.$$

For $k = 4$, by expanding along the first row we have

$$y_4 = |A_4| = \begin{vmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{vmatrix} = (-2) \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} - (1) \begin{vmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} + (0) + (0)$$

that is,

$$(\star) \quad y_4 = (-2) \cdot y_3 - (1)(1) \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}.$$

This suggests that it would be good to also define

$$A_2 \triangleq \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad y_2 \triangleq |A_2| = 3.$$

Then we see from $(\star)$ that

$$y_4 = -2\,y_3 - y_2.$$

In fact, in general, by expanding the determinant along the first row, we have

$$y_k = \left| \begin{array}{cccccccccc} -2 & 1 & 0 & 0 & . & . & . & & & 0 \\ 1 & -2 & 1 & 0 & & & & & & . \\ 0 & 1 & -2 & 1 & & & & & & . \\ . & & & . & . & . & & & & . \\ . & & & & . & . & . & & & \\ . & & & & & . & . & -2 & 1 & 0 \\ & & & & & & & 1 & -2 & 1 \\ 0 & . & . & . & & & & 0 & 1 & -2 \end{array} \right|$$

$$= (-2) \left| \begin{array}{ccccccccc} -2 & 1 & 0 & . & . & . & & & 0 \\ 1 & -2 & 1 & & & & & & . \\ 0 & 1 & -2 & & & & & & . \\ . & & . & . & . & & & & . \\ . & & & . & . & . & & & \\ . & & & & & . & 1 & -2 & 1 \\ 0 & . & . & . & & & 0 & 1 & -2 \end{array} \right| - (1) \left| \begin{array}{cccccccc} 1 & 1 & 0 & . & . & . & & 0 \\ 0 & -2 & 1 & & & & & . \\ & 1 & -2 & & & & & . \\ . & & . & . & . & & & . \\ . & & & . & . & . & & \\ . & & & & & . & -2 & 1 \\ 0 & . & . & . & & & 1 & -2 \end{array} \right|.$$

Now, expand the second term along its first column to get

$$y_k = -2 \cdot y_{k-1} \; - \; (1)(1) \left| \begin{array}{ccccccc} -2 & 1 & 0 & . & . & . & 0 \\ 1 & -2 & 1 & & & & . \\ & . & . & . & & & . \\ . & & . & . & . & & \\ . & & & & 1 & -2 & 1 \\ 0 & . & . & . & 0 & 1 & -2 \end{array} \right| = -2y_{k-1} - y_{k-2}.$$

So, y_k satisfies the second order difference equation $y_k = -2y_{k-1} - y_{k-2}$, for $k \geq 3$, that is,

$$y_{k+2} = -2y_{k+1} - y_k, \text{ for } k \geq 1,$$

whose characteristic polynomial, $r^2 + 2r + 1 = (r+1)^2$ has a repeated real root $r = -1, -1$.

The general solution of the difference equation is

$$y_k = c_1\,(-1)^k + c_2\,k\,(-1)^k,$$

where c_1, c_2 are arbitrary constants.

To satisfy the initial conditions,

$$\left\{ \begin{array}{rcl} 3 = y_2 = & c_1 & +\,2\,c_2 \\ -4 = y_3 = & -c_1 & -\,3\,c_2 \end{array} \right\},$$

we get

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So,

$$\det(A_k) = y_k = 1 \cdot (-1)^k + 1 \cdot k\,(-1)^k = (1+k)(-1)^k, \; k \geq 3.$$

4.6.4.17. As in Example 4.38, we will find a LCCHΔE satisfied by $y_k \triangleq |A_k|$. First, look at some small sized examples of the matrix A_k and their determinants.

For $k = 3$,

$$A_3 = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -3 \end{bmatrix}.$$

By expanding along the first row

$$y_3 = |A_3| = (-3)\begin{vmatrix} -3 & 2 \\ 2 & -3 \end{vmatrix} - (2)\begin{vmatrix} 2 & 2 \\ 0 & -3 \end{vmatrix} + (0)\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} = \ldots = -3.$$

For $k = 4$, by expanding along the first row we have

$$y_4 = |A_4| = \begin{vmatrix} -3 & 2 & 0 & 0 \\ 2 & -3 & 2 & 0 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 2 & -3 \end{vmatrix} = (-3)\begin{vmatrix} -3 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -3 \end{vmatrix} - (2)\begin{vmatrix} 2 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & -3 \end{vmatrix} + (0) + (0)$$

that is,

$$(\star) \quad y_4 = (-3)\cdot y_3 - (2)(2)\begin{vmatrix} -3 & 2 \\ 2 & -3 \end{vmatrix}.$$

This suggests that it would be good to also define

$$A_2 \triangleq \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}, \quad y_2 \triangleq |A_2| = 5.$$

Then we see from $(\star)$ that

$$y_4 = -3\,y_3 - 4y_2.$$

In fact, in general, by expanding the determinant along the first row, we have

$$y_k = \begin{vmatrix} -3 & 2 & 0 & 0 & \cdot & \cdot & \cdot & & & 0 \\ 2 & -3 & 2 & 0 & & & & & & \cdot \\ 0 & 2 & -3 & 2 & & & & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & & & \\ \cdot & & & & & \cdot & \cdot & -3 & 2 & 0 \\ & & & & & & & 2 & -3 & 2 \\ 0 & \cdot & \cdot & \cdot & & & & 0 & 2 & -3 \end{vmatrix}$$

$$= (-3)\begin{vmatrix} -3 & 2 & 0 & \cdot & \cdot & \cdot & & & 0 \\ 2 & -3 & 2 & & & & & & \cdot \\ 0 & 2 & -3 & & & & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & & \\ \cdot & & & & & \cdot & 2 & -3 & 2 \\ 0 & \cdot & \cdot & \cdot & & & 0 & 2 & -3 \end{vmatrix} - (2)\begin{vmatrix} 2 & 2 & 0 & \cdot & \cdot & \cdot & & 0 \\ 0 & -3 & 2 & & & & & \cdot \\ & 2 & -3 & & & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & \\ \cdot & & & & & \cdot & -3 & 2 \\ 0 & \cdot & \cdot & \cdot & & & 2 & -3 \end{vmatrix}.$$

Now, expand the second term along its first column to get

$$y_k = -3\cdot y_{k-1} \;-\; (2)(2)\begin{vmatrix} -3 & 2 & 0 & \cdot & \cdot & \cdot & 0 \\ 2 & -3 & 2 & & & & \cdot \\ & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & & & & 2 & -3 & 2 \\ 0 & \cdot & \cdot & \cdot & 0 & 2 & -3 \end{vmatrix} = -3y_{k-1} - 4y_{k-2}.$$

So, y_k satisfies the second order difference equation $y_k = -3y_{k-1} - 4y_{k-2},$ for $k \ge 3$, that is,

$$y_{k+2} = -3y_{k+1} - 4y_k, \text{ for } k \ge 1,$$

whose characteristic polynomial, $r^2 + 3r + 4 = \left(r + \frac{3}{2}\right)^2 + \frac{7}{4}$ has a complex conjugate pair of roots

$$r = -\frac{3}{2} \pm i\,\frac{\sqrt{7}}{2} = \rho\,\cos\omega\, +\, i\,\rho\,\sin\omega.$$

As in equation (4.64), $\rho = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} = 2$. Because $\tan\omega = \frac{\sqrt{7}/2}{-3/2} = -\frac{\sqrt{7}}{3}$ and ω is in the second quadrant of the $(\alpha,\nu)-$plane, we have

$$\omega = \pi + \arctan\left(-\frac{\sqrt{7}}{3}\right) = \pi - \arctan\left(\frac{\sqrt{7}}{3}\right).$$

Note, for future reference, that

$$\sin\omega = \frac{\sqrt{7}/2}{2} = \frac{\sqrt{7}}{4},\quad \cos\omega = \frac{-3/2}{2} = -\frac{3}{4},\quad \sin 2\omega = 2\sin\omega\,\cos\omega = 2\cdot\frac{\sqrt{7}}{4}\cdot\left(-\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8},$$

$$\cos 2\omega = \cos^2\omega - \sin^2\omega = \left(-\frac{3}{4}\right)^2 - \left(\frac{\sqrt{7}}{4}\right)^2 = \frac{1}{8},$$

$$\cos 3\omega = \cos(2\omega+\omega) = \cos 2\omega\,\cos\omega - \sin 2\omega\,\sin\omega = \left(\frac{1}{8}\right)\left(-\frac{3}{4}\right) - \left(-\frac{3\sqrt{7}}{8}\right)\left(\frac{\sqrt{7}}{4}\right) = \frac{9}{16},$$

and

$$\sin 3\omega = \sin(2\omega+\omega) = \sin 2\omega\,\cos\omega + \cos 2\omega\,\sin\omega = \left(-\frac{3\sqrt{7}}{8}\right)\left(-\frac{3}{4}\right) + \left(\frac{1}{8}\right)\left(\frac{\sqrt{7}}{4}\right) = \frac{5\sqrt{7}}{16}.$$

The general solution of the difference equation is

$$y_k = c_1\,\rho^k\,\cos(\omega k) + c_2\,\rho^k\,\sin(\omega k) = c_1\,2^k\,\cos(\omega k) + c_2\,2^k\,\sin(\omega k),$$

where c_1, c_2 are arbitrary constants.

To satisfy the initial conditions,

$$\left\{\begin{array}{r} 5 = y_2 = 4c_1\cos 2\omega + 4c_2\sin 2\omega \\ -3 = y_3 = 8c_1\cos 3\omega + 8c_2\sin 3\omega \end{array}\right\},$$

we get

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4\cos 2\omega & 4\sin 2\omega \\ 8\cos 3\omega & 8\sin 3\omega \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \frac{1}{32\cos 2\omega\sin 3\omega - 32\sin 2\omega\cos 3\omega}\begin{bmatrix} 8\sin 3\omega & -4\sin 2\omega \\ -8\cos 3\omega & 4\cos 2\omega \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$= \frac{1}{32\sin(3\omega - 2\omega)}\begin{bmatrix} 40\,\sin 3\omega + 12\,\sin 2\omega \\ -40\,\cos 3\omega - 12\,\cos 2\omega \end{bmatrix} = \frac{1}{32\sin\omega}\begin{bmatrix} (40)\left(\frac{5\sqrt{7}}{16}\right) + (12)\left(-\frac{3\sqrt{7}}{8}\right) \\ -(40)\left(\frac{9}{16}\right) - (12)\left(\frac{1}{8}\right) \end{bmatrix} = \frac{1}{32\cdot\frac{\sqrt{7}}{4}}\begin{bmatrix} 8\sqrt{7} \\ -24 \end{bmatrix}$$

So,

$$\det(A_k) = y_k = 2^k\cdot\left(\cos(\omega\,k)\, -\, \frac{3}{\sqrt{7}}\,\sin(\omega\,k)\right),\ k \ge 3,$$

where $\omega = \pi - \arctan\left(\frac{\sqrt{7}}{3}\right)$.

4.6.4.19. y_k, the fraction of $U^{235}F_6$ after the mixture has been run through the $k-th$ centrifuge, satisfies the difference equation $y_{k+1} = r\,y_k$, where $r = 1.0014$. The solution of the difference equation is

$$y_k = r^k\,y_0$$

and we are also given that $y_0 = 0.01 = 1\%$. So, $y_k = 0.01\,r^k$.

After running through a cascade of 1024 centrifuges, the mixture contains

$$y_{1024} = (0.01)(1.0014)^{1024} \approx 0.04189567119 = 4.189567119\% \approx 4.19\%$$

of $U^{235}F_6$.

4.6.4.21. (a) If $\{y^{(j)}_{k+1}\}$, $j = 1, 2, 3$ are solutions of LCCHΔE (4.57), that is, $y_{k+3} = a_{1,k}y_{k+2} + a_{2,k}y_{k+1} + a_{3,k}y_k$,

$$C_{k+1} = \begin{vmatrix} y^{(1)}_{k+1} & y^{(2)}_{k+1} & y^{(3)}_{k+1} \\ y^{(1)}_{k+2} & y^{(2)}_{k+2} & y^{(3)}_{k+2} \\ y^{(1)}_{k+3} & y^{(2)}_{k+3} & y^{(3)}_{k+3} \end{vmatrix}$$

$$= \begin{vmatrix} y^{(1)}_{k+1} & y^{(2)}_{k+1} & y^{(3)}_{k+1} \\ y^{(1)}_{k+2} & y^{(2)}_{k+2} & y^{(3)}_{k+2} \\ a_{1,k}y^{(1)}_{k+2} + a_{2,k}y^{(1)}_{k+1} + a_{3,k}y^{(1)}_k & a_{1,k}y^{(2)}_{k+2} + a_{2,k}y^{(2)}_{k+1} + a_{3,k}y^{(2)}_k & a_{1,k}y^{(3)}_{k+2} + a_{2,k}y^{(3)}_{k+1} + a_{3,k}y^{(3)}_k \end{vmatrix}$$

$$\underset{\substack{-a_{2,k}R_1 + R_3 \to R_3 \\ -a_{1,k}R_2 + R_3 \to R_3}}{=} \begin{vmatrix} y^{(1)}_{k+1} & y^{(2)}_{k+1} & y^{(3)}_{k+1} \\ y^{(1)}_{k+2} & y^{(2)}_{k+2} & y^{(3)}_{k+2} \\ a_{3,k}y^{(1)}_k & a_{3,k}y^{(2)}_k & a_{3,k}y^{(3)}_k \end{vmatrix} \underset{R_3 \leftarrow a_{3,k}R_3}{=} a_{3,k} \begin{vmatrix} y^{(1)}_{k+1} & y^{(2)}_{k+1} & y^{(3)}_{k+1} \\ y^{(1)}_{k+2} & y^{(2)}_{k+2} & y^{(3)}_{k+2} \\ y^{(1)}_k & y^{(2)}_k & y^{(3)}_k \end{vmatrix}$$

$$\underset{R_2 \leftrightarrow R_3}{=} -\,a_{3,k} \begin{vmatrix} y^{(1)}_{k+1} & y^{(2)}_{k+1} & y^{(3)}_{k+1} \\ y^{(1)}_k & y^{(2)}_k & y^{(3)}_k \\ y^{(1)}_{k+2} & y^{(2)}_{k+2} & y^{(3)}_{k+2} \end{vmatrix} \underset{R_1 \leftrightarrow R_2}{=} (-1)^2 a_{3,k} \begin{vmatrix} y^{(1)}_k & y^{(2)}_k & y^{(3)}_k \\ y^{(1)}_{k+1} & y^{(2)}_{k+1} & y^{(3)}_{k+1} \\ y^{(1)}_{k+2} & y^{(2)}_{k+2} & y^{(3)}_{k+2} \end{vmatrix} = (-1)^2 a_{2,k} C_k,$$

as was desired.

(b) If $\{y^{(j)}_{k+1}\}$, $j = 1, ..., n$ are solutions of LCCHΔE (4.57), that is, $y_{k+n} = a_1 y_{k+n-1} + a_2 y_{k+n-2} + ... + a_n y_k$,

$$C(y^{(1)}_{k+1}, y^{(2)}_{k+1}, ..., y^{(n)}_{k+1}) = \begin{vmatrix} y^{(1)}_{k+1} & \cdots & y^{(n)}_{k+1} \\ y^{(1)}_{k+2} & & y^{(n)}_{k+2} \\ \vdots & \ddots & \vdots \\ y^{(1)}_{k+n} & \cdots & y^{(n)}_{k+n} \end{vmatrix}$$

$$= \begin{vmatrix} y^{(1)}_{k+1} & \cdots & y^{(n)}_{k+1} \\ y^{(1)}_{k+2} & & y^{(n)}_{k+2} \\ \vdots & \ddots & \vdots \\ a_1 y^{(1)}_{k+n-1} + a_2 y^{(1)}_{k+n-2} + ... + a_n y^{(1)}_k & \cdots & a_1 y^{(n)}_{k+n-1} + a_2 y^{(n)}_{k+n-2} + ... + a_n y^{(n)}_k \end{vmatrix}$$

$$\underset{\substack{-a_{n-1,k}R_1 + R_n \to R_n \\ -a_{n-2,k}R_2 + R_n \to R_n \\ \vdots \\ -a_{1,k}R_{n-1} + R_n \to R_n}}{=} \begin{vmatrix} y^{(1)}_{k+1} & \cdots & y^{(n)}_{k+1} \\ y^{(1)}_{k+2} & \ddots & y^{(n)}_{k+2} \\ \vdots & \ddots & \vdots \\ a_n y^{(1)}_k & \cdots & a_n y^{(n)}_k \end{vmatrix} \underset{R_n \leftarrow a_{n,k}R_n}{=} a_{n,k} \begin{vmatrix} y^{(1)}_{k+1} & \cdots & y^{(n)}_{k+1} \\ y^{(1)}_{k+2} & \ddots & y^{(n)}_{k+2} \\ \vdots & \ddots & \vdots \\ y^{(1)}_k & \cdots & y^{(n)}_k \end{vmatrix}$$

$$\underset{R_{n-1} \leftrightarrow R_n}{=} -a_{n,k} \begin{vmatrix} y^{(1)}_{k+1} & \cdots & y^{(n)}_{k+1} \\ y^{(1)}_{k+2} & \ddots & y^{(n)}_{k+2} \\ \vdots & \ddots & \vdots \\ y^{(1)}_k & & y^{(n)}_k \\ y^{(1)}_{k+n-1} & \cdots & y^{(n)}_{k+n-1} \end{vmatrix} \underset{R_{n-2} \leftrightarrow R_{n-1}}{=} (-1)^2 a_{n,k} \begin{vmatrix} y^{(1)}_{k+1} & \cdots & y^{(n)}_{k+1} \\ \vdots & \ddots & \vdots \\ y^{(1)}_k & \ddots & y^{(n)}_k \\ y^{(1)}_{k+n-2} & & y^{(n)}_{k+n-2} \\ y^{(1)}_{k+n-1} & \cdots & y^{(n)}_{k+n-1} \end{vmatrix}$$

$$= ... \underset{R_1 \leftrightarrow R_2}{=} (-1)^{n-1} a_{n,k} \begin{vmatrix} y^{(1)}_k & \cdots & y^{(n)}_k \\ y^{(1)}_{k+1} & & y^{(n)}_{k+1} \\ y^{(1)}_{k+2} & \ddots & y^{(n)}_{k+2} \\ \vdots & \ddots & \vdots \\ y^{(1)}_{k+n-1} & \cdots & y^{(n)}_{k+n-1} \end{vmatrix} = (-1)^{n-1} a_{n,k} C_k,$$

as was desired.

Section 4.7.5

4.7.5.1. Denote $X(z) \triangleq \mathcal{Z}\big[\,x[n]\,\big]$. Take the $z-$transforms of both sides of the difference equation and use (4.78) to get

$$zX(z) - z\,x[0] = -X(z) + z \cdot \frac{1}{z-\alpha},$$

hence

$$(z+1)X(z) = z \cdot \Big(x[0] + \frac{1}{z-\alpha}\Big).$$

Using the initial condition and dividing both sides by $(z+1)$ gives

$$X(z) = z \cdot \Big(\frac{1}{z+1} + \frac{1}{(z+1)(z-\alpha)}\Big).$$

Partial fractions for the last term is

$$\frac{1}{(z+1)(z-\alpha)} = \frac{A}{z+1} + \frac{B}{z-\alpha}\,.$$

Multiply both sides by $(z+1)(z-\alpha)$ to get

$$(\star) \quad 1 = A(z-\alpha) + B(z+1).$$

Substitute in $z = -1$ and $z = \alpha$ to get, respectively,

$$1 = A(-1-\alpha) \quad \text{and} \quad 1 = B(\alpha+1),$$

hence, respectively,

$$A = -\frac{1}{1+\alpha} \quad \text{and} \quad B = \frac{1}{1+\alpha}\,.$$

So,

$$X(z) = z \cdot \Big(\frac{1}{z+1} - \frac{1}{1+\alpha} \cdot \frac{1}{z+1} + \frac{1}{1+\alpha} \cdot \frac{1}{z-\alpha}\Big) = z \cdot \Big(\frac{\alpha}{\alpha+1} \cdot \frac{1}{z+1} + \frac{1}{1+\alpha} \cdot \frac{1}{z-\alpha}\Big).$$

The solution of the problem is

$$x[n] = \mathcal{Z}^{-1}[\,X(z)\,] = \frac{\alpha}{\alpha+1} \cdot (-1)^n + \frac{1}{1+\alpha} \cdot \alpha^n.$$

4.7.5.3. Denote $X(z) \triangleq \mathcal{Z}\big[\,x[n]\,\big]$. Take the $z-$transforms of both sides of the difference equation $S^2x[n] = -\frac{1}{2}\,Sx[n] + \frac{1}{2}x[n] + \cos\frac{\pi n}{3}$. Using (4.80) with $\omega = \frac{\pi}{3}$ we get

$$z^2X(z) - z^2x[0] - z\,x[1] = -\frac{1}{2}\big(zX(z) - z\,x[0]\big) + \frac{1}{2}\,X(z) + z \cdot \frac{z - \cos\frac{\pi}{3}}{z^2 - 2(\cos\frac{\pi}{3})z + 1}\,.$$

Using the initial conditions $x[0] = x[1] = 0$ and the exact value $\cos\frac{\pi}{3} = \frac{1}{2}$, we get

$$\big(z^2 + \frac{1}{2}\,z - \frac{1}{2}\big)X(z) = z \cdot \frac{z - \frac{1}{2}}{z^2 - z + 1}\,.$$

Rewrite this is as

$$X(z) = z \cdot \frac{z - \frac{1}{2}}{\big(z^2 + \frac{1}{2}\,z - \frac{1}{2}\big)(z^2 - z + 1)}.$$

The quadratic polynomial $\big(z^2 + \frac{1}{2}\,z - \frac{1}{2}\big)$ has roots

$$z = \frac{-\frac{1}{2} \pm \sqrt{(\frac{1}{2})^2 - 4 \cdot 1 \cdot \big(-\frac{1}{2}\big)}}{2} = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{2} = \frac{-\frac{1}{2} \pm \frac{3}{2}}{2} = \left\{\begin{array}{c} \frac{1}{2} \\ \\ -1 \end{array}\right\},$$

so we can factor $\left(z^2+\frac{1}{2}\,z-\frac{1}{2}\right)=(z+1)\left(z-\frac{1}{2}\right)$.

So,

$$X(z)=z\cdot\frac{\left(z-\frac{1}{2}\right)}{(z+1)\left(z-\frac{1}{2}\right)(z^2-z+1)}$$

hence

$$X(z)=z\cdot\frac{1}{(z+1)(z^2-z+1)}\,.$$

The partial fractions expansion is

$$(\star)\quad \frac{1}{(z+1)(z^2-z+1)}=\frac{Az+B}{z^2-z+1}+\frac{C}{z+1}\,.$$

Multiply both sides by $(z+1)(z^2-z+1)$ to get

$$(\star\star)\;\; 1=(Az+B)(z+1)+C(z^2-z+1)$$

Substitute in $z=-1$ to get

$$1=(-A+B)\cdot 0+C\big((-1)^2-(-1)+1\big),$$

hence $C=\dfrac{1}{3}$. Substitute the value of C into $(\star\star)$ to get

$$1=(Az+B)(z+1)+\frac{1}{3}(z^2-z+1)$$

hence

$$1-\frac{1}{3}(z^2-z+1)=(Az+B)(z+1).$$

That is,

$$-\frac{1}{3}(z^2-z-2)=(Az+B)(z+1).$$

It follows that

$$Az+B=\frac{-\frac{1}{3}(z^2-z-2)}{(z+1)}=-\frac{1}{3}(z-2).$$

Putting everything together,

$$X(z)=z\cdot\Big(\frac{-\frac{1}{3}(z-2)}{z^2-z+1}+\frac{\frac{1}{3}}{z+1}\Big).$$

Recall that

$$\mathcal{Z}[\,\cos\frac{\pi n}{3}\,]=z\cdot\frac{z-\frac{1}{2}}{z^2-z+1}$$

was used at the beginning of the work. We also have, from Table 4.3's entry Z.9, that

$$\mathcal{Z}[\,\sin\frac{\pi n}{3}\,]=z\cdot\frac{\frac{\sqrt{3}}{2}}{z^2-z+1}$$

We rewrite

$$X(z)=z\cdot\Big(\frac{-\frac{1}{3}(z-\frac{1}{2}-\frac{3}{2})}{z^2-z+1}+\frac{\frac{1}{3}}{z+1}\Big).=-\frac{1}{3}\,z\cdot\frac{z-\frac{1}{2}}{z^2-z+1}+z\cdot\frac{\frac{1}{2}}{z^2-z+1}+\frac{1}{3}\,z\cdot\frac{1}{z+1}\,.$$

So,

$$x[n]=\mathcal{Z}^{-1}[\,X(z)\,]$$

implies

$$x[n]=-\frac{1}{3}\cos\frac{\pi n}{3}+\frac{1}{\sqrt{3}}\,\sin\frac{\pi n}{3}+\frac{1}{3}\,(-1)^n.$$

There is no transient and all of $x[n]$ is bounded as $n \to \infty$. The steady state solution is

$$x_S[n] = -\frac{1}{3}\cos\frac{\pi n}{3} + \frac{1}{\sqrt{3}}\sin\frac{\pi n}{3} + \frac{1}{3}(-1)^n.$$

4.7.5.5. Denote $X(z) \triangleq \mathcal{Z}\big[\,x[n]\,\big]$. Take the $z-$transforms of both sides of the difference equation $S^2x[n] = -x[n] + \alpha^n$. Using (4.75) we get

$$z^2X(z) - z^2x[0] - z\,x[1] = -X(z) + z\cdot\frac{1}{z-\alpha}\,.$$

Using the initial conditions we get

$$(z^2+1)X(z) = z\cdot\frac{1}{z-\alpha}\,.$$

Rewrite this is as

$$X(z) = z\cdot\frac{1}{(z^2+1)(z-\alpha)}.$$

The partial fractions expansion is

$$(\star)\quad \frac{1}{(z^2+1)(z-\alpha)} = \frac{Az+B}{z^2+1} + \frac{C}{z-\alpha}\,.$$

Multiply both sides by $(z^2+1)(z-\alpha)$ to get

$$(\star\star)\ \ 1 = (Az+B)(z-\alpha) + C(z^2+1)\,.$$

Substitute in $z=\alpha$ to get

$$1 = (\alpha A + B)\cdot 0 + C(\alpha^2+1)$$

hence $C = \dfrac{1}{\alpha^2+1}$. Substitute the value of C into $(\star\star)$ to get

$$1 = (Az+B)(z-\alpha) + \frac{1}{\alpha^2+1}\cdot(z^2+1)$$

hence

$$1 - \frac{1}{\alpha^2+1}\cdot(z^2+1) = (Az+B)(z-\alpha).$$

That is,

$$-\frac{1}{\alpha^2+1}\Big(z^2-\alpha^2\Big) = (Az+B)(z-\alpha)\,.$$

It follows that

$$Az+B = -\frac{z+\alpha}{\alpha^2+1}\,.$$

Putting everything together,

$$X(z) = z\cdot\frac{1}{\alpha^2+1}\cdot\Big(-\frac{z+\alpha}{z^2+1} + \frac{1}{z-\alpha}\Big)\,.$$

Table entry Z.9 with $\omega = \frac{\pi}{2}$ gives

$$\mathcal{Z}[\sin\frac{\pi n}{2}\,] = z\cdot\frac{1}{z^2+1}\,.$$

So,

$$x[n]{=}\mathcal{Z}^{-1}[X(z)]{=}\frac{1}{\alpha^2+1}\mathcal{Z}^{-1}\Big[z\cdot\Big(\frac{z+\alpha}{z^2+1} - \frac{1}{z-\alpha}\Big)\Big] = \frac{1}{\alpha^2+1}\mathcal{Z}^{-1}\Big[z\cdot\Big(-\frac{z}{z^2+1} - \frac{\alpha}{z^2+1} + \frac{1}{z-\alpha}\Big)\Big]$$

hence, using Table entries Z.9 and Z.10,

$$x[n] = \frac{1}{\alpha^2+1}\Big(-\cos\frac{\pi n}{2} - \alpha\sin\frac{\pi n}{2} + \alpha^n\Big).$$

We were given that $|\alpha| < 1$, so the term $\frac{1}{\alpha^2+1}\,\alpha^n$ is transient. The steady state solution is

$$x_S[n] = \frac{1}{\alpha^2+1}\Big(-\cos\frac{\pi n}{2} - \alpha\sin\frac{\pi n}{2}\Big).$$

$x[n+2] = -x[n] + \alpha^n$, with initial conditions $x[0] = x[1] = 0$, assuming the constant α satisfies $|\alpha| < 1$.
4.7.5.7. Define $y[n] \triangleq \sin\omega n$. Using table entry Z.11,

$$\mathcal{Z}\big[\,\alpha^n\sin\omega n\,\big] = \mathcal{Z}\big[\,\alpha^n y[n]\,\big] = Y\Big(\frac{z}{\alpha}\Big) = \frac{z}{\alpha}\cdot\frac{\sin\omega}{\Big(\frac{z}{\alpha}\Big)^2 - 2\,\frac{z}{\alpha}\,\cos\omega + 1} = \frac{z}{\alpha}\cdot\frac{\alpha}{\alpha}\cdot\frac{\sin\omega}{\Big(\frac{z}{\alpha}\Big)^2 - 2\,\frac{z}{\alpha}\,\cos\omega + 1}$$

$$= z\cdot\frac{\alpha\sin\omega}{z^2 - 2\,\alpha z\,\cos\omega + \alpha^2}\,.$$

4.7.5.9. Denote $X(z) \triangleq \mathcal{Z}\big[\,x[n]\,\big]$. Take the $z-$transforms of both sides of the difference equation $S^2x[n] = Sx[n] + 2x[n] + \cos\frac{\pi n}{3}$. Using (4.78), and (4.80) with $\omega = \frac{\pi}{3}$, hence $\cos\omega = \frac{1}{2}$ and $\sin\omega = \frac{\sqrt{3}}{2}$, we calculate that the difference equation implies

$$z^2X(z) - z^2x[0] - z\,x[1] = \big(zX(z) - z\,x[0]\big) + 2X(z) + z\cdot\frac{z-\frac{1}{2}}{z^2-z+1},$$

hence, using the initial conditions,

$$(z^2-z-2)X(z) = z\cdot\frac{z-\frac{1}{2}}{z^2-z+1}.$$

So,

$$X(z) = z\cdot\frac{z-\frac{1}{2}}{(z^2-z+1)(z^2-z-2)}.$$

Using the quadratic formula, as in Example 4.37, we factor $z^2-z-2 = (z-2)(z+1)$. Using partial fractions in the form

$$\frac{z-\frac{1}{2}}{(z^2-z+1)(z-2)(z+1)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{Cz+D}{z^2-z+1}$$

where A, B, C and D are constants to be determined, we multiply through by $(z^2-z+1)(z-2)(z+1)$ to get

$$(\star)\quad z-\frac{1}{2} = A(z^2-z+1)(z+1) + B(z^2-z+1)(z-2) + (Cz+D)(z-2)(z+1).$$

Substitute in $z = 2$ and $z = -1$ to get, respectively,

$$\frac{3}{2} = A(2^2-2+1)(2+1) \quad \text{hence} \quad A = \frac{1}{6}$$

and

$$-\frac{3}{2} = B(1^2-(-1)+1)(-1-2), \quad \text{hence} \quad B = \frac{1}{6}.$$

Substitute these values of A and B into $(\star)$ to get

$$z-\frac{1}{2} = \frac{1}{6}(z^2-z+1)(z+1) + \frac{1}{6}(z^2-z+1)(z-2) + (Cz+D)(z-2)(z+1)$$

hence

$$z-\frac{1}{2} - \frac{1}{6}(z^2-z+1)(z+1) - \frac{1}{6}(z^2-z+1)(z-2) = (Cz+D)(z-2)(z+1)$$

that is,

$$-\frac{1}{6}(2z^3 - 3z^2 - 3z + 2) = (Cz + D)(z - 2)(z + 1),$$

So,

$$Cz + D = -\frac{1}{6} \cdot \frac{2z^3 - 3z^2 - 3z + 2}{(z - 2)(z + 1)} = -\frac{1}{6} \cdot (2z - 1).$$

This gives

$$X(z) = z \cdot \left(\frac{1}{6} \cdot \frac{1}{z - 2} + \frac{1}{6} \cdot \frac{1}{z + 1} - \frac{1}{3} \cdot \frac{z - \frac{1}{2}}{z^2 - z + 1} \right).$$

Using (4.75) and (4.80) in reverse, that is, taking $x[n] = \mathcal{Z}^{-1}\left[\, X(z) \,\right]$, gives that the solution is

$$x[n] = \frac{1}{6} \cdot 2^n + \frac{1}{6} \cdot (-1)^n - \frac{1}{3} \cdot \cos \frac{\pi n}{3}.$$

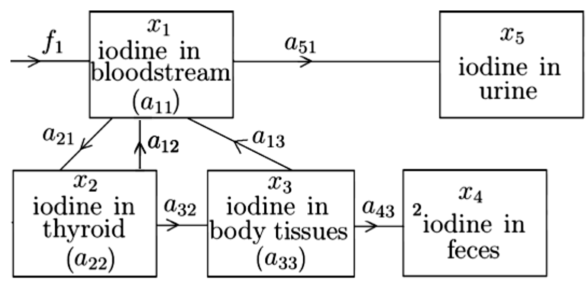
Chapter Five

Section 5.1.3

5.1.3.1. $x_1(t)$ is the amount of iodine in the the bloodstream and $x_2(t)$ is the amount of iodine in the thyroid. Including the assumption that iodine also flows into the bloodstream from the thyroid in the form of hormone means including a term $a_{12}x_2$ in the ODE for x_1 and a term $-a_{12}x_2$ in the ODE for x_2, where $a_{12} > 0$. This is illustrated in the figure below. Note that $a_{21} < 0$.

With this new assumption, the system of ODEs becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} - a_{12} & a_{22} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 \\ a_{51} & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$



Problem 5.1.3.1: Modification of Example 5.6

Figure 1: Problem 5.1.3.1: Modification of Example 5.6

5.1.3.3. As in Example 5.5, assume $x_1 > 0$ when the first object is to the right of its equilibrium position and similarly for $x_2 > 0$ and x_3.

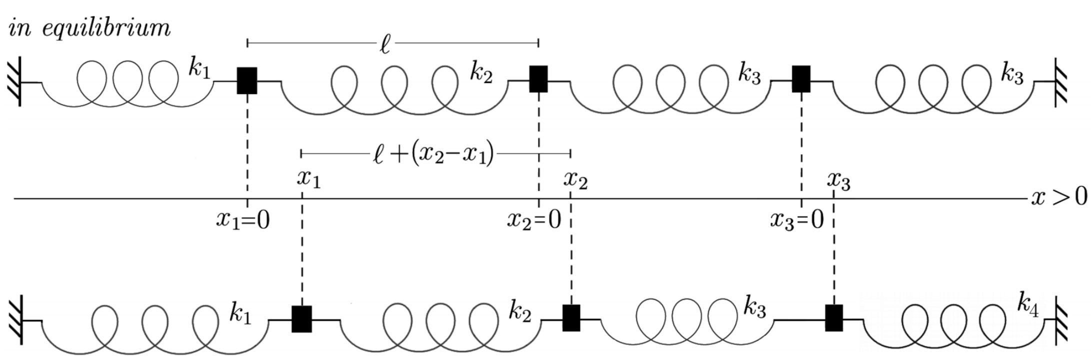
The easiest forces to understand are (a) the force the first spring exerts only on the first object, and (b) the force the fourth spring exerts only on the third object.

The first spring is stretched a distance of x_1, if $x_1 > 0$ and, conversely, the first spring is compressed a distance of $-x_1$, if $x_1 < 0$. The first spring exerts a force of $-k_1x_1$ on the first object, so the first spring acts to bring the first object back to equilibrium.

The fourth spring is compressed by a distance of x_3; equivalently, the fourth spring is stretched by a distance of $-x_3$. In the picture, $x_3 > 0$ the position of the third object contributes a positive compression to the length of the fourth spring. The fourth spring exerts on the third object a force of $-k_4x_3$. [If $x_3 > 0$ then the fourth spring's force acts to bring the third object back to equilibrium.]

The second spring is compressed by a distance of x_1 if $x_1 > 0$; equivalently, the second spring is stretched by a distance of $-x_1$. The second spring is also stretched by a distance of x_2, if $x_2 > 0$; equivalently, the second spring is compressed by a distance $-x_2$. So, the second spring has $(\textit{net compression}) = x_1 + (-x_2) = (x_1 - x_2)$, that is, the second spring has $(\textit{net stretch}) = -(\textit{net compression}) = (x_2 - x_1)$. The second spring exerts on the second object a force of $-k_2(x_2 - x_1)$. [In the picture, $x_1 > x_2$, so the second spring pushes the second object to the right.] The second spring exerts on the first object the opposite force of $k_2(\textit{net stretch})$, that is, $k_2(x_2 - x_1)$. [For example, the picture has $x_1 > x_2$, so the second spring pulls the first object to the right as the second spring tries to shrink to its unstretched length, ℓ.]

The third spring is compressed by a distance of x_2 if $x_2 > 0$; equivalently, the third spring is stretched by a distance of $-x_2$. The third spring is also stretched by a distance of x_3, if $x_3 > 0$; equivalently, the third spring is compressed by a distance $-x_3$. So, the third spring has $(\textit{net compression}) = x_2 + (-x_3) = (x_2 - x_3)$, that is, the third spring has $(\textit{net stretch}) = -(\textit{net compression}) = (x_3 - x_2)$. The third spring exerts on



Problem 5.1.3.3: Three masses and four springs

Figure 2: Problem 5.1.3.3: Three masses and four springs

the third object a force of $-k_3(x_3-x_2)$. The third spring exerts on the second object the opposite force of $k_3(\textit{net stretch})$, that is, $k_3(x_3-x_2)$.

Newton's second law of motion gives us the ODEs

$$m_1\ddot{x_1} = \Sigma Forces\ on\ first\ object = -k_1x_1 + k_2(x_2-x_1),$$

$$m_2\ddot{x_2} = \Sigma Forces\ on\ second\ object = -k_2(x_2-x_1) + k_3(x_3-x_2),$$

and

$$m_3\ddot{x_3} = \Sigma Forces\ on\ third\ object = -k_3(x_3-x_2) - k_4x_3.$$

Later we will divide through these three equations by m_1, m_2, or m_3, respectively. Recall that we assumed this system has no damping forces.

We can write this system of second order ODEs in terms of the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$:

$$\ddot{\mathbf{x}} = \begin{bmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} & \dfrac{k_3}{m_2} \\ 0 & \dfrac{k_3}{m_3} & -\dfrac{k_3+k_4}{m_3} \end{bmatrix} \mathbf{x} \triangleq A\mathbf{x}.$$

5.1.3.5. Let V_1, V_2 be the volumes, in gallons, of the mixtures in the two tanks. Figure 5.5 in the textbook has flow rates of mixtures into or out of the tanks. We have

$$\dot{V}_1 = \text{total rate of change of mixture volume in tank \#1} = +4-1-5+2 = 0$$

and

$$\dot{V}_2 = \text{total rate of change of mixture volume in tank \#2} = +5-2-3 = 0.$$

It follows that $V_1(t) \equiv V(0)$ and $V_2(t) \equiv V_2(0)$.

Also let A_1, A_2 be the amounts of dye, in pounds, in the mixtures in the two tanks. Figure 5.5 in the textbook has flow rates, and possibly dye concentrations of mixtures, into or out of the tanks. If a dye concentration is not given, then the assumption that the mixture in each tank is well-mixed implies that the dye concentration in any flow out of a certain tank equals the concentration of dye in the mixture in that tank.

We have that the amount of dye that flows in or out equals the flow rate times the dye concentration in the flow, so

$$\dot{A}_1 = \text{total rate of change of dye amount in tank \#1} = 4\cdot 2 - 1\cdot\frac{A_1}{V_1} - 5\cdot\frac{A_1}{V_1} + 2\cdot\frac{A_2}{V_2} = 8 - \frac{6}{V_1(0)}\,A_1 + \frac{2}{V_2(0)}\,A_2$$

and

$$\dot{A}_2 = \text{total rate of change of dye amount in tank \#2} = +5\cdot\frac{A_1}{V_1} - 2\cdot\frac{A_2}{V_2} - 3\cdot\frac{A_2}{V_2} = \frac{5}{V_1(0)}\,A_1 - \frac{5}{V_2(0)}\,A_2.$$

The system of ODEs for the amounts of dye in the mixtures in the tanks is

$$\left\{\begin{array}{l} \dot{A}_1 = 8 - \dfrac{6}{V_1(0)}\,A_1 + \dfrac{2}{V_2(0)}\,A_2 \\ \\ \dot{A}_2 = \qquad \dfrac{5}{V_1(0)}\,A_1 - \dfrac{5}{V_2(0)}\,A_2 \end{array}\right\}.$$

5.1.3.7. Let T_1, T_2, M be the temperatures, respectively, of the two objects and the medium. Apply Newton's Law of Cooling to each of the two objects to get

$$\dot{T}_1(t) = -k_{T,1}(T_1 - M)$$

$$\dot{T}_2(t) = -k_{T,2}(T_2 - M)$$

where $k_{T,1}$ and $k_{T,2}$ are constants dependent on the material natures of the two objects, respectively.

Apply Newton's Law of Cooling to the medium to get

$$\dot{M}(t) = -k_M(M - T_1) - k_M(M - T_2),$$

where k_M is a constant dependent on the medium's material nature.

So, the temperature of the medium affects the temperature of the two objects, which in turn affects the temperature of the medium: The temperatures of the objects and the medium are intertwined. Note that because the first object affects the medium and the medium affects the second object, indirectly the first object affects the second object.

To summarize, the three temperatures satisfy the system of ODEs

$$\frac{d}{dt}\begin{bmatrix} T_1 \\ T_2 \\ M \end{bmatrix} = \begin{bmatrix} -k_{T,1} & 0 & k_{T,1} \\ 0 & -k_{T,2} & k_{T,2} \\ k_M & k_M & -2k_M \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ M \end{bmatrix}.$$

We'll assume that $k_{T,1}, k_{T,2}, k_M$ are constants.

Section 5.2.5

5.2.5.1. $0 = \begin{vmatrix} 5-\lambda & 4 \\ 4 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) - 16 = \lambda^2 - 4\lambda - 21 = (\lambda+3)(\lambda-7)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -3, \lambda_2 = 7$

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 8 & 4 & 0 \\ 4 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-\frac{1}{2}\,R_1 + R_2 \to R_2,\ -\frac{1}{8}\,R_1 \to R_1$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -3$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 4 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $2R_1 + R_2 \to R_2\ -\frac{1}{2}\,R_1 \to R_1$,

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 7$

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

5.2.5.3. Using the fact the determinant of an upper triangular matrix is the product of the diagonal entries, the characteristic equation is $0 = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(-1-\lambda)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 3$

$[A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/12 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, after $\frac{1}{4}R_2 \to R_2, -R_2 + R_1 \to R_1, \frac{1}{3}R_1 \to R_1$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} 1 \\ -3 \\ 12 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -1$

$[A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, after $-R_1 + R_2 \to R_2, 3R_2 + R_3 \to R_3$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 2$

$[A - \lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, after $-R_1 \to R_1, 4R_2 + R_3 \to R_3$

$\Rightarrow \mathbf{v}_3 = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_3 = 3$

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -3 \\ 12 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{where } c_1, c_2, c_3 = \text{arbitrary constants.}$$

5.2.5.5. $0 = \begin{vmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda) - 2 = \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -4, \lambda_2 = -1$

$[\, A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-\sqrt{2}\, R_1 + R_2 \to R_2$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -4$

$[\, A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $\frac{1}{\sqrt{2}}\, R_1 + R_2 \to R_2\ -\frac{1}{2}\, R_1 \to R_1$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = -1$

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

Because all eigenvalues are negative, all solutions $\to 0$ as $t \to \infty$. So, there is a time constant. Its value is $\tau = \dfrac{1}{\min\{4, 1\}} = 1$.

5.2.5.7. $0 = \begin{vmatrix} a-\lambda & 0 \\ b & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda) \Rightarrow$ eigenvalues are $\lambda_1 = a, \lambda_2 = c$

Case 1: If $b \neq 0$,

$$[\, A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ b & c-a & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & b^{-1}(c-a) & 0 \\ 0 & 0 & 0 \end{array}\right], \text{after } R_1 \leftrightarrow R_2, b^{-1}R_1 \to R_1.$$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} a-c \\ b \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = a$.

Case 2: If $b = 0$,

$[\, A-\lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ b & c-a & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & c-a & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2, (c-a)^{-1}R_1 \to R_1$.

$\Rightarrow \mathbf{v}_1 = \tilde{c}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for any constant $\tilde{c}_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = a$, if $b = 0$.

But, if $b = 0$ then the eigenvectors corresponding to eigenvalue $\lambda_1 = a$ can, instead, be written as $\mathbf{v}_1 = c_1 \begin{bmatrix} a-c \\ 0 \end{bmatrix}$, because $a - c \neq 0$.

We also need an eigenvector(s) corresponding to eigenvalue λ_2:

$[\, A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} a-c & 0 & 0 \\ b & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $(a-c)^{-1}\, R_1 \to R_1, -bR_1 + R_2 \to R_2$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = c$

Whether or not $b \neq 0$, the general solution of the system can be written as

$$\mathbf{x}(t) = c_1 e^{at} \begin{bmatrix} a-c \\ b \end{bmatrix} + c_2 e^{ct} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

There is a time constant if, and only if, both a and c are negative. If they are, the time constant is $\tau = \dfrac{1}{\min\{|a|,|c|\}}$.

5.2.5.9. This is the same system of ODEs as in problem 5.2.5.3, where we found the general solution to be

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -3 \\ 12 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{where } c_1, c_2, c_3 = \text{arbitrary constants.}$$

So, a fundamental matrix is given by $Z(t) = \begin{bmatrix} e^{-t} & e^{2t} & e^{3t} \\ -3e^{-t} & 0 & e^{3t} \\ 12e^{-t} & 0 & 0 \end{bmatrix}$.

5.2.5.11. $0 = \begin{vmatrix} \sqrt{3}-\lambda & -\sqrt{3} \\ -2\sqrt{3} & -\sqrt{3}-\lambda \end{vmatrix} = (\sqrt{3}-\lambda)(-\sqrt{3}-\lambda) - 6 = \lambda^2 - 9 = (\lambda+3)(\lambda-3)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -3, \lambda_2 = 3$

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} \sqrt{3}+3 & -\sqrt{3} & 0 \\ -2\sqrt{3} & -\sqrt{3}+3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1-\sqrt{3}}{2} & 0 \\ \sqrt{3}+3 & -\sqrt{3} & 0 \end{array}\right], \text{ after } R_1 \leftrightarrow R_2, -\tfrac{1}{2\sqrt{3}}R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & \frac{1-\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right], \quad \text{after } \; -(3+\sqrt{3})R_1 + R_2 \to R_2.$$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -1+\sqrt{3} \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -3$

$$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} \sqrt{3}-3 & -\sqrt{3} & 0 \\ -2\sqrt{3} & -\sqrt{3}-3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1+\sqrt{3}}{2} & 0 \\ \sqrt{3}-3 & -\sqrt{3} & 0 \end{array}\right], \text{ after } R_1 \leftrightarrow R_2, -\tfrac{1}{2\sqrt{3}}R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & \frac{1+\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right], \quad \text{after } \; (3-\sqrt{3})R_1 + R_2 \to R_2$$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} -1-\sqrt{3} \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 3$

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{bmatrix} -1+\sqrt{3} \\ 2 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -1-\sqrt{3} \\ 2 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

So, a fundamental matrix is given by $Z(t) = \begin{bmatrix} (-1+\sqrt{3})e^{-3t} & (-1-\sqrt{3})e^{3t} \\ 2e^{-3t} & 2e^{3t} \end{bmatrix}$. We find e^{tA} by calculating

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = \begin{bmatrix} (-1+\sqrt{3})e^{-3t} & (-1-\sqrt{3})e^{3t} \\ 2e^{-3t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} (-1+\sqrt{3}) & (-1-\sqrt{3}) \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (-1+\sqrt{3})e^{-3t} & (-1-\sqrt{3})e^{3t} \\ 2e^{-3t} & 2e^{3t} \end{bmatrix} \frac{1}{4\sqrt{3}} \begin{bmatrix} 2 & (1+\sqrt{3}) \\ -2 & (-1+\sqrt{3}) \end{bmatrix}$$

$$= \frac{1}{4\sqrt{3}}\begin{bmatrix} 2(-1+\sqrt{3})e^{-3t}-2(-1-\sqrt{3})e^{3t} & 2e^{-3t}-2e^{3t} \\ \\ 4e^{-3t}-4e^{3t} & 2(1+\sqrt{3})e^{-3t}+2(-1+\sqrt{3})e^{3t}\end{bmatrix}$$

$$= \frac{1}{2\sqrt{3}}\begin{bmatrix} (-1+\sqrt{3})e^{-3t}+(1+\sqrt{3})e^{3t} & e^{-3t}-e^{3t} \\ \\ 2e^{-3t}-2e^{3t} & (1+\sqrt{3})e^{-3t}+(-1+\sqrt{3})e^{3t}\end{bmatrix}.$$

5.2.5.13. The characteristic equation is

$$0 = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & -1-\lambda & 3 \\ -1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & -1-\lambda & 3 \\ -\lambda & 0 & -\lambda \end{vmatrix}, \text{ after the row operation } R_1+R_3\to R_3,$$

$$= -\lambda\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & -1-\lambda & 3 \\ 1 & 0 & 1 \end{vmatrix}, \text{ after the row operation } R_3 \leftarrow -\lambda R_3.$$

Expanding along the third row, we get that the characteristic equation is

$$0 = |\,A-\lambda\, I\,| = -\lambda\left(\begin{vmatrix} -1 & 0 \\ -1-\lambda & 3\end{vmatrix} + \begin{vmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda\end{vmatrix}\right) = -\lambda\big(-3+(1-\lambda)(-1-\lambda)\big) = -\lambda\big(-4+\lambda^2\big)$$

$$= -\lambda\big(-4+\lambda^2\big) = -\lambda(-2+\lambda)(2+\lambda) \quad \Rightarrow \text{eigenvalues are } \lambda_1=-2, \lambda_2=0, \lambda_3=2$$

$$[\,A-\lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{rrr|r} 3 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & 0 \end{array}\right], \text{ after } R_1\leftrightarrow R_3, 3R_1+R_3\to R_3, -R_1\to R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ after } -2R_2+R_3\to R_3, R_2+R_1\to R_1$$

$$\Rightarrow \mathbf{v}_1 = c_1\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}, \text{ for any constant } c_1\neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_1=-2$$

$$[\,A-\lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ after } R_1+R_3\to R_3, -R_2\to R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ after } R_2+R_1\to R_1$$

$$\Rightarrow \mathbf{v}_2 = c_1\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \text{ for any constant } c_1\neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_2=0$$

$$[\,A-\lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{rrr|r} -1 & -1 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ -1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right], \text{ after } -R_1\to R_1, R_1+R_3\to R_3, -\tfrac{1}{3}R_2\to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ after } -R_2+R_1 \to R_1, -2R_2+R_3 \to R_3$$

$\Rightarrow \mathbf{v}_3 = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_3 = 2$

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \text{where } c_1, c_2, c_3 = \text{arbitrary constants.}$$

So, a fundamental matrix is given by $Z(t) = \begin{bmatrix} -e^{-2t} & 3 & -e^{2t} \\ -3e^{-2t} & 3 & e^{2t} \\ e^{-2t} & 1 & e^{2t} \end{bmatrix}$. We find e^{tA} by calculating

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = \begin{bmatrix} -e^{-2t} & 3 & -e^{2t} \\ -3e^{-2t} & 3 & e^{2t} \\ e^{-2t} & 1 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ -3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -e^{-2t} & 3 & -e^{2t} \\ -3e^{-2t} & 3 & e^{2t} \\ e^{-2t} & 1 & e^{2t} \end{bmatrix} \frac{1}{8} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 2 \\ -3 & 2 & 3 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -e^{-2t}+6+3e^{2t} & 2e^{-2t}-2e^{2t} & -3e^{-2t}+6-3e^{2t} \\ -3e^{-2t}+6-3e^{2t} & 6e^{-2t}+2e^{2t} & -9e^{-2t}+6+3e^{2t} \\ e^{-2t}+2-3e^{2t} & -2e^{-2t}+2e^{2t} & 3e^{-2t}+2+3e^{2t} \end{bmatrix}.$$

5.2.5.15. $0 = \begin{vmatrix} -a-\lambda & b \\ b & -a-\lambda \end{vmatrix} = (-a-\lambda)^2 - b^2 = \big((-a-\lambda)-b\big)\big((-a-\lambda)+b\big) = (-a-b-\lambda)(-a+b-\lambda)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -a-b$, $\lambda_2 = -a+b$. Because $b > 0$, $\lambda_1 \neq \lambda_2$, and

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} b & b & 0 \\ b & b & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad \text{after } -R_1+R_2 \to R_2, b^{-1}R_1 \to R_1.$$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -a-b$.

$$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -b & b & 0 \\ b & -b & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad \text{after } R_1+R_2 \to R_2, -b^{-1}R_1 \to R_1.$$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = -a+b$.

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-(a+b)t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-(a-b)t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

So, a fundamental matrix is given by

$$Z(t) = \begin{bmatrix} -e^{-(a+b)t} & e^{-(a-b)t} \\ e^{-(a+b)t} & e^{-(a-b)t} \end{bmatrix} = e^{-at} \begin{bmatrix} -e^{-bt} & e^{bt} \\ e^{-bt} & e^{bt} \end{bmatrix}.$$

We find e^{tA} by calculating

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = e^{-at} \begin{bmatrix} -e^{-bt} & e^{bt} \\ e^{-bt} & e^{bt} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= e^{-at}\begin{bmatrix} -e^{-bt} & e^{bt} \\ e^{-bt} & e^{bt} \end{bmatrix} \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{-2}\,e^{-at}\begin{bmatrix} -e^{-bt}-e^{bt} & e^{-bt}-e^{bt} \\ e^{-bt}-e^{bt} & -e^{-bt}-e^{bt} \end{bmatrix}$$

$$= \frac{1}{2}\,e^{-at}\begin{bmatrix} e^{-bt}+e^{bt} & -e^{-bt}+e^{bt} \\ -e^{-bt}+e^{bt} & e^{-bt}+e^{bt} \end{bmatrix}.$$

5.2.5.17. (a) Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -3 & 2 \end{bmatrix}$

(b) Because the matrix $(A - \lambda I)$ is lower triangular, the characteristic equation is

$$0 = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & -\lambda & 0 \\ -2 & -3 & 2-\lambda \end{vmatrix} = (1-\lambda)(-\lambda)(2-\lambda) \Rightarrow \text{eigenvalues are } \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 0$$

$$[A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right], \text{ after } R_1 \leftrightarrow R_2, -R_1 \to R_1, 2R_1 + R_3 \to R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ after } R_2 \leftrightarrow R_3, R_2 + R_1 \to R_1, -R_2 \to R_2$$

$$\Rightarrow \mathbf{v}_1 = c_1\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_1 = 1$$

$$[A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right],$$

after $-R_1 + R_2 \to R_2, -2R_1 + R_3 \to R_3,\ -R_1 \to R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ after } -\tfrac{1}{2}R_2 \to R_2, 3R_2 + R_3 \to R_3$$

$$\Rightarrow \mathbf{v}_2 = c_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_2 = 2$$

$$[A - \lambda_3 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & -3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 \end{array}\right], \text{ after } R_1 + R_2 \to R_2, 2R_1 + R_3 \to R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ after } R_2 \leftrightarrow R_3, -\tfrac{1}{3}R_2 \to R_2$$

$$\Rightarrow \mathbf{v}_3 = c_1\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_3 = 0$$

The general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \text{ where } c_1, c_2, c_3 = \text{arbitrary constants}$$

5.2.5.19. (a) Using the new given eigenvectors, Theorem 5.4 says that

$$X(t) \triangleq \left[e^{-2t} \begin{bmatrix} -\frac{1}{2} \\ \\ 1 \end{bmatrix} \;\vdots\; e^{-3t} \begin{bmatrix} -\frac{1}{3} \\ \\ 1 \end{bmatrix} \right] = \begin{bmatrix} -\frac{1}{2}e^{-2t} & -\frac{1}{3}e^{-3t} \\ \\ e^{-2t} & e^{-3t} \end{bmatrix}$$

is a fundamental matrix for $\dot{\mathbf{x}} = A\mathbf{x}$. Then

$$e^{tA} = X(t)\big(X(0)\big)^{-1} = \begin{bmatrix} -\frac{1}{2}e^{-2t} & -\frac{1}{3}e^{-3t} \\ \\ e^{-2t} & e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{1}{3} \\ \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2}e^{-2t} & -\frac{1}{3}e^{-3t} \\ \\ e^{-2t} & e^{-3t} \end{bmatrix} \begin{bmatrix} -6 & -2 \\ \\ 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-3t} + 3e^{-2t} & -e^{-3t} + e^{-2t} \\ \\ 6e^{-3t} - 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix},$$

which is the same conclusion as in Example 5.13.

5.2.5.21. Define $y(t) = x_1(t)$, so $\dot{y}(t) = \dot{x}_1(t) = x_2(t)$ and $\ddot{y}(t) = \dot{x}_2(t) = -2t^{-2}x_1 + 2t^{-1}x_2 = -2t^{-2}y + 2t^{-1}\dot{y}$, that is, $t^2\ddot{y} - 2t\dot{y} + 2y = 0$. Substituting $y(t) = t^m$ gives characteristic equation $0 = m(m-1) - 2m + 2 = m^2 - 3m + 2 = (m-1)(m-2)$. So, the general solution of the equivalent second order ODE is $x_1(t) = y(t) = c_1 t + c_2 t^2$, where c_1, c_2 =arbitrary constants.

The solution of the original system is

$$\mathbf{x}(t) = \begin{bmatrix} c_1 t & c_2 t^2 \\ c_1 & 2c_2 t \end{bmatrix} = \begin{bmatrix} c_1 t \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 t^2 \\ 2c_2 t \end{bmatrix} = c_1 \begin{bmatrix} t \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t^2 \\ 2t \end{bmatrix},$$

so a fundamental matrix is given by

$$X(t) \triangleq \left[\begin{bmatrix} t \\ \\ 1 \end{bmatrix} \;\vdots\; \begin{bmatrix} t^2 \\ \\ 2t \end{bmatrix} \right] = \begin{bmatrix} t & t^2 \\ \\ 1 & 2t \end{bmatrix}.$$

5.2.5.23. Write the matrix in terms of its columns: $Z(t) = \left[\mathbf{z}^{(1)}(t) \;\vdots\; \cdots \;\vdots\; \mathbf{z}^{(n)}(t)\right]$. We are given that $\dot{Z}(t) = A(t)Z(t)$, hence, using Theorem 1.9 in Section 1.2,

$$\left[\dot{\mathbf{z}}^{(1)}(t) \;\vdots\; \cdots \;\vdots\; \dot{\mathbf{z}}^{(n)}(t)\right] = \dot{Z}(t) = A(t)Z(t) = A(t)\left[\mathbf{z}^{(1)}(t) \;\vdots\; \cdots \;\vdots\; \mathbf{z}^{(n)}(t)\right] = \left[A(t)\mathbf{z}^{(1)}(t) \;\vdots\; \cdots \;\vdots\; A(t)\mathbf{z}^{(n)}(t)\right]$$

hence $\dot{\mathbf{z}}^{(j)}(t) = A(t)\mathbf{z}^{(j)}(t)$ for $j = 1, .., n$. This says that every column of $Z(t)$ is a solution of the same linear homogeneous system $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$, as we were asked to show.

5.2.5.25. Yes, because $X(t)$ is a fundamental matrix for $(\star)$ $\dot{\mathbf{x}} = A(t)\mathbf{x}$, we have $\dot{X}(t) = A(t)X(t)$, so

$$\dot{Y}(t) = \frac{d}{dt}[\,X(t)B\,] = \dot{X}(t)B = \big(A(t)X(t)\big)B = A(t)\big(X(t)B\big) = A(t)Y(t).$$

Because $X(t)$ is a fundamental matrix, it is invertible on some open interval of time. We were given that B is invertible, so $X(t)B$ is also invertible. This and the differential equation that $Y(t)$ satisfies implies that $Y(t)$ is also a fundamental matrix for $(\star)$.

5.2.5.27. Take the hint and define $Y(t) \triangleq X(-t)$, where $X(t)$ is a fundamental matrix for a system $(\star)$ $\dot{\mathbf{x}} = A(t)\mathbf{x}$. The latter is equivalent to $\dot{X}(t) = A(t)X(t)$. By replacing t by $-t$ throughout, it follows that $(\star\star)\dot{X}(-t) = A(-t)X(-t)$

The chain rule, and after that, $(\star\star)$, imply

$$\dot{Y}(t) \triangleq \frac{d}{dt}[\,Y(t)\,] = \frac{d}{dt}[\,X(-t)\,] = \dot{X}(-t)\cdot\frac{d}{dt}[\,(-t)\,] = -\dot{X}(-t) = -A(-t)X(-t).$$

But, $A(-t) \equiv -A(t)$, so

$$\dot{Y}(t) = A(t)X(-t) = A(t)Y(t).$$

So, both

$$X(t) = \left[\mathbf{x}^{(1)}(t) \;\vdots\; \cdots \;\vdots\; \mathbf{x}^{(n)}(t)\right] \quad\text{and}\quad Y(t) = \left[\mathbf{y}^{(1)}(t) \;\vdots\; \cdots \;\vdots\; \mathbf{y}^{(n)}(t)\right]$$

are fundamental matrices for $(\star)$ $\dot{\mathbf{x}} = A(t)\mathbf{x}$, and $X(0) = I = \left[\mathbf{e}^{(1)} \;\vdots\; \cdots \;\vdots\; \mathbf{e}^{(n)}\right] = Y(0)$. For $j = 1, ..., n$, uniqueness of solutions for each of the IVPs $(\star)$ $\dot{\mathbf{x}} = A(t)\mathbf{x}$, $\quad \mathbf{x}(0) = \mathbf{e}^{(j)}$, implies that $\mathbf{x}^{(j)}(t) \equiv \mathbf{y}^{(j)}(t)$. So, $X(t) \equiv Y(t) \triangleq X(-t)$, that is, $X(t)$ is an even function of t.

5.2.5.29. $0 = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)(-3-\lambda) - 1 = (-3-\lambda)^2 - 1$

$\Rightarrow$ eigenvalues are $\lambda_1 = -4, \lambda_2 = -2$

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -4$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = -2$

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

So, a fundamental matrix is given by $Z(t) = \begin{bmatrix} -e^{-4t} & e^{-2t} \\ e^{-4t} & e^{-2t} \end{bmatrix}$. We find e^{tA} by calculating

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = \begin{bmatrix} -e^{-4t} & e^{-2t} \\ e^{-4t} & e^{-2t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -e^{-4t} & e^{-2t} \\ e^{-4t} & e^{-2t} \end{bmatrix}\frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{-4t} + e^{-2t} & -e^{-4t} + e^{-2t} \\ -e^{-4t} + e^{-2t} & e^{-4t} + e^{-2t} \end{bmatrix}$$

We calculate that the improper integral is

$$\int_0^\infty e^{tA^T} e^{tA} dt \triangleq \lim_{b\to\infty}\int_0^b e^{tA^T} e^{tA} dt = \lim_{b\to\infty}\int_0^b \left(e^{tA}\right)^T e^{tA} dt =$$

$$= \lim_{b\to\infty}\int_0^b \left(\frac{1}{2}\begin{bmatrix} e^{-4t} + e^{-2t} & -e^{-4t} + e^{-2t} \\ -e^{-4t} + e^{-2t} & e^{-4t} + e^{-2t} \end{bmatrix}\right)^T \frac{1}{2}\begin{bmatrix} e^{-4t} + e^{-2t} & -e^{-4t} + e^{-2t} \\ -e^{-4t} + e^{-2t} & e^{-4t} + e^{-2t} \end{bmatrix} dt$$

$$= \frac{1}{4}\lim_{b\to\infty}\int_0^b \begin{bmatrix} 2e^{-8t}+2e^{-4t} & -2e^{-8t}+2e^{-4t} \\ -2e^{-8t}+2e^{-4t} & 2e^{-8t}+2e^{-4t} \end{bmatrix} dt$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\begin{bmatrix} -\frac{1}{8}\,e^{-8t}-\frac{1}{4}\,e^{-4t} & \frac{1}{8}\,e^{-8t}-\frac{1}{4}\,e^{-4t} \\ \frac{1}{8}\,e^{-8t}-\frac{1}{4}\,e^{-4t} & -\frac{1}{8}\,e^{-8t}-\frac{1}{4}\,e^{-4t} \end{bmatrix} \right]_0^b$$

$$= \frac{1}{16}\lim_{b\to\infty}\begin{bmatrix} -e^{-8b}+1-2\,e^{-4b}+2 & e^{-8b}-1-2e^{-4b}+2 \\ e^{-8b}-1-2e^{-4b}+2 & -e^{-8b}+1-2e^{-4b}+2 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

5.2.5.31. Take the hint and define $Y(t) = (X(t))^{-1}$, where $X(t)$ is a fundamental matrix for a system $(\star)$ $\dot{\mathbf{x}} = A(t)\mathbf{x}$. The latter is equivalent to $\dot{X}(t) = A(t)X(t)$.

To find the ODE that $Y(t)$ satisfies, note that $I = X(t)\,(X(t))^{-1} = X(t)\,Y(t)$. So, using the product rule to differentiate both sides with respect to t, we get

$$O = \frac{d}{dt}[\,I\,] = \frac{d}{dt}[\,X(t)\,Y(t)\,] = X(t)\dot{Y}(t) + \dot{X}(t)Y(t),$$

hence $X(t)\dot{Y}(t) = -\dot{X}(t)Y(t)$. It follows that

$$\dot{Y}(t) = -\big(X(t)\big)^{-1}\dot{X}(t)Y(t) = -\big(X(t)\big)^{-1}A(t)X(t)Y(t) = -Y(t)A(t)\big(X(t)Y(t)\big) = -Y(t)A(t)\big(I\big),$$

hence

$$(\star\star)\quad \dot{Y}(t) = -Y(t)A(t).$$

On the other hand, define $Z(t) \triangleq X(t)^T$ and note that $Z(t)$ satisfies the ODE system

$$\dot{Z}(t) = \big(\dot{X}(t)\big)^T = \big(A(t)X(t)\big)^T = \big(X(t)\big)^T\big(A(t)\big)^T = Z(t)\big(A(t)\big)^T.$$

But, we were given that $A(t)^T \equiv -A(t)$, so

$$\dot{Z}(t) = Z(t)\big(A(t)\big)^T = Z(t)\big(-A(t)\big) = -Z(t)A(t).$$

So, both

$$Y(t) = \begin{bmatrix} \big(\mathbf{y}^{(1)}(t)\big)^T \\ \vdots \\ \big(\mathbf{y}^{(n)}(t)\big)^T \end{bmatrix} \quad \text{and} \quad Z(t) = \begin{bmatrix} \big(\mathbf{z}^{(1)}(t)\big)^T \\ \vdots \\ \big(\mathbf{z}^{(n)}(t)\big)^T \end{bmatrix}$$

are fundamental matrices for $(\star\star\star)$ $(\dot{\mathbf{x}})^T = -(\mathbf{x})^T A(t)$, and $Y(0) = I = \begin{bmatrix} \big(\mathbf{e}^{(1)})\big)^T \\ \vdots \\ \big(\mathbf{e}^{(n)})\big)^T \end{bmatrix} = Z(0)$. For $j = 1, ..., n$, uniqueness of solutions for each of the IVPs $(\star)$ $(\dot{\mathbf{x}})^T = -(\mathbf{x})^T A(t)$, $\quad (\mathbf{x})^T(0) = (\mathbf{e}^{(j)})^T$, implies that $\big(\mathbf{y}^{(j)}(t)\big)^T \equiv \big(\mathbf{z}^{(j)}(t)\big)^T$. So, $Y(t) \equiv Z(t)$, that is, $\big(X(t)\big)^{-1} \equiv \big(X(t)\big)^T$, as we wished to show.

5.2.5.33. (a) Given a third order scalar ODE $\dddot{y} + p\ddot{y} + q\dot{y} + ry = 0$, define $x_1 = y$, $x_2 = \dot{y}$, and $x_3 = \ddot{y}$. Equivalent to the third order scalar ODE is the system

$$\left\{ \begin{array}{l} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = x_3 \\ \dot{x}_3 = \dddot{y} = -p\ddot{y} - q\dot{y} - ry = -px_3 - qx_2 - rx_1 \end{array} \right\},$$

that is,
$$\frac{d}{dt}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -r & -q & -p\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}.$$

So, in $\mathbb{R}^3$ the generalization of companion form is $A=\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ \star & \star & \star\end{bmatrix}$.

(b) Given an $n-$th order scalar ODE $y^{(n)}+a_1y^{(n-1)}+...+a_{n-1}\dot{y}+a_ny=0$, define $x_1=y,\ ...\ ,x_n=y^{(n-1)}$. Equivalent to the $n-$th order scalar ODE is the system

$$\left\{\begin{array}{l}\dot{x}_1=\dot{y}=x_2\\ \dot{x}_2=\ddot{y}=x_3\\ \vdots\\ \dot{x}_n=y^{(n)}=-\Big(a_1y^{(n-1)}+...+a_{n-1}\dot{y}+a_ny\Big)\end{array}\right\},$$

that is,
$$\frac{d}{dt}\begin{bmatrix} x_1\\ x_2\\ \\ \vdots\\ \\ x_n\end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 & ... & 0\\ 0 & 0 & 1 & 0 & ... & 0\\ . & & & . & & .\\ . & & & & . & .\\ . & & & & . & .\\ 0 & . & . & . & & 1\\ -a_n & -a_{n-1} & & & ... & -a_1\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ \\ \vdots\\ \\ x_n\end{bmatrix}.$$

So, in $\mathbb{R}^n$ the generalization of companion form is $A=\begin{bmatrix} 0 & 1 & 0 & 0 & ... & 0\\ 0 & 0 & 1 & 0 & ... & 0\\ . & & & . & & .\\ . & & & & . & .\\ . & & & & . & .\\ 0 & . & . & . & & 1\\ \star & \star & & & ... & \star\end{bmatrix}$.

5.2.5.35. Yes, if A is a real, symmetric $n\times n$ matrix, then e^{tA} must be real and symmetric because of the Maclaurin series formula
$$e^{tA}\triangleq I+tA+\frac{t^2}{2!}A^2+\frac{t^3}{3!}A^3+...$$
just before equation (5.29). Why? From the Maclaurin series and the assumption that A is symmetric, that is, $A^T=A$, it follows that

$$\left(e^{tA}\right)^T=\left(I+tA+\frac{t^2}{2!}A^2+\frac{t^3}{3!}A^3+...\right)^T=I^T+\left(tA\right)^T+\left(\frac{t^2}{2!}A^2\right)^T+\left(\frac{t^3}{3!}A^3\right)^T+...$$
$$=I+tA^T+\frac{t^2}{2!}\left(A^2\right)^T+\frac{t^3}{3!}\left(A^3\right)^T+...=I+tA^T+\frac{t^2}{2!}\left(A^T\right)^2+\frac{t^3}{3!}\left(A^T\right)^3+...$$
$$=I+tA+\frac{t^2}{2!}A^2+\frac{t^3}{3!}A^3+...=e^{tA},$$

that is, e^{tA} is symmetric.

Also, from the Maclaurin series we see that if A is real then all of the terms in the series for e^{tA} are real, hence e^{tA} is real.

Section 5.3.6

5.3.6.1. $0 = |\,A-\lambda I\,| = \begin{vmatrix} -2-\lambda & -5 \\ 1 & 0-\lambda \end{vmatrix} = (-2-\lambda)(-\lambda)+5 = \lambda^2+2\lambda+5 = (\lambda+1)^2+4$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = -1 \pm 2i$. Corresponding to eigenvalue $\lambda_1 = -1+i2$, eigenvectors are found by

$$\left[\, A-(-1+i2)I \mid \mathbf{0} \,\right] = \left[\begin{array}{cc|c} -1-i2 & -5 & 0 \\ 1 & 1-i2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1-i2 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 \leftrightarrow R_2, (1+i2)R_1+R_2 \rightarrow R_2$. Corresponding to eigenvalue $\lambda_1 = -1+i2$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} -1+i2 \\ 1 \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(-1+i2)t}\begin{bmatrix} -1+i2 \\ 1 \end{bmatrix}\right) = \mathcal{R}e\left(e^{-t}(\cos 2t + i\sin 2t)\begin{bmatrix} -1+i2 \\ 1 \end{bmatrix}\right)$$

$$= \mathcal{R}e\left(e^{-t}\begin{bmatrix} -\cos 2t - 2\sin 2t - i\sin 2t + i2\cos 2t \\ \cos 2t + i\sin 2t \end{bmatrix}\right) = e^{-t}\begin{bmatrix} -\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(-1+i2)t}\begin{bmatrix} -1+i2 \\ 1 \end{bmatrix}\right) = \mathcal{I}m\left(e^{-t}\begin{bmatrix} -\cos 2t - 2\sin 2t - i\sin 2t + i2\cos 2t \\ \cos 2t + i\sin 2t \end{bmatrix}\right)$$

$$= e^{-t}\begin{bmatrix} 2\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix}.$$

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} -\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

5.3.6.3. $0 = |\,A-\lambda I\,| = \begin{vmatrix} 3-\lambda & 2 \\ -4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)+8 = \lambda^2-2\lambda+5 = (\lambda-1)^2+4$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = 1 \pm 2i$. Corresponding to eigenvalue $\lambda_1 = 1+i2$, eigenvectors are found by

$$\left[\, A-(1+i2)I \mid \mathbf{0} \,\right] = \left[\begin{array}{cc|c} 2-i2 & 2 & 0 \\ -4 & -2-i2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1+i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 \leftrightarrow R_2, -\frac{1}{4}R_1 \rightarrow R_1, (-2+i2)R_1+R_2 \rightarrow R_2$. Corresponding to eigenvalue $\lambda_1 = 1+i2$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(1+i2)t}\begin{bmatrix} -1-i \\ 2 \end{bmatrix}\right) = \mathcal{R}e\left(e^{t}(\cos 2t + i\sin 2t)\begin{bmatrix} -1-i \\ 2 \end{bmatrix}\right)$$

$$= \mathcal{R}e\left(e^{t}\begin{bmatrix} -\cos 2t + \sin 2t - i\sin 2t - i\cos 2t \\ 2\cos 2t + i2\sin 2t \end{bmatrix}\right) = e^{t}\begin{bmatrix} -\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(1+i2)t}\begin{bmatrix} -1-i \\ 2 \end{bmatrix}\right) = \mathcal{I}m\left(e^{t}\begin{bmatrix} -\cos 2t + \sin 2t - i\sin 2t - i\cos 2t \\ 2\cos 2t + i2\sin 2t \end{bmatrix}\right) = e^{t}\begin{bmatrix} -\cos 2t - \sin 2t \\ 2\sin 2t \end{bmatrix}.$$

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{t} \begin{bmatrix} -\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -\cos 2t - \sin 2t \\ 2\sin 2t \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

5.3.6.5. $0 = |\, A - \lambda I \,| = \begin{vmatrix} 5-\lambda & 0 & -10 \\ 0 & -2-\lambda & 0 \\ 4 & 0 & -7-\lambda \end{vmatrix} = (-2-\lambda) \begin{vmatrix} 5-\lambda & -10 \\ 4 & -7-\lambda \end{vmatrix}$

$$= (-2-\lambda)\big((5-\lambda)(-7-\lambda)+40\big) = (-2-\lambda)(\lambda^2+2\lambda+5)$$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = -1 \pm 2i$ and the real eigenvalue $\lambda_3 = -2$.

Corresponding to eigenvalue $\lambda_1 = -1 + i2$, eigenvectors are found by

$$\big[\, A - (1+i2)I \mid \mathbf{0} \,\big] = \left[\begin{array}{ccc|c} 6-i2 & 0 & -10 & 0 \\ 0 & -1-i2 & 0 & 0 \\ 4 & 0 & -6-i2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3+i}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 \leftrightarrow R_3, \frac{1}{4}R_1 \rightarrow R_1, (-6+i2)R_1 + R_3 \rightarrow R_3, \frac{1}{-1-i2}R_2 \rightarrow R_2$. Corresponding to eigenvalue $\lambda_1 = -1 + i2$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} 3+i \\ 0 \\ 2 \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(-1+i2)t} \begin{bmatrix} 3+i \\ 0 \\ 2 \end{bmatrix} \right) = \mathcal{R}e\left(e^{-t}(\cos 2t + i\sin 2t) \begin{bmatrix} 3+i \\ 0 \\ 2 \end{bmatrix} \right)$$

$$= \mathcal{R}e\left(e^{-t} \begin{bmatrix} 3\cos 2t - \sin 2t + i3\sin 2t + i\cos 2t \\ 0 \\ 2\cos 2t + i2\sin 2t \end{bmatrix} \right) = e^{-t} \begin{bmatrix} 3\cos 2t - \sin 2t \\ 0 \\ 2\cos 2t \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(-1+i2)t} \begin{bmatrix} 3+i \\ 0 \\ 2 \end{bmatrix} \right) = \mathcal{I}m\left(e^{-t} \begin{bmatrix} 3\cos 2t - \sin 2t + i3\sin 2t + i\cos 2t \\ 0 \\ 2\cos 2t + i2\sin 2t \end{bmatrix} \right)$$

$$= e^{-t} \begin{bmatrix} 3\sin 2t + \cos 2t \\ 0 \\ 2\sin 2t \end{bmatrix}.$$

Corresponding to eigenvalue $\lambda_3 = -2$, eigenvectors are found by

$$\big[\, A - (-2)I \mid \mathbf{0} \,\big] = \left[\begin{array}{ccc|c} 7 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

after row operations $\frac{1}{7}R_1 \rightarrow R_1, -4R_1 + R_3 \rightarrow R_3, \frac{7}{5}R_3 \rightarrow R_3, \frac{10}{7}R_3 + R_1 \rightarrow R_1$. Corresponding to eigenvalue $\lambda_3 = -2$ we have an eigenvector $\mathbf{v}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 3\cos 2t - \sin 2t \\ 0 \\ 2\cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 3\sin 2t + \cos 2t \\ 0 \\ 2\sin 2t \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where c_1, c_2, c_3 =arbitrary constants.

5.3.6.7. $0 = |\,A - \lambda I\,| = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 5 = \lambda^2 + 2\lambda + 2 = (\lambda+1)^2 + 1$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = -1 \pm i$. Corresponding to eigenvalue $\lambda_1 = -1+i$, eigenvectors are found by

$$\left[\,A - (-1+i)I \mid \mathbf{0}\,\right] = \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2-i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 \leftrightarrow R_2, (-2+i)R_1 + R_2 \to R_2$. Corresponding to eigenvalue $\lambda_1 = -1+i$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(-1+i)t}\begin{bmatrix} 2+i \\ 1 \end{bmatrix}\right) = \mathcal{R}e\left(e^{-t}(\cos t + i\sin t)\begin{bmatrix} 2+i \\ 1 \end{bmatrix}\right)$$

$$= \mathcal{R}e\left(e^{-t}\begin{bmatrix} 2\cos t - \sin t + i2\sin t + i\cos t \\ \cos t + i\sin t \end{bmatrix}\right) = e^{-t}\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(-1+i)t}\begin{bmatrix} 2+i \\ 1 \end{bmatrix}\right) = \mathcal{I}m\left(e^{-t}\begin{bmatrix} 2\cos t - \sin t + i2\sin t + i\cos t \\ \cos t + i\sin t \end{bmatrix}\right) = e^{-t}\begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix}.$$

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

A fundamental matrix is given by $X(t) = e^{-t}\begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{bmatrix}$.

5.3.6.9. $0 = |\,A - \lambda I\,| = \begin{vmatrix} -12-\lambda & -25 \\ 4 & 8-\lambda \end{vmatrix} = (-12-\lambda)(8-\lambda) + 100 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$

$\Rightarrow$ the eigenvalues of A are $\lambda_1 = \lambda_2 = -2$. Corresponding to eigenvalue $\lambda_1 = -2$, eigenvectors are found by

$$\left[\,A - (-2)I \mid \mathbf{0}\,\right] = \left[\begin{array}{cc|c} -10 & -25 & 0 \\ 4 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $-\frac{1}{10}R_1 \to R_1, -4R_1 + R_2 \to R_2$. Corresponding to eigenvalue $\lambda_1 = -2$ we have *only* eigenvectors $\mathbf{v} = c_1\begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}$, $c_1 \neq 0$. Define $\mathbf{v}^{(1)} = \begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}$.

Because $\lambda_1 = \lambda_2 = -2$ is a deficient eigenvalue, we need to also find a generalized eigenvector $\mathbf{w}$ that should satisfy the system $(A - (-2)I)\mathbf{w} = \mathbf{v}^{(1)}$:

$$\left[\,A - (-2)I \mid \mathbf{v}^{(1)}\,\right] = \left[\begin{array}{cc|c} -10 & -25 & -\frac{5}{2} \\ 4 & 10 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{5}{2} & \frac{1}{4} \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $-\frac{1}{10}R_1 \to R_1, -4R_1 + R_2 \to R_2$. So, corresponding to eigenvalue $\lambda_1 = -2$ we have a generalized eigenvector $\mathbf{w} = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$.

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{-2t}\mathbf{v}^{(1)} + c_2 e^{-2t}(t\mathbf{v}^{(1)} + \mathbf{w}) = c_1 e^{-2t}\begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -\frac{5}{2}t + \frac{1}{4} \\ t \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

A fundamental matrix is given by $X(t) = e^{-2t} \begin{bmatrix} \frac{5}{2} & \frac{5}{2}\,t - \frac{1}{4} \\ -1 & -t \end{bmatrix}$.

5.3.6.11. Because A is in companion form, the easiest way to do this problem is to first solve the equivalent scalar second order ODE, $\ddot{y} + 2\dot{y} + y = 0$, where $y(t) = x_1(t)$ and $\dot{y}(t) = x_2(t)$: The characteristic equation is $0 = s^2 + 2s + 1 = (s+1)^2$, hence $s = -1, -1$ is a repeated root. The general solution of the scalar second order ODE is $y(t) = c_1\,e^{-t} + c_2\,t\,e^{-t}$, where c_1, c_2 =arbitrary constants.

So, the general solution of the original LCCHS in companion form is

$$\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} c_1\,e^{-t} + c_2\,t\,e^{-t} \\ -c_1\,e^{-t} + c_2(1-t)e^{-t} \end{bmatrix} = c_1\,e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\,e^{-t} \begin{bmatrix} t \\ 1-t \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

A fundamental matrix is given by $X(t) = e^{-t} \begin{bmatrix} 1 & t \\ -1 & 1-t \end{bmatrix}$.

5.3.6.13. $0 = \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = a \pm i\,b$.

Corresponding to eigenvalue $\lambda_1 = a + i\,b$, eigenvectors are found by

$$\left[\, A - (a+i\,b)I \mid \mathbf{0} \,\right] = \left[\begin{array}{cc|c} -i\,b & b & 0 \\ -b & -i\,b & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $\frac{i}{b}\,R_1 \to R_1,\ bR_1 + R_2 \to R_2$. Corresponding to eigenvalue $\lambda_1 = a + i\,b$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(a+i\,b)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} \right) = \mathcal{R}e\left(e^{at}(\cos bt + i \sin bt) \begin{bmatrix} -i \\ 1 \end{bmatrix} \right) = \mathcal{R}e\left(e^{at} \begin{bmatrix} \sin bt - i\,\cos bt \\ \cos bt + i \sin bt \end{bmatrix} \right)$$

$$= e^{-t} \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(a+i\,b)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} \right) = \mathcal{I}m\left(e^{at} \begin{bmatrix} \sin bt - i\,\cos bt \\ \cos bt + i \sin bt \end{bmatrix} \right) = e^{at} \begin{bmatrix} -\cos bt \\ \sin bt \end{bmatrix}.$$

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{at} \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix} + c_2 e^{at} \begin{bmatrix} -\cos bt \\ \sin bt \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

A fundamental matrix is given by

$$Z(t) = e^{at} \begin{bmatrix} \sin bt & -\cos bt \\ \cos bt & \sin bt \end{bmatrix}.$$

Alternatively, if we switch the columns and multiply a column by -1 we get another possible fundamental matrix given by $X(t) = e^{at} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$.

5.3.6.15. $0 = |\,A - \lambda I\,| = \begin{vmatrix} -2-\lambda & -3 \\ 2 & -4-\lambda \end{vmatrix} = (-2-\lambda)(-4-\lambda)+6 = \lambda^2+6\lambda+14 = (\lambda+3)^2+5$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = -3 \pm i\sqrt{5}$. Corresponding to eigenvalue $\lambda_1 = -3+i\sqrt{5}$, eigenvectors are found by

$$\left[\, A-(-3+i\sqrt{5})I \mid \mathbf{0}\,\right] = \left[\begin{array}{cc|c} 1-i\sqrt{5} & -3 & 0 \\ 2 & -1-i\sqrt{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{1+i\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 \leftrightarrow R_2$, $\frac{1}{2}R_1 \to R_1$, $(-1+i\sqrt{5})R_1+R_2 \to R_2$. Corresponding to eigenvalue $\lambda_1 = -3+i\sqrt{5}$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} 1+i\sqrt{5} \\ 2 \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(-3+i\sqrt{5})t}\begin{bmatrix} 1+i\sqrt{5} \\ 2 \end{bmatrix}\right) = \mathcal{R}e\left(e^{-3t}\big(\cos(\sqrt{5}\,t)+i\sin(\sqrt{5}\,t)\big)\begin{bmatrix} 1+i\sqrt{5} \\ 2 \end{bmatrix}\right)$$

$$= \mathcal{R}e\left(e^{-3t}\begin{bmatrix} \cos(\sqrt{5}\,t)-\sqrt{5}\,\sin(\sqrt{5}\,t)+i\sin(\sqrt{5}\,t)+i\sqrt{5}\,\cos(\sqrt{5}\,t) \\ 2\cos(\sqrt{5}\,t)+i2\sin(\sqrt{5}\,t) \end{bmatrix}\right)$$

$$= e^{-3t}\begin{bmatrix} \cos(\sqrt{5}\,t)-\sqrt{5}\,\sin(\sqrt{5}\,t) \\ 2\cos(\sqrt{5}\,t) \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(-3+i\sqrt{5})t}\begin{bmatrix} 1+i\sqrt{5} \\ 2 \end{bmatrix}\right)$$

$$= \mathcal{I}m\left(e^{-3t}\begin{bmatrix} \cos(\sqrt{5}\,t)-\sqrt{5}\,\sin(\sqrt{5}\,t)+i\sin(\sqrt{5}\,t)+i\sqrt{5}\,\cos(\sqrt{5}\,t) \\ 2\cos(\sqrt{5}\,t)+i2\sin(\sqrt{5}\,t) \end{bmatrix}\right) = e^{-3t}\begin{bmatrix} \sin(\sqrt{5}\,t)+\sqrt{5}\,\cos(\sqrt{5}\,t) \\ 2\sin(\sqrt{5}\,t) \end{bmatrix}.$$

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{bmatrix} \cos(\sqrt{5}\,t)-\sqrt{5}\,\sin(\sqrt{5}\,t) \\ 2\cos(\sqrt{5}\,t) \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} \sin(\sqrt{5}\,t)+\sqrt{5}\,\cos(\sqrt{5}\,t) \\ 2\sin(\sqrt{5}\,t) \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

A fundamental matrix is given by $Z(t) = e^{-3t}\begin{bmatrix} \cos(\sqrt{5}\,t)-\sqrt{5}\,\sin(\sqrt{5}\,t) & \sin(\sqrt{5}\,t)+\sqrt{5}\,\cos(\sqrt{5}\,t) \\ 2\cos(\sqrt{5}\,t) & 2\sin(\sqrt{5}\,t) \end{bmatrix}$.

Further,

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = e^{-3t}\begin{bmatrix} \cos(\sqrt{5}\,t)-\sqrt{5}\sin(\sqrt{5}\,t) & \sin(\sqrt{5}\,t)+\sqrt{5}\cos(\sqrt{5}\,t) \\ 2\cos(\sqrt{5}\,t) & 2\sin(\sqrt{5}\,t) \end{bmatrix}\begin{bmatrix} 1 & \sqrt{5} \\ 2 & 0 \end{bmatrix}^{-1}$$

$$= -\frac{1}{2\sqrt{5}}\,e^{-3t}\begin{bmatrix} \cos(\sqrt{5}\,t)-\sqrt{5}\sin(\sqrt{5}\,t) & \sin(\sqrt{5}\,t)+\sqrt{5}\cos(\sqrt{5}\,t) \\ 2\cos(\sqrt{5}\,t) & 2\sin(\sqrt{5}\,t) \end{bmatrix}\begin{bmatrix} 0 & -\sqrt{5} \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}}\,e^{-3t}\begin{bmatrix} 2\sqrt{5}\cos(\sqrt{5}\,t)+2\sin(\sqrt{5}\,t) & -6\sin(\sqrt{5}\,t) \\ 4\sin(\sqrt{5}\,t) & 2\sqrt{5}\cos(\sqrt{5}\,t)-2\sin(\sqrt{5}\,t) \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\,e^{-3t}\begin{bmatrix} \sqrt{5}\cos(\sqrt{5}\,t)+\sin(\sqrt{5}\,t) & -3\sin(\sqrt{5}\,t) \\ 2\sin(\sqrt{5}\,t) & \sqrt{5}\cos(\sqrt{5}\,t)-\sin(\sqrt{5}\,t) \end{bmatrix}.$$

Aside: We get the same e^{tA} even if the middle of our work has a different fundamental matrix $X(t)$, for example, from using an eigenvector

$$\check{\mathbf{v}}^{(1)} \triangleq \frac{3}{(1-i\sqrt{5})}\begin{bmatrix} 1+i\sqrt{5} \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{6}{(1-i\sqrt{5})} \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{6}{(1-i\sqrt{5})}\cdot\frac{(1+i\sqrt{5})}{(1+i\sqrt{5})} \end{bmatrix} = \begin{bmatrix} 3 \\ 1+i\sqrt{5} \end{bmatrix}$$

corresponding to eigenvalue $\lambda_1 = -3 + i\sqrt{5}$.

5.3.6.17. $0 = |\,A - \lambda I\,| = \begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & -1-\lambda & 0 \\ 4 & 0 & 3-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ 4 & 3-\lambda \end{vmatrix}$

$$= (-1-\lambda)\big((3-\lambda)^2 + 8\big)$$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = 3 \pm i\sqrt{8}$ and the real eigenvalue $\lambda_3 = -1$.

Corresponding to eigenvalue $\lambda_1 = 3 + i\sqrt{8}$, eigenvectors are found by

$$\big[\,A - (3+i\sqrt{8})I \mid \mathbf{0}\,\big] = \left[\begin{array}{ccc|c} -i\sqrt{8} & 0 & -2 & 0 \\ 0 & -4-i\sqrt{8} & 0 & 0 \\ 4 & 0 & -i\sqrt{8} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 \leftrightarrow R_3$, $\frac{1}{4}R_1 \to R_1$, $i\sqrt{8}\,R_1 + R_3 \to R_3$, $\frac{1}{-4-i\sqrt{8}}R_2 \to R_2$. Corresponding to eigenvalue $\lambda_1 = 3 + i\sqrt{8}$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} i \\ 0 \\ \sqrt{2} \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(3+i2)t}\begin{bmatrix} i \\ 0 \\ \sqrt{2} \end{bmatrix}\right) = \mathcal{R}e\left(e^{3t}(\cos(\sqrt{8}\,t) + i\sin(\sqrt{8}\,t))\begin{bmatrix} i \\ 0 \\ \sqrt{2} \end{bmatrix}\right)$$

$$= \mathcal{R}e\left(e^{3t}\begin{bmatrix} -\sin(\sqrt{8}\,t) + i\cos(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\cos(\sqrt{8}\,t) + i\sqrt{2}\sin(\sqrt{8}\,t) \end{bmatrix}\right) = e^{3t}\begin{bmatrix} -\sin(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\cos(\sqrt{8}\,t) \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(3+i2)t}\begin{bmatrix} i \\ 0 \\ \sqrt{2} \end{bmatrix}\right) = \mathcal{I}m\left(e^{3t}\begin{bmatrix} -\sin(\sqrt{8}\,t) + i\cos(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\cos(\sqrt{8}\,t) + i\sqrt{2}\sin(\sqrt{8}\,t) \end{bmatrix}\right) = e^{3t}\begin{bmatrix} \cos(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\sin(\sqrt{8}\,t) \end{bmatrix}.$$

Corresponding to eigenvalue $\lambda_3 = -1$, eigenvectors are found by

$$\big[\,A - (-1)I \mid \mathbf{0}\,\big] = \left[\begin{array}{ccc|c} 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

after row operations $-R_1 + R_3 \to R_3$, $\frac{1}{4}R_1 \to R_1$, $\frac{1}{6}R_3 \to R_3$, $\frac{1}{2}R_3 + R_1 \to R_1$. Corresponding to eigenvalue $\lambda_3 = -1$ we have an eigenvector $\mathbf{v}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{3t}\begin{bmatrix} -\sin(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\cos(\sqrt{8}\,t) \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} \cos(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\sin(\sqrt{8}\,t) \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where c_1, c_2, c_3 =arbitrary constants.

A fundamental matrix is given by $X(t) = \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ 0 & 0 & e^{-t} \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix}$.

Further,

$$e^{t\,A} = X(t)\big(X(0)\big)^{-1} = \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ 0 & 0 & e^{-t} \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ 0 & 0 & e^{-t} \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t}\cos(\sqrt{8}\,t) & 0 & -\frac{1}{\sqrt{2}}\,e^{3t}\sin(\sqrt{8}\,t) \\ & & \\ 0 & e^{-t} & 0 \\ & & \\ \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 & e^{3t}\cos(\sqrt{8}\,t) \end{bmatrix}.$$

5.3.6.19. (a) $0 = |\,A - \lambda I\,| = \begin{vmatrix} 10-\lambda & 11 \\ -11 & -12-\lambda \end{vmatrix} = (10-\lambda)(-12-\lambda) + 121 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$ $\Rightarrow$ the eigenvalues of A are $\lambda_1 = \lambda_2 = -1$. Corresponding to eigenvalue $\lambda_1 = -1$, eigenvectors are found by

$$\big[\,A - (-1)I \mid \mathbf{0}\,\big] = \left[\begin{array}{cc|c} 11 & 11 & 0 \\ -11 & -11 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 + R_2 \rightarrow R_2$, $\frac{1}{11}R_1 \rightarrow R_1$. Corresponding to eigenvalue $\lambda_1 = -1$ we have *only* eigenvectors $\mathbf{v} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, where $c_1 \neq 0$. Define $\mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Because $\lambda_1 = \lambda_2 = -1$ is a deficient eigenvalue, we need to also find a generalized eigenvector $\mathbf{w}$ that should satisfy the system $(A - (-1)I)\mathbf{w} = \mathbf{v}^{(1)}$:

$$\big[\,A - (-1)I \mid \mathbf{v}^{(1)}\,\big] = \left[\begin{array}{cc|c} 11 & 11 & -1 \\ -11 & -11 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -\frac{1}{11} \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 + R_2 \rightarrow R_2$, $\frac{1}{11}R_1 \rightarrow R_1$. So, corresponding to eigenvalue $\lambda_1 = -1$ we have a generalized eigenvector $\mathbf{w} = \begin{bmatrix} -\frac{1}{11} \\ 0 \end{bmatrix}$.

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{-t}\mathbf{v}^{(1)} + c_2 e^{-t}(t\mathbf{v}^{(1)} + \mathbf{w}) = c_1 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -t - \frac{1}{11} \\ t \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

A fundamental matrix is given by

$$X(t) = e^{-t}\begin{bmatrix} -1 & -t - \frac{1}{11} \\ & \\ 1 & t \end{bmatrix}.$$

Further,

$$e^{t\,A} = X(t)\big(X(0)\big)^{-1} = e^{-t}\begin{bmatrix} -1 & -t - \frac{1}{11} \\ & \\ 1 & t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ & \\ -11 & -11 \end{bmatrix} = e^{-t}\begin{bmatrix} 11t+1 & 11t \\ & \\ -11t & -11t+1 \end{bmatrix}.$$

(b) Define $\mathbf{X}(s) = \mathcal{L}[\,\mathbf{x}(t)\,](s)$. Take the Laplace transforms of both sides of the system of ODEs to get

$$s\,\mathbf{X} - \mathbf{x}(0) = A\mathbf{X},$$

hence

$$\mathbf{X}=(sI-A)^{-1}\mathbf{x}(0)=\begin{bmatrix} s-10 & -11 \\ 11 & s+12 \end{bmatrix}^{-1}\mathbf{x}(0)=\frac{1}{(s-10)(s+12)+121}\begin{bmatrix} s-10 & 11 \\ -11 & s+12 \end{bmatrix}\mathbf{x}(0)$$

$$= \frac{1}{s^2+2s+1}\begin{bmatrix} s+12 & 11 \\ -11 & s-11 \end{bmatrix}\mathbf{x}(0) = \begin{bmatrix} \dfrac{s+12}{(s+1)^2} & \dfrac{11}{(s+1)^2} \\[2ex] -\dfrac{11}{(s+1)^2} & \dfrac{s-11}{(s+1)^2} \end{bmatrix}\mathbf{x}(0)$$

$$= \begin{bmatrix} \dfrac{(s+1)+11}{(s+1)^2} & \dfrac{11}{(s+1)^2} \\[2ex] -\dfrac{11}{(s+1)^2} & \dfrac{(s+1)-11}{(s+1)^2} \end{bmatrix}\mathbf{x}(0).$$

So,

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\,\mathbf{X}(s)\,](t) = \mathcal{L}^{-1}\left[\begin{bmatrix} \dfrac{(s+1)+11}{(s+1)^2} & \dfrac{11}{(s+1)^2} \\[2ex] -\dfrac{11}{(s+1)^2} & \dfrac{(s+1)-11}{(s+1)^2} \end{bmatrix}\right](t)\;\mathbf{x}(0)$$

$$= \mathcal{L}^{-1}\left[\begin{bmatrix} \dfrac{1}{(s+1)}+\dfrac{11}{(s+1)^2} & \dfrac{11}{(s+1)^2} \\[2ex] -\dfrac{11}{(s+1)^2} & \dfrac{1}{s+1}-\dfrac{11}{(s+1)^2} \end{bmatrix}\right](t)\;\mathbf{x}(0)$$

$$= \begin{bmatrix} e^{-t}+11\,t\,e^{-t} & 11\,t\,e^{-t} \\[1ex] -11\,t\,e^{-t} & e^{-t}-11\,t\,e^{-t} \end{bmatrix}\mathbf{x}(0) = e^{-t}\begin{bmatrix} 1+11\,t & 11\,t \\[1ex] -11\,t & 1-11\,t \end{bmatrix}\mathbf{x}(0) = e^{t\,A}\mathbf{x}(0).$$

So,

$$e^{t\,A} = e^{-t}\begin{bmatrix} 1+11\,t & 11\,t \\[1ex] -11\,t & 1-11\,t \end{bmatrix}.$$

This conclusion agrees with the conclusion in part (a).

5.3.6.21. We are given that the eigenvalues of A are the complex conjugate pair $\lambda = -1 \pm i$ and the real eigenvalue $\lambda_3 = -1$.

Corresponding to eigenvalue $\lambda_1 = -1+i$, eigenvectors are found by

$$\left[\,A-(-1+i)I \mid \mathbf{0}\,\right] = \left[\begin{array}{ccc|c} -i & -1 & 0 & 0 \\ 2 & -i & 1 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 1 & -i & 0 \end{array}\right],$$

after row operations $iR_1 \to R_1,\ -2R_1 + R_2 \to R_1$. Continuing,

$$\left[\, A - (-1+i)I \mid \mathbf{0} \,\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after row operations $R_2 + R_1 \to R_1,\ -iR_2 \to R_2,\ -R_2 + R_3 \to R_3$. Corresponding to eigenvalue $\lambda_1 = -1 + i$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(-1+i)t} \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} \right) = \mathcal{R}e\left(e^{-t}(\cos t + i \sin t) \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} \right)$$

$$= \mathcal{R}e\left(e^{-t} \begin{bmatrix} -\cos t - i\sin t \\ -\sin t + i\cos t \\ \cos t + i \sin t \end{bmatrix} \right) = e^{-t} \begin{bmatrix} -\cos t \\ -\sin t \\ \cos t \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(-1+i)t} \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} \right) = \mathcal{I}m\left(e^{-t} \begin{bmatrix} -\cos t - i\sin t \\ -\sin t + i\cos t \\ \cos t + i \sin t \end{bmatrix} \right) = e^{-t} \begin{bmatrix} -\sin t \\ \cos t \\ \sin t \end{bmatrix}.$$

Corresponding to eigenvalue $\lambda_3 = -1$, eigenvectors are found by

$$\left[\, A - (-1)I \mid \mathbf{0} \,\right] = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 + R_3 \to R_3,\ R_1 \leftrightarrow R_2,\ \frac{1}{2}R_1 \to R_1,\ -R_2 \to R_2$. Corresponding to eigenvalue $\lambda_3 = -1$ we have an eigenvector $\mathbf{v}^{(3)} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} -\cos t \\ -\sin t \\ \cos t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -\sin t \\ \cos t \\ \sin t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},$$

where c_1, c_2, c_3 =arbitrary constants.

A fundamental matrix is given by

$$X(t) = e^{-t} \begin{bmatrix} -\cos t & -\sin t & -1 \\ -\sin t & \cos t & 0 \\ \cos t & \sin t & 2 \end{bmatrix}.$$

Further,

$$e^{tA} = X(t)\big(X(0)\big)^{-1} = e^{-t} \begin{bmatrix} -\cos t & -\sin t & -1 \\ -\sin t & \cos t & 0 \\ \cos t & \sin t & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1}$$

$$= e^{-t} \begin{bmatrix} -\cos t & -\sin t & -1 \\ -\sin t & \cos t & 0 \\ \cos t & \sin t & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = e^{-t} \begin{bmatrix} -1 + 2\cos t & -\sin t & -1 + \cos t \\ 2\sin t & \cos t & \sin t \\ 2 - 2\cos t & \sin t & 2 - \cos t \end{bmatrix}.$$

5.3.6.23. $0 = \begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 + 2^2 \Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = -1 \pm i\,2$.

Corresponding to eigenvalue $\lambda_1 = -1 + i\,2$, eigenvectors are found by

$$[\, A - (-1+i\,2)I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -i\,2 & 2 & 0 \\ -2 & -i\,2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $\frac{i}{2}\,R_1 \to R_1,\ 2R_1 + R_2 \to R_2$. Corresponding to eigenvalue $\lambda_1 = -1 + i\,2$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$. This gives two solutions of the LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(-1+i\,2)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}\right) = \mathcal{R}e\left(e^{-t}(\cos 2t + i \sin 2t)\begin{bmatrix} -i \\ 1 \end{bmatrix}\right) = \mathcal{R}e\left(e^{-t}\begin{bmatrix} \sin 2t - i\,\cos 2t \\ \cos 2t + i \sin 2t \end{bmatrix}\right)$$

$$= e^{-t}\begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(-1+i\,2)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}\right) = \mathcal{I}m\left(e^{-t}\begin{bmatrix} \sin 2t - i\,\cos 2t \\ \cos 2t + i \sin 2t \end{bmatrix}\right) = e^{-t}\begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}.$$

The general solution of the LCCHS is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix},$$

where $c_1, c_2 =$arbitrary constants.

The ICs require

$$\begin{bmatrix} \pi \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

which implies $c_1 = 2$ and $c_2 = -\pi$. The solution of the IVP is

$$\mathbf{x}(t) = 2e^{-t}\begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} - \pi\,e^{-t}\begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix} = e^{-t}\begin{bmatrix} 2\sin 2t + \pi\,\cos 2t \\ 2\cos 2t\ -\pi\,\sin 2t \end{bmatrix}.$$

5.3.6.25. Because A is in companion form, the easiest way to do this problem is to first solve the equivalent scalar second order ODE, $\ddot{y} + 2\dot{y} + 5y = 0$, where $\dot{y}(t) = v(t)$: The characteristic equation is $0 = s^2 + 2s + 5 = (s+1)^2 + 4$, hence the roots are the complex conjugate pair $s = -1 \pm i\,2$. The general solution of the scalar second order ODE is $y(t) = c_1\,e^{-t}\cos 2t + c_2\,t\,e^{-t}\sin 2t$, where $c_1, c_2 =$arbitrary constants.

So, the general solution of the original LCCHS in companion form is

$$\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} c_1\,e^{-t}\cos 2t + c_2\,e^{-t}\sin 2t \\ -c_1\,e^{-t}\cos 2t - 2c_1\,e^{-t}\sin 2t - c_2\,e^{-t}\sin 2t + 2c_2\,e^{-t}\cos 2t \end{bmatrix}$$

$$= c_1\,e^{-t}\begin{bmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{bmatrix} + c_2\,e^{-t}\begin{bmatrix} \sin 2t \\ -\sin 2t + 2\cos 2t \end{bmatrix},$$

where $c_1, c_2 =$arbitrary constants.

The ICs require

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

which implies $c_1 = 1$, which implies $0 = -c_1 + 2c_2 = -1 + 2c_2$, hence $c_2 = \frac{1}{2}$. The solution of the IVP is

$$\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \mathbf{x}(t) = e^{-t}\begin{bmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{bmatrix} + \frac{1}{2}\,e^{-t}\begin{bmatrix} \sin 2t \\ -\sin 2t + 2\cos 2t \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} \cos 2t + \frac{1}{2}\sin 2t \\ \\ -\cos 2t - 2\sin 2t - \frac{1}{2}\sin 2t + \cos 2t \end{bmatrix} = e^{-t}\begin{bmatrix} \cos 2t + \frac{1}{2}\sin 2t \\ \\ -\frac{5}{2}\sin 2t \end{bmatrix}.$$

5.3.6.27. Solutions $\mathbf{x}(t) = e^{\sigma t}\mathbf{v}$ of the second order system $\ddot{\mathbf{x}}(t) = A\mathbf{x}$ imply, where $\lambda = \sigma^2$, that

$$0 = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda) - 2 = \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1)$$

$\Rightarrow$ the eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = -1$.

Corresponding to eigenvalue $\lambda_1 = -4$, eigenvectors are found by

$$\left[\, A - (-4)I \mid \mathbf{0} \,\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operation $2R_1 + R_2 \rightarrow R_2$. Corresponding to eigenvalue $\lambda_1 = -4$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Corresponding to eigenvalue $\lambda_2 = -1$, eigenvectors are found by

$$\left[\, A - (-1)I \mid \mathbf{0} \,\right] = \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 + R_2 \rightarrow R_2, -\frac{1}{2}R_1 \rightarrow R_1$. Corresponding to eigenvalue $\lambda_2 = -1$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Corresponding to $-\nu_1^2 = \sigma^2 = \lambda_1 = -4$, hence $\nu_1 = 2$, we get solutions of the system in the forms

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{i\nu_1 t}\mathbf{v}_1\right) = \cos 2t\mathbf{v}_1 \quad \text{and} \quad \mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{i\nu_1 t}\mathbf{v}_1\right) = \sin 2t\mathbf{v}_1.$$

Corresponding to $-\nu_2^2 = \sigma^2 = \lambda_2 = -1$, hence $\nu_2 = 1$, we get solutions of the system in the forms

$$\mathbf{x}^{(3)}(t) = \mathcal{R}e\left(e^{i\nu_1 t}\mathbf{v}_2\right) = \cos t\mathbf{v}_1 \quad \text{and} \quad \mathbf{x}^{(4)}(t) = \mathcal{I}m\left(e^{i\nu_2 t}\mathbf{v}_2\right) = \sin t\mathbf{v}_2.$$

The general solution of the second order ODE system is

$$\mathbf{x}(t) = (c_1\cos 2t + d_1\sin 2t)\begin{bmatrix} -1 \\ 1 \end{bmatrix} + (c_2\cos t + d_2\sin t)\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where c_1, c_2, d_1, d_2 are arbitrary constants.

5.3.6.29. The eigenvalues of the upper triangular matrix $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ are $\lambda_1 = -1$ and $\lambda_1 = -2$. Because both have negative real part, the system $\dot{\mathbf{x}} = A\mathbf{x}$ is asymptotically stable.

5.3.6.31. $0 = |\, A - \lambda I \,| = \begin{vmatrix} -12-\lambda & -25 \\ 4 & 8-\lambda \end{vmatrix} = (-12-\lambda)(8-\lambda) + 100 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$

$\Rightarrow$ the eigenvalues of A are $\lambda_1 = \lambda_2 = -2$. Because both eigenvalues have negative real part, the system $\dot{\mathbf{x}} = A\mathbf{x}$ is asymptotically stable.

[Aside: Even though there is a repeated eigenvalue, because it has negative real part it does not matter whether it is deficient, according to Theorem 5.11.]

5.3.6.33. Expanding the determinant along the second column, we have characteristic equation

$$0 = |\, A - \lambda I \,| = \begin{vmatrix} -3-\lambda & 0 & -1 \\ -1 & -4-\lambda & 1 \\ -1 & 0 & -3-\lambda \end{vmatrix} = (-4-\lambda)\begin{vmatrix} -3-\lambda & -1 \\ -1 & -3-\lambda \end{vmatrix} = (-4-\lambda)\big((-3-\lambda)^2 - 1\big)$$

$\Rightarrow$ the eigenvalues of A are $\lambda_1 = -4$, $\lambda_2 = -3 - 1 = -4$, and $\lambda_3 = -3 + 1 = -2$. Because all eigenvalues have negative real part, the system $\dot{\mathbf{x}} = A\mathbf{x}$ is asymptotically stable.

5.3.6.35. (a) must be false, because there are eigenvalues whose real part is zero,

(b) may be true and maybe false, depending upon whether the repeated eigenvalues $\pm\, i$, $\pm\, i$ are not deficient or deficient, respectively

(c) must be true, by the same reasoning as for part (b)

(d) must be true, because by themselves the first pair of eigenvalues $\pm\, i \cdot 1 = \pm\, i\,\omega$ give *some* periodic solutions whose period is $\frac{2\pi}{1} = 2\pi$. [The problem did not ask whether *all* solutions are periodic.]

(e) may be true, because the repeated eigenvalues $\pm\, i$, $\pm\, i$ may be deficient.

Section 5.4.1

5.4.1.1. The solution of $\dot{\mathbf{x}} = A\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ can be written in the form (5.40):

$$\mathbf{x}(t) = X(t)\left(\mathbf{c} + \int (X(t))^{-1}\,\mathbf{f}(t)dt\right) = \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix}\left(\mathbf{c} + \int \left(\begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} dt\right)$$

$$= \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix}\left(\mathbf{c} + \int \frac{1}{-e^{-5t}}\left(\begin{bmatrix} -3e^{-3t} & -e^{-3t} \\ 2e^{-2t} & e^{-2t} \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} dt\right)$$

$$= \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix}\left(\mathbf{c} + \int \begin{bmatrix} 3e^{2t} \\ -2e^{3t} \end{bmatrix} dt\right) = \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix}\left(\mathbf{c} + \begin{bmatrix} \frac{3}{2}\,e^{2t} \\ -\frac{2}{3}\,e^{3t} \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix}\mathbf{c} + \begin{bmatrix} \frac{5}{6} \\ -1 \end{bmatrix},$$

where $\mathbf{c}$ is a vector of arbitrary constants.

5.4.1.3. The solution of $\dot{\mathbf{x}} = A\mathbf{x} + \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$ can be written in the form (5.40):

$$\mathbf{x}(t) = X(t)\left(\mathbf{c} + \int (X(t))^{-1}\,\mathbf{f}(t)dt\right) =$$

$$= e^{-3t}\begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\sin t \end{bmatrix}\left(\mathbf{c} + \int \left(e^{-3t}\begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\sin t \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix} dt\right)$$

$$= e^{-3t}\begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\sin t \end{bmatrix}\left(\mathbf{c} + \int \frac{1}{-2e^{-3t}}\begin{bmatrix} 2\sin t & -\cos t - \sin t \\ -2\cos t & \cos t - \sin t \end{bmatrix}\begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix} dt\right)$$

$$= e^{-3t}\begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\sin t \end{bmatrix}\left(\mathbf{c} + \int \frac{1}{2}\begin{bmatrix} \cos t + \sin t \\ -\cos t + \sin t \end{bmatrix} dt\right)$$

$$= e^{-3t}\begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\sin t \end{bmatrix}\left(\mathbf{c} + \frac{1}{2}\begin{bmatrix} \sin t - \cos t \\ -\sin t - \cos t \end{bmatrix}\right)$$

$$= \ldots = e^{-3t}\begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\sin t \end{bmatrix}\mathbf{c} \;+\; e^{-3t}\begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

where $\mathbf{c}$ is a vector of arbitrary constants.

5.4.1.5. $0 = |\,A - \lambda I\,| = \begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & -1-\lambda & 0 \\ 4 & 0 & 3-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ 4 & 3-\lambda \end{vmatrix} = (-1-\lambda)\big((3-\lambda)^2+8\big)$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = 3 \pm i\sqrt{8}$ and the real eigenvalue $\lambda_3 = -1$.

Corresponding to eigenvalue $\lambda_1 = 3 + i\sqrt{8}$, eigenvectors are found by

$$\left[\,A - (3 + i\sqrt{8})I \mid \mathbf{0}\,\right] = \left[\begin{array}{ccc|c} -i\sqrt{8} & 0 & -2 & 0 \\ 0 & -4 - i\sqrt{8} & 0 & 0 \\ 4 & 0 & -i\sqrt{8} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 \leftrightarrow R_3$, $\frac{1}{4}R_1 \to R_1$, $i\sqrt{8}\,R_1 + R_3 \to R_3$, $\frac{1}{-4-i\sqrt{8}}R_2 \to R_2$. Corresponding to eigenvalue $\lambda_1 = 3+i\sqrt{8}$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} i \\ 0 \\ \sqrt{2} \end{bmatrix}$. This gives two solutions of the corresponding LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(3+i\sqrt{8})t}\begin{bmatrix} i \\ 0 \\ \sqrt{2} \end{bmatrix}\right) = \mathcal{R}e\left(e^{3t}(\cos(\sqrt{8}\,t) + i\sin(\sqrt{8}\,t))\begin{bmatrix} i \\ 0 \\ \sqrt{2} \end{bmatrix}\right)$$

$$= \mathcal{R}e\left(e^{3t}\begin{bmatrix} -\sin(\sqrt{8}\,t) + i\cos(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\cos(\sqrt{8}\,t) + i\sqrt{2}\sin(\sqrt{8}\,t) \end{bmatrix}\right) = e^{3t}\begin{bmatrix} -\sin(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\,\cos(\sqrt{8}\,t) \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(3+i\sqrt{8})t}\begin{bmatrix} i \\ 0 \\ \sqrt{2} \end{bmatrix}\right) = \mathcal{I}m\left(e^{3t}\begin{bmatrix} -\sin(\sqrt{8}\,t) + i\cos(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\,\cos(\sqrt{8}\,t) + i\sqrt{2}\,\sin(\sqrt{8}\,t) \end{bmatrix}\right) = e^{3t}\begin{bmatrix} \cos(\sqrt{8}\,t) \\ 0 \\ \sqrt{2}\,\sin(\sqrt{8}\,t) \end{bmatrix}.$$

Corresponding to eigenvalue $\lambda_3 = -1$, eigenvectors are found by

$$\left[\, A - (-1)I \mid \mathbf{0} \,\right] = \left[\begin{array}{ccc|c} 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after row operations $-R_1 + R_3 \to R_3$, $\frac{1}{4}R_1 \to R_1$, $\frac{1}{6}R_3 \to R_3$, $\frac{1}{2}R_3 + R_1 \to R_1$, $R_2 \leftrightarrow R_3$. Corresponding to eigenvalue $\lambda_3 = -1$ we have an eigenvector $\mathbf{v}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

A fundamental matrix is given by

$$X(t) = \left[\, \mathbf{x}^{(1)}(t) \;\vdots\; \mathbf{x}^{(2)}(t) \;\vdots\; \mathbf{x}^{(3)}(t) \,\right] = \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ 0 & 0 & e^{-t} \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix}.$$

The solution of $\dot{\mathbf{x}} = A\mathbf{x} + \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$ can be written in the form (5.40). We can calculate $(X(t))^{-1}$ by using the adjugate matrix, to get

$$\mathbf{x}(t) = X(t)\left(\mathbf{c} + \int (X(t))^{-1}\,\mathbf{f}(t)dt\right) = \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ 0 & 0 & e^{-t} \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix}.$$

$$\cdot\left(\mathbf{c} + \int \left(\begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ 0 & 0 & e^{-t} \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ e^{-t} \\ 7 \end{bmatrix} dt\right)$$

$$= \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ 0 & 0 & e^{-t} \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix}.$$

$$\cdot\left(\mathbf{c} + \int \frac{1}{\sqrt{2}e^{5t}}\begin{bmatrix} -\sqrt{2}\,e^{2t}\sin(\sqrt{8}\,t) & 0 & e^{2t}\cos(\sqrt{8}\,t) \\ \sqrt{2}\,e^{2t}\cos(\sqrt{8}\,t) & 0 & e^{2t}\sin(\sqrt{8}\,t) \\ 0 & \sqrt{2}\,e^{6t} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ e^{-t} \\ 7 \end{bmatrix} dt\right)$$

$$= \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ 0 & 0 & e^{-t} \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix} \left(\mathbf{c} + \int \begin{bmatrix} \frac{7}{\sqrt{2}}\,e^{-3t}\cos(\sqrt{8}\,t) \\ \frac{7}{\sqrt{2}}\,e^{-3t}\sin(\sqrt{8}\,t) \\ 1 \end{bmatrix} dt \right)$$

$$= \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ \\ 0 & 0 & e^{-t} \\ \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix} \left(\mathbf{c} + \begin{bmatrix} \frac{7}{17\sqrt{2}}\,e^{-3t}(-3\cos(\sqrt{8}\,t) + \sqrt{8}\,\sin(\sqrt{8}\,t)) \\ \\ \frac{7}{17\sqrt{2}}\,e^{-3t}(-3\sin(\sqrt{8}\,t) - \sqrt{8}\,\cos(\sqrt{8}\,t)) \\ \\ t \end{bmatrix} \right)$$

$$= \begin{bmatrix} -\frac{14}{17} \\ \\ t\,e^{-t} \\ \\ -\frac{21}{17} \end{bmatrix} + \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ \\ 0 & 0 & e^{-t} \\ \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix} \mathbf{c},$$

where $\mathbf{c}$ is a vector of arbitrary constants.

5.4.1.7. *Method 1*: The system of ODEs, $\left\{ \begin{array}{rcrcr} \dot{A}_1 & = & 5 & - & \frac{A_1}{10} & & \\ \\ \dot{A}_2 & = & & & \frac{A_1}{10} & - & \frac{A_2}{6} \end{array} \right\}$, can most easily be solved by first solving the first ODE, that is, the first oder linear ODE $\dot{A}_1 + \frac{A_1}{10} = 5$, using the integrating factor $\mu(t) = e^{t/10}$:

$$\frac{d}{dt}\left[e^{t/10}A_1 \right] = e^{t/10}\cdot 5 \quad \Leftrightarrow \quad e^{t/10}A_1 = \int 5\,e^{t/10}\,dt = 50\,e^{t/10} + c_1$$

$$\Leftrightarrow \quad A_1 = 50 + c_1\,e^{-t/10},$$

where c_1 is an arbitrary constant. After that, substitute A_1 into the second ODE, $\dot{A}_2 = \frac{A_1}{10} - \frac{A_2}{6}$, to get

$$\dot{A}_2 = 5 + \frac{c_1}{10}\,e^{-t/10} - \frac{A_2}{6}, \quad \text{that is,} \quad \dot{A}_2 + \frac{A_2}{6} = 5 + \frac{c_1}{10}\,e^{-t/10}.$$

This first order linear ODE can be solved using the integrating factor $\mu(t) = e^{t/6}$:

$$\frac{d}{dt}\left[e^{t/6}A_2 \right] = e^{t/6}\cdot(5 + \frac{c_1}{10}\,e^{-t/10}) \quad \Leftrightarrow \quad e^{t/6}A_2 = \int 5\,e^{t/6} + \frac{c_1}{10}\,e^{t/15}dt = 30\,e^{t/6} + 1.5\,c_1\,e^{t/15} + c_2$$

$$\Leftrightarrow \quad A_2 = 30 + 1.5\,c_1\,e^{-t/10} + c_2\,e^{-t/6}.$$

So, the general solution of the system is

$$\mathbf{A}(t) = \begin{bmatrix} 50 + c_1\,e^{-t/10} \\ \\ 30 + 1.5\,c_1\,e^{-t/10} + c_2\,e^{-t/6} \end{bmatrix}$$

where c_1, c_2 =arbitrary constants.

Method 2: The corresponding homogeneous system of ODEs, $\left\{ \begin{array}{rcrcr} \dot{A}_1 & = & - & \frac{A_1}{10} & & \\ \\ \dot{A}_2 & = & & \frac{A_1}{10} & - & \frac{A_2}{6} \end{array} \right\}$, can most easily be solved by first solving the first ODE to get $A_1 = c_1 e^{-t/10}$, where c_1 is an arbitrary constant, and then substituting that into the second ODE to get

$$\dot{A}_2 = \frac{c_1}{10}\,e^{-t/10} - \frac{A_2}{6}, \quad \text{that is,} \quad \dot{A}_2 + \frac{A_2}{6} = \frac{c_1}{10}\,e^{-t/10}.$$

This first order linear ODE can be solved using the integrating factor $\mu(t) = e^{t/6}$:

$$\frac{d}{dt}\left[e^{t/6}A_2\right] = e^{t/6}\cdot\frac{c_1}{10}\,e^{-t/10} \quad \Leftrightarrow \quad e^{t/6}A_2 = \int \frac{c_1}{10}\,e^{t/15}\,dt = 1.5c_1\,e^{t/15} + c_2$$

$$\Leftrightarrow \quad A_2 = 1.5c_1\,e^{-t/10} + c_2\,e^{-t/6}.$$

So, the general solution of the corresponding homogeneous system is

$$\mathbf{A}(t) = \begin{bmatrix} c_1 e^{-t/10} \\ 1.5c_1\,e^{-t/10} + c_2\,e^{-t/6} \end{bmatrix} = c_1\begin{bmatrix} e^{-t/10} \\ 1.5e^{-t/10} \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{-t/6} \end{bmatrix},$$

where c_1, c_2 =arbitrary constants. It follows that a fundamental matrix is given by

$$X(t) = \begin{bmatrix} e^{-t/10} & 0 \\ 1.5e^{-t/10} & e^{-t/6} \end{bmatrix}.$$

The solution of the original, non-homogeneous system can be written in the form (5.40):

$$\mathbf{A}(t)= X(t)\left(\mathbf{c}+\int (X(t))^{-1}\,\mathbf{f}(t)dt\right)= \begin{bmatrix} e^{-t/10} & 0 \\ 1.5e^{-t/10} & e^{-t/6} \end{bmatrix}\left(\mathbf{c}+\int\left(\begin{bmatrix} e^{-t/10} & 0 \\ 1.5e^{-t/10} & e^{-t/6} \end{bmatrix}\right)^{-1}\begin{bmatrix} 5 \\ 0 \end{bmatrix}dt\right)$$

$$= \begin{bmatrix} e^{-t/10} & 0 \\ 1.5e^{-t/10} & e^{-t/6} \end{bmatrix}\left(\mathbf{c}+\int\begin{bmatrix} e^{t/10} & 0 \\ -1.5e^{t/6} & e^{t/6} \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix}dt\right) = \begin{bmatrix} e^{-t/10} & 0 \\ 1.5e^{-t/10} & e^{-t/6} \end{bmatrix}\left(\mathbf{c}+\int\begin{bmatrix} 5\,e^{t/10} \\ -7.5\,e^{t/6} \end{bmatrix}dt\right)$$

$$= \begin{bmatrix} e^{-t/10} & 0 \\ 1.5e^{-t/10} & e^{-t/6} \end{bmatrix}\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 50\,e^{t/10} \\ -45\,e^{t/6} \end{bmatrix}dt\right) = \begin{bmatrix} c_1\,e^{-t/10} + 50 \\ 1.5\,c_1\,e^{-t/10} + c_2\,e^{-t/6} + 30 \end{bmatrix},$$

which agrees with the conclusion using Method 1.

5.4.1.9. (a) This homogeneous system is in companion form, so it is equivalent to the scalar second order Cauchy-Euler ODE, $\ddot{y} - 5t^{-1}\dot{y} + 8t^{-2}y = 0$, where $y(t) = x_1(t)$ and $\dot{y}(t) = x_2(t)$. Substitute $y = t^m$ to get the characteristic equation $0 = m(m-1) - 5r + 8 = m^2 - 6m + 8 = (m-2)(m-4)$, hence the roots are $m = 2,\,4$. The general solution of the scalar second order ODE is $y(t) = c_1\,t^2 + c_2\,t^4$, where c_1, c_2 =arbitrary constants.

The general solution of this homogeneous system is

$$\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} c_1\,t^2 + c_2\,t^4 \\ c_1\,2t + c_2\,4t^3 \end{bmatrix} = c_1\begin{bmatrix} t^2 \\ 2t \end{bmatrix} + c_2\begin{bmatrix} t^4 \\ 4t^3 \end{bmatrix}$$

so, a fundamental matrix is given by

$$X(t) = \begin{bmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{bmatrix}.$$

(b) The solution of the non-homogeneous ODE system can be written in the form (5.40):

$$\mathbf{x}(t)= X(t)\left(\mathbf{c}+\int (X(t))^{-1}\,\mathbf{f}(t)dt\right)= \begin{bmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{bmatrix}\left(\mathbf{c}+\int\left(\begin{bmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{bmatrix}\right)^{-1}\begin{bmatrix} 3 \\ 0 \end{bmatrix}dt\right)$$

$$= \begin{bmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{bmatrix}\left(\mathbf{c}+\int\frac{1}{2t^5}\begin{bmatrix} 4t^3 & -t^4 \\ -2t & t^2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}dt\right) = \begin{bmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{bmatrix}\left(\mathbf{c}+\int\begin{bmatrix} 6t^{-2} \\ -3t^{-4} \end{bmatrix}dt\right)$$

$$= \begin{bmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{bmatrix} \left(\mathbf{c} + \begin{bmatrix} -6t^{-1} \\ t^{-3} \end{bmatrix} \right) = \begin{bmatrix} c_1 t^2 + c_2 t^4 - 5t \\ 2c_1 t + 4c_2 t^3 - 8 \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

The ICs require

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{x}(1) = \begin{bmatrix} c_1 + c_2 - 5 \\ 2c_1 + 4c_2 - 8 \end{bmatrix},$$

hence

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 13 \\ -3 \end{bmatrix}.$$

The solution of the IVP system is

$$\mathbf{x}(t) = \begin{bmatrix} \frac{13}{2} t^2 - \frac{3}{2} t^4 - 5t \\ 13\, t - 6t^3 - 8 \end{bmatrix}.$$

5.4.1.11. The non-homogeneous, scalar, second order ODE $\ddot{y} + y = f(t)$ is equivalent to the ODE system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{g}(t)$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{g}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$. The corresponding homogeneous ODE system,

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x},$$

is equivalent to the homogeneous, scalar, second order ODE $\ddot{y} + y = 0$, whose general solution is

$$\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{bmatrix} = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix},$$

where c_1, c_2 =arbitrary constants. So, the corresponding homogeneous ODE system has a fundamental matrix

$$X(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

As it happens, this matrix satisfies $X(0) = I$, so $e^{t\,A} = X(t)$.

The solution of the non-homogeneous ODE system can be written in the form (5.43):

$$\mathbf{x}(t) = e^{tA}\mathbf{c} + \int_{t_0}^{t} e^{(t-\tau)A}\mathbf{g}(\tau)d\tau = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{c} + \int_{t_0}^{t} \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix} \mathbf{g}(\tau)d\tau$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{c} + \int_{t_0}^{t} \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ f(\tau) \end{bmatrix} d\tau.$$

So,

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \mathbf{x}(t) = \begin{bmatrix} c_1 \cos t + c_2 \sin t + \int_{t_0}^{t} \sin(t-\tau)\, f(\tau)\, d\tau \\ -c_1 \sin t + c_2 \cos t + \int_{t_0}^{t} \cos(t-\tau)\, f(\tau)\, d\tau \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

Changing the integration variable name from τ to u, specifying $t_0 = 0$, and leaving out the homogeneous solution, we find that there is a particular solution of ODE $\ddot{y} + y = f(t)$ given by $y(t) = \displaystyle\int_0^t \sin(t-u) f(u) du$. This agrees with the result of Example 4.33 in Section 4.5 when $\omega = 1$.

5.4.1.13. Suppose $\{y_1(t), y_2(t)\}$ is a complete set of basic solutions of the corresponding linear homogeneous ODE, $\ddot{y} + p(t)\dot{y} + q(t)y = 0$ on some open interval containing $t = 0$.

The non-homogeneous, scalar, second order ODE $\ddot{y}+p(t)\dot{y}+q(t)y = f(t)$ is equivalent to the ODE system $\dot{\mathbf{x}} = A\mathbf{x}+\mathbf{g}(t)$, where $A = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}$ and $\mathbf{g}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$. The corresponding homogeneous ODE system,

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\mathbf{x},$$

is equivalent to the homogeneous, scalar, second order ODE $\ddot{y}+p(t)\dot{y}+q(t)y = 0$, whose general solution is

$$\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} c_1\,y_1(t)+c_2\,y_2(t) \\ c_1\,\dot{y}_1(t)+c_2\,\dot{y}_2(t) \end{bmatrix} = c_1\begin{bmatrix} y_1(t) \\ \dot{y}_1(t) \end{bmatrix} + c_2\begin{bmatrix} y_2(t) \\ \dot{y}_2(t) \end{bmatrix},$$

where c_1, c_2 =arbitrary constants. So, the corresponding homogeneous ODE system has a fundamental matrix

$$X(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix}.$$

Note that $\{y_1(t), y_2(t)\}$ being a complete set of basic solutions of $\ddot{y}+p(t)\dot{y}+q(t)y = 0$ on some open interval containing $t = 0$ implies that $W\big((y_1(t), y_2(t)\big) = |X(t)| \neq 0$ on that interval.

The solution of the non-homogeneous ODE system can be written in the form (5.41), with $t_0 = 0$:

$$\mathbf{x}(t) = X(t)\left(\mathbf{c}+\int_0^t (X(\tau))^{-1}\,\mathbf{f}(\tau)d\tau\right) = \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix}\left(\mathbf{c}+\int_0^t \begin{bmatrix} y_1(\tau) & y_2(\tau) \\ \dot{y}_1(\tau) & \dot{y}_2(\tau) \end{bmatrix}^{-1}\mathbf{g}(\tau)d\tau\right)$$

$$= \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix}\left(\mathbf{c}+\int_0^t \frac{1}{|X(\tau)|}\begin{bmatrix} \dot{y}_2(\tau) & -y_2(\tau) \\ -\dot{y}_1(\tau) & y_1(\tau) \end{bmatrix}\mathbf{g}(\tau)d\tau\right)$$

$$= \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix}\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}+\int_0^t \frac{1}{W\big((y_1(\tau), y_2(\tau)\big)}\begin{bmatrix} \dot{y}_2(\tau) & -y_2(\tau) \\ -\dot{y}_1(\tau) & y_1(\tau) \end{bmatrix}\begin{bmatrix} 0 \\ f(\tau) \end{bmatrix}d\tau\right).$$

Replacing the integration variable τ by s and multiplying, we have

$$= \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix}\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}+\int_0^t \frac{1}{W\big((y_1(s), y_2(s)\big)}\begin{bmatrix} -y_2(s)f(s) \\ y_1(s)f(s) \end{bmatrix}ds\right),$$

that is,

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \mathbf{x}(t) = \begin{bmatrix} c_1y_1(t)+c_2y_2(t) \\ c_1\dot{y}_1(t)+c_2\dot{y}_2(t) \end{bmatrix} + \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix}\int_0^t \frac{1}{W\big((y_1(s), y_2(s)\big)}\begin{bmatrix} -y_2(s)f(s) \\ y_1(s)f(s) \end{bmatrix}ds,$$

where c_1, c_2 =arbitrary constants.

The first component, $y(t)$, of the solution vector can be rewritten as

$$y(t) = \left(c_1+\int_0^t -\frac{y_2(s)f(s)}{W(y_1,y_2)(s)}\,ds\right)y_1(t)+\left(c_2+\int_0^t \frac{y_1(s)f(s)}{W(y_1,y_2)(s)}\,ds\right)y_2(t),$$

which is equation (4.39), a formula for all solutions of $\ddot{y}+p(t)\dot{y}+q(t)y = f(t)$, as we wanted to show.

5.4.1.15. Take Laplace transforms of both sides of the system of ODEs and use the ICs to get

$$s\mathcal{L}[\,\mathbf{x}(t)\,]-\mathbf{x}(0) = \mathcal{L}[\,\dot{\mathbf{x}}(t)\,] = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\mathcal{L}[\,\mathbf{x}\,]+\mathcal{L}\left[\begin{bmatrix} -\cos 2t \\ \sin 3t \end{bmatrix}\right],$$

hence

$$s\mathcal{L}[\,\mathbf{x}(t)\,]-\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\mathcal{L}[\,\mathbf{x}\,]+\begin{bmatrix} -\frac{s}{s^2+4} \\ \frac{3}{s^2+9} \end{bmatrix},$$

that is,

$$\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \right) \mathcal{L}[\,\mathbf{x}(t)\,] = \begin{bmatrix} -\frac{s}{s^2+4} \\ \frac{3}{s^2+9} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So,

$$\mathcal{L}[\,\mathbf{x}(t)\,] = \begin{bmatrix} s-2 & 5 \\ -1 & s+2 \end{bmatrix}^{-1} \left(\begin{bmatrix} -\frac{s}{s^2+4} \\ \frac{3}{s^2+9} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= \frac{1}{(s-2)(s+2)+5} \begin{bmatrix} s+2 & -5 \\ 1 & s-2 \end{bmatrix} \left(\begin{bmatrix} -\frac{s}{s^2+4} \\ \frac{3}{s^2+9} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{s+2}{s^2+1} - \frac{s(s+2)}{(s^2+1)(s^2+4)} - \frac{15}{(s^2+1)(s^2+9)} \\ \frac{1}{s^2+1} - \frac{s}{(s^2+1)(s^2+4)} + \frac{3(s-2)}{(s^2+1)(s^2+9)} \end{bmatrix}.$$

This leads to four partial fractions expansions problems, which we may do in any order.

$$\frac{s(s+2)}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+E}{(s^2+4)}$$

$$\Rightarrow \quad s^2+2s = s(s+2) = (As+B)(s^2+4)+(Cs+E)(s^2+1) = (A+C)s^3+(B+E)s^2+(4A+C)s+(4B+E).$$

Sorting by powers of s gives a system of four equations in the four unknown constants A, B, C, E:

$$\Rightarrow \left\{ \begin{array}{llllll} s^3: & 0 & = A & & +C & \\ s^2: & 1 & = & B & & +E \\ s^1: & 2 & = 4A & & +C & \\ s^0: & 0 & = & 4B & & +E \end{array} \right\},$$

that is,

$$\left\{ \begin{array}{lll} 1 & = B & +E \\ 0 & = 4B & +E \end{array} \right\} \text{ and } \left\{ \begin{array}{lll} 0 & = A & +C \\ 2 & = 4A & +C \end{array} \right\},$$

whose solutions are, respectively,

$$\begin{bmatrix} B \\ E \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

and

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

So,

$$\frac{s(s+2)}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+E}{(s^2+4)} = \frac{1}{3} \left(\frac{2s-1}{s^2+1} + \frac{-2s+4}{s^2+4} \right).$$

The second expansion is

$$\frac{15}{(s^2+1)(s^2+9)} = \frac{\alpha s+\beta}{(s^2+1)} + \frac{\gamma s+\delta}{(s^2+9)}$$

$$\Rightarrow \quad 15 = s(s+2) = (\alpha s+\beta)(s^2+9) + (\gamma s+\delta)(s^2+1) = (\alpha+\gamma)s^3 + (\beta+\delta)s^2 + (9\alpha+\gamma)s + (9\beta+\delta)$$

$$\Rightarrow \left\{ \begin{array}{llllll} 0 & = \alpha & & +\gamma & \\ 0 & = & \beta & & +\delta \\ 0 & = 9\alpha & & +\gamma & \\ 15 & = & 9\beta & & +\delta \end{array} \right\},$$

that is,

$$\left\{ \begin{array}{lll} 0 & = \beta & +\delta \\ 15 & = 9\beta & +\delta \end{array} \right\} \text{ and } \left\{ \begin{array}{lll} 0 & = \alpha & +\gamma \\ 0 & = 9\alpha & +\gamma \end{array} \right\},$$

whose solutions are, respectively,

$$\begin{bmatrix} \beta \\ \delta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 15 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 1 & -1 \\ -9 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 15 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 15 \\ -15 \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 1 & -1 \\ -9 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So,

$$\frac{15(s+2)}{(s^2+1)(s^2+9)} = \frac{\alpha s+\beta}{(s^2+1)} + \frac{\gamma s+\delta}{(s^2+9)} = \frac{1}{8}\left(\frac{15}{s^2+1} + \frac{-15}{s^2+9}\right).$$

The third expansion is

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{ps+q}{(s^2+1)} + \frac{rs+u}{(s^2+4)}$$

$$\Rightarrow \quad s = (ps+q)(s^2+4) + (rs+u)(s^2+1) = (p+r)s^3 + (q+u)s^2 + (4p+r)s + (4q+u)$$

$$\Rightarrow \quad \left\{ \begin{array}{llllll} 0 & = p & & +r & \\ 0 & = & q & & +u \\ 1 & = 4p & & +r & \\ 0 & = & 4q & & +u \end{array} \right\},$$

that is,

$$\left\{ \begin{array}{lll} 0 & = q & +u \\ 0 & = 4q & +u \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{lll} 0 & = p & +r \\ 1 & = 4p & +r \end{array} \right\},$$

whose solutions are, respectively,

$$\begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} q \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So,

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{ps+q}{s^2+1} + \frac{rs+u}{s^2+4} = \frac{1}{3}\left(\frac{s}{s^2+1} + \frac{-s}{s^2+4}\right).$$

The fourth expansion is

$$\frac{3(s-2)}{(s^2+1)(s^2+9)} = \frac{as+b}{(s^2+1)} + \frac{cs+e}{(s^2+9)}$$

$$\Rightarrow \quad 3s-6 = 3(s-2) = (as+b)(s^2+9) + (cs+e)(s^2+1) = (a+c)s^3 + (b+e)s^2 + (9a+c)s + (9b+e)$$

$$\Rightarrow \quad \left\{ \begin{array}{llllll} 0 & = a & & +c & \\ 0 & = & b & & +e \\ 3 & = 9a & & +c & \\ -6 & = & 9b & & +e \end{array} \right\},$$

that is,

$$\left\{ \begin{array}{lll} 0 & = b & +e \\ -6 & = 9b & +e \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{lll} 0 & = a & +c \\ 3 & = 9a & +c \end{array} \right\},$$

whose solutions are, respectively,

$$\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -6 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 1 & -1 \\ -9 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -6 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -6 \\ 6 \end{bmatrix}$$

and

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 1 & -1 \\ -9 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

So,

$$\frac{3(s-2)}{(s^2+1)(s^2+9)} = \frac{as+b}{(s^2+1)} + \frac{cs+e}{(s^2+9)} = \frac{1}{8}\left(\frac{3s-6}{s^2+1} + \frac{-3s+6}{s^2+9}\right).$$

Altogether,

$$\mathcal{L}[\,\mathbf{x}(t)\,] = \begin{bmatrix} \frac{s+2}{s^2+1} - \frac{1}{3}\left(\frac{2s-1}{s^2+1} + \frac{-2s+4}{s^2+4}\right) - \frac{1}{8}\left(\frac{15}{s^2+1} + \frac{-15}{s^2+9}\right) \\ \\ \frac{1}{s^2+1} - \frac{1}{3}\left(\frac{s}{s^2+1} + \frac{-s}{s^2+4}\right) + \frac{1}{8}\left(\frac{3s-6}{s^2+1} + \frac{-3s+6}{s^2+9}\right) \end{bmatrix} = \frac{1}{24}\begin{bmatrix} \frac{8s+11}{s^2+1} + \frac{16s-32}{s^2+4} + \frac{45}{s^2+9} \\ \\ \frac{s+6}{s^2+1} + \frac{8s}{s^2+4} + \frac{-9s+18}{s^2+9} \end{bmatrix}.$$

So,

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[\frac{1}{24}\begin{bmatrix} \frac{8s+11}{s^2+1} + \frac{16s-32}{s^2+4} + \frac{45}{s^2+9} \\ \\ \frac{s+6}{s^2+1} + \frac{8s}{s^2+4} + \frac{-9s+18}{s^2+9} \end{bmatrix}\right],$$

hence

$$\mathbf{x}(t) = \frac{1}{24}\begin{bmatrix} 8\cos t + 11\sin t + 16\cos 2t - 16\sin 2t + 15\sin 3t \\ \\ \cos t + 6\sin t + 8\cos 2t - 9\cos 3t + 6\sin 3t \end{bmatrix}.$$

Section 5.5.2

5.5.2.1. Define $A = \begin{bmatrix} 2 & 1 \\ 7 & -4 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} 0 \\ -e^{-t} \end{bmatrix} = e^{-t}\,\mathbf{w}$, where $\mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Let's try for a particular solution in the form $\mathbf{x}_p(t) = e^{-t}\mathbf{a}$. We substitute $\mathbf{x}_p(t)$ into the non-homogeneous system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t)$ to get

$$-e^{-t}\mathbf{a} = \dot{\mathbf{x}}_p = A\mathbf{x}_p + \mathbf{f}(t) = A\left(e^{-t}\mathbf{a}\right) + e^{-t}\mathbf{w} = e^{-t}\left(A\mathbf{a} + \mathbf{w}\right).$$

As in Example 5.26, we get $\mathbf{a} = -\left(A - (-1)I\right)^{-1}\mathbf{w}$, *as long as* $(A - (-1)I)$ is invertible. Here, this gives

$$\mathbf{a} = -\left(A - (-1)I\right)^{-1}\mathbf{w} = -\begin{bmatrix} 3 & 1 \\ 7 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} -3 & -1 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

So, a particular solution is given by

$$\mathbf{x}_p(t) = e^{-t}\mathbf{a} = \frac{1}{16}\,e^{-t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

We can see already that the method of Section 5.5 gave the same particular solution as the variation of constants method in Section 5.4.

For the rest of the problem, we can use the result of Example 5.23 in Section 5.4 that

$$e^{tA} = \frac{1}{8}\begin{bmatrix} e^{-5t} + 7e^{3t} & -e^{-5t} + e^{3t} \\ -7e^{-5t} + 7e^{3t} & 7e^{-5t} + e^{3t} \end{bmatrix},$$

hence the general solution of the system of ODEs is

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t) = \frac{1}{16}\,e^{-t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + e^{tA}\mathbf{c} = \frac{1}{16}\,e^{-t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} e^{-5t} + 7e^{3t} & -e^{-5t} + e^{3t} \\ -7e^{-5t} + 7e^{3t} & 7e^{-5t} + e^{3t} \end{bmatrix}\mathbf{c},$$

where $\mathbf{c}$ is a vector of arbitrary constants.

The ICs require

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \mathbf{x}(0) = \frac{1}{16}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \mathbf{c} \quad \Rightarrow \quad \mathbf{c} = \frac{1}{16}\begin{bmatrix} 47 \\ -29 \end{bmatrix}.$$

So, the solution of the IVP is given by

$$\mathbf{x}(t) = \frac{1}{8}\begin{bmatrix} e^{-5t} + 7e^{3t} & -e^{-5t} + e^{3t} \\ -7e^{-5t} + 7e^{3t} & 7e^{-5t} + e^{3t} \end{bmatrix}\cdot\frac{1}{16}\begin{bmatrix} 47 \\ -29 \end{bmatrix} + \frac{1}{16}\,e^{-t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= \frac{1}{32}\begin{bmatrix} 19e^{-5t} + 75e^{3t} \\ -133e^{-5t} + 75e^{3t} \end{bmatrix} + \frac{1}{16}\begin{bmatrix} e^{-t} \\ -3e^{-t} \end{bmatrix},$$

which agrees with the final conclusion of Example 5.23 in Section 5.4.

5.5.2.3. Define $A = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} e^{-3t} \\ 0 \end{bmatrix} = e^{-3t}\,\mathbf{w}$, where $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Let's try for a particular solution in the form $\mathbf{x}_p(t) = e^{-3t}\mathbf{a}$. We substitute $\mathbf{x}_p(t)$ into the non-homogeneous system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t)$ to get

$$-3e^{-3t}\mathbf{a} = \dot{\mathbf{x}}_p = A\mathbf{x}_p + \mathbf{f}(t) = A\left(e^{-3t}\mathbf{a}\right) + e^{-3t}\mathbf{w} = e^{-3t}\left(A\mathbf{a} + \mathbf{w}\right).$$

As in Example 5.26, we get $\mathbf{a} = -\left(A - (-3)I\right)^{-1}\mathbf{w}$, *as long as* $(A - (-3)I)$ is invertible. Here, this gives

$$\mathbf{a} = -\left(A - (-3)I\right)^{-1}\mathbf{w} = -\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

So, a particular solution is given by

$$\mathbf{x}_p(t) = e^{-3t}\mathbf{a} = e^{-3t}\begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

For the rest of the problem, we first find $e^{t\,A}$:

$$0 = \begin{vmatrix} -4-\lambda & 3 \\ -2 & 1-\lambda \end{vmatrix} = (-4-\lambda)(1-\lambda)+6 = \lambda^2+3\lambda+2 = (\lambda+2)(\lambda+1)$$

$\Rightarrow$ eigenvalues are $\lambda_1 = -2, \lambda_2 = -1$

$[\,A-\lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-R_1+R_2 \rightarrow R_2,\ -\frac{1}{2}R_1 \rightarrow R_1$

$\Rightarrow \mathbf{v}_1 = c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -2$

$[\,A-\lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-\frac{1}{3}R_1 \rightarrow R_1,\ 2R_1+R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_2 = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = -1$

The general solution of the corresponding LCCHS is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants},$$

so a fundamental matrix is given by

$$Z(t) = \begin{bmatrix} 3e^{-2t} & e^{-t} \\ 2e^{-2t} & e^{-t} \end{bmatrix}.$$

This implies that

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = \begin{bmatrix} 3e^{-2t} & e^{-t} \\ 2e^{-2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3e^{-2t} & e^{-t} \\ 2e^{-2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-2t}-2e^{-t} & -3e^{-2t}+3e^{-t} \\ \\ 2e^{-2t}-2e^{-t} & -2e^{-2t}+3e^{-t} \end{bmatrix}.$$

Hence the general solution of the system of ODEs is

$$\mathbf{x}(t) = \mathbf{x}_p(t)+\mathbf{x}_h(t) = e^{-3t}\begin{bmatrix} -2 \\ \\ -1 \end{bmatrix} + e^{tA}\mathbf{c} = e^{-3t}\begin{bmatrix} -2 \\ \\ -1 \end{bmatrix} + \begin{bmatrix} 3e^{-2t}-2e^{-t} & -3e^{-2t}+3e^{-t} \\ \\ 2e^{-2t}-2e^{-t} & -2e^{-2t}+3e^{-t} \end{bmatrix}\mathbf{c},$$

where $\mathbf{c}$ is a vector of arbitrary constants.

[Note: Alternatively, we could have used $Z(t)$ instead of e^{tA} to help write the general solution.]

5.5.2.5. As in Example 5.28, rather than solve the system in the form $(\star)$ $\dot{\mathbf{x}} = A\mathbf{x} + (\cos t)\mathbf{w}$, we will first solve its complexification $(\star\star)$ $\dot{\mathbf{x}} = A\mathbf{x} + e^{it}\mathbf{w}$. The relationship between $\widetilde{\mathbf{x}}_p(t)$, the solution of $(\star\star)$, and $\mathbf{x}_p(t)$, the solution of $(\star)$, is

$$\mathbf{x}_p(t) = \mathcal{R}e\left(\widetilde{\mathbf{x}}_p(t)\right),$$

because $\cos t = \mathcal{R}e\left(e^{it}\right)$.

Define $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We look for a particular solution of $(\star\star)$ in the form

$$\widetilde{\mathbf{x}}_p(t) = e^{it}\widetilde{\mathbf{a}},$$

where $\widetilde{\mathbf{a}}$ is constant vector, possibly complex. Substituting this into $(\star\star)$ we get

$$ie^{it}\widetilde{\mathbf{a}} = \dot{\widetilde{\mathbf{x}}}_p(t) = A\widetilde{\mathbf{x}}_p(t) + e^{it}\mathbf{w} = A\left(e^{it}\widetilde{\mathbf{a}}\right) + e^{it}\mathbf{w} = e^{it}\left(A\widetilde{\mathbf{a}} + \mathbf{w}\right), \quad \text{that is,} \ -\mathbf{w} = (A - iI)\,\widetilde{\mathbf{a}}.$$

The solution for $\widetilde{\mathbf{a}}$ is given by

$$\widetilde{\mathbf{a}} = -\left(A - iI\right)^{-1}\mathbf{w} = -\left(\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} - iI\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} -1-i & 2 \\ -2 & -1-i \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{4+2i}\begin{bmatrix} -1-i & -2 \\ 2 & -1-i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4+2i}\begin{bmatrix} 1+i \\ -2 \end{bmatrix} = \frac{1}{4+2i}\cdot\frac{4-2i}{4-2i}\begin{bmatrix} 1+i \\ -2 \end{bmatrix}$$

$$= \frac{4-2i}{20}\begin{bmatrix} 1+i \\ \\ -2 \end{bmatrix} = \frac{2-i}{10}\begin{bmatrix} 1+i \\ \\ -2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} (2-i)(1+i) \\ \\ -2(2-i) \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 3+i \\ \\ -4+2i \end{bmatrix}$$

so

$$\widetilde{\mathbf{x}}_p(t) = e^{it}\widetilde{\mathbf{a}} = \frac{1}{10}\left(\cos t + i\sin t\right)\begin{bmatrix} 3+i \\ \\ -4+2i \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 3\cos t - \sin t + i(\cos t + 3\sin t) \\ \\ -4\cos t - 2\sin t + i(2\cos t - 4\sin t) \end{bmatrix}.$$

A particular solution is given by

$$\mathbf{x}_p(t) = \mathcal{R}e\left(\widetilde{\mathbf{x}}_p(t)\right) = \frac{1}{10}\begin{bmatrix} 3\cos t - \sin t \\ \\ -4\cos t - 2\sin t \end{bmatrix}.$$

For the rest of the problem, we first find $e^{t\,A}$: $0 = \begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 + 4$

$\Rightarrow$ the eigenvalues of A are the complex conjugate pair $\lambda = -1 \pm 2i$.
Corresponding to eigenvalue $\lambda_1 = -1+2i$, eigenvectors are found by

$$\left[\, A - (-1+2i)I \mid \mathbf{0} \,\right] = \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $\frac{i}{2}R_1 \to R_1,\ 2R_1 + R_2 \to R_2$. Corresponding to eigenvalue $\lambda_1 = -1+2i$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$. This gives two solutions of the corresponding LCCHS: The first is

$$\mathbf{x}^{(1)}(t) = \mathcal{R}e\left(e^{(-1+i2)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}\right) = \mathcal{R}e\left(e^{-t}(\cos 2t + i\sin 2t\begin{bmatrix} -i \\ 1 \end{bmatrix}\right) = \mathcal{R}e\left(e^{-t}\begin{bmatrix} \sin 2t - i\cos 2t \\ \cos 2t + i\sin 2t \end{bmatrix}\right)$$

$$= e^{-t}\begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}.$$

For the second, we don't have to do all of the algebra steps again:

$$\mathbf{x}^{(2)}(t) = \mathcal{I}m\left(e^{(-1+i2)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}\right) = \mathcal{I}m\left(e^{-t}\begin{bmatrix} \sin 2t - i\cos 2t \\ \cos 2t + i\sin 2t \end{bmatrix}\right) = e^{-t}\begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}.$$

A fundamental matrix is given by

$$Z(t) = e^{-t}\begin{bmatrix} \sin 2t & -\cos 2t \\ \cos 2t & \sin 2t \end{bmatrix}.$$

This implies that

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = e^{-t}\begin{bmatrix} \sin 2t & -\cos 2t \\ \cos 2t & \sin 2t \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = e^{-t}\begin{bmatrix} \sin 2t & -\cos 2t \\ \cos 2t & \sin 2t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}.$$

Hence the general solution of the system of ODEs is

$$\mathbf{x}(t) = \mathbf{x}_p(t)+\mathbf{x}_h(t) = \frac{1}{10}\begin{bmatrix} 3\cos t - \sin t \\ \\ -4\cos t - 2\sin t \end{bmatrix} + e^{tA}\mathbf{c} = \frac{1}{10}\begin{bmatrix} 3\cos t - \sin t \\ \\ -4\cos t - 2\sin t \end{bmatrix} + e^{-t}\begin{bmatrix} \cos 2t & \sin 2t \\ \\ -\sin 2t & \cos 2t \end{bmatrix}\mathbf{c},$$

where $\mathbf{c}$ is a vector of arbitrary constants.

5.5.2.7. Define $A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} = e^{-t}\,\mathbf{w}$, where $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Let's try for a particular solution in the form $\mathbf{x}_p(t) = e^{-t}\mathbf{a}$. We substitute $\mathbf{x}_p(t)$ into the non-homogeneous system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t)$ to get

$$-e^{-t}\mathbf{a} = \dot{\mathbf{x}}_p = A\mathbf{x}_p + \mathbf{f}(t) = A\left(e^{-t}\mathbf{a}\right) + e^{-t}\mathbf{w} = e^{-t}\left(A\mathbf{a} + \mathbf{w}\right).$$

As in Example 5.26, we get $\mathbf{a} = -\left(A - (-1)I\right)^{-1}\mathbf{w}$, *as long as* $(A - (-1)I)$ is invertible. Here, this gives

$$\mathbf{a} = -\left(A - (-1)I\right)^{-1}\mathbf{w} = -\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\frac{1}{-4}\begin{bmatrix} -1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So, a particular solution is given by

$$\mathbf{x}_p(t) = e^{-t}\mathbf{a} = \frac{1}{4}\,e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the rest of the problem, we first find $e^{t\,A}$:

$$0 = \begin{vmatrix} 0-\lambda & 3 \\ 1 & -2-\lambda \end{vmatrix} = (-\lambda)(-2-\lambda) - 3 = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1)$$

$\Rightarrow$ eigenvalues are $\lambda_1 = -3, \lambda_2 = 1$

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $\frac{1}{3}R_1 \to R_1,\ -R_1 + R_2 \to R_2$

$\Rightarrow \mathbf{v}_1 = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -3$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 + R_2 \to R_2,\ -R_1 \to R_1$

$\Rightarrow \mathbf{v}_2 = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 1$

The general solution of the corresponding LCCHS is

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants},$$

so a fundamental matrix is given by

$$Z(t) = \begin{bmatrix} -e^{-3t} & 3e^{t} \\ e^{-3t} & e^{t} \end{bmatrix}.$$

This implies that

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = \begin{bmatrix} -e^{-3t} & 3e^{t} \\ e^{-3t} & e^{t} \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -e^{-3t} & 3e^{t} \\ e^{-3t} & e^{t} \end{bmatrix}\cdot\frac{1}{-4}\begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} e^{-3t}+3e^{t} & -3e^{-3t}+3e^{t} \\ \\ -e^{-3t}+e^{t} & 3e^{-3t}+e^{t} \end{bmatrix}.$$

Hence the general solution of the system of ODEs is

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t) = \frac{1}{4}\,e^{-t}\begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} + e^{tA}\mathbf{c} = \frac{1}{4}\,e^{-t}\begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} e^{-3t}+3e^{t} & -3e^{-3t}+3e^{t} \\ \\ -e^{-3t}+e^{t} & 3e^{-3t}+e^{t} \end{bmatrix}\mathbf{c},$$

where $\mathbf{c}$ is a vector of arbitrary constants.

The ICs require

$$\begin{bmatrix} 4 \\ \\ 5 \end{bmatrix} = \mathbf{x}(0) = \frac{1}{4}\begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} + \mathbf{c} \quad \Rightarrow \quad \mathbf{c} = \frac{1}{4}\begin{bmatrix} 15 \\ \\ 19 \end{bmatrix}.$$

So, the solution of the IVP is given by

$$\mathbf{x}(t) = \frac{1}{4}\,e^{-t}\begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} e^{-3t}+3e^{t} & -3e^{-3t}+3e^{t} \\ \\ -e^{-3t}+e^{t} & 3e^{-3t}+e^{t} \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 15 \\ \\ 19 \end{bmatrix}$$

that is,

$$\mathbf{x}(t) = \frac{1}{16}\begin{bmatrix} 4e^{-t} - 42e^{-3t} + 102e^{t} \\ \\ 4e^{-t} + 42e^{-3t} + 34e^{t} \end{bmatrix}.$$

5.5.2.9. Define $A = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} = e^{-t}\,\mathbf{w}$, where $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Let's try for a particular solution in the form $\mathbf{x}_p(t) = e^{-t}\mathbf{a}$. We substitute $\mathbf{x}_p(t)$ into the non-homogeneous system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t)$ to get

$$-e^{-t}\mathbf{a} = \dot{\mathbf{x}}_p = A\mathbf{x}_p + \mathbf{f}(t) = A\left(e^{-t}\mathbf{a}\right) + e^{-t}\mathbf{w} = e^{-t}\left(A\mathbf{a} + \mathbf{w}\right).$$

As in Example 5.26, we get $\mathbf{a} = -\left(A - (-1)I\right)^{-1}\mathbf{w}$, *as long as* $(A - (-1)I)$ is invertible. Unfortunately,

$$A - (-1)I = \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix}$$

is not invertible. By using row reduction on an augmented matrix, it also turns out that the system of equations $\left(A - (-1)I\right)\mathbf{a} = -\mathbf{w}$ has no solution.

At this point we could revert to using the method of Section 5.4. Instead, similarly to the remarks after Example 5.26, we can try to find a particular solution of the form

$$\mathbf{x}_p(t) = e^{-t}\left(t\mathbf{v} + \mathbf{u}\right).$$

[This is like the derivation of the solution of a homogeneous ODE system in the deficient eigenvalue case.] Substitute this into the non-homogeneous system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t)$ to get

$$-te^{-t}\mathbf{v} + e^{-t}\mathbf{v} - e^{-t}\mathbf{u} = \frac{d}{dt}\left[e^{-t}\left(t\mathbf{v} + \mathbf{u}\right)\right] = \dot{\mathbf{x}}_p = A\mathbf{x}_p + \mathbf{f}(t) = Ae^{-t}\left(t\mathbf{v} + \mathbf{u}\right) + e^{-t}\mathbf{w}$$

$$= e^{-t}\left(tA\mathbf{v} + A\mathbf{u} + \mathbf{w}\right).$$

Multiplying through by e^{t} gives

$$-t\mathbf{v} + \mathbf{v} - \mathbf{u} = tA\mathbf{v} + A\mathbf{u} + \mathbf{w}.$$

Sorting by powers of t yields

$$-\mathbf{v} = A\mathbf{v} \quad \text{and} \quad \mathbf{v} - \mathbf{u} = A\mathbf{u} + \mathbf{w}$$

So, we want $\mathbf{v}$ to satisfy $\big(A-(-1)I\big)\mathbf{v}=\mathbf{0}$, that is, $\mathbf{v}$ is some eigenvector of A corresponding to eigenvalue -1, and
$$\big(A-(-1)I\big)\mathbf{u}=\mathbf{v}-\mathbf{w}.$$
[If we used $\mathbf{v}=\mathbf{0}$ then the attempted solution would be $\mathbf{x}_p(t)=e^{-t}\mathbf{u}$, which we already saw wouldn't succeed because $A-(-1)I$ is not invertible.]

$$[\,A-(-1)I \mid \mathbf{0}\,]=\left[\begin{array}{rr|r}-3 & 3 & 0\\ -2 & 2 & 0\end{array}\right]\sim\left[\begin{array}{rr|r}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right], \text{ after } -\tfrac{1}{3}R_1\to R_1,\ 2R_1+R_2\to R_2$$

$\Rightarrow \mathbf{v}=c_1\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector corresponding to A's eigenvalue -1, as long as $c_1\neq 0$.

We solve for $\mathbf{u}$ satisfying $\big(A-(-1)I\big)\mathbf{u}=\mathbf{v}-\mathbf{w}=\begin{bmatrix}c_1-1\\c_1\end{bmatrix}$ using the augmented matrix

$$[\,A-(-1)I \mid \mathbf{v}-\mathbf{w}\,]=\left[\begin{array}{rr|r}-3 & 3 & c_1-1\\ & & \\ -2 & 2 & c_1\end{array}\right]\sim\left[\begin{array}{rr|r}1 & -1 & -\frac{1}{3}c_1+\frac{1}{3}\\ & & \\ 0 & 0 & \frac{1}{3}c_1+\frac{2}{3}\end{array}\right],$$

after $-\frac{1}{3}R_1\to R_1$, $2R_1+R_2\to R_2$.

In order for there to be a solution for $\mathbf{u}$, we must choose $c_1=-2$. In this case,
$$\mathbf{v}=\begin{bmatrix}-2\\-2\end{bmatrix}$$
and $\mathbf{u}$ is found using the augmented matrix, in RREF,
$$\left[\begin{array}{rr|r}1 & -1 & 1\\ 0 & 0 & 0\end{array}\right].$$

One solution is $\mathbf{u}=\begin{bmatrix}1\\0\end{bmatrix}$.

So, a particular solution is given by
$$\mathbf{x}_p(t)=e^{-t}\,(t\mathbf{v}+\mathbf{u})=e^{-t}\left(t\begin{bmatrix}-2\\-2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\right)=e^{-t}\begin{bmatrix}-2t+1\\-2t\end{bmatrix}.$$

For the rest of the problem, we first find e^{tA}:
$$0=\begin{vmatrix}-4-\lambda & 3\\ -2 & 1-\lambda\end{vmatrix}=(-4-\lambda)(1-\lambda)+6=\lambda^2+3\lambda+2=(\lambda+2)(\lambda+1)$$

$\Rightarrow$ eigenvalues are $\lambda_1=-2,\lambda_2=-1$

$$[\,A-\lambda_1 I \mid \mathbf{0}\,]=\left[\begin{array}{rr|r}-2 & 3 & 0\\ -2 & 3 & 0\end{array}\right]\sim\left[\begin{array}{rr|r}1 & -\frac{3}{2} & 0\\ 0 & 0 & 0\end{array}\right], \text{ after } R_1+R_2\to R_2,\ -\tfrac{1}{2}R_1\to R_1,$$

$\Rightarrow \mathbf{v}_1=c_1\begin{bmatrix}3\\2\end{bmatrix}$, for any constant $c_1\neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1=-2$

As for the eigenvalue -1, not "incidentally" we already found the eigenvectors above when we solved for a particular solution: $\Rightarrow \mathbf{v}_2=c_1\begin{bmatrix}1\\1\end{bmatrix}$, for any constant $c_1\neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2=-1$.

The general solution of the corresponding homogeneous system can be written as
$$\mathbf{x}_h(t)=c_1e^{-2t}\begin{bmatrix}3\\2\end{bmatrix}+c_2e^{-t}\begin{bmatrix}1\\1\end{bmatrix}, \text{where } c_1,c_2=\text{arbitrary constants.}$$

So, the general solution of the original, non-homogeneous problem can be written as

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}(t)_h = e^{-t}\begin{bmatrix} -2t+1 \\ -2t \end{bmatrix} + c_1 e^{-2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where c_1, c_2 =arbitrary constants.

5.5.2.11. We are given that $(A - \lambda I)\mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. For the non-homogenous problem $\dot{\mathbf{x}} = A\mathbf{x} + e^{\lambda t}\mathbf{v}$, assume a solution of the form $\mathbf{x}_p(t) = e^{\lambda t}(t\mathbf{v} + \mathbf{w})$. Substitute that into the ODE to get

$$\lambda t\, e^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{v} + \lambda e^{\lambda t}\mathbf{w} = \frac{d}{dt}\left[e^{\lambda t}\left(t\mathbf{v} + \mathbf{w}\right)\right] = \dot{\mathbf{x}}_p = A\mathbf{x}_p + e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\left(t\mathbf{v} + \mathbf{w}\right) + e^{\lambda t}\mathbf{v}$$

$$= e^{\lambda t}\left(tA\mathbf{v} + A\mathbf{w} + \mathbf{v}\right) = e^{\lambda t}\left(t\lambda v + A\mathbf{w} + \mathbf{v}\right).$$

Multiplying through by e^t gives

$$\cancel{\lambda t\mathbf{v}} + \mathbf{v} + \lambda\mathbf{w} = \cancel{t\lambda\mathbf{v}} + A\mathbf{w} + \mathbf{v}.$$

This is equivalent to

$$(A - \lambda I)\mathbf{w} = \mathbf{0},$$

that is, that $\mathbf{w}$ is in the eigenspace of A corresponding to eigenvalue λ.

So, for any $\mathbf{w}$ in the eigenspace of A corresponding to eigenvalue λ, $\mathbf{x}_p(t) = e^{\lambda t}(t\mathbf{v} + \mathbf{w})$ solves the non-homogeneous problem. As a special choice, $\mathbf{w} = \mathbf{0}$ works, so $\mathbf{x}_p(t) = t\,e^{\lambda t}\mathbf{v}$ is one such particular solution.

5.5.2.15. Suppose that α is not an eigenvalue of the constant matrix A and $\mathbf{w}$ is a constant vector. Substitute into $\dot{\mathbf{x}} = A\mathbf{x} + e^{\alpha t}\mathbf{w}$, that is, (5.50), a solution in the form (5.51), that is, $\mathbf{x}_p(t) = e^{\alpha t}\mathbf{a}$. This gives

$$\alpha e^{\alpha t}\mathbf{a} = \dot{\mathbf{x}}_p = A\mathbf{x}_p + \mathbf{f}(t) = A\left(e^{\alpha t}\mathbf{a}\right) + e^{\alpha t}\mathbf{w} = e^{\alpha t}\left(A\mathbf{a} + \mathbf{w}\right).$$

Multiplying through by $e^{-\alpha t}$ gives

$$\alpha\mathbf{a} = A\mathbf{a} + \mathbf{w},$$

that is,

$$\left(A - \alpha I\right)\mathbf{a} = \mathbf{w}.$$

because we assumed that α is not an eigenvalue of A, there exists $\left(A - \alpha I\right)^{-1}$. So, the solution for $\mathbf{a}$ exists and is given by

$$\mathbf{a} = -\left(A - \alpha I\right)^{-1}\mathbf{w},$$

hence

$$\mathbf{x}_p(t) = -e^{\alpha t}\left(A - \alpha I\right)^{-1}\mathbf{w},$$

solves (5.50), as we wanted to show.

Section 5.6.2

5.6.2.1. $\mathcal{V}(A,\mathbf{b}) = Span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} = Span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = Span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

Because $\mathbf{x}_0 = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in $\mathcal{V}(A,\mathbf{b})$, yes, the system can be driven from $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ to $\mathbf{0}$.

5.6.2.3. Because $|\, \mathbf{b} \;\vdots\; A\mathbf{b} \,| = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$, the system is completely controllable.

5.6.2.5. Because $|\, \mathbf{b} \;\vdots\; A\mathbf{b} \,| = \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0$, the system is not completely controllable.

5.6.2.7. Because $[\, B \;\vdots\; AB \,] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in RREF and has $rank = 1$, the system is not completely controllable.

Section 5.7.7

5.7.7.1. Define $A = \begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix}$. We can use eigenvalues and eigenvectors to find the general solution of the homogeneous system of difference equations (LCCHSΔ) $\mathbf{x}_{k+1} = A\mathbf{x}_k$:

$0 = \begin{vmatrix} 3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 4 = \lambda^2 - 13 = (\lambda + \sqrt{13})(\lambda - \sqrt{13})$

$\Rightarrow$ eigenvalues are $\lambda_1 = -\sqrt{13}, \lambda_2 = \sqrt{13}$

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 3+\sqrt{13} & 2 & 0 \\ 2 & -3+\sqrt{13} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{-3+\sqrt{13}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $\frac{1}{2}R_1 \rightarrow R_1$, $-(3+\sqrt{13})R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} 3-\sqrt{13} \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -\sqrt{13}$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 3-\sqrt{13} & 2 & 0 \\ 2 & -3-\sqrt{13} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{-3-\sqrt{13}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $\frac{1}{2}R_1 \rightarrow R_1$, $-(3-\sqrt{13})R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 3+\sqrt{13} \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = \sqrt{13}$

The general solution of the system is

$$\mathbf{x}_k = c_1(-\sqrt{13})^k \begin{bmatrix} 3-\sqrt{13} \\ 2 \end{bmatrix} + c_2(\sqrt{13})^k \begin{bmatrix} 3+\sqrt{13} \\ 2 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

5.7.7.3. Define $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. We can use eigenvalues and eigenvectors to find the general solution of the homogeneous system of difference equations (LCCHSΔ) $\mathbf{x}_{k+1} = A\mathbf{x}_k$:

$0 = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -1, \lambda_2 = 4$

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $\frac{1}{2}R_1 \rightarrow R_1$, $-3R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -1$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-\frac{1}{3}R_1 \rightarrow R_1$, $-3R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 4$

The general solution of the system is

$$\mathbf{x}_k = c_1(-1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 4^k \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

5.7.7.5. Define $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We can use eigenvalues and eigenvectors to find the general solution of the homogeneous system of difference equations (LCCHSΔ) $\mathbf{x}_{k+1} = A\mathbf{x}_k$:

$0 = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \quad \Rightarrow$ eigenvalues are $\lambda_1 = i, \lambda_2 = -i$

$[\,A-\lambda_1 I \mid \mathbf{0}\,]=\left[\begin{array}{cc|c} -i & -1 & 0\\ 1 & -i & 0\end{array}\right]\sim\left[\begin{array}{cc|c} 1 & -i & 0\\ 0 & 0 & 0\end{array}\right]$, after $iR_1\leftrightarrow R_1,\ -R_1+R_2\rightarrow R_2$,

$\Rightarrow \mathbf{v}_1=c_1\begin{bmatrix} i\\ 1\end{bmatrix}$, for any constant $c_1\neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1=i$. Because the eigenvalues are a complex conjugate pair, we don't need to find eigenvectors corresponding to $\lambda_2=\overline{\lambda_1}$.

As in Example 5.33, we calculate that

$$\alpha+i\nu=i=\rho(\cos\omega+i\sin\omega)=1\cdot\Big(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\Big),$$

where

$$\rho=1\quad\text{and}\quad\omega=\frac{\pi}{2}.$$

We have

$$(\cos\omega k+i\sin\omega k)\mathbf{v}=\Big(\cos\frac{\pi k}{2}+i\,\sin\frac{\pi k}{2}\Big)\begin{bmatrix} i\\ \\ 1\end{bmatrix}=\begin{bmatrix} i\cos\frac{\pi k}{2}-\sin\frac{\pi k}{2}\\ \\ \cos\frac{\pi k}{2}+i\sin\frac{\pi k}{2}\end{bmatrix}.$$

As in the discussion before Example 5.33, there are solutions

$$\mathbf{x}_k^{(1)}=\rho^k\ \mathcal{R}e\left((\cos\omega k+i\sin\omega k)\mathbf{v}\right)=\begin{bmatrix} -\sin\frac{\pi k}{2}\\ \\ \cos\frac{\pi k}{2}\end{bmatrix}$$

and

$$\mathbf{x}_k^{(2)}=\rho^k\ \mathcal{I}m\left((\cos\omega k+i\sin\omega k)\mathbf{v}\right)=\begin{bmatrix} \cos\frac{\pi k}{2}\\ \\ \sin\frac{\pi k}{2}\end{bmatrix}.$$

The general solution is

$$\mathbf{x}_k=c_1\begin{bmatrix} -\sin\frac{\pi k}{2}\\ \\ \cos\frac{\pi k}{2}\end{bmatrix}+c_2\begin{bmatrix} \cos\frac{\pi k}{2}\\ \\ \sin\frac{\pi k}{2}\end{bmatrix}\text{where } c_1,c_2=\text{arbitrary constants.}$$

5.7.7.7. Using the results of Example 5.35, with $n=5$, $\mathbf{x}_k\triangleq\begin{bmatrix} V_k\\ I_k\end{bmatrix}$, for $k=0,...,5$, $Y_0=Y_1=...=Y_4=\frac{1}{2}$, and $Z_0=Z_1=...=Z_4=1$, we get that

$$\mathbf{x}_{k+1}=A_k\mathbf{x}_k,$$

where for $k=0,...,3$,

$$A_k=\begin{bmatrix} 1 & -Z_k\\ -Y_k & 1+Y_kZ_k\end{bmatrix}=\begin{bmatrix} 1 & -1\\ -\frac{1}{2} & 1+\frac{1}{2}\cdot 1\end{bmatrix}=\begin{bmatrix} 1 & -1\\ -\frac{1}{2} & \frac{3}{2}\end{bmatrix},$$

and

$$A_4=\begin{bmatrix} 1 & -Z_4\\ 0 & 1\end{bmatrix}=\begin{bmatrix} 1 & -1\\ 0 & 1\end{bmatrix}.$$

(a) *Method 1*: To find the equivalent impedance, using the above, we calculate

$$A_0^{-1}A_1^{-1}\cdots A_4^{-1}\begin{bmatrix} 1\\ Y_4\end{bmatrix}=\begin{bmatrix} 1 & -1\\ -\frac{1}{2} & \frac{3}{2}\end{bmatrix}^{-1}\begin{bmatrix} 1 & -1\\ -\frac{1}{2} & \frac{3}{2}\end{bmatrix}^{-1}\begin{bmatrix} 1 & -1\\ -\frac{1}{2} & \frac{3}{2}\end{bmatrix}^{-1}\begin{bmatrix} 1 & -1\\ -\frac{1}{2} & \frac{3}{2}\end{bmatrix}^{-1}\begin{bmatrix} 1 & -1\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix} 1\\ \frac{1}{2}\end{bmatrix}$$

$$=\left(\frac{1}{1}\begin{bmatrix} \frac{3}{2} & 1\\ \frac{1}{2} & 1\end{bmatrix}\right)^4\begin{bmatrix} 1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix} 1\\ \frac{1}{2}\end{bmatrix}=\frac{1}{16}\begin{bmatrix} 171 & 170\\ 85 & 86\end{bmatrix}\begin{bmatrix} 1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix} 1\\ \frac{1}{2}\end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 171 & 170 \\ 85 & 86 \end{bmatrix}\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 683 \\ 341 \end{bmatrix},$$

whose $(1,1)$ entry is $\eta = \frac{683}{32}$, and the equivalent impedance is $Z = \frac{\eta}{Y_4} = \frac{683/32}{1/2} = \frac{683}{16}$.

Method 2: To find the equivalent impedance, it helps to use eigenvalues and eigenvectors of A_k, for $k = 0, ..., 3$. In part (b), below, we will diagonalize A_0:

$$\begin{bmatrix} 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = A_0 = PDP^{-1},$$

where $P = [\,\mathbf{v}_1 \,\vdots\, \mathbf{v}_2\,] = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = diag(\lambda_1, \lambda_2) = diag\Big(2,\ \frac{1}{2}\Big)$.

It follows that

$$A_0^{-1}A_1^{-1}\cdots A_4^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix} = \left(PDP^{-1}\right)^{-4}A_4^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix} = PD^{-4}P^{-1}A_4^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^{-4} & 0 \\ 0 & \lambda_2^{-4} \end{bmatrix}\frac{1}{-3}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}A_4^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -\lambda_1^{-4} & 2\lambda_2^{-4} \\ \lambda_1^{-4} & \lambda_2^{-4} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -\lambda_1^{-4} - 2\lambda_2^{-4} & 2\lambda_1^{-4} - 2\lambda_2^{-4} \\ \lambda_1^{-4} - \lambda_2^{-4} & -2\lambda_1^{-4} - \lambda_2^{-4} \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = -\frac{1}{3\,\lambda_1^4\lambda_2^4}\begin{bmatrix} -\lambda_2^4 - 2\lambda_1^4 & 2\lambda_2^4 - 2\lambda_1^4 \\ \lambda_2^4 - \lambda_1^4 & -2\lambda_2^4 - \lambda_1^4 \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= -\frac{1}{3\cdot 2^4\left(\frac{1}{2}\right)^4}\begin{bmatrix} -\left(\frac{1}{2}\right)^4 - 2\cdot 2^4 & 2\left(\frac{1}{2}\right)^4 - 2\cdot 2^4 \\ \left(\frac{1}{2}\right)^4 - 2^4 & -2\left(\frac{1}{2}\right)^4 - 2^4 \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = -\frac{1}{48}\begin{bmatrix} -513 & -510 \\ -255 & -258 \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = -\frac{1}{96}\begin{bmatrix} -2049 \\ -1023 \end{bmatrix}$$

$$= \frac{1}{32}\begin{bmatrix} 683 \\ 341 \end{bmatrix},$$

whose $(1,1)$ entry is $\eta = \frac{683}{32}$, and the equivalent impedance is $Z = \frac{\eta}{Y_4} = \frac{683/32}{1/2} = \frac{683}{16}$.

(b) To find a formula for V_k it helps to use eigenvalues and eigenvectors of A_k, for $k = 0, ..., 3$. We calculate

$0 = \begin{vmatrix} 1-\lambda & -1 \\ -\frac{1}{2} & \frac{3}{2}-\lambda \end{vmatrix} = (1-\lambda)(\frac{3}{2}-\lambda) - \frac{1}{2} = \lambda^2 - \frac{5}{2}\lambda + 1 = (\lambda - \frac{5}{4})^2 - \frac{9}{16}$

$\Rightarrow$ eigenvalues are $\lambda_1 = \frac{5}{4} + \frac{3}{4} = 2$, $\lambda_2 = \frac{1}{2}$

$[\,A_0 - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $-R_1 \rightarrow R_1$, $\frac{1}{2}R_1 + R_2 \rightarrow R_2$.

$\Rightarrow \mathbf{v}_1 = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = 2$

$[\,A_0 - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} \frac{1}{2} & -1 & 0 \\ -\frac{1}{2} & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 + R_2 \rightarrow R_2$, $2R_1 \rightarrow R_1$.

$\Rightarrow \mathbf{v}_2 = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = \frac{1}{2}$.

So, we can diagonalize A_0:

$$A_0 = PDP^{-1}, \quad \text{where} \quad P = \left[\,\mathbf{v}_1 \,\vdots\, \mathbf{v}_2\,\right] = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = diag(\lambda_1, \lambda_2) = diag\Big(2,\ \frac{1}{2}\Big).$$

As in Example 5.33, for $k = 1, ..., 4$, using the fact that $A_k = A_0 = \begin{bmatrix} 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$, for $k = 0, ..., 3$,

$$\mathbf{x}_k = \begin{bmatrix} V_k \\ I_k \end{bmatrix} = A_{k-1}A_{k-2}\cdots A_1A_0 \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \big(PDP^{-1}\big)^k \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = PD^kP^{-1} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \frac{1}{-3} \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -\lambda_1^k & 2\lambda_2^k \\ \lambda_1^k & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} -\lambda_1^k - 2\lambda_2^k & 2\lambda_1^k - 2\lambda_2^k \\ \lambda_1^k - \lambda_2^k & -2\lambda_1^k - \lambda_2^k \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}.$$

Further, using the fact that

$$A_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

we have

$$\mathbf{x}_5 = \begin{bmatrix} V_5 \\ I_5 \end{bmatrix} = A_4(A_3A_2A_1A_0) \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\lambda_1^4 - 2\lambda_2^4 & 2\lambda_1^4 - 2\lambda_2^4 \\ \lambda_1^4 - \lambda_2^4 & -2\lambda_1^4 - \lambda_2^4 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} -2\lambda_1^4 - \lambda_2^4 & 4\lambda_1^4 - \lambda_2^4 \\ \lambda_1^4 - \lambda_2^4 & -2\lambda_1^4 - \lambda_2^4 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -2\cdot 2^4 - \big(\frac{1}{2}\big)^4 & 4\cdot 2^4 - \big(\frac{1}{2}\big)^4 \\ 2^4 - \big(\frac{1}{2}\big)^4 & -2\cdot 2^4 - \big(\frac{1}{2}\big)^4 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{48} \begin{bmatrix} -513 & 1023 \\ 255 & -513 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = -\frac{1}{16} \begin{bmatrix} -171 & 341 \\ 85 & -171 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 171V_0 - 341I_0 \\ -85V_0 + 171I_0 \end{bmatrix}.$$

From part (a) and (5.101) in the textbook, we have

$$\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \mathbf{x}_0 = V_5A_0^{-1}A_1^{-1}\cdots A_4^{-1} \begin{bmatrix} 1 \\ Y_4 \end{bmatrix} = V_5 \left(\frac{1}{1} \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & 1 \end{bmatrix}\right)^4 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = ... = V_5\, \frac{1}{32} \begin{bmatrix} 683 \\ 341 \end{bmatrix}.$$

hence $I_0 = \frac{341}{32}V_5$. By definition, $\eta = \frac{V_0}{V_5}$, so $I_0 = \frac{341}{32}V_5 = \frac{341/32}{683/32}V_0 = \frac{341}{683}V_0$.

From earlier work in part (b), we have, for $k = 0, 1, ..., 4$

$$\mathbf{x}_k = \begin{bmatrix} V_k \\ I_k \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -\lambda_1^k - 2\lambda_2^k & 2\lambda_1^k - 2\lambda_2^k \\ \lambda_1^k - \lambda_2^k & -2\lambda_1^k - \lambda_2^k \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = -\frac{V_0}{3} \begin{bmatrix} -\lambda_1^k - 2\lambda_2^k & 2\lambda_1^k - 2\lambda_2^k \\ \lambda_1^k - \lambda_2^k & -2\lambda_1^k - \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 \\ \frac{341}{683} \end{bmatrix},$$

hence, for $k = 1, ..., 4$.

$$V_k = -\frac{V_0}{2049}\big((-683 + 682)\lambda_1^k - (1366 + 682)\lambda_2^k\big) = -\frac{V_0}{2049}\Big(-2^k - 2048\big(\frac{1}{2}\big)^k\Big) = \frac{V_0}{2049}\big(2^k + 2048\cdot 2^{-k}\big),$$

Further, using the fact that in this problem $Y_4 = \frac{1}{2}$,

$$V_5 = Y_4^{-1}I_4 = -\frac{2V_0}{2049}\big((683 - 682)\lambda_1^4 - (683 + 341)\lambda_2^4\big) = \frac{2V_0}{2049}\big(-2^4 + 1024\cdot 2^{-4}\big) = \frac{192}{4098}\,V_0 = \frac{32}{683}\,V_0.$$

Note that, by definition of η, we also have $V_5 = \eta^{-1} V_0 = \frac{32}{683}\, V_0$, which agrees with our final conclusion.

5.7.7.9. Using the results of Example 5.35, with $n = 5$, and $\mathbf{x}_k \triangleq \begin{bmatrix} V_k \\ I_k \end{bmatrix}$, for $k = 0, ..., 5$. Because the input voltage source is sinusoidal, each resistor has impedance $Z = R$. In this problem, $Y_0 = Y_1 = ... = Y_5 = 1$, and $Z_0 = Z_1 = ... = Z_5 = 1$. We get that

$$\mathbf{x}_{k+1} = A_k \mathbf{x}_k,$$

where for $k = 0, ..., 3$,

$$A_k = \begin{bmatrix} 1 & -Z_k \\ -Y_k & 1 + Y_k Z_k \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

and

$$A_4 = \begin{bmatrix} 1 & -Z_4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

(a) *Method 1*: To find the equivalent impedance, using the above, we calculate

$$A_0^{-1} A_1^{-1} \cdots A_4^{-1} \begin{bmatrix} 1 \\ Y_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left(\frac{1}{1} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right)^4 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 34 & 55 \\ 21 & 34 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 89 \\ 55 \end{bmatrix},$$

whose $(1,1)$ entry is $\eta = 89$, and the equivalent impedance is $Z = \frac{\eta}{Y_5} = \frac{89}{1} = 89$.

Method 2: To find the equivalent impedance, it helps to use eigenvalues and eigenvectors of A_k, for $k = 0, ..., 3$. In part (b), below, we will diagonalize A_0:

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = A_0 = PDP^{-1},$$

where $P = [\, \mathbf{v}_1 \; \vdots \; \mathbf{v}_2 \,] = \begin{bmatrix} 1 - \sqrt{5} & 1 + \sqrt{5} \\ 2 & 2 \end{bmatrix}$ and $D = diag(\lambda_1, \lambda_2) = diag\big(\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\big)$.

It follows that

$$A_0^{-1} A_1^{-1} \cdots A_4^{-1} \begin{bmatrix} 1 \\ Y_5 \end{bmatrix} = \big(PDP^{-1}\big)^{-4} A_4^{-1} \begin{bmatrix} 1 \\ Y_5 \end{bmatrix} = PD^{-4} P^{-1} A_4^{-1} \begin{bmatrix} 1 \\ Y_5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \sqrt{5} & 1 + \sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1^{-5} & 0 \\ 0 & \lambda_2^{-5} \end{bmatrix} \frac{1}{-4\sqrt{5}} \begin{bmatrix} 2 & -1 - \sqrt{5} \\ -2 & 1 - \sqrt{5} \end{bmatrix} A_4^{-1} \begin{bmatrix} 1 \\ Y_5 \end{bmatrix}$$

$$= -\frac{1}{4\sqrt{5}} \begin{bmatrix} (1 - \sqrt{5})\lambda_1^{-4} & (1 + \sqrt{5})\lambda_2^{-4} \\ 2\lambda_1^{-4} & 2\lambda_2^{-4} \end{bmatrix} \begin{bmatrix} 2 & -1 - \sqrt{5} \\ -2 & 1 - \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{4\sqrt{5}} \begin{bmatrix} 2(1 - \sqrt{5})\lambda_1^{-4} - 2(1 + \sqrt{5})\lambda_2^{-4} & 4\lambda_1^{-4} - 4\lambda_2^{-4} \\ 4\lambda_1^{-4} - 4\lambda_2^{-4} & -2(1 + \sqrt{5})\lambda_1^{-4} + 2(1 - \sqrt{5})\lambda_2^{-4} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{4\sqrt{5}} \begin{bmatrix} 4(2 - \sqrt{5})\lambda_1^{-4} - 4(2 + \sqrt{5})\lambda_2^{-4} \\ 2(3 - \sqrt{5})\lambda_1^{-4} - 2(3 + \sqrt{5})\lambda_2^{-4} \end{bmatrix} = -\frac{1}{2\sqrt{5}\, \lambda_1^4 \lambda_2^4} \begin{bmatrix} 2(2 - \sqrt{5})\lambda_2^4 - 2(2 + \sqrt{5})\lambda_1^4 \\ (3 - \sqrt{5})\lambda_2^4 - (3 + \sqrt{5})\lambda_1^4 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{5}\big(\frac{3+\sqrt{5}}{2}\big)^4\big(\frac{3-\sqrt{5}}{2}\big)^4}\begin{bmatrix} 2(2-\sqrt{5})\big(\frac{3-\sqrt{5}}{2}\big)^4 - 2(2+\sqrt{5})\big(\frac{3+\sqrt{5}}{2}\big)^4 \\ \\ (3-\sqrt{5})\big(\frac{3-\sqrt{5}}{2}\big)^4 - (3+\sqrt{5})\big(\frac{3+\sqrt{5}}{2}\big)^4 \end{bmatrix}$$

hence

$$A_0^{-1}A_1^{-1}\cdots A_4^{-1}\begin{bmatrix} 1 \\ Y_5 \end{bmatrix} = -\frac{1}{2\sqrt{5}\big(\frac{(3+\sqrt{5})(3-\sqrt{5})}{4}\big)^4}\begin{bmatrix} -178\sqrt{5} \\ \\ -110\sqrt{5} \end{bmatrix} = -\frac{1}{2\sqrt{5}}\begin{bmatrix} -178\sqrt{5} \\ \\ -110\sqrt{5} \end{bmatrix} = \begin{bmatrix} 89 \\ \\ 55 \end{bmatrix},$$

whose $(1,1)$ entry is $\eta = 89$, and the equivalent impedance is $Z = \frac{\eta}{Y_5} = \frac{89}{1} = 89$.

(b) To find a formula for V_k it helps to use eigenvalues and eigenvectors of A_k, for $k = 0, ..., 3$. We calculate

$0 = \begin{vmatrix} 1-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = \big(\lambda - \frac{3}{2}\big)^2 - \frac{5}{4}$

$\Rightarrow$ eigenvalues are $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$

$[\,A_0 - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} \frac{-1-\sqrt{5}}{2} & -1 & 0 \\ -1 & \frac{1-\sqrt{5}}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{-1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-R_1 \rightarrow R_1$, $\frac{1+\sqrt{5}}{2}R_1 + R_2 \rightarrow R_2$.

$\Rightarrow \mathbf{v}_1 = c_1\begin{bmatrix} 1-\sqrt{5} \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = \frac{3+\sqrt{5}}{2}$

$[\,A_0 - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} \frac{-1+\sqrt{5}}{2} & -1 & 0 \\ -1 & \frac{1+\sqrt{5}}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-R_1 \rightarrow R_1$, $\frac{1-\sqrt{5}}{2}R_1 + R_2 \rightarrow R_2$.

$\Rightarrow \mathbf{v}_2 = c_1\begin{bmatrix} 1+\sqrt{5} \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = \frac{3-\sqrt{5}}{2}$.

So, we can diagonalize A_0:

$A_0 = PDP^{-1}$, where $P = [\,\mathbf{v}_1 \mid \mathbf{v}_2\,] = \begin{bmatrix} 1-\sqrt{5} & 1+\sqrt{5} \\ 2 & 2 \end{bmatrix}$ and $D = diag(\lambda_1, \lambda_2) = diag\Big(\dfrac{3+\sqrt{5}}{2}, \dfrac{3-\sqrt{5}}{2}\Big)$.

As in Example 5.33, for $k = 1, ..., 5$, using the fact that $A_k = A_0 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, for $k = 0, ..., 3$,

$$\mathbf{x}_k = \begin{bmatrix} V_k \\ I_k \end{bmatrix} = A_{k-1}A_{k-2}\cdots A_1A_0\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \big(PDP^{-1}\big)^k\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = PD^kP^{-1}\begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\sqrt{5} & 1+\sqrt{5} \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\frac{1}{-4\sqrt{5}}\begin{bmatrix} 2 & -1-\sqrt{5} \\ -2 & 1-\sqrt{5} \end{bmatrix}\begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{4\sqrt{5}}\begin{bmatrix} (1-\sqrt{5})\lambda_1^k & (1+\sqrt{5})\lambda_2^k \\ 2\lambda_1^k & 2\lambda_2^k \end{bmatrix}\begin{bmatrix} 2 & -1-\sqrt{5} \\ -2 & 1-\sqrt{5} \end{bmatrix}\begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{4\sqrt{5}}\begin{bmatrix} 2(1-\sqrt{5})\lambda_1^k - 2(1+\sqrt{5})\lambda_2^k & 4\lambda_1^k - 4\lambda_2^k \\ \\ 4\lambda_1^k - 4\lambda_2^k & -2(1+\sqrt{5})\lambda_1^k + 2(1-\sqrt{5})\lambda_2^k \end{bmatrix}\begin{bmatrix} V_0 \\ \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{5}}\begin{bmatrix}(1-\sqrt{5})\lambda_1^k-(1+\sqrt{5})\lambda_2^k & 2\lambda_1^k-2\lambda_2^k\\ \\ 2\lambda_1^k-2\lambda_2^k & -(1+\sqrt{5})\lambda_1^k+(1-\sqrt{5})\lambda_2^k\end{bmatrix}\begin{bmatrix}V_0\\ \\ I_0\end{bmatrix}$$

Further, using the fact that

$$A_4=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix},$$

we have

$$\mathbf{x}_5=\begin{bmatrix}V_5\\ I_5\end{bmatrix}=A_4(A_3A_2A_1A_0)\begin{bmatrix}V_0\\ I_0\end{bmatrix}$$

$$= -\frac{1}{2\sqrt{5}}\begin{bmatrix}1 & -1\\ \\ 0 & 1\end{bmatrix}\begin{bmatrix}(1-\sqrt{5})\lambda_1^4-(1+\sqrt{5})\lambda_2^4 & 2\lambda_1^4-2\lambda_2^4\\ \\ 2\lambda_1^4-2\lambda_2^4 & -(1+\sqrt{5})\lambda_1^4+(1-\sqrt{5})\lambda_2^4\end{bmatrix}\begin{bmatrix}V_0\\ \\ I_0\end{bmatrix}$$

$$= -\frac{1}{2\sqrt{5}}\begin{bmatrix}-(1+\sqrt{5})\lambda_1^4+(1-\sqrt{5})\lambda_2^4 & (3+\sqrt{5})\lambda_1^4-(3-\sqrt{5})\lambda_2^4\\ \\ 2\lambda_1^4-2\lambda_2^4 & -(1+\sqrt{5})\lambda_1^4+(1-\sqrt{5})\lambda_2^4\end{bmatrix}\begin{bmatrix}V_0\\ \\ I_0\end{bmatrix}$$

$$= -\frac{1}{2\sqrt{5}}\begin{bmatrix}-(1+\sqrt{5})\left(\frac{3+\sqrt{5}}{2}\right)^4+(1-\sqrt{5})\left(\frac{3-\sqrt{5}}{2}\right)^4 & (3+\sqrt{5})\left(\frac{3+\sqrt{5}}{2}\right)^4-(3-\sqrt{5})\left(\frac{3-\sqrt{5}}{2}\right)^4\\ \\ 2\left(\frac{3+\sqrt{5}}{2}\right)^4-2\left(\frac{3-\sqrt{5}}{2}\right)^4 & -(1+\sqrt{5})\left(\frac{3+\sqrt{5}}{2}\right)^4+(1-\sqrt{5})\left(\frac{3-\sqrt{5}}{2}\right)^4\end{bmatrix}\begin{bmatrix}V_0\\ \\ I_0\end{bmatrix}$$

$$= -\frac{1}{2\sqrt{5}}\begin{bmatrix}-68\sqrt{5} & 110\sqrt{5}\\ 42\sqrt{5} & -68\sqrt{5}\end{bmatrix}\begin{bmatrix}V_0\\ I_0\end{bmatrix}=\begin{bmatrix}34 & -55\\ -21 & 34\end{bmatrix}\begin{bmatrix}V_0\\ I_0\end{bmatrix}=\begin{bmatrix}34V_0-55I_0\\ -21V_0+34I_0\end{bmatrix}.$$

From part (a) and (5.101) in the textbook, we have

$$\begin{bmatrix}V_0\\ I_0\end{bmatrix}=\mathbf{x}_0=V_4A_0^{-1}A_1^{-1}\cdots A_4^{-1}\begin{bmatrix}1\\ Y_5\end{bmatrix}=V_4\left(\frac{1}{1}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\right)^4\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=...=V_4\begin{bmatrix}89\\ 55\end{bmatrix}.$$

hence $I_0=55V_4$. By definition, $\eta=\frac{V_0}{V_4}$, so $I_0=55V_4=\frac{55}{89}V_0$.

From earlier work in part (b), we have, for $k=0,1,...,4$

$$\mathbf{x}_k=\begin{bmatrix}V_k\\ I_k\end{bmatrix}=-\frac{1}{2\sqrt{5}}\begin{bmatrix}(1-\sqrt{5})\lambda_1^k-(1+\sqrt{5})\lambda_2^k & 2\lambda_1^k-2\lambda_2^k\\ \\ 2\lambda_1^k-2\lambda_2^k & -(1+\sqrt{5})\lambda_1^k+(1-\sqrt{5})\lambda_2^k\end{bmatrix}\begin{bmatrix}V_0\\ \\ I_0\end{bmatrix}$$

$$= -\frac{V_0}{2\sqrt{5}}\begin{bmatrix}(1-\sqrt{5})\lambda_1^k-(1+\sqrt{5})\lambda_2^k & 2\lambda_1^k-2\lambda_2^k\\ \\ 2\lambda_1^k-2\lambda_2^k & -(1+\sqrt{5})\lambda_1^k+(1-\sqrt{5})\lambda_2^k\end{bmatrix}\begin{bmatrix}1\\ \\ \frac{55}{89}\end{bmatrix},$$

hence

$$V_k=-\frac{V_0}{178\sqrt{5}}\Big((89(1-\sqrt{5})+110)\lambda_1^k-(89(1+\sqrt{5})+110)\lambda_2^k\Big).$$

Further, using the fact that in this problem $Y_5=1$,

$$V_5=Y_5^{-1}I_4=-\frac{V_0}{178\sqrt{5}}\Big(\big(178-55(1+\sqrt{5})\big)\lambda_1^4+\big(-178+55(1-\sqrt{5})\big)\lambda_2^4\Big)$$

$$= -\frac{V_0}{178\sqrt{5}}\left(\big(178-55(1+\sqrt{5})\big)\left(\frac{3+\sqrt{5}}{2}\right)^4+\big(-178+55(1-\sqrt{5})\big)\left(\frac{3-\sqrt{5}}{2}\right)^4\right)=-\frac{V_0}{178\sqrt{5}}\cdot\Big(-2\sqrt{5}\Big)$$

$$= \frac{V_0}{89}\,.$$

Note that, by definition of η, we also have $V_5 = \eta^{-1}V_0 = \frac{1}{89}\,V_0$, which agrees with our final conclusion.

5.7.7.11. Use the methods of Example 5.35, with $n = 4$, and $\mathbf{x}_k \triangleq \begin{bmatrix} V_k \\ I_k \end{bmatrix}$, for $k = 0, ..., 4$. Because the input voltage source is sinusoidal, each resistor has impedance $Z = R$. In this problem, $Y_0 = Y_1 = ... = Y_4 = 1$, and $Z_0 = Z_1 = ... = Z_4 = 2$. We get that

$$\mathbf{x}_{k+1} = A_k\mathbf{x}_k,$$

where for $k = 0, 1, 2$

$$A_k = \begin{bmatrix} 1 & -Z_k \\ -Y_k & 1 + Y_kZ_k \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 1 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix},$$

and

$$A_3 = \begin{bmatrix} 1 & -Z_4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

(a) *Method 1*: To find the equivalent impedance, using the above, we calculate

$$A_0^{-1}A_1^{-1}\cdots A_3^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left(\frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\right)^3\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 41 & 112 \\ 15 & 41 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 153 \\ 56 \end{bmatrix},$$

whose $(1,1)$ entry is $\eta = 153$, and the equivalent impedance is $Z = \frac{\eta}{Y_4} = \frac{\eta}{1} = 153$.

Method 2: To find the equivalent impedance, it helps to use eigenvalues and eigenvectors of A_k, for $k = 0, 1, 2$. In part (b), below, we will diagonalize A_0:

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = A_0 = PDP^{-1},$$

where $P = [\,\mathbf{v}_1 \,\vdots\, \mathbf{v}_2\,] = \begin{bmatrix} 1-\sqrt{3} & 1+\sqrt{3} \\ 1 & 1 \end{bmatrix}$ and $D = diag(\lambda_1, \lambda_2) = diag\big(2+\sqrt{3}, 2-\sqrt{3}\big)$.

It follows that

$$A_0^{-1}A_1^{-1}\cdots A_3^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix} = \big(PDP^{-1}\big)^{-3}A_3^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix} = PD^{-3}P^{-1}A_3^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\sqrt{3} & 1+\sqrt{3} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^{-3} & 0 \\ 0 & \lambda_2^{-3} \end{bmatrix}\frac{1}{-2\sqrt{3}}\begin{bmatrix} 1 & -1-\sqrt{3} \\ -1 & 1-\sqrt{3} \end{bmatrix}A_3^{-1}\begin{bmatrix} 1 \\ Y_4 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}}\begin{bmatrix} (1-\sqrt{3})\lambda_1^{-3} & (1+\sqrt{3})\lambda_2^{-3} \\ \lambda_1^{-3} & \lambda_2^{-3} \end{bmatrix}\begin{bmatrix} 1 & -1-\sqrt{3} \\ -1 & 1-\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}}\begin{bmatrix} (1-\sqrt{3})\lambda_1^{-3} - (1+\sqrt{3})\lambda_2^{-3} & 2\lambda_1^{-3} - 2\lambda_2^{-3} \\ \\ \lambda_1^{-3} - \lambda_2^{-3} & -(1+\sqrt{3})\lambda_1^{-3} + (1-\sqrt{3})\lambda_2^{-3} \end{bmatrix}\begin{bmatrix} 3 \\ \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}}\begin{bmatrix} (5-3\sqrt{3})\lambda_1^{-3} - (5+3\sqrt{3})\lambda_2^{-3} \\ \\ (2-\sqrt{3})\lambda_1^{-3} - (2+\sqrt{3})\lambda_2^{-3} \end{bmatrix} = -\frac{1}{2\sqrt{3}\,\lambda_1^3\lambda_2^3}\begin{bmatrix} (5-3\sqrt{3})\lambda_2^3 - (5+3\sqrt{3})\lambda_1^3 \\ \\ (2-\sqrt{3})\lambda_2^3 - (2+\sqrt{3})\lambda_1^3 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}\left(2+\sqrt{3}\right)^3\left(2-\sqrt{3}\right)^3}\begin{bmatrix} (5-3\sqrt{3})\left(2-\sqrt{3}\right)^3-(5+\sqrt{3})\left(2+\sqrt{3}\right)^3 \\ \\ (2-\sqrt{3})\left(2-\sqrt{3}\right)^3-(2+\sqrt{3})\left(2+\sqrt{3}\right)^3 \end{bmatrix}$$

hence

$$= -\frac{1}{2\sqrt{3}\left((2+\sqrt{3})(2-\sqrt{3})\right)^3}\begin{bmatrix} -306\sqrt{3} \\ \\ -112\sqrt{3} \end{bmatrix} = -\frac{1}{2\sqrt{3}}\begin{bmatrix} -306\sqrt{3} \\ \\ -112\sqrt{3} \end{bmatrix} = \begin{bmatrix} 153 \\ \\ 56 \end{bmatrix},$$

whose $(1,1)$ entry is $\eta = 153$, and the equivalent impedance is $Z = \frac{\eta}{Y_4} = \frac{153}{1} = 153$.

(b) To find a formula for V_k it helps to use eigenvalues and eigenvectors of A_k, for $k = 0, 1, 2$. We calculate

$0 = \begin{vmatrix} 1-\lambda & -2 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 2 = \lambda^2 - 4\lambda + 1 = (\lambda-2)^2 - 3$

$\Rightarrow$ eigenvalues are $\lambda_1 = 2+\sqrt{3}, \lambda_2 = 2-\sqrt{3}$

$[\,A_0 - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -1-\sqrt{3} & -2 & 0 \\ -1 & 1-\sqrt{3} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1+\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-R_1 \rightarrow R_1$, $(1+\sqrt{3})R_1 + R_2 \rightarrow R_2$.

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} 1-\sqrt{3} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = 2+\sqrt{3}$

$[\,A_0 - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -1+\sqrt{3} & -2 & 0 \\ -1 & 1+\sqrt{3} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1-\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-R_1 \rightarrow R_1$, $(1-\sqrt{3})R_1 + R_2 \rightarrow R_2$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1+\sqrt{3} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 2-\sqrt{3}$.

So, we can diagonalize A_0:

$A_0 = PDP^{-1}$, where $P = [\,\mathbf{v}_1 \mid \mathbf{v}_2\,] = \begin{bmatrix} 1-\sqrt{3} & 1+\sqrt{3} \\ 1 & 1 \end{bmatrix}$ and $D = diag(\lambda_1, \lambda_2) = diag(2+\sqrt{3}, 2-\sqrt{3})$.

As in Example 5.33, for $k = 1, ..., 4$, using the fact that $A_k = A_0 = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, for $k = 1, 2, 3$,

$$\mathbf{x}_k = \begin{bmatrix} V_k \\ I_k \end{bmatrix} = A_{k-1}A_{k-2}\cdots A_1 A_0 \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \left(PDP^{-1}\right)^k \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = PD^kP^{-1}\begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\sqrt{3} & 1+\sqrt{3} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\frac{1}{-2\sqrt{3}}\begin{bmatrix} 1 & -1-\sqrt{3} \\ -1 & 1-\sqrt{3} \end{bmatrix}\begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}}\begin{bmatrix} (1-\sqrt{3})\lambda_1^k & (1+\sqrt{3})\lambda_2^k \\ \lambda_1^k & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -1-\sqrt{3} \\ -1 & 1-\sqrt{3} \end{bmatrix}\begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}}\begin{bmatrix} (1-\sqrt{3})\lambda_1^k - (1+\sqrt{3})\lambda_2^k & 2\lambda_1^k - 2\lambda_2^k \\ \\ \lambda_1^k - \lambda_2^k & -(1+\sqrt{3})\lambda_1^k + (1-\sqrt{3})\lambda_2^k \end{bmatrix}\begin{bmatrix} V_0 \\ \\ I_0 \end{bmatrix}.$$

Further, using the fact that

$$A_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

we have

$$\mathbf{x}_4 = \begin{bmatrix} V_4 \\ I_4 \end{bmatrix} = A_3A_2A_1A_0 \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1-\sqrt{3})\lambda_1^3 - (1+\sqrt{3})\lambda_2^3 & 2\lambda_1^3 - 2\lambda_2^3 \\ \lambda_1^3 - \lambda_2^3 & -(1+\sqrt{3})\lambda_1^3 + (1-\sqrt{3})\lambda_2^3 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}} \begin{bmatrix} -(1+\sqrt{3})(2+\sqrt{3})^3 + (1-\sqrt{3})(2-\sqrt{3})^3 & (4+2\sqrt{3})(2+\sqrt{3})^3 + (-4+2\sqrt{3})(2-\sqrt{3})^3 \\ (2+\sqrt{3})^3 - (2-\sqrt{3})^3 & -(1+\sqrt{3})(2+\sqrt{3})^3 + (1-\sqrt{3})(2-\sqrt{3})^3 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{1}{2\sqrt{3}} \begin{bmatrix} -82\sqrt{3} & 224\sqrt{3} \\ 30\sqrt{3} & -82\sqrt{3} \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} 41 & -112 \\ -15 & 41 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} 41V_0 - 112I_0 \\ -15V_0 + 41I_0 \end{bmatrix}.$$

From part (a) and (5.101) in the textbook, we have

$$\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \mathbf{x}_0 = V_4 A_0^{-1} A_1^{-1} \cdots A_3^{-1} \begin{bmatrix} 1 \\ Y_4 \end{bmatrix} = V_4 \left(\frac{1}{1} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \right)^3 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = ... = V_4 \begin{bmatrix} 153 \\ 56 \end{bmatrix}.$$

hence $I_0 = 56V_4$. By definition, $\eta = \frac{V_0}{V_4}$, so $I_0 = 56V_4 = \frac{56}{153}V_0$.

From earlier work in part (b), we have, for $k = 0, 1, 2, 3$,

$$\mathbf{x}_k = \begin{bmatrix} V_k \\ I_k \end{bmatrix} = -\frac{1}{2\sqrt{3}} \begin{bmatrix} (1-\sqrt{3})\lambda_1^k - (1+\sqrt{3})\lambda_2^k & 2\lambda_1^k - 2\lambda_2^k \\ \lambda_1^k - \lambda_2^k & -(1+\sqrt{3})\lambda_1^k + (1-\sqrt{3})\lambda_2^k \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$= -\frac{V_0}{2\sqrt{3}} \begin{bmatrix} (1-\sqrt{3})\lambda_1^k - (1+\sqrt{3})\lambda_2^k & 2\lambda_1^k - 2\lambda_2^k \\ \lambda_1^k - \lambda_2^k & -(1+\sqrt{3})\lambda_1^k + (1-\sqrt{3})\lambda_2^k \end{bmatrix} \begin{bmatrix} 1 \\ \frac{56}{153} \end{bmatrix},$$

hence

$$V_k = -\frac{V_0}{306\sqrt{3}} \Big((153(1-\sqrt{3}) + 112)\lambda_1^k - (153(1+\sqrt{3}) + 112)\lambda_2^k \Big).$$

Further, using the fact that in this problem $Y_4 = 1$,

$$V_4 = Y_4^{-1} I_3 = -\frac{V_0}{306\sqrt{3}} \Big(\big(153 - 56(1+\sqrt{3})\big)\lambda_1^3 + \big(-153 + 56(1-\sqrt{3})\big)\lambda_2^3 \Big)$$

$$= -\frac{V_0}{306\sqrt{3}} \Big(\big(153 - 56(1+\sqrt{3})\big)(2+\sqrt{3})^3 + \big(-153 + 56(1-\sqrt{3})\big)(2-\sqrt{3})^3 \Big) = -\frac{V_0}{306\sqrt{3}} \cdot (-2\sqrt{3})$$

$$= \frac{V_0}{153}.$$

Note that, by definition of η, we also have $V_4 = \eta^{-1} V_0 = \frac{1}{153} V_0$, which agrees with our final conclusion.

5.7.7.13. Define $A = \begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{f}_k = \left(\frac{1}{2}\right)^k \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \left(\frac{1}{2}\right)^k \mathbf{w}$, where $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

First, we will use a method of undetermined coefficients analogous to a method for non-resonant non-homogeneous linear systems of ODEs: Let's try for a particular solution of $\mathbf{x}_{p,k+1} = A\mathbf{x}_p + \mathbf{f}_k$ in the form $\mathbf{x}_{p,k} = \left(\frac{1}{2}\right)^k \mathbf{a}$, where $\mathbf{a}$ is a constant vector. We substitute $\mathbf{x}_{p,k}$ into the non-homogeneous system to get

$$\left(\frac{1}{2}\right)^{k+1} \mathbf{a} = \mathbf{x}_{p,k+1} = A\mathbf{x}_p + \mathbf{f}_k = A\left(\left(\frac{1}{2}\right)^k \mathbf{a} \right) + \left(\frac{1}{2}\right)^k \mathbf{w} = \left(\frac{1}{2}\right)^k A\mathbf{a} + \left(\frac{1}{2}\right)^k \mathbf{w},$$

Similar to work in Example 5.26 in Section 5.5, we get $\frac{1}{2}\,\mathbf{a} = A\mathbf{a} + \mathbf{w}$ hence $\mathbf{a} = -\left(A - \frac{1}{2}\,I\right)^{-1}\mathbf{w}$, *as long as* $\left(A - \frac{1}{2}\,I\right)$ is invertible. Here, this gives

$$\mathbf{a} = -\left(A - \frac{1}{2}\,I\right)^{-1}\mathbf{w} = -\begin{bmatrix} \frac{1}{2} & 4 \\ 1 & -\frac{3}{2} \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\frac{1}{-19/4}\begin{bmatrix} -\frac{3}{2} & -4 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{4}{19}\begin{bmatrix} -\frac{17}{2} \\ -\frac{5}{2} \end{bmatrix}.$$

So, a particular solution is given by

$$\mathbf{x}_{p,k} = -\frac{2}{19}\left(\frac{1}{2}\right)^k \begin{bmatrix} 17 \\ 5 \end{bmatrix}.$$

We can use eigenvalues and eigenvectors to find the general solution of the corresponding homogeneous system of difference equations (LCCHSΔ) $\mathbf{x}_{k+1} = A\mathbf{x}_k$:

$$0 = \begin{vmatrix} 1-\lambda & 4 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 4 = \lambda^2 - 5$$

$\Rightarrow$ eigenvalues are $\lambda_1 = -\sqrt{5}, \lambda_2 = \sqrt{5}$

$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 1+\sqrt{5} & 4 & 0 \\ 1 & -1+\sqrt{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1+\sqrt{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-(1+\sqrt{5})R_1 + R_2 \to R_2$

$\Rightarrow \mathbf{v}_1 = c_1\begin{bmatrix} 1-\sqrt{5} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -\sqrt{5}$

$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 1-\sqrt{5} & 4 & 0 \\ 1 & -1-\sqrt{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1-\sqrt{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$, after $R_1 \leftrightarrow R_2$, $-(1-\sqrt{5})R_1 + R_2 \to R_2$

$\Rightarrow \mathbf{v}_2 = c_1\begin{bmatrix} 1+\sqrt{5} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = \sqrt{5}$

The general solution of the corresponding homogeneous system is

$$\mathbf{x}_k = c_1(-\sqrt{5})^k\begin{bmatrix} 1-\sqrt{5} \\ 1 \end{bmatrix} + c_2(\sqrt{5})^k\begin{bmatrix} 1+\sqrt{5} \\ 1 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

The general solution of the original problem is

$$\mathbf{x}_k = -\frac{2}{19}\left(\frac{1}{2}\right)^k\begin{bmatrix} 17 \\ 5 \end{bmatrix} + c_1(-\sqrt{5})^k\begin{bmatrix} 1-\sqrt{5} \\ 1 \end{bmatrix} + c_2(\sqrt{5})^k\begin{bmatrix} 1+\sqrt{5} \\ 1 \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

5.7.7.15. Define $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ -1 & -2 & 5 \end{bmatrix}$ and $\mathbf{f}_k = \left(\frac{1}{2}\right)^k\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \left(\frac{1}{2}\right)^k\mathbf{w}$, where $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

First, we will use a method of undetermined coefficients analogous to a method for non-resonant non-homogeneous linear systems of ODEs: Let's try for a particular solution of $\mathbf{x}_{p,k+1} = A\mathbf{x}_p + \mathbf{f}_k$ in the form $\mathbf{x}_{p,k} = \left(\frac{1}{2}\right)^k\mathbf{a}$, where $\mathbf{a}$ is a constant vector. We substitute $\mathbf{x}_{p,k}$ into the non-homogeneous system to get

$$\left(\frac{1}{2}\right)^{k+1}\mathbf{a} = \mathbf{x}_{p,k+1} = A\mathbf{x}_p + \mathbf{f}_k = A\left(\left(\frac{1}{2}\right)^k\mathbf{a}\right) + \left(\frac{1}{2}\right)^k\mathbf{w} = \left(\frac{1}{2}\right)^k A\mathbf{a} + \left(\frac{1}{2}\right)^k\mathbf{w},$$

Similar to work in Example 5.26 in Section 5.5, we get $\frac{1}{2}\,\mathbf{a} = A\mathbf{a} + \mathbf{w}$ hence $\mathbf{a} = -\left(A - \frac{1}{2}\,I\right)^{-1}\mathbf{w}$, *as long as* $\left(A - \frac{1}{2}\,I\right)$ is invertible. Here, this gives

$$\mathbf{a} = -\left(A - \frac{1}{2}\,I\right)^{-1}\mathbf{w} = -\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 2 & \frac{5}{2} & 1 \\ -1 & -2 & \frac{9}{2} \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -\frac{1}{53}\begin{bmatrix} 106 & 0 & 0 \\ -80 & 18 & -4 \\ -12 & 8 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{53}\begin{bmatrix} -106 \\ 56 \\ -34 \end{bmatrix}.$$

So, a particular solution is given by

$$\mathbf{x}_{p,k} = \frac{1}{53}\left(\frac{1}{2}\right)^k \begin{bmatrix} -106 \\ 56 \\ -34 \end{bmatrix}.$$

We can use eigenvalues and eigenvectors to find the general solution of the corresponding homogeneous system of difference equations (LCCHSΔ) $\mathbf{x}_{k+1} = A\mathbf{x}_k$:

$$0 = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 1 \\ -1 & -2 & 5-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -2 & 5-\lambda \end{vmatrix} = (1-\lambda)\big((3-\lambda)(5-\lambda)+2\big)$$

$$= (1-\lambda)\big(\lambda^2 - 8\lambda + 17\big) = (1-\lambda)\big((\lambda-4)^2+1\big).$$

$\Rightarrow$ eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 4+i$, $\lambda_3 = 4-i$

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ -1 & -2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -2 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -\frac{9}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after $R_1 \leftrightarrow R_3$, $-R_1 \to R_1$, $-2R_1 + R_2 \to R_2$, followed by $R_2 + R_1 \to R_1$, $-\frac{1}{2}R_2 \to R_2$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -10 \\ 9 \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = 1$.

A solution of the corresponding homogeneous system of difference equations is given by

$$\mathbf{x}_k^{(1)} = 1^k \begin{bmatrix} -10 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} -10 \\ 9 \\ 2 \end{bmatrix}.$$

$$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} -3-i & 0 & 0 & 0 \\ 2 & -1-i & 1 & 0 \\ -1 & -2 & 1-i & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1+i & 0 \\ 0 & -5-i & 3-i2 & 0 \\ 0 & 2(3+i) & -4+i2 & 0 \end{array}\right]$$

after $R_1 \leftrightarrow R_3$, $-R_1 \to R_1$, $-2R_1 + R_2 \to R_2$,

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{3-i2}{5+i} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-1+i}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

after $-\frac{1}{5+i}R_2 \to R_2$, $-2R_2 + R_1 \to R_1$, $-2(3+i)R_2 + R_3 \to R_3$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 0 \\ 1-i \\ 2 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 4+i$.

Because the eigenvalues are a complex conjugate pair, we don't need to find eigenvectors corresponding to $\lambda_3 = \overline{\lambda_3}$.

As in Example 5.33, we calculate that $\lambda_2 = 4 + i = \alpha + i\nu = \rho(\cos\omega + i\sin\omega)$, where

$$\rho = \sqrt{4^2 + 1^2} = \sqrt{17} \quad \text{and} \quad \tan\omega = \frac{1}{4}.$$

Because $(\alpha, \nu) = (4, 1)$ is in the first quadrant, $\omega = \arctan\frac{1}{4}$. We have

$$\rho^k(\cos\omega k + i\sin\omega k)\mathbf{v}_2 = 17^{k/2}\left(\cos\left(k\arctan\frac{1}{4}\right) + i\,\sin\left(k\arctan\frac{1}{4}\right)\right)\begin{bmatrix} 0 \\ 1-i \\ 2 \end{bmatrix}$$

$$= 17^{k/2}\begin{bmatrix} 0 \\ \cos\left(k\arctan\frac{1}{4}\right)+\sin\left(k\arctan\frac{1}{4}\right)-i\cos(k\arctan\frac{1}{4}\right)+i\sin\left(k\arctan\frac{1}{4}\right) \\ 2\cos\left(k\arctan\frac{1}{4}\right)+i2\sin\left(k\arctan\frac{1}{4}\right)\end{bmatrix}.$$

As in the discussion before Example 5.33, there are solutions

$$\mathbf{x}_k^{(2)} = 17^{k/2}\ \mathcal{R}e\left((\cos\omega k + i\sin\omega k)\mathbf{v}_2\right) = 17^{k/2}\begin{bmatrix} 0 \\ \cos\left(k\arctan\frac{1}{4}\right)+\sin\left(k\arctan\frac{1}{4}\right) \\ 2\cos\left(k\arctan\frac{1}{4}\right)\end{bmatrix}$$

and

$$\mathbf{x}_k^{(3)} = 17^{k/2}\ \mathcal{I}m\left((\cos\omega k + i\sin\omega k)\mathbf{v}_2\right) = 17^{k/2}\begin{bmatrix} 0 \\ -\cos\left(k\arctan\frac{1}{4}\right)+\sin\left(k\arctan\frac{1}{4}\right) \\ 2\sin\left(k\arctan\frac{1}{4}\right)\end{bmatrix}$$

The general solution of the corresponding homogeneous system is

$$\mathbf{x}_k{=}c_1\begin{bmatrix}-10\\9\\2\end{bmatrix}+c_2 17^{k/2}\begin{bmatrix} 0 \\ \cos\left(k\arctan\frac{1}{4}\right)+\sin\left(k\arctan\frac{1}{4}\right) \\ 2\cos\left(k\arctan\frac{1}{4}\right)\end{bmatrix}+c_3 17^{k/2}\begin{bmatrix} 0 \\ -\cos\left(k\arctan\frac{1}{4}\right)+\sin\left(k\arctan\frac{1}{4}\right) \\ 2\sin\left(k\arctan\frac{1}{4}\right)\end{bmatrix},$$

where c_1, c_2, c_3 =arbitrary constants.

The general solution of the original problem is

$$\mathbf{x}_k = \frac{1}{53}\left(\frac{1}{2}\right)^k\begin{bmatrix}-106\\56\\-34\end{bmatrix}+c_1\begin{bmatrix}-10\\9\\2\end{bmatrix}+c_2 17^{k/2}\begin{bmatrix} 0 \\ \cos\left(k\arctan\frac{1}{4}\right)+\sin\left(k\arctan\frac{1}{4}\right) \\ 2\cos\left(k\arctan\frac{1}{4}\right)\end{bmatrix}$$

$$+c_3 17^{k/2}\begin{bmatrix} 0 \\ -\cos\left(k\arctan\frac{1}{4}\right)+\sin\left(k\arctan\frac{1}{4}\right) \\ 2\sin\left(k\arctan\frac{1}{4}\right)\end{bmatrix},$$

where c_1, c_2, c_3 =arbitrary constants.

5.7.7.17. A's eigenvalues $\frac{\sqrt{3}}{2}\pm\frac{i}{2} = e^{\pm i\pi/6}$ have modulus one, so LCCHSΔ $(\star)$ $\mathbf{x}_{k+1} = A\mathbf{x}_k$ cannot be asymptotically stable.

(a) The system may be neutrally stable. If the repeated eigenvalue $\frac{\sqrt{3}}{2}+\frac{i}{2}$ is not deficient then the system is neutrally stable.

(b) The system may be unstable. If the repeated eigenvalue $\frac{\sqrt{3}}{2}+\frac{i}{2}$ is deficient then the system is unstable.

(c) By the reasoning in parts (a) and (b) combined, the system may be neutrally stable, depending upon more information concerning A.

(d) A's eigenvalue $\lambda = \frac{\sqrt{3}}{2}+\frac{i}{2} = e^{i\pi/6}$ satisfies $\lambda^6 = \left(e^{i\pi/6}\right)^6 = e^{6\cdot i\pi/6} = e^{i\pi} = -1$ and $\lambda^{12} = \left(\lambda^6\right)^2 = (-1)^2 = 1$. It follows that $(\star)$ has solutions $\mathbf{x}_k$ that are periodic in k with period 12. But, there is no eigenvalue μ satisfying $\mu^6 = 1$, so $(\star)$ does not have solutions $\mathbf{x}_k$ that are periodic in k with period 6. So, it must be false to claim that, "$(\star)$ has solutions $\mathbf{x}_k$ that are periodic in k with period 6, that is, $\mathbf{x}_{k+6} \equiv \mathbf{x}_k$."

5.7.7.19. Equivalent to the scalar difference equation $y_{k+2} = a_{1,k}y_{k+1} + a_{2,k}y_k$ is the LCCHSΔ $(\star)$ $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where $\mathbf{x}_k = [\,y_k \quad y_{k+1}\,]^T$ and, for all ℓ, $A_\ell = A \triangleq \begin{bmatrix}0 & 1\\ a_{2,k} & a_{1,k}\end{bmatrix}$ is in companion form.

Abel's Theorem 5.19 says that if $\mathbf{x}_k^{(1)}$ and $\mathbf{x}_k^{(2)}$ solve the LCCHSΔ $(\star)$ $\mathbf{x}_{k+1} = A\mathbf{x}_k$ and $C(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}) \triangleq \left| \mathbf{x}_k^{(1)} \ \vdots\ \mathbf{x}_k^{(2)} \right|$, then

$$C(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}) = \left(\prod_{\ell=0}^{k-1} |A_\ell| \right) C(\mathbf{x}_0^{(1)}, \mathbf{x}_0^{(2)})$$

for any $k \geq 1$.

Let's restate this in terms of $\mathbf{x}_k^{(j)} = [\, y_k^{(j)} \quad y_{k+1}^{(j)} \,]^T$, for $j = 1, 2$: We have that $C(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}) = \begin{vmatrix} y_k^{(1)} & y_k^{(2)} \\ y_{k+1}^{(1)} & y_{k+1}^{(2)} \end{vmatrix}$, so

$$\begin{vmatrix} y_k^{(1)} & y_k^{(2)} \\ y_{k+1}^{(1)} & y_{k+1}^{(2)} \end{vmatrix} = \left(\prod_{\ell=0}^{k-1} |A_\ell| \right) \begin{vmatrix} y_0^{(1)} & y_0^{(2)} \\ y_1^{(1)} & y_1^{(2)} \end{vmatrix} = \left(\prod_{\ell=0}^{k-1} (-a_{2,k}) \right) \begin{vmatrix} y_0^{(1)} & y_0^{(2)} \\ y_1^{(1)} & y_1^{(2)} \end{vmatrix},$$

which is (4.68) in the textbook if $n = 2$. This explains why Abel's Theorem 4.15 (in Section 4.6) for the scalar difference equation $y_{k+2} = a_{1,k} y_{k+1} + a_{2,k} y_k$ follows from Abel's Theorem 5.19 for the LCCHSΔ $(\star)$ $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

5.7.7.23. The system of problem 5.7.7.3 has eigenvalues $\lambda_1 = -1, \lambda_2 = 4$. Because $|\lambda_2| = 4 > 1$, the system is unstable.

5.7.7.25. The system of problem 5.7.7.5 has eigenvalues $\lambda_1 = i, \lambda_2 = -i$. Because $|\lambda_1| = |\lambda_2| = 1$ and the eigenvalues of A are not deficient (because the 2×2 matrix has two distinct eigenvalues), the system is neutrally stable.

5.7.7.27. Expanding the determinant along the third row,

$$0 = \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{2} - \lambda & 0 \\ 0 & 0 & \frac{1}{2} - \lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda \right) \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} - \lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda \right) \left(\left(\frac{1}{2} - \lambda \right)^2 - \left(\frac{1}{\sqrt{6}} \right)^2 \right),$$

so the system of problem 5.7.7.27 has eigenvalues $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2} - \frac{1}{\sqrt{6}}$, and $\lambda_3 = \frac{1}{2} + \frac{1}{\sqrt{6}}$. Because $|\lambda_1| = \frac{1}{2} < 1$, $|\lambda_2| = \frac{1}{2} - \frac{1}{\sqrt{6}} < \frac{1}{2} < 1$, and $|\lambda_3| = \frac{1}{2} + \frac{1}{\sqrt{6}} < \frac{1}{2} + \frac{1}{2} < 1$, the system is asymptotically stable.

Section 5.8.6

5.8.6.1. (a) $\dot{Z}(t)=\begin{bmatrix} e^{t-\cos t}(1+\sin t) & 0\\ e^{t} & 0\end{bmatrix}$, and

$$A(t)Z(t)=\begin{bmatrix} 1+\sin t & 0\\ e^{\cos t} & 0\end{bmatrix}\begin{bmatrix} e^{t-\cos t} & 0\\ e^{t} & 1\end{bmatrix}=\begin{bmatrix} (1+\sin t)e^{t-\cos t} & 0\\ e^{\cos t}e^{t-\cos t} & 0\end{bmatrix}=\begin{bmatrix} e^{t-\cos t}(1+\sin t) & 0\\ e^{t} & 0\end{bmatrix}=\dot{Z}(t).$$

Also, $\det\big(Z(t)\big)=\begin{vmatrix} e^{t-\cos t} & 0\\ e^{t} & 1\end{vmatrix}=e^{t-\cos t}$ is never zero.

By Theorem 5.5 in Section 5.2, $Z(t)$ is a fundamental matrix for $(\star)$ $\dot{\mathbf{x}}=A(t)\mathbf{x}$.

(b) The principal fundamental matrix at $t=0$ is

$$X(t)=Z(t)\big(Z(0)\big)^{-1}=\begin{bmatrix} e^{t-\cos t} & 0\\ e^{t} & 1\end{bmatrix}\begin{bmatrix} e^{-1} & 0\\ 1 & 1\end{bmatrix}^{-1}=\begin{bmatrix} e^{t-\cos t} & 0\\ e^{t} & 1\end{bmatrix}\frac{1}{e^{-1}}\begin{bmatrix} 1 & 0\\ -1 & e^{-1}\end{bmatrix}$$

$$=\begin{bmatrix} e^{1+t-\cos t} & 0\\ e\cdot(e^{t}-1) & 1\end{bmatrix}$$

In this problem, the period of the coefficient matrix $A(t)=\begin{bmatrix} 1+\sin t & 0\\ e^{\cos t} & 0\end{bmatrix}$ is $\mathsf{T}=2\pi$. The monodromy matrix is

$$X(2\pi)=\begin{bmatrix} e^{1+2\pi-\cos 2\pi} & 0\\ e\cdot(e^{2\pi}-1) & 1\end{bmatrix}=\begin{bmatrix} e^{2\pi} & 0\\ e\cdot(e^{2\pi}-1) & 1\end{bmatrix}.$$

Its eigenvalues are the characteristic multipliers μ, which satisfy

$$0=\begin{vmatrix} e^{2\pi}-\mu & 0\\ e\cdot(e^{2\pi}-1) & 1-\mu\end{vmatrix}=(e^{2\pi}-\mu)(1-\mu),$$

hence $\mu_1=1$ and $\mu_2=e^{2\pi}$. We also need to find the corresponding eigenvectors.

$[\,X(2\pi)-\mu_1 I\mid \mathbf{0}\,]=\left[\begin{array}{cc|c} e^{2\pi}-1 & 0 & 0\\ e\cdot(e^{2\pi}-1) & 0 & 0\end{array}\right]\sim\left[\begin{array}{cc|c} 1 & 0 & 0\\ 0 & 0 & 0\end{array}\right]$, after $-e\cdot R_1+R_2\to R_2$, $\frac{1}{e^{2\pi}-1}R_1\to R_1$

$\Rightarrow \mathbf{v}_1=c_1\begin{bmatrix}0\\1\end{bmatrix}$, for any constant $c_1\neq 0$, are the eigenvectors corresponding to eigenvalue $\mu_1=1$.

$[\,X(2\pi)-\mu_2 I\mid \mathbf{0}\,]=\left[\begin{array}{cc|c} 0 & 0 & 0\\ e\cdot(e^{2\pi}-1) & 1-e^{2\pi} & 0\end{array}\right]\sim\left[\begin{array}{cc|c} 1 & -e^{-1} & 0\\ 0 & 0 & 0\end{array}\right]$, after $R_1\leftrightarrow R_2$, $\frac{1}{e\cdot(e^{2\pi}-1)}R_1\to R_1$

$\Rightarrow \mathbf{v}_2=c_1\begin{bmatrix}1\\e\end{bmatrix}$, for any constant $c_1\neq 0$, are the eigenvectors corresponding to eigenvalue $\mu_2=e^{2\pi}$.

Let $D=diag(\mu_1,\mu_2)=(1,e^{2\pi})$ and $Q=[\,\mathbf{v}_1\;\vdots\;\mathbf{v}_2\,]=\begin{bmatrix}0 & 1\\ 1 & e\end{bmatrix}$. As in the discussion on pages 441-442, we can use these results to calculate

$$E\triangleq diag\left(\frac{1}{2\pi}\ln(\mu_1),\frac{1}{2\pi}\ln(\mu_2)\right)=diag\big(0,1\big),$$

and

$$C=QEQ^{-1}=\begin{bmatrix}0 & 1\\ 1 & e\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & e\end{bmatrix}^{-1}=\begin{bmatrix}0 & 1\\ 1 & e\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-e & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ e & 0\end{bmatrix}.$$

Using eigenvalues and eigenvectors of C, or using the MacLaurin series on page 373 in Section 5.2, we can calculate e^{tC}. Note that $C^2=C$, so

$$e^{tC}=I+tC+\frac{t^2}{2!}C^2+\frac{t^3}{3!}C^3+...=I+tC+\frac{1}{2!}t^2C^2+\frac{1}{3!}t^3C^3+...=I-C+(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...)C$$

$$= I - C + e^t C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ e & 0 \end{bmatrix} + e^t \begin{bmatrix} 1 & 0 \\ e & 0 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ e\cdot(e^t-1) & 1 \end{bmatrix}.$$

We have

$$P(t) = X(t)e^{(-t)C} = \begin{bmatrix} e^{1+t-\cos t} & 0 \\ e\cdot(e^t-1) & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ e\cdot(e^{-t}-1) & 1 \end{bmatrix} = \begin{bmatrix} e^{1-\cos t} & 0 \\ 0 & 1 \end{bmatrix}.$$

The Floquet representation is

$$X(t) = P(t)e^{tC} = \begin{bmatrix} e^{1-\cos t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ e\cdot(e^t-1) & 1 \end{bmatrix}.$$

5.8.6.3. (a) For a system in $\mathbb{R}^1$, all of the matrices are 1×1. The principal fundamental matrix at $t=0$ for the scalar, linear, homogeneous ODE $\dot{y} - p(t)y = 0$ is $X(t) = [\, y_1(t)\,]$, where $y_1(t)$ solves

$$\left\{ \begin{array}{c} \dot{y} - p(t)y = 0 \\ y(0) = 1 \end{array} \right\}.$$

We can solve the ODE using the method of integrating factor $\nu(t) \triangleq \exp\left(\int_0^t -p(s)ds\right)$. [We call the integrating factor $\nu(t)$ instead of $\mu(t)$ so as not to confuse notations with characteristic multipliers.] Using the Fundamental Theorem of Calculus to find $\dot{\nu}(t)$, we have

$$\frac{d}{dt}\left[\,\nu(t)y\,\right] = \dot{y}\nu(t) + \dot{\nu}(t)y = \dot{y}\nu(t) - p(t)\nu(t)y = \nu(t)\big(\dot{y} - p(t)y\big) \equiv 0,$$

so $\nu(t)y(t) = c$, where c is an arbitrary constant. The solutions of the ODE are

$$y(t) = c\big(\nu(t)\big)^{-1} = c\exp\left(\int_0^t p(s)ds\right) \triangleq c y_1(t),$$

We have $X(t) = [\, y_1(t)\,] = \left[\exp\left(\int_0^t p(s)ds\right)\right]$. In this problem, the period of the 1×1 coefficient matrix $A(t) = [\,p(t)\,]$ is T. The monodromy matrix is $X(2\pi) = \left[\exp\left(\int_0^{\mathsf{T}} p(s)ds\right)\right]$. The only characteristic multiplier is $\mu_1 = \exp\left(\int_0^{\mathsf{T}} p(s)ds\right)$ and the corresponding eigenvector is $[\,1\,]$.

The 1×1 matrix C satisfies $e^{2\pi C} = X(2\pi) = \left[\exp\Big(\int_0^{\mathsf{T}} p(s)ds\Big)\right]$, so $C = \left[\int_0^{\mathsf{T}} p(s)ds\right]$, and

$$P(t) = X(t)\exp(-tC) = \left[\exp\Big(\int_0^t p(s)ds\Big)\right]\left[\exp\Big(-\int_0^T p(s)ds\Big)\right] = \left[\exp\Big(\int_0^t p(s)ds - \int_0^T p(s)ds\Big)\right]$$

$$= \left[\exp\Big(\int_0^t p(s)ds + \int_T^0 p(s)ds\Big)\right] = \left[\exp\int_T^t p(s)ds\right]$$

The Floquet representation is

$$X(t) = P(t)e^{tC} = \left[\exp\int_T^t p(s)ds\right]\left[\exp\Big(\int_0^T p(s)ds\Big)\right].$$

The only characteristic multiplier is $\mu_1 = \exp\left(\int_0^T p(t)dt\right)$.

The problem is a little vague as to whether the question, "Also, if $\int_0^{\mathsf{T}} p(t)dt < 0$, what can you say about the solutions $y(t)$?" refers to the homogeneous or to the non-homogeneous ODE. [The placement of the question coming after a question about the non- homogeneous ODE suggests that the "Also,..." question refers to the non- homogeneous ODE.] So, let's answer the question for both. [If someone is grading the solution of this problem, they should probably insist on having an answer for at least one of the two ODEs, either the homogeneous ODE or the non-homogeneous ODE.]

If $\int_0^{\mathsf{T}} p(t)dt < 0$, then the only characteristic multiplier satisfies $|\mu_1| < 1$, hence the homogeneous ODE is asymptotically stable.

(b) For the scalar, linear, nonhomogeneous ODE $\dot{y} - p(t)y = f(t)$, we can solve the ODE using the method of integrating factor $\nu(t) \triangleq \exp\left(\int_0^t -p(s)ds\right)$. Using the Fundamental Theorem of Calculus to find $\dot{\nu}(t)$, we have

$$\frac{d}{dt}\left[\,\nu(t)y\,\right] = \dot{y}\nu(t) + \dot{\nu}(t)y = \dot{y}\nu(t) - p(t)\nu(t)y = \nu(t)\big(\dot{y} - p(t)y\big) = \nu(t)f(t),$$

so $\nu(t)y(t) = c + \int_0^t \nu(s)f(s)ds$, where c is an arbitrary constant. The solutions of the ODE are

$$\begin{aligned} y(t) &= \frac{1}{\nu(t)}\left(c + \int_0^t \nu(s)f(s)ds\right) = \exp\left(\int_0^t p(s)ds\right)\left(c + \int_0^t \nu(s)f(s)ds\right) \\ &= cy_1(t) + \exp\left(\int_0^t p(s)ds\right)\int_0^t \nu(s)f(s)ds \triangleq cy_1(t) + y_p(t), \end{aligned}$$

where $y_1(t)$ was found in part (a) and c is an arbitrary constant.

Further, a natural question is whether the non-homogeneous ODE has a T-periodic solution. Using the fundamental matrix $X(t) = [\,y_1(t)\,]$, existence of a T-periodic solution $y(t)$ is equivalent to the existence of a solution of the scalar version of (5.114) in the textbook,

$$(1 - y_1(\mathsf{T}))\, y_0 = y_1(\mathsf{T}) \int_0^{\mathsf{T}} (y_1(s))^{-1} f(s)ds,$$

that is,

$$\left(1 - \exp\Big(\int_0^{\mathsf{T}} p(s)ds\Big)\right) y_0 = \exp\Big(\int_0^{\mathsf{T}} p(s)ds\Big)\int_0^{\mathsf{T}} \exp\Big(-\int_0^s p(u)du\Big)f(s)ds.$$

So, the necessary and sufficient condition for he non-homogeneous ODE to have a T-periodic solution is $1 - \exp\Big(\int_0^{\mathsf{T}} p(s)ds\Big) \neq 0$, that is,

$$\int_0^{\mathsf{T}} p(s)ds \neq 0.$$

In particular, if $\int_0^{\mathsf{T}} p(t)dt < 0$, then the non-homogeneous ODE has a T-periodic solution given by the scalar version of (5.115) in the textbook, that is,

$$y(t) = \exp\Big(\int_0^t p(s)ds\Big)\cdot\left(\int_0^t \exp\Big(-\int_0^s p(u)du\Big)f(s)ds + \frac{\exp\Big(\int_0^{\mathsf{T}} p(s)ds\Big)\cdot\int_0^{\mathsf{T}} \exp\Big(-\int_0^s p(u)du\Big)f(s)ds}{1 - \exp\Big(\int_0^{\mathsf{T}} p(s)ds\Big)}\right).$$

5.8.6.5. For the T-periodic Hill's equation (5.111), that is, $(\star)\ \ddot{y} + (\lambda + q(t))y = 0$, the corresponding system in $\mathbb{R}^2$ has principal fundamental matrix at $t = 0$ given by $X(t;\lambda) \triangleq \begin{bmatrix} y_1(t;\lambda) & y_2(t;\lambda) \\ \dot{y}_1(t;\lambda) & \dot{y}_2(t;\lambda) \end{bmatrix}$. It satisfies

$|X(t;\lambda)| \equiv 1$, so the characteristic multipliers μ satisfy

$$0 = |\,X(\mathsf{T};\lambda) - \mu I\,| = \begin{vmatrix} y_1(\mathsf{T};\lambda) - \mu & y_2(\mathsf{T};\lambda) \\ \dot{y}_1(\mathsf{T};\lambda) & \dot{y}_2(\mathsf{T};\lambda) - \mu \end{vmatrix} = (y_1(\mathsf{T};\lambda) - \mu)(\dot{y}_2(\mathsf{T};\lambda) - \mu) - y_2(\mathsf{T};\lambda)\dot{y}_1(\mathsf{T};\lambda)$$

$$= y_1(\mathsf{T};\lambda)\dot{y}_2(\mathsf{T};\lambda) - y_2(\mathsf{T};\lambda)\dot{y}_1(\mathsf{T};\lambda) - \big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)\mu + \mu^2 = |X(\mathsf{T};\lambda)| - \big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)\mu + \mu^2$$
$$= 1 - \big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)\mu + \mu^2,$$

so

$$\mu_{1,2} = \frac{\big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big) \pm \sqrt{\big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)^2 - 4}}{2}.$$

Because we are assuming that $|\dot{y}_2(\mathsf{T};\lambda) + y_1(\mathsf{T};\lambda)| < 2$, we have that $\mu_{1,2}$ are not real (because $\sqrt{4 - \big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)^2} \neq 0$) and are the complex conjugate pair

$$\mu_{1,2} = \frac{\big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big) \pm i\sqrt{4 - \big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)^2}}{2},$$

whose moduli are

$$|\,\mu_{1,2}\,| = \frac{1}{2}\sqrt{\big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)^2 + 4 - \big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)^2} = \frac{1}{2}\sqrt{4} = 1.$$

Since there are only two characteristic multipliers and they are distinct, by Theorem (5.25)(b), the system is neutrally stable.

5.8.6.7. (a) If $X(\mathsf{T}) = I$ then all of the characteristic multipliers are equal to one and are not deficient, because the characteristic equation is $0 = |\,X(\mathsf{T}) - \mu\,| = |\,I - \mu I\,| = (1-\mu)^n$. We can construct the Floquet representation as on pages 441-442 of the textbook:

$$X(t) = P(t)e^{tC} = P(t)Qe^{tE}Q^{-1} = P(t)Qe^{tO}Q^{-1} = P(t)Qe^{O}Q^{-1} = P(t)Q\,I\,Q^{-1} = P(t),$$

because $E = diag\left(\frac{1}{\mathsf{T}}\ln(\mu_1), ..., \frac{1}{\mathsf{T}}\ln(\mu_n)\right) = diag\left(\frac{1}{\mathsf{T}}\ln(1), ..., \frac{1}{\mathsf{T}}\ln(1)\right) = diag(0, ..., 0)$. Because $X(t) = P(t)$ is periodic with period T, all solutions of $(\star)$ $\dot{\mathbf{x}} = A(t)\mathbf{x}$ are periodic with period T.

(b) The result of problem 5.2.5.27 says that, because $A(-t) \equiv -A(t)$, it follows that $X(t)$ is an *even* function, that is, satisfies $X(-t) \equiv X(t)$. Replace t by $\frac{\mathsf{T}}{2} - \mathsf{T}$ to see that $X\Big(-\big(\frac{\mathsf{T}}{2} - \mathsf{T}\big)\Big) \equiv X\Big(\frac{\mathsf{T}}{2} - \mathsf{T}\Big)$, that is, $X\big(\mathsf{T} - \frac{\mathsf{T}}{2}\big) \equiv X\big(\frac{\mathsf{T}}{2} - \mathsf{T}\big)$.

(c) Lemma 5.6 states that $X(t + \mathsf{T}) \equiv X(t)X(\mathsf{T})$. With $t = -\frac{\mathsf{T}}{2}$, this gives $X(-\frac{\mathsf{T}}{2} + \mathsf{T}) \equiv X(-\frac{\mathsf{T}}{2})X(\mathsf{T})$.

(d) Proceeding from the right hand side of the result of part (c), and then using the result of part (b), we have

$$X\Big(-\frac{\mathsf{T}}{2}\Big)X(\mathsf{T}) \equiv X\Big(-\frac{\mathsf{T}}{2} + \mathsf{T}\Big) \equiv X\Big(\frac{\mathsf{T}}{2} - \mathsf{T}\Big) \equiv X\Big(-\frac{\mathsf{T}}{2}\Big).$$

(e) Because $X(t)$ is a fundamental matrix, it is invertible for all t. In particular, $X\big(-\frac{\mathsf{T}}{2}\big)$ is invertible. Using the result of part (d), we have

$$X(\mathsf{T}) \equiv \bigg(X\Big(-\frac{\mathsf{T}}{2}\Big)\bigg)^{-1} X\Big(-\frac{\mathsf{T}}{2}\Big)X(\mathsf{T}) \equiv \bigg(X\Big(-\frac{\mathsf{T}}{2}\Big)\bigg)^{-1} X\Big(-\frac{\mathsf{T}}{2}\Big) = I.$$

Using the result of part (a), this implies that all of the solutions of $(\star)$ $\dot{\mathbf{x}} = A(t)\mathbf{x}$ are periodic with period T.

5.8.6.9. The system is periodic with period $\mathsf{T} = 2\pi$. The first equation in the system, $\dot{x}_1 = \left(-1 - \frac{\cos t}{2 + \sin t}\right)x_1$, is solvable using the method of separation of variables:

$$\ln|x_1| = c + \int \left(-1 - \frac{\cos t}{2 + \sin t}\right)dt = -t - \ln|2 + \sin t|,$$

where c is an arbitrary constant, yields $x_1(t) = c_1 e^{-t}\,(2+\sin t)^{-1}$. The initial value is $x_1(0) = \frac{1}{2}\,c_1$, so the general solution of the first ODE in the system can be written as

$$x_1(t) = e^{-t} \cdot \frac{2}{2+\sin t} \cdot x_1(0).$$

Substitute that into the second ODE in the system, rearrange terms to put it into the standard form of a first order linear ODE, $\dot{x}_2 + x_2 = 2(\cos t) \cdot e^{-t} \cdot \frac{2}{2+\sin t} \cdot x_1(0)$. After that, multiply through by the integrating factor of e^t to get

$$\frac{d}{dt}\left[\, e^t x_2 \,\right] = e^t(\dot{x}_2 + x_2) = e^t \cdot e^{-t} \cdot \frac{4\cos t}{2+\sin t} \cdot x_1(0) = \frac{4\cos t}{2+\sin t} \cdot x_1(0).$$

Indefinite integration with respect to t of both sides yields

$$e^t x_2(t) = c_2 + 4\ln(2+\sin t)x_1(0),$$

so

$$x_2(t) = e^{-t}\Big(c_2 + 4\ln(2+\sin t)x_1(0)\Big).$$

The initial value is

$$x_2(0) = c_2 + (4\ln 2)x_1(0),$$

so the solutions of the second ODE can be written in the form

$$x_2(t) = e^{-t}\Big(x_2(0) + 4\big(-\ln 2 + \ln(2+\sin t)\big)x_1(0)\Big) = e^{-t}\left(x_2(0) + 4\ln\Big(\frac{2+\sin t}{2}\Big)x_1(0)\right).$$

To find a fundamental matrix, first summarize the general solution by writing it in matrix times vector form:

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} \cdot \frac{2}{2+\sin t} \cdot x_1(0) \\ \\ e^{-t}\left(x_2(0) + 4\ln\Big(\frac{2+\sin t}{2}\Big)x_1(0)\right) \end{bmatrix} = e^{-t}\begin{bmatrix} \dfrac{2}{2+\sin t} & 0 \\ \\ 4\ln\Big(\dfrac{2+\sin t}{2}\Big) & 1 \end{bmatrix}\mathbf{x}(0).$$

So, a fundamental matrix is given by

$$X(t) = e^{-t}\begin{bmatrix} \dfrac{2}{2+\sin t} & 0 \\ \\ 4\ln\Big(\dfrac{2+\sin t}{2}\Big) & 1 \end{bmatrix}.$$

In particular, substitute in $t = 2\pi$ to see that in this problem $X(\mathsf{T}) = X(2\pi) = e^{-2\pi}I$, so the characteristic multipliers, being the eigenvalues of $X(\mathsf{T})$, are $\mu = e^{-2\pi}, e^{-2\pi}$. We can take $Q = I, D = e^{-2\pi}I$ in the calculation of the Floquet representation, hence $S = Q = I$,

$$E = diag\left(\frac{1}{2\pi}\ln(e^{-2\pi}), \frac{1}{2\pi}\ln(e^{-2\pi})\right) = -I,$$

and $C = -I$. The Floquet representation here is $X(t) = P(t)e^{tC} = P(t)\,(e^{-t}I)$, so in this example

$$P(t) = e^t X(t) = \begin{bmatrix} \dfrac{2}{2+\sin t} & 0 \\ \\ 4\ln\Big(\dfrac{2+\sin t}{2}\Big) & 1 \end{bmatrix}$$

To summarize, a Floquet representation is given by

$$X(t) = P(t)e^{tC} = \left(\begin{bmatrix} \dfrac{2}{2+\sin t} & 0 \\ \\ 4\ln\left(\dfrac{2+\sin t}{2}\right) & 1 \end{bmatrix} \right) \left(e^{-t}I\right).$$

5.8.6.11. Using the product rule and the fact that $\dfrac{d}{dt}\left[\, e^{tC}\,\right] = Ce^{tC}$, we calculate $\dot{X}(t) = \dot{P}(t)e^{tC} + P(t)Ce^{tC}$. Using the information given about $P(t)$, C, A_0, and Ω, along with (5.29) in Section 5.2 in the textbook, we have

$$\dot{X}(t) = \left(\frac{d}{dt}\left[\, e^{t\Omega}\,\right]\right)e^{tC} + e^{t\Omega}Ce^{tC} = \Omega e^{t\Omega}e^{tC} + e^{t\Omega}Ce^{tC} = e^{t\Omega}\Omega e^{tC} + e^{t\Omega}Ce^{tC} = e^{t\Omega}(\Omega + C)e^{tC}$$

$$= e^{t\Omega}(\Omega + A_0 - \Omega)e^{tC} = e^{t\Omega}A_0e^{tC},$$

while, on the other hand, we have

$$A(t)X(t) = (e^{t\Omega}A_0e^{-t\Omega})e^{t\Omega}e^{tC} = e^{t\Omega}A_0(e^{-t\Omega}e^{t\Omega})e^{tC} = e^{t\Omega}A_0(I)e^{tC} = e^{t\Omega}A_0e^{tC}.$$

So, $\dot{X}(t) = A(t)X(t)$. Moreover, $|X(t)| = \left|e^{t\Omega}e^{tC}\right| = \left|e^{t\Omega}\right|\;\left|e^{tC}\right| \neq 0$. So, $X(t) \triangleq e^{t\Omega}e^{tC}$ is a fundamental matrix of solutions for $\dot{\mathbf{x}} = A(t)\mathbf{x}$. In fact, $X(t)$ is the principal fundamental matrix of solutions at $t = 0$, because $X(0) = e^{0\cdot\Omega}e^{0\cdot C} = I \cdot I = I$.

Because $P(t) \triangleq e^{t\Omega}$ is T-periodic and C is a constant matrix, $X(t) = P(t)e^{tC}$ is a Floquet representation for $\dot{\mathbf{x}} = A(t)\mathbf{x}$.

Note: At *no* time in the above work did we assume that the matrices A_0 and Ω commute, that is, we did *not* assume that $A_0\Omega = \Omega A_0$. So, at no time did we claim, without justification, that e^{tC} could be rewritten as $e^{tA_0}e^{-tC}$ or as $e^{-tC}e^{tA_0}$.

5.8.6.13. We will study this problem in nine cases:

(1) $\delta+\epsilon>0$ and $\delta-\epsilon>0$, (2) $\delta+\epsilon>0$ and $\delta-\epsilon=0$, (3) $\delta+\epsilon>0$ and $\delta-\epsilon<0$,

(4) $\delta+\epsilon=0$ and $\delta-\epsilon>0$, (5) $\delta+\epsilon=0$ and $\delta-\epsilon=0$, (6) $\delta+\epsilon=0$ and $\delta-\epsilon<0$,

(7) $\delta+\epsilon<0$ and $\delta-\epsilon>0$, (8) $\delta+\epsilon<0$ and $\delta-\epsilon=0$, (9) $\delta+\epsilon<0$ and $\delta-\epsilon<0$.

In Cases (1), (2), and (3), we assume $\delta+\epsilon>0$ and denote $\omega \triangleq \sqrt{\delta+\epsilon}$. Let $y_1(t;\delta,\epsilon)$ solve $(a)\quad \ddot{y} + (\delta+\epsilon)y = 0,\ 0<t<\pi$ with initial data $y_1(0;\delta,\epsilon)=1, \dot{y}_1(0;\delta,\epsilon)=0$. The general solution of the ODE is $y(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$, so the ICs require $1 = y(0) = c_1$ and $0 = \dot{y}(0) = \omega c_2$. These conditions imply $c_1 = 1$ and $c_2 = 0$, so $y_1(t;\delta,\epsilon) = \cos(\omega t)$.

Let $y_2(t;\delta,\epsilon)$ solve $(a)\quad \ddot{y} + (\delta+\epsilon)y = 0,\ 0<t<\pi$ with initial data $y_2(0;\delta,\epsilon)=0, \dot{y}_2(0;\delta,\epsilon)=1$. The general solution of the ODE is $y(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$, so the ICs require $0 = y(0) = c_1$ and $1 = \dot{y}(0) = \omega c_2$. These conditions imply $c_1 = 0$ and $c_2 = \omega^{-1}$, so $y_2(t;\delta,\epsilon) = \omega^{-1}\sin(\omega t)$.

Case 1: We assume $\delta+\epsilon>0$. We also assume $\delta-\epsilon>0$ and denote $\nu = \sqrt{\delta-\epsilon}$. The general solution of the ODE (b) $\ddot{y} + (\delta-\epsilon)y = 0$ on the interval $\pi<t<2\pi$ can be written as $y(t;\delta,\epsilon) = \mathit{constant}\cdot\cos\big(\nu(t-\pi)\big) + \mathit{constant}\cdot\sin\big(\nu(t-\pi)\big)$.

The ODE implies $y_1(t;\delta,\epsilon) = A\cos\big(\nu(t-\pi)\big) + B\sin\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_1(t) = \cos(\omega t)$ and its first derivative, $\dot{y}_1(t) = -\omega\sin(\omega t)$, for $0<t<\pi$, requires

$$\cos(\omega\pi) = y_1(\pi;\delta,\epsilon) = A \quad \text{and} \quad -\omega\sin(\omega\pi) = \dot{y}_1(\pi;\delta,\epsilon) = B\nu.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon) = \left\{ \begin{array}{ll} \cos(\omega t), & 0<t<\pi \\ \\ \cos(\omega\pi)\cos\big(\nu(t-\pi)\big) - \frac{\omega}{\nu}\,\sin(\omega\pi)\sin\big(\nu(t-\pi)\big), & \pi<t<2\pi \end{array} \right\},$$

Note that $\dot{y}_1(t) = -\nu\cos(\omega\pi)\sin\big(\nu(t-\pi)\big) - \omega\sin(\omega\pi)\cos\big(\nu(t-\pi)\big)$, for $\pi < t < 2\pi$.

The ODE implies $y_2(t;\delta,\epsilon) = c_1\cos\big(\nu(t-\pi)\big) + c_2\sin\big(\nu(t-\pi)\big)$ on $\pi < t < 2\pi$, so continuity of the function $y_2(t) = \omega^{-1}\sin(\omega t)$ and its first derivative, $\dot{y}_2(t) = \cos(\omega t)$, for $0 < t < \pi$, requires

$$\omega^{-1}\sin(\omega\pi) = y_2(\pi;\delta,\epsilon) = c_1 \quad \text{and} \quad \cos(\omega\pi) = \dot{y}_2(\pi;\delta,\epsilon) = c_2\nu.$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon) = \left\{\begin{array}{ll} \frac{1}{\omega}\sin(\omega t), & 0 < t < \pi \\ \\ \frac{1}{\omega}\sin(\omega\pi)\cos\big(\nu(t-\pi)\big) + \frac{1}{\nu}\cos(\omega\pi)\sin\big(\nu(t-\pi)\big), & \pi < t < 2\pi \end{array}\right\},$$

Note that $\dot{y}_2(t) = -\frac{\nu}{\omega}\sin(\omega\pi)\sin\big(\nu(t-\pi)\big) + \cos(\omega\pi)\cos\big(\nu(t-\pi)\big)$, for $\pi < t < 2\pi$.

In this case, $X(t)$, the principal fundamental matrix at $t = 0$, satisfies

$$X(2\pi) - \mu I = \begin{bmatrix} \cos(\omega\pi)\cos(\nu\pi) - \frac{\omega}{\nu}\sin(\omega\pi)\sin(\nu\pi) - \mu & \frac{1}{\omega}\sin(\omega\pi)\cos(\nu\pi) + \frac{1}{\nu}\cos(\omega\pi)\sin(\nu\pi) \\ \\ -\nu\cos(\omega\pi)\sin(\nu\pi) - \omega\sin(\omega\pi)\cos(\nu\pi) & -\frac{\nu}{\omega}\sin(\omega\pi)\sin(\nu\pi) + \cos(\omega\pi)\cos(\nu\pi) - \mu \end{bmatrix},$$

so the characteristic equation is

$$\begin{aligned} 0 &= \Big(\cos(\omega\pi)\cos(\nu\pi) - \frac{\omega}{\nu}\sin(\omega\pi)\sin(\nu\pi) - \mu\Big)\Big(-\frac{\nu}{\omega}\sin(\omega\pi)\sin(\nu\pi) + \cos(\omega\pi)\cos(\nu\pi) - \mu\Big) \\ &\quad - \Big(\frac{1}{\omega}\sin(\omega\pi)\cos(\nu\pi) + \frac{1}{\nu}\cos(\omega\pi)\sin(\nu\pi)\Big)\big(-\nu\cos(\omega\pi)\sin(\nu\pi) - \omega\sin(\omega\pi)\cos(\nu\pi)\big) \\ &= \mu^2 - \mu\Big(\cos(\omega\pi)\cos(\nu\pi) - \frac{\omega}{\nu}\sin(\omega\pi)\sin(\nu\pi) - \frac{\nu}{\omega}\sin(\omega\pi)\sin(\nu\pi) + \cos(\omega\pi)\cos(\nu\pi)\Big) \\ &\quad -\frac{\nu}{\omega}\cos(\omega\pi)\sin(\omega\pi)\cos(\nu\pi)\sin(\nu\pi) + \sin^2(\omega\pi)\sin^2(\nu\pi) + \cos^2(\omega\pi)\cos^2(\nu\pi) \\ &\quad -\frac{\omega}{\nu}\cos(\omega\pi)\sin(\omega\pi)\cos(\nu\pi)\sin(\nu\pi) + \sin^2(\omega\pi)\cos^2(\nu\pi) + \cos^2(\omega\pi)\sin^2(\nu\pi) \\ &\quad +\frac{\nu}{\omega}\cos(\omega\pi)\sin(\omega\pi)\cos(\nu\pi)\sin(\nu\pi) + \frac{\omega}{\nu}\cos(\omega\pi)\sin(\omega\pi)\cos(\nu\pi)\sin(\nu\pi) \end{aligned}$$

Using $\cos^2\theta + \sin^2\theta \equiv 1$ and canceling four of the terms, this becomes

$$0 = \mu^2 - \mu\left(2\cos(\omega\pi)\cos(\nu\pi) - \left(\frac{\omega}{\nu} + \frac{\nu}{\omega}\right)\sin(\omega\pi)\sin(\nu\pi)\right) + 1$$

that is,

$$0 = \mu^2 - \mu\left(2\cos(\omega\pi)\cos(\nu\pi) - \frac{2\delta}{\sqrt{\delta^2 - \epsilon^2}}\sin(\omega\pi)\sin(\nu\pi)\right) + 1,$$

So, in the case that $\delta + \epsilon > 0$ and $\delta - \epsilon > 0$, the characteristic multipliers are

$$\mu = \cos(\omega\pi)\cos(\nu\pi) - \frac{\delta}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sin(\nu\pi) \pm \sqrt{\left(\cos(\omega\pi)\cos(\nu\pi) - \frac{\delta}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sin(\nu\pi)\right)^2 - 1}\,.$$

By the way, the constant term, 1, in the characteristic equation follows from the fact that $|\,X(\mathsf{T}\,| = 1$, which follows from Abel's Theorem 5.7 in Section 5.2.

Case 2: We assume $\delta+\epsilon > 0$. We also assume $\delta-\epsilon = 0$ hence the general solution of the ODE (b) $\ddot{y} + (\delta-\epsilon)y = 0$ on the interval $\pi < t < 2\pi$ can be written as $y_1(t;\delta,\epsilon) = c_1 + c_2(t-\pi)$. So, continuity of the function $y_1(t) = \cos(\omega t)$ and its first derivative, $\dot{y}_1(t) = -\omega\sin(\omega t)$, for $0 < t < \pi$, requires

$$\cos(\omega\pi) = y_1(\pi;\delta,\epsilon) = c_1 \quad \text{and} \quad -\omega\sin(\omega\pi) = \dot{y}_1(\pi;\delta,\epsilon) = c_2.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon) = \left\{ \begin{array}{ll} \cos(\omega t), & 0 < t < \pi \\ \\ \cos(\omega\pi) - \omega\ \sin(\omega\pi)\,(t-\pi), & \pi < t < 2\pi \end{array} \right\},$$

Note that $\dot{y}_1(t) = -\omega\ \sin(\omega\pi)$, for $\pi < t < 2\pi$.

The general solution of the ODE (b) $\ddot{y} + (\delta - \epsilon)y = 0$ on the interval $\pi < t < 2\pi$ can be written as $y_2(t;\delta,\epsilon) = A + B(t-\pi)$. Continuity of the function $y_2(t) = \omega^{-1}\sin(\omega t)$ and its first derivative, $\dot{y}_2(t) = \cos(\omega t)$, for $0 < t < \pi$, requires

$$\omega^{-1}\sin(\omega\pi) = y_2(\pi;\delta,\epsilon) = A \quad \text{and} \quad \cos(\omega\pi) = \dot{y}_2(\pi;\delta,\epsilon) = B$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon) = \left\{ \begin{array}{ll} \frac{1}{\omega}\ \sin(\omega t), & 0 < t < \pi \\ \\ \frac{1}{\omega}\ \sin(\omega\pi) + \cos(\omega\pi)(t-\pi), & \pi < t < 2\pi \end{array} \right\},$$

Note that $\dot{y}_1(t) = \cos(\omega\pi)$, for $\pi < t < 2\pi$.

If $\delta + \epsilon > 0$ and $\delta = \epsilon$, then

$$X(2\pi) - \mu I = \begin{bmatrix} \cos(\omega\pi) - \pi\omega\ \sin(\omega\pi) - \mu & \frac{1}{\omega}\ \sin(\omega\pi) + \pi\cos(\omega\pi) \\ \\ -\omega\sin(\omega\pi) & \cos(\omega\pi) - \mu \end{bmatrix}$$

so the characteristic equation is

$$0 = (\cos(\omega\pi) - \pi\omega\ \sin(\omega\pi)\ -\ \mu)\,(\cos(\omega\pi)\ -\ \mu) - \left(\frac{1}{\omega}\ \sin(\omega\pi) + \pi\cos(\omega\pi)\right)(-\omega\sin(\omega\pi))$$

$$= \mu^2 - \mu\Big(\cos(\omega\pi) - \pi\omega\ \sin(\omega\pi) + \cos(\omega\pi)\Big) + \cos^2(\omega\pi) - \cancel{\pi\omega\cos(\omega\pi)\sin(\omega\pi)} + \sin^2(\omega\pi) + \cancel{\pi\omega\cos(\omega\pi)\sin(\omega\pi)}$$

$$= \mu^2 - \mu\Big(2\cos(\omega\pi) - \pi\omega\sin(\omega\pi)\Big) + 1.$$

So, in the case that $\delta + \epsilon > 0$ and $\delta - \epsilon > 0$, the characteristic multipliers are

$$\mu = \cos(\omega\pi) \pm \frac{1}{2}\sqrt{\Big(2\cos(\omega\pi) - \pi\omega\sin(\omega\pi)\Big)^2 - 4}\,.$$

Case 3: We assume $\delta + \epsilon > 0$. We also assume $\delta - \epsilon < 0$ and denote $\nu = \sqrt{\epsilon - \delta}$. The general solution of the ODE (b) $\ddot{y} + (\delta - \epsilon)y = 0$ on the interval $\pi < t < 2\pi$ can be written as $y(t;\delta,\epsilon) = \mathit{constant}\cdot\cosh\big(\nu(t-\pi)\big) + \mathit{constant}\cdot\sinh\big(\nu(t-\pi)\big)$.

The ODE implies $y_1(t;\delta,\epsilon) = A\cosh\big(\nu(t-\pi)\big) + B\sinh\big(\nu(t-\pi)\big)$ on $\pi < t < 2\pi$, so continuity of the function $y_1(t) = \cos(\omega t)$ and its first derivative, $\dot{y}_1(t) = -\omega\sin(\omega t)$, for $0 < t < \pi$, requires

$$\cos(\omega\pi) = y_1(\pi;\delta,\epsilon) = A \quad \text{and} \quad -\omega\sin(\omega\pi) = \dot{y}_1(\pi;\delta,\epsilon) = B\nu.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon) = \left\{ \begin{array}{ll} \cos(\omega t), & 0 < t < \pi \\ \\ \cos(\omega\pi)\cosh\big(\nu(t-\pi)\big) - \frac{\omega}{\nu}\ \sin(\omega\pi)\sinh\big(\nu(t-\pi)\big), & \pi < t < 2\pi \end{array} \right\},$$

Note that $\dot{y}_1(t) = \nu\cos(\omega\pi)\sinh\big(\nu(t-\pi)\big) - \omega\sin(\omega\pi)\cosh\big(\nu(t-\pi)\big)$, for $\pi < t < 2\pi$.

The ODE implies $y_2(t;\delta,\epsilon)=c_1\cosh\big(\nu(t-\pi)\big)+c_2\sinh\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_2(t)=\omega^{-1}\sin(\omega t)$ and its first derivative, $\dot y_2(t)=\cos(\omega t)$, for $0<t<\pi$, requires

$$\omega^{-1}\sin(\omega\pi)=y_2(\pi;\delta,\epsilon)=c_1\quad\text{and}\quad\cos(\omega\pi)=\dot y_2(\pi;\delta,\epsilon)=c_2\nu.$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon)=\left\{\begin{array}{ll}\frac{1}{\omega}\sin(\omega t), & 0<t<\pi\\ \\ \frac{1}{\omega}\sin(\omega\pi)\cosh\big(\nu(t-\pi)\big)+\frac{1}{\nu}\cos(\omega\pi)\sinh\big(\nu(t-\pi)\big), & \pi<t<2\pi\end{array}\right\},$$

Note that $\dot y_2(t)=\frac{\nu}{\omega}\sin(\omega\pi)\sinh\big(\nu(t-\pi)\big)+\cos(\omega\pi)\cosh\big(\nu(t-\pi)\big)$, for $\pi<t<2\pi$.

If $\delta+\epsilon>0$ and $\delta-\epsilon<0$, then

$$X(2\pi)-\mu I=\begin{bmatrix}\cos(\omega\pi)\cosh(\nu\pi)-\frac{\omega}{\nu}\sin(\omega\pi)\sinh(\nu\pi)-\mu & \frac{1}{\omega}\sin(\omega\pi)\cosh(\nu\pi)+\frac{1}{\nu}\cos(\omega\pi)\sinh(\nu\pi)\\ \\ \nu\cos(\omega\pi)\sinh(\nu\pi)-\omega\sin(\omega\pi)\cosh(\nu\pi) & \frac{\nu}{\omega}\sin(\omega\pi)\sinh(\nu\pi)+\cos(\omega\pi)\cosh(\nu\pi)-\mu\end{bmatrix}$$

so the characteristic equation is

$$\begin{aligned}0=&\Big(\cos(\omega\pi)\cosh(\nu\pi)-\frac{\omega}{\nu}\sin(\omega\pi)\sinh(\nu\pi)-\mu\Big)\Big(\frac{\nu}{\omega}\sin(\omega\pi)\sinh(\nu\pi)+\cos(\omega\pi)\cosh(\nu\pi)-\mu\Big)\\ &-\Big(\frac{1}{\omega}\sin(\omega\pi)\cosh(\nu\pi)+\frac{1}{\nu}\cos(\omega\pi)\sinh(\nu\pi)\Big)\big(\nu\cos(\omega\pi)\sinh(\nu\pi)-\omega\sin(\omega\pi)\cosh(\nu\pi)\big)\\ =&\ \mu^2-\mu\Big(\cos(\omega\pi)\cosh(\nu\pi)-\frac{\omega}{\nu}\sin(\omega\pi)\sinh(\nu\pi)+\frac{\nu}{\omega}\sin(\omega\pi)\sinh(\nu\pi)+\cos(\omega\pi)\cosh(\nu\pi)\Big)\\ &+\frac{\nu}{\omega}\cos(\omega\pi)\sin(\omega\pi)\cosh(\nu\pi)\sinh(\nu\pi)-\sin^2(\omega\pi)\sinh^2(\nu\pi)+\cos^2(\omega\pi)\cosh^2(\nu\pi)\\ &-\frac{\omega}{\nu}\cos(\omega\pi)\sin(\omega\pi)\cosh(\nu\pi)\sinh(\nu\pi)+\sin^2(\omega\pi)\cosh^2(\nu\pi)-\cos^2(\omega\pi)\sinh^2(\nu\pi)\\ &-\frac{\nu}{\omega}\cos(\omega\pi)\sin(\omega\pi)\cosh(\nu\pi)\sinh(\nu\pi)+\frac{\omega}{\nu}\cos(\omega\pi)\sin(\omega\pi)\cosh(\nu\pi)\sinh(\nu\pi).\end{aligned}$$

Using $\cosh^2\theta-\sinh^2\theta\equiv 1$ and $\cos^2\theta+\sin^2\theta\equiv 1$, this becomes

$$0=\mu^2-\mu\left(2\cos(\omega\pi)\cosh(\nu\pi)-\left(\frac{\omega}{\nu}-\frac{\nu}{\omega}\right)\sin(\omega\pi)\sinh(\nu\pi)\right)+1$$

that is,

$$0=\mu^2-\mu\left(2\cos(\omega\pi)\cosh(\omega\pi)-\frac{2\epsilon}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sinh(\omega\pi)\right)+1,$$

So, in the case that $\delta+\epsilon>0$ and $\delta-\epsilon>0$, the characteristic multipliers are

$$\mu=\cos(\omega\pi)\cosh(\nu\pi)-\frac{\epsilon}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sinh(\nu\pi)\pm\sqrt{\left(\cos(\omega\pi)\cosh(\nu\pi)-\frac{\epsilon}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sinh(\nu\pi)\right)^2-1}\,.$$

In Cases (4), (5), and (6), we assume $\delta+\epsilon=0$. Let $y_1(t;\delta,\epsilon)$ solve $(a)\quad \ddot y+(0)y=0,\ 0<t<\pi$ with initial data $y_1(0;\delta,\epsilon)=1,\dot y_1(0;\delta,\epsilon)=0$. The general solution of the ODE is $y(t)=c_1+c_2t$, so the ICs require $1=y(0)=c_1$ and $0=\dot y(0)=c_2$, so $y_1(t;\delta,\epsilon)=1$.

Let $y_2(t;\delta,\epsilon)$ solve $(a)\quad \ddot y+(0)y=0,\ 0<t<\pi$ with initial data $y_2(0;\delta,\epsilon)=0,\dot y_2(0;\delta,\epsilon)=1$. The general solution of the ODE is $y(t)=c_1+c_2t$, so the ICs require $0=y(0)=c_1$ and $1=\dot y(0)=c_2$, so $y_2(t;\delta,\epsilon)=t$.

Case 4: We assume $\delta+\epsilon=0$. We also assume $\delta-\epsilon>0$ and denote $\nu \triangleq \sqrt{\delta-\epsilon}$. The general solution of the ODE (b) $\ddot{y}+(\delta-\epsilon)y=0$ on the interval $\pi<t<2\pi$ can be written as $y(t;\delta,\epsilon)=\textit{constant}\cdot\cos\big(\nu(t-\pi)\big)+\textit{constant}\cdot\sin\big(\nu(t-\pi)\big)$.

The ODE implies $y_1(t;\delta,\epsilon)=A\cos\big(\nu(t-\pi)\big)+B\sin\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_1(t)\equiv 1$ and its first derivative, $\dot{y}_1(t)=0$, for $0<t<\pi$, requires

$$1=y_1(\pi;\delta,\epsilon)=A \quad\text{and}\quad 0=\dot{y}_1(\pi;\delta,\epsilon)=B\nu.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon)=\left\{\begin{array}{ll}1, & 0<t<\pi\\ \\ \cos\big(\nu(t-\pi)\big), & \pi<t<2\pi\end{array}\right\},$$

Note that $\dot{y}_1(t;\delta,\epsilon)=-\nu\sin\big(\nu(t-\pi)\big)$ for $\pi<t<2\pi$.

The ODE implies $y_2(t;\delta,\epsilon)=c_1\cos\big(\nu(t-\pi)\big)+c_2\sin\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_2(t)=t$ and its first derivative, $\dot{y}_2(t)=1$, for $0<t<\pi$, requires

$$\pi=y_2(\pi;\delta,\epsilon)=c_1 \quad\text{and}\quad 1=\dot{y}_2(\pi;\delta,\epsilon)=c_2\nu.$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon)=\left\{\begin{array}{ll}t, & 0<t<\pi\\ \\ \pi\cos\big(\nu(t-\pi)\big)+\frac{1}{\nu}\sin\big(\nu(t-\pi)\big), & \pi<t<2\pi\end{array}\right\},$$

Note that $\dot{y}_2(t;\delta,\epsilon)=-\pi\nu\sin\big(\nu(t-\pi)\big)+\cos\big(\nu(t-\pi)\big)$ for $\pi<t<2\pi$.

In this case, $X(t)$, the principal fundamental matrix at $t=0$, satisfies

$$X(2\pi)-\mu I=\begin{bmatrix}\cos(\nu\pi)-\mu & \pi\cos(\nu\pi)+\frac{1}{\nu}\sin(\nu\pi)\\ \\ -\nu\sin(\nu\pi) & -\pi\nu\sin(\nu\pi)+\cos(\nu\pi)-\mu\end{bmatrix},$$

so the characteristic equation is

$$0=(\cos(\nu\pi)-\mu)\,(-\pi\nu\sin(\nu\pi)+\cos(\nu\pi)-\mu)-\left(\pi\cos(\nu\pi)+\frac{1}{\nu}\sin\nu\pi)\right)\Big(-\nu\sin(\nu\pi)\Big)$$

$$=\mu^2-\mu\Big(\cos(\nu\pi)-\pi\nu\sin(\nu\pi)+\cos(\nu\pi)\Big)-\pi\nu\sin(\nu\pi)\cos(\nu\pi)+\cos^2(\nu\pi)+\pi\nu\sin(\nu\pi)\cos(\nu\pi)+\sin^2(\nu\pi)$$

Using $\cos^2\theta+\sin^2\theta\equiv 1$, this becomes

$$0=\mu^2-\mu\Big(2\cos(\nu\pi)-\pi\nu\sin(\nu\pi)\Big)\mu+1.$$

So, in the case that $\delta+\epsilon=0$ and $\delta-\epsilon>0$, the characteristic multipliers are

$$\mu=\frac{2\cos(\nu\pi)-\pi\nu\sin(\nu\pi)\pm\sqrt{\Big(2\cos(\nu\pi)-\pi\nu\sin(\nu\pi)\Big)^2-4}}{2}.$$

By the way, the constant term, 1, in the characteristic equation follows from the fact that $|\,X(\mathsf{T}\,|=1$, which follows from Abel's Theorem 5.7 in Section 5.2.

Case 5: We assume $\delta+\epsilon=0$. We also assume $\delta-\epsilon=0$, hence the general solution of the ODE (b) $\ddot{y}+(0)y=0$ on the interval $\pi<t<2\pi$ can be written as $y_1(t;\delta,\epsilon)=c_1+c_2(t-\pi)$. So, continuity of the function $y_1(t)\equiv 1$ and its first derivative, $\dot{y}_1(t)=0$, for $0<t<\pi$, requires

$$1=y_1(\pi;\delta,\epsilon)=c_1 \quad\text{and}\quad 0=\dot{y}_1(\pi;\delta,\epsilon)=c_2.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon) = \left\{ \begin{array}{ll} 1, & 0<t<\pi \\ \\ 1, & \pi<t<2\pi \end{array} \right\} \equiv 1,$$

Note that $\dot y_1(t;\delta,\epsilon)=0$, for $\pi<t<2\pi$.

The general solution of the ODE (b) $\ddot y + (0)y = 0$ on the interval $\pi<t<2\pi$ can be written as $y_2(t;\delta,\epsilon) = c_1 + c_2(t-\pi)$, so continuity of the function $y_2(t)=t$ and its first derivative, $\dot y_2(t)=1$, for $0<t<\pi$, requires

$$\pi = y_2(\pi;\delta,\epsilon) = c_1 \quad \text{and} \quad 1 = \dot y_2(\pi;\delta,\epsilon) = c_2$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon) = \left\{ \begin{array}{ll} t, & 0<t<\pi \\ \\ \pi+(t-\pi), & \pi<t<2\pi \end{array} \right\} \equiv t,$$

Note that $\dot y_2(t;\delta,\epsilon)=1$, for $\pi<t<2\pi$.

If $\delta+\epsilon=0$ and $\delta=\epsilon$, that is, if $\delta=\epsilon=1$, then

$$X(2\pi)-\mu I = \left[\begin{array}{cc} 1-\mu & 2\pi \\ \\ 0 & 1-\mu \end{array} \right]$$

so the characteristic equation is

$$0 = (1-\mu)^2.$$

So, in the case that $\delta+\epsilon=0$ and $\delta=\epsilon$, the characteristic multipliers are

$$\mu = 1 \pm 0\,.$$

Case 6: We assume $\delta+\epsilon=0$. We also assume $\delta-\epsilon<0$ and denote $\nu=\sqrt{\epsilon-\delta}$. The general solution of the ODE (b) $\ddot y + (\delta-\epsilon)y = 0$ on the interval $\pi<t<2\pi$ can be written as $y(t;\delta,\epsilon) = \mathit{constant}\cdot\cosh\big(\nu(t-\pi)\big) + \mathit{constant}\cdot\sinh\big(\nu(t-\pi)\big)$.

The ODE implies $y_1(t;\delta,\epsilon) = A\cosh\big(\nu(t-\pi)\big) + B\sinh\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_1(t)\equiv 1$ and its first derivative, $\dot y_1(t)=0$, for $0<t<\pi$, requires

$$1 = y_1(\pi;\delta,\epsilon) = A \quad \text{and} \quad 0 = \dot y_1(\pi;\delta,\epsilon) = B\nu.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon) = \left\{ \begin{array}{ll} 1, & 0<t<\pi \\ \\ \cosh\big(\nu(t-\pi)\big), & \pi<t<2\pi \end{array} \right\},$$

Note that $\dot y_1(t;\delta,\epsilon) = \nu\sinh\big(\nu(t-\pi)\big)$, for $\pi<t<2\pi$.

The ODE implies $y_2(t;\delta,\epsilon) = c_1\cosh\big(\nu(t-\pi)\big) + c_2\sinh\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_2(t)=t$ and its first derivative, $\dot y_2(t)=1$, for $0<t<\pi$, requires

$$\pi = y_2(\pi;\delta,\epsilon) = c_1 \quad \text{and} \quad 1 = \dot y_2(\pi;\delta,\epsilon) = c_2\nu.$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon) = \left\{ \begin{array}{ll} t, & 0<t<\pi \\ \\ \pi\cosh\big(\nu(t-\pi)\big) + \frac{1}{\nu}\sinh\big(\nu(t-\pi)\big), & \pi<t<2\pi \end{array} \right\},$$

Note that $\dot{y}_2(t;\delta,\epsilon)=\pi\nu\sinh\big(\nu(t-\pi)\big)+\cosh\big(\nu(t-\pi)\big)$, for $\pi<t<2\pi$.

If $\delta+\epsilon>0$ and $\delta-\epsilon<0$, then

$$X(2\pi)-\mu I=\begin{bmatrix}\cosh(\nu\pi)-\mu & \pi\cosh(\nu\pi)+\frac{1}{\nu}\sinh(\nu\pi)\\ \nu\sinh(\nu\pi) & \pi\nu\sinh(\nu\pi)+\cosh(\nu\pi)-\mu\end{bmatrix}$$

so the characteristic equation is

$$0=\Big(\cosh(\nu\pi)-\mu\Big)\Big(\pi\nu\sinh(\nu\pi)+\cosh(\nu\pi)-\mu\Big)-\Big(\pi\cosh(\nu\pi)+\frac{1}{\nu}\sinh(\nu\pi)\Big)\Big(\nu\sinh(\nu\pi)\Big).$$

Using $\cosh^2\theta-\sinh^2\theta\equiv 1$ and $\cos^2\theta+\sin^2\theta\equiv 1$, this becomes

$$0=\mu^2-\mu\Big(2\cosh(\nu\pi)+\pi\nu\sinh(\nu\pi)\Big)+\cancel{\pi\nu\cosh(\nu\pi)\sinh(\nu\pi)}+\cosh^2(\nu\pi)-\cancel{\pi\nu\cosh(\nu\pi)\sinh(\nu\pi)}-\sinh^2(\nu\pi)$$

that is,

$$0=\mu^2-\mu\Big(2\cosh(\nu\pi)+\pi\nu\sinh(\nu\pi)\Big)+1.$$

So, in the case that $\delta+\epsilon>0$ and $\delta-\epsilon>0$, the characteristic multipliers are

$$\mu=\frac{2\cosh(\nu\pi)+\pi\nu\sinh(\nu\pi)\pm\sqrt{\Big(2\cosh(\nu\pi)+\pi\nu\sinh(\nu\pi)\Big)^2-4}}{2}.$$

In Cases (7), (8), and (9), we assume $\delta+\epsilon<0$ and denote $\omega\triangleq\sqrt{\delta+\epsilon}$. Let $y_1(t;\delta,\epsilon)$ solve $(a)\quad \ddot{y}+(\delta+\epsilon)y=0,\ 0<t<\pi$ with initial data $y_1(0;\delta,\epsilon)=1, \dot{y}_1(0;\delta,\epsilon)=0$. The general solution of the ODE is $y(t)=c_1\cosh(\omega t)+c_2\sinh(\omega t)$, so the ICs require $1=y(0)=c_1$ and $0=\dot{y}(0)=\omega c_2$. These conditions imply $c_1=1$ and $c_2=0$, so $y_1(t;\delta,\epsilon)=\cosh(\omega t)$.

Let $y_2(t;\delta,\epsilon)$ solve $(a)\quad \ddot{y}+(\delta+\epsilon)y=0,\ 0<t<\pi$ with initial data $y_2(0;\delta,\epsilon)=0, \dot{y}_2(0;\delta,\epsilon)=1$. The general solution of the ODE is $y(t)=c_1\cosh(\omega t)+c_2\sinh(\omega t)$, so the ICs require $0=y(0)=c_1$ and $1=\dot{y}(0)=\omega c_2$. These conditions imply $c_1=0$ and $c_2=\omega^{-1}$, so $y_2(t;\delta,\epsilon)=\omega^{-1}\sinh(\omega t)$.

Case 7: We assume $\delta+\epsilon<0$. We also assume $\delta-\epsilon>0$ and denote $\nu=\sqrt{\delta-\epsilon}$. The general solution of the ODE (b) $\ddot{y}+(\delta-\epsilon)y=0$ on the interval $\pi<t<2\pi$ can be written as $y(t;\delta,\epsilon)=\mathit{constant}\cdot\cos\big(\nu(t-\pi)\big)+\mathit{constant}\cdot\sin\big(\nu(t-\pi)\big)$.

The ODE implies $y_1(t;\delta,\epsilon)=A\cos\big(\nu(t-\pi)\big)+B\sin\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_1(t)=\cosh(\omega t)$ and its first derivative, $\dot{y}_1(t)=\omega\sinh(\omega t)$, for $0<t<\pi$, requires

$$\cosh(\omega\pi)=y_1(\pi;\delta,\epsilon)=A\quad\text{and}\quad\omega\sinh(\omega\pi)=\dot{y}_1(\pi;\delta,\epsilon)=B\nu.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon)=\left\{\begin{array}{ll}\cosh(\omega t), & 0<t<\pi\\ \cosh(\omega\pi)\cos\big(\nu(t-\pi)\big)+\frac{\omega}{\nu}\sinh(\omega\pi)\sin\big(\nu(t-\pi)\big), & \pi<t<2\pi\end{array}\right\},$$

Note that $\dot{y}_1(t)=-\nu\cosh(\omega\pi)\sin\big(\nu(t-\pi)\big)+\omega\sinh(\omega\pi)\cos\big(\nu(t-\pi)\big)$, for $\pi<t<2\pi$.

The ODE implies $y_2(t;\delta,\epsilon)=c_1\cos\big(\nu(t-\pi)\big)+c_2\sin\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_2(t)=\omega^{-1}\sinh(\omega t)$ and its first derivative, $\dot{y}_2(t)=\cosh(\omega t)$, for $0<t<\pi$, requires

$$\omega^{-1}\sinh(\omega\pi)=y_2(\pi;\delta,\epsilon)=c_1\quad\text{and}\quad\cosh(\omega\pi)=\dot{y}_2(\pi;\delta,\epsilon)=c_2\nu.$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon)=\left\{\begin{array}{ll}\frac{1}{\omega}\sinh(\omega t), & 0<t<\pi\\ \frac{1}{\omega}\sinh(\omega\pi)\cos\big(\nu(t-\pi)\big)+\frac{1}{\nu}\cosh(\omega\pi)\sin\big(\nu(t-\pi)\big), & \pi<t<2\pi\end{array}\right\},$$

Note that $\dot y_2(t) = -\frac{\nu}{\omega}\,\sinh(\omega\pi)\sin\big(\nu(t-\pi)\big) + \cosh(\omega\pi)\cos\big(\nu(t-\pi)\big)$, for $\pi < t < 2\pi$.

In this case, $X(t)$, the principal fundamental matrix at $t = 0$, satisfies

$$X(2\pi)-\mu I = \begin{bmatrix} \cosh(\omega\pi)\cos\big(\nu\pi) + \frac{\omega}{\nu}\,\sinh(\omega\pi)\sin(\nu\pi) - \mu & \frac{1}{\omega}\,\sinh(\omega\pi)\cos(\nu\pi) + \frac{1}{\nu}\,\cosh(\omega\pi)\sin(\nu\pi) \\ -\nu\cosh(\omega\pi)\sin(\nu\pi) + \omega\sinh(\omega\pi)\cos(\nu\pi) & -\frac{\nu}{\omega}\,\sinh(\omega\pi)\sin(\nu\pi) + \cosh(\omega\pi)\cos(\nu\pi) - \mu \end{bmatrix},$$

so the characteristic equation is

$$0 = \Big(\cosh(\omega\pi)\cos(\nu\pi) + \frac{\omega}{\nu}\,\sinh(\omega\pi)\sin(\nu\pi) - \mu\Big)\Big(-\frac{\nu}{\omega}\,\sinh(\omega\pi)\sin(\nu\pi) + \cosh(\omega\pi)\cos(\nu\pi) - \mu\Big)$$

$$-\left(\frac{1}{\omega}\,\sinh(\omega\pi)\cos(\nu\pi) + \frac{1}{\nu}\,\cosh(\omega\pi)\sin(\nu\pi)\right)\big(-\nu\cosh(\omega\pi)\sin(\nu\pi) + \omega\sinh(\omega\pi)\cos(\nu\pi)\big)$$

$$= \mu^2 - \mu\Big(\cosh(\omega\pi)\cos(\nu\pi) - \frac{\nu}{\omega}\,\sinh(\omega\pi)\sin(\nu\pi) + \frac{\omega}{\nu}\,\sinh(\omega\pi)\sin(\nu\pi) + \cosh(\omega\pi)\cos(\nu\pi)\Big)$$

$$-\frac{\nu}{\omega}\,\cosh(\omega\pi)\sinh(\omega\pi)\cos(\nu\pi)\sin(\nu\pi) - \sinh^2(\omega\pi)\sin^2(\nu\pi) + \cosh^2(\omega\pi)\cos^2(\nu\pi)$$

$$+\frac{\omega}{\nu}\,\cosh(\omega\pi)\sinh(\omega\pi)\cos(\nu\pi)\sin(\nu\pi) - \sinh^2(\omega\pi)\cos^2(\nu\pi) + \cosh^2(\omega\pi)\sin^2(\nu\pi)$$

$$+\frac{\nu}{\omega}\,\cosh(\omega\pi)\sinh(\omega\pi)\cos(\nu\pi)\sin(\nu\pi) - \frac{\omega}{\nu}\,\cosh(\omega\pi)\sinh(\omega\pi)\cos(\nu\pi)\sin(\nu\pi)$$

Using $\cos^2\theta + \sin^2\theta \equiv 1$ and $\cosh^2\theta - \sinh^2\theta \equiv 1$ and canceling four of the terms, this becomes

$$0 =\mu^2 - \mu\left(2\cosh(\omega\pi)\cos(\nu\pi) - \left(\frac{\omega}{\nu} + \frac{\nu}{\omega}\right)\sinh(\omega\pi)\sin(\nu\pi)\right) + 1$$

that is,

$$0 = \mu^2 - \mu\left(2\cosh(\omega\pi)\cos(\nu\pi) - \frac{2\delta}{\sqrt{\delta^2-\epsilon^2}}\sinh(\omega\pi)\sin(\nu\pi)\right) + 1,$$

So, in the case that $\delta + \epsilon > 0$ and $\delta - \epsilon > 0$, the characteristic multipliers are

$$\mu{=}\cosh(\omega\pi)\cos(\nu\pi) - \frac{\delta}{\sqrt{\delta^2-\epsilon^2}}\sinh(\omega\pi)\sin(\nu\pi) \pm \sqrt{\left(\cosh(\omega\pi)\cos(\nu\pi) - \frac{\delta}{\sqrt{\delta^2-\epsilon^2}}\sinh(\omega\pi)\sin(\nu\pi)\right)^2 - 1}\,.$$

By the way, the constant term, 1, in the characteristic equation follows from the fact that $|\,X(\mathsf{T}\,| = 1$, which follows from Abel's Theorem 5.7 in Section 5.2.

Case 8: We assume $\delta{+}\epsilon < 0$. We also assume $\delta{-}\epsilon = 0$ hence the general solution of the ODE (b) $\ddot y{+}(\delta{-}\epsilon)y = 0$ on the interval $\pi < t < 2\pi$ can be written as $y_1(t;\delta,\epsilon) = c_1 + c_2(t-\pi)$. So, continuity of the function $y_1(t) = \cos(\omega t)$ and its first derivative, $\dot y_1(t) = -\omega\sin(\omega t)$, for $0 < t < \pi$, requires

$$\cosh(\omega\pi) = A = y_1(\pi;\delta,\epsilon) = c_1 \quad\text{and}\quad \omega\sinh(\omega\pi) = B = \dot y_1(\pi;\delta,\epsilon) = c_2.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon) = \left\{\begin{array}{ll} \cosh(\omega t), & 0 < t < \pi \\ \cosh(\omega\pi) + \omega\,\sinh(\omega\pi)\,(t-\pi), & \pi < t < 2\pi \end{array}\right\}.$$

Note that $\dot y_1(t) = \omega\,\sinh(\omega\pi)$, for $\pi < t < 2\pi$.

The general solution of the ODE (b) $\ddot{y}+(\delta-\epsilon)y=0$ on the interval $\pi<t<2\pi$ can be written as $y_2(t;\delta,\epsilon)=A+B(t-\pi)$. Continuity of the function $y_2(t)=\omega^{-1}\sinh(\omega t)$ and its first derivative, $\dot{y}_2(t)=\cosh(\omega t)$, for $0<t<\pi$, requires

$$\omega^{-1}\sinh(\omega\pi)=y_2(\pi;\delta,\epsilon)=A \quad \text{and} \quad \cosh(\omega\pi)=\dot{y}_2(\pi;\delta,\epsilon)=B$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon)=\left\{\begin{array}{ll} \frac{1}{\omega}\,\sinh(\omega t), & 0<t<\pi \\ & \\ \frac{1}{\omega}\,\sinh(\omega\pi)+\cosh(\omega\pi)(t-\pi), & \pi<t<2\pi \end{array}\right\},$$

Note that $\dot{y}_2(t)=\cosh(\omega\pi)$, for $\pi<t<2\pi$.

If $\delta+\epsilon>0$ and $\delta=\epsilon$, then

$$X(2\pi)-\mu I=\begin{bmatrix} \cosh(\omega\pi)+\pi\omega\,\sinh(\omega\pi)-\mu & \frac{1}{\omega}\,\sinh(\omega\pi)+\pi\cosh(\omega\pi) \\ & \\ \omega\,\sinh(\omega\pi) & \cosh(\omega\pi)-\mu \end{bmatrix}$$

so the characteristic equation is

$$0=\Big(\cosh(\omega\pi)+\pi\omega\,\sinh(\omega\pi)\;-\;\mu\Big)\Big(\cosh(\omega\pi)\;-\;\mu\Big)-\Big(\frac{1}{\omega}\,\sinh(\omega\pi)+\pi\cosh(\omega\pi)\Big)\Big(\omega\sinh(\omega\pi)\Big)$$

$$=\mu^2-\mu\Big(2\cosh(\omega\pi)+\pi\omega\,\sinh(\omega\pi)\Big)+\cosh^2(\omega\pi)+\cancel{\pi\omega\cosh(\omega\pi)\sinh(\omega\pi)}-\sinh^2(\omega\pi)-\cancel{\pi\omega\cosh(\omega\pi)\sinh(\omega\pi)}$$

$$=\mu^2-\mu\Big(2\cosh(\omega\pi)+\pi\omega\sinh(\omega\pi)\Big)+1.$$

So, in the case that $\delta+\epsilon>0$ and $\delta-\epsilon>0$, the characteristic multipliers are

$$\mu=\cos(\omega\pi)\pm\frac{1}{2}\sqrt{\Big(2\cosh(\omega\pi)+\pi\omega\sinh(\omega\pi)\Big)^2-4}\,.$$

Case 9: We assume $\delta+\epsilon<0$. We also assume $\delta-\epsilon<0$ and denote $\nu=\sqrt{\epsilon-\delta}$. The general solution of the ODE (b) $\ddot{y}+(\delta-\epsilon)y=0$ on the interval $\pi<t<2\pi$ can be written as $y(t;\delta,\epsilon)=\textit{constant}\cdot\cosh\big(\nu(t-\pi)\big)+\textit{constant}\cdot\sinh\big(\nu(t-\pi)\big)$.

The ODE implies $y_1(t;\delta,\epsilon)=A\cosh\big(\nu(t-\pi)\big)+B\sinh\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_1(t)=\cos(\omega t)$ and its first derivative, $\dot{y}_1(t)=-\omega\sin(\omega t)$, for $0<t<\pi$, requires

$$\cos(\omega\pi)=y_1(\pi;\delta,\epsilon)=A \quad \text{and} \quad -\omega\sin(\omega\pi)=\dot{y}_1(\pi;\delta,\epsilon)=B\nu.$$

Putting our results, so far, together we have

$$y_1(t;\delta,\epsilon)=\left\{\begin{array}{ll} \cos(\omega t), & 0<t<\pi \\ & \\ \cos(\omega\pi)\cosh\big(\nu(t-\pi)\big)-\frac{\omega}{\nu}\,\sin(\omega\pi)\sinh\big(\nu(t-\pi)\big), & \pi<t<2\pi \end{array}\right\},$$

Note that $\dot{y}_1(t)=\nu\cos(\omega\pi)\sinh\big(\nu(t-\pi)\big)-\omega\sin(\omega\pi)\cosh\big(\nu(t-\pi)$, for $\pi<t<2\pi$.

The ODE implies $y_2(t;\delta,\epsilon)=c_1\cosh\big(\nu(t-\pi)\big)+c_2\sinh\big(\nu(t-\pi)\big)$ on $\pi<t<2\pi$, so continuity of the function $y_2(t)=\omega^{-1}\sin(\omega t)$ and its first derivative, $\dot{y}_2(t)=\cos(\omega t)$, for $0<t<\pi$, requires

$$\omega^{-1}\sin(\omega\pi)=y_2(\pi;\delta,\epsilon)=c_1 \quad \text{and} \quad \cos(\omega\pi)=\dot{y}_2(\pi;\delta,\epsilon)=c_2\nu.$$

Putting our results, so far, together we have

$$y_2(t;\delta,\epsilon)=\left\{\begin{array}{ll} \frac{1}{\omega}\,\sin(\omega t), & 0<t<\pi \\ & \\ \frac{1}{\omega}\,\sin(\omega\pi)\cosh\big(\nu(t-\pi)\big)+\frac{1}{\nu}\,\cos(\omega\pi)\sinh\big(\nu(t-\pi)\big), & \pi<t<2\pi \end{array}\right\},$$

Note that $\dot{y}_2(t) = \frac{\nu}{\omega}\,\sin(\omega\pi)\sinh\big(\nu(t-\pi)\big) + \cos(\omega\pi)\cosh\big(\nu(t-\pi)\big)$, for $\pi < t < 2\pi$.

If $\delta + \epsilon > 0$ and $\delta - \epsilon < 0$, then

$$X(2\pi) - \mu I = \begin{bmatrix} \cos(\omega\pi)\cosh(\nu\pi) - \frac{\omega}{\nu}\,\sin(\omega\pi)\sinh(\nu\pi) - \mu & \frac{1}{\omega}\,\sin(\omega\pi)\cosh(\nu\pi) + \frac{1}{\nu}\,\cos(\omega\pi)\sinh(\nu\pi) \\ \nu\cos(\omega\pi)\sinh(\nu\pi) - \omega\sin(\omega\pi)\cosh(\nu\pi) & \frac{\nu}{\omega}\,\sin(\omega\pi)\sinh(\nu\pi) + \cos(\omega\pi)\cosh(\nu\pi) - \mu \end{bmatrix}$$

so the characteristic equation is

$$\begin{aligned} 0 &= \Big(\cos(\omega\pi)\cosh(\nu\pi) - \frac{\omega}{\nu}\,\sin(\omega\pi)\sinh(\nu\pi) - \mu\Big)\Big(\frac{\nu}{\omega}\,\sin(\omega\pi)\sinh(\nu\pi) + \cos(\omega\pi)\cosh(\nu\pi) - \mu\Big) \\ &\quad - \Big(\frac{1}{\omega}\,\sin(\omega\pi)\cosh(\nu\pi) + \frac{1}{\nu}\,\cos(\omega\pi)\sinh(\nu\pi)\Big)\Big(\nu\cos(\omega\pi)\sinh(\nu\pi) - \omega\sin(\omega\pi)\cosh(\nu\pi)\Big) \\ &= \mu^2 - \mu\Big(\cos(\omega\pi)\cosh(\nu\pi) - \frac{\omega}{\nu}\,\sin(\omega\pi)\sinh(\nu\pi) + \frac{\nu}{\omega}\,\sin(\omega\pi)\sinh(\nu\pi) + \cos(\omega\pi)\cosh(\nu\pi)\Big) \\ &\quad + \frac{\nu}{\omega}\,\cos(\omega\pi)\sin(\omega\pi)\cosh(\nu\pi)\sinh(\nu\pi) - \sin^2(\omega\pi)\sinh^2(\nu\pi) + \cos^2(\omega\pi)\cosh^2(\nu\pi) \\ &\quad - \frac{\omega}{\nu}\,\cos(\omega\pi)\sin(\omega\pi)\cosh(\nu\pi)\sinh(\nu\pi) + \sin^2(\omega\pi)\cosh^2(\nu\pi) - \cos^2(\omega\pi)\sinh^2(\nu\pi) \\ &\quad - \frac{\nu}{\omega}\,\cos(\omega\pi)\sin(\omega\pi)\cosh(\nu\pi)\sinh(\nu\pi) + \frac{\omega}{\nu}\,\cos(\omega\pi)\sin(\nu\pi)\cosh(\omega\pi)\sinh(\nu\pi). \end{aligned}$$

Using $\cosh^2\theta - \sinh^2\theta \equiv 1$ and $\cos^2\theta + \sin^2\theta \equiv 1$, this becomes

$$0 = \mu^2 - \mu\left(2\cos(\omega\pi)\cosh(\nu\pi) - \left(\frac{\omega}{\nu} - \frac{\nu}{\omega}\right)\sin(\omega\pi)\sinh(\nu\pi)\right) + 1$$

that is,

$$0 = \mu^2 - \mu\left(2\cos(\omega\pi)\cosh(\nu\pi) - \frac{2\epsilon}{\sqrt{\delta^2-\epsilon^2}}\,\sin(\omega\pi)\sinh(\nu\pi)\right) + 1,$$

So, in the case that $\delta + \epsilon > 0$ and $\delta - \epsilon > 0$, the characteristic multipliers are

$$\mu = \cos(\omega\pi)\cosh(\nu\pi) - \frac{\epsilon}{\sqrt{\delta^2-\epsilon^2}}\,\sin(\omega\pi)\sinh(\nu\pi) \pm \sqrt{\left(\cos(\omega\pi)\cosh(\nu\pi) - \frac{\epsilon}{\sqrt{\delta^2-\epsilon^2}}\,\sin(\omega\pi)\sinh(\nu\pi)\right)^2 - 1}\,.$$

5.8.6.15. We are given that $\dot{\mathbf{y}} = -A(t)^T\mathbf{y}$ has a T-periodic solution $\mathbf{y}(t)$ and that the system $\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{f}(t)$ has a T-periodic solution $\mathbf{x}(t)$.

Somehow, we want to use the periodicity, which implies $\mathbf{y}(\mathsf{T}) = \mathbf{y}(0)$ and $\mathbf{x}(\mathsf{T}) = \mathbf{x}(0)$.

Consider the scalar quantity $z(t) \triangleq \mathbf{y}^T(t)\mathbf{x}(t)$. The product rule implies that

$$\begin{aligned} \dot{z}(t) = \dot{\mathbf{y}}^T(t)\mathbf{x}(t) + \mathbf{y}^T(t)\dot{\mathbf{x}}(t) &= \big(\dot{\mathbf{y}}(t)\big)^T\mathbf{x}(t) + \mathbf{y}^T(t)\big(A(t)\mathbf{x} + \mathbf{f}(t)\big) = \big(-A(t)^T\mathbf{y}\big)^T\mathbf{x}(t) + \mathbf{y}^T(t)\big(A(t)\mathbf{x} + \mathbf{f}(t)\big) \\ &= -\mathbf{y}^T A(t)\mathbf{x}(t) + \mathbf{y}^T(t)A(t)\mathbf{x} + \mathbf{y}^T(t)\mathbf{f}(t) = \mathbf{y}^T(t)\mathbf{f}(t). \end{aligned}$$

It follows that

$$0 = z(\mathsf{T}) - z(0) = \int_0^{\mathsf{T}} \dot{z}(s)ds = \int_0^{\mathsf{T}} \mathbf{y}^T(s)\mathbf{f}(s)\,ds,$$

which gives the desired result.

5.8.6.17. For the system of problem 5.8.6.10, the characteristic multipliers are -1 and -1 but are not deficient, so the system is neutrally stable.

Chapter Six

Section 6.1

6.1.4.1. (a) $\mathbf{A} \bullet \mathbf{B} = \left(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}\right) \bullet \left(2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right) = -2$, (b) $\mathbf{A} \times \mathbf{B} = \left(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}\right) \times \left(2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right) = ... = -\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$
(c) the angle between $\mathbf{A}$ and $\mathbf{B}$ is $\cos^{-1}\left(\frac{\mathbf{A}\bullet\mathbf{B}}{||\mathbf{A}||\ ||\mathbf{B}||}\right) = \cos^{-1}\left(\frac{-2}{\sqrt{10}}\right) \approx 2.25551553$. [In Calculus, angles are measured in radians.]

6.1.4.3. $\mathbf{v} = (2-0)\hat{\boldsymbol{\imath}} + (1-3)\hat{\boldsymbol{\jmath}} + (-1-5)\hat{\boldsymbol{k}} = 2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}}$, so $x = 2t, y = 3-2t, z = 5-6t,\ -\infty < t < \infty$ are parametric equations of the line

6.1.4.5. the line has a direction vector $\mathbf{v} = \mathbf{n} = -\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}$ in order to be perpendicular to the given plane, so $x = 2-t, y = 4+2t, z = 1+3t,\ -\infty < t < \infty$ are parametric equations of the line

6.1.4.7. normal vector $\mathbf{n} = (\overrightarrow{P_2 - P_1}) \times (\overrightarrow{P_2 - P_3}) = (\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) \times (-\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}) = \hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$, so an equation of the plane is $(x-0) - 3(y-1) - 4(z-1) = 0$ or $x - 3y - 4z = -7$ or other versions of the same

6.1.4.9. $\mathbf{n} = 3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ is a normal to the plane, so an equation of the plane is $3(x-2) - 2(y-4) + (z-1) = 0$ or $3x - 2y + z = -1$ or other versions of the same

6.1.4.11. $\mathbf{n} = (-1-0)\hat{\boldsymbol{\imath}} + (2-1)\hat{\boldsymbol{\jmath}} + (4-3)\hat{\boldsymbol{k}} = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ is a normal to the plane, so an equation of the plane is $-(x-2) + (y-4) + (z-1) = 0$ or $-x + y + z = 3$ or other versions of the same

6.1.4.13. $\mathbf{v} = [x \quad y \quad z]^T$ being perpendicular to the two given vectors gives augmented matrix and, after elementary row operations $\frac{1}{2}R_1 \rightarrow R_1$, $-4R_1 + R_2 \rightarrow R_2$, $\frac{1}{2}R_2 \rightarrow R_2$, $\frac{1}{2}R_2 + R_1 \rightarrow R_1$, its RREF

$$\left[\begin{array}{rrrr} 2 & -1 & 1 & |\ 0 \\ 4 & 0 & 1 & |\ 0 \end{array}\right] \sim \left[\begin{array}{rrrr} 1 & 0 & 1/4 & |\ 0 \\ 0 & 1 & -1/2 & |\ 0 \end{array}\right]$$

so $\mathbf{v}$ is a normalized version of $\pm[-0.25 \quad 0.5 \quad 1]^T$, that is, $\mathbf{v} = \pm\dfrac{1}{\sqrt{21}}[-1 \quad 2 \quad 4]^T$.

6.1.4.15. $\theta = 0$, to get $\cos\theta = 1$

6.1.4.17. Note that the instructions for this problem have been changed on the Errata page.

The assumptions that $\mathbf{v}(0)$ and $\mathbf{B}_0$ are non-zero and not parallel imply that the cross product $\mathbf{a}(0) \triangleq -\frac{q}{m}\mathbf{v}_0 \times \mathbf{B}_0$ is non-zero and automatically orthogonal to both $\mathbf{v}_0$ and $\mathbf{B}_0$.

Expressing the solution of the system in the form $\mathbf{r}(t) = \mathbf{r}_0 + g(t)\mathbf{v}(0) + h(t)\mathbf{a}(0)$, we have

$$\dot{\mathbf{r}}(t) = \mathbf{v}(t) = \dot{g}(t)\mathbf{v}(0) + \dot{h}(t)\mathbf{a}(0) \quad \text{and} \quad \dot{\mathbf{v}}(t) = \ddot{g}(t)\mathbf{v}(0) + \ddot{h}(t)\mathbf{a}(0).$$

Substituting the latter into the ODEs $m\dot{\mathbf{v}} = -q\mathbf{v} \times \mathbf{B}_0$ gives

$$\ddot{g}(t)\mathbf{v}(0) + \ddot{h}(t)\mathbf{a}(0) = \dot{\mathbf{v}}(t) = -\frac{q}{m}\mathbf{v}(t) \times \mathbf{B}_0 = -\frac{q}{m}\left(\dot{g}(t)\mathbf{v}(0) + \dot{h}(t)\mathbf{a}(0)\right) \times \mathbf{B}_0.$$

Using the definition of $\mathbf{a}(0)$ reduces the ODEs to

$$(1) \quad \ddot{g}(t)\mathbf{v}(0) + \ddot{h}(t)\mathbf{a}(0) = -\dot{g}(t)\Big(\frac{q}{m}\Big)\mathbf{v}(0) \times \mathbf{B}_0 - \dot{h}(t)\Big(\frac{q}{m}\Big)\mathbf{a}(0) \times \mathbf{B}_0 = -\dot{g}(t)\mathbf{a}(0) - \dot{h}(t)\Big(\frac{q}{m}\Big)\mathbf{a}(0) \times \mathbf{B}_0.$$

Recalling that $\mathbf{v}(0)$ is orthogonal to $\mathbf{a}(0)$, taking the dot product of (1) with $\mathbf{a}(0)$ gives

$$(2) \quad 0 + ||\mathbf{a}(0)||^2\ddot{h}(t) = -||\mathbf{a}(0)||^2\dot{g}(t) - \dot{h}(t)\Big(\frac{q}{m}\Big)\mathbf{a}(0) \bullet \Big(\mathbf{a}(0) \times \mathbf{B}_0\Big).$$

The result of problem 6.8.4.15, namely the identity $\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\bullet(\mathbf{A}\times\mathbf{B})$, implies that $\mathbf{a}(0)\bullet\Big(\mathbf{a}(0)\times\mathbf{B}_0\Big) = \mathbf{B}_0 \bullet \Big(\mathbf{a}(0) \times \mathbf{a}(0)\Big) = \mathbf{B}_0 \bullet \big(\mathbf{0}\big) = 0$. So, equation (2) reduces to $||\mathbf{a}(0)||^2 \ddot{h}(t) = -||\mathbf{a}(0)||^2 \dot{g}(t)$, hence

$$(3) \quad \ddot{h}(t) = -\dot{g}(t).$$

On the other hand, recalling that $\mathbf{v}(0)$ is orthogonal to $\mathbf{a}(0)$, taking the dot product of (1) with $\mathbf{v}(0)$ gives

$$(4) \quad ||\mathbf{v}(0)||^2 \ddot{g}(t) + 0 = 0 - \dot{h}(t)\frac{q}{m}\,\mathbf{v}(0) \bullet \Big(\mathbf{a}(0) \times \mathbf{B}_0\Big)$$

Using the anti-symmetry of the cross product, and then using again the result of problem 6.8.4.15 as well as the definition of $\mathbf{a}(0)$, we have

$$\mathbf{v}(0) \bullet \Big(\mathbf{a}(0) \times \mathbf{B}_0\Big) = -\mathbf{v}(0) \bullet \Big(\mathbf{B}_0 \times \mathbf{a}(0)\Big) = -\mathbf{a}(0) \bullet \Big(\mathbf{v}(0) \times \mathbf{B}_0\Big) = -\mathbf{a}(0) \bullet \Big(\frac{m}{q}\,\mathbf{a}(0)\Big) = -\frac{m}{q}\,||\mathbf{a}(0)||^2.$$

Substituting this into (4) gives

$$||\mathbf{v}(0)||^2 \ddot{g}(t) = ||\mathbf{a}(0)||^2 \dot{h}(t),$$

hence

$$\dot{h}(t) = \big(||\mathbf{v}(0)||^2/||\mathbf{a}(0)||^2\big)\ddot{g}(t).$$

Taking the time derivative of both sides and substituting this into (3) gives

$$\big(||\mathbf{v}(0)||^2/||\mathbf{a}(0)||^2\big)\,\dddot{g} = -\dot{g}(t).$$

It follows that $g(t) = c + y(t)$ where $y(t)$ satisfies $\ddot{y} + \big(||\mathbf{a}(0)||^2/||\mathbf{v}(0)||^2\big)y = 0$, the ODE of an undamped harmonic oscillator.

Section 6.2

6.2.6.1. $r = \sqrt{(\sqrt{6})^2 + (-\sqrt{2})^2} = \sqrt{6+2} = 2\sqrt{2}$ and $\tan\theta = \dfrac{-\sqrt{2}}{\sqrt{6}} = -\dfrac{1}{\sqrt{3}}$ is in quadrant IV, so θ is co-terminal with $-\dfrac{\pi}{6}$. But, $0 \le \theta \le 2\pi$, so the polar coordinates are $(r,\theta) = (2\sqrt{2}, \dfrac{11\pi}{6})$.

6.2.6.3. (a) $r = \sqrt{x^2+y^2} = 2$ and $\tan\theta = \dfrac{1}{-\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$ is in quadrant II, so θ is $\dfrac{5\pi}{6}$. The cylindrical coordinates are $(r,\theta,z) = (2, \dfrac{5\pi}{6}, 5)$.

(b) $r = \sqrt{x^2+y^2} = 2$ and $\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$ is in quadrant II, so θ is $\dfrac{2\pi}{3}$. The cylindrical coordinates are $(r,\theta,z) = (2, \dfrac{2\pi}{3}, 4)$.

6.2.6.5. (a) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{1}{2}+\frac{3}{2}+2} = 2. \quad \Longrightarrow -\sqrt{2} = z = \rho\cos\phi = 2\cos\phi \quad \Longrightarrow \cos\phi = -\dfrac{1}{\sqrt{2}}$

$\Longrightarrow \phi = \frac{3\pi}{4}$ because $0 \le \phi \le \pi \quad \Longrightarrow -\frac{\sqrt{2}}{2} = x = \rho\,\sin\phi\cos\theta = \sqrt{2}\cdot\frac{1}{\sqrt{2}}\cos\theta \quad \Longrightarrow \cos\theta = -\frac{1}{2}$;

Also, $\frac{\sqrt{6}}{2} = y = \rho\,\sin\phi\sin\theta = \sqrt{2}\cdot\frac{1}{\sqrt{2}}\sin\theta \quad \Longrightarrow \sin\theta = -\frac{\sqrt{3}}{2}$.

$\cos\theta = -\frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$ together imply $\theta = \dfrac{4\pi}{3}$. The spherical coordinates are $(\rho,\phi,\theta) = (2, \dfrac{3\pi}{4}, \dfrac{4\pi}{3})$.

(b) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{3}{2}+\frac{1}{2}+2} = 2. \quad \Longrightarrow -\sqrt{2} = z = \rho\cos\phi = 2\cos\phi \Longrightarrow \cos\phi = -\dfrac{1}{\sqrt{2}}$

$\Longrightarrow \phi = \frac{3\pi}{4}$ because $0 \le \phi \le \pi \quad \Longrightarrow -\frac{\sqrt{6}}{2} = x = \rho\,\sin\phi\cos\theta = 2\cdot\frac{1}{\sqrt{2}}\cos\theta \quad \Longrightarrow \cos\theta = -\frac{\sqrt{3}}{2}$;

Also, $\frac{\sqrt{2}}{2} = y = \rho\,\sin\phi\sin\theta = 2\cdot\frac{1}{\sqrt{2}}\sin\theta \quad \Longrightarrow \sin\theta = \frac{1}{2}$.

$\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$ together imply $\theta = \dfrac{5\pi}{6}$. The spherical coordinates are $(\rho,\phi,\theta) = (2, \dfrac{3\pi}{4}, \dfrac{5\pi}{6})$.

6.2.6.7. Using the results at the end of Section 6.2,

$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} = \cos\theta(\sin\phi\,\cos\theta\,\hat{\mathbf{e}}_\rho + \cos\phi\,\cos\theta\,\hat{\mathbf{e}}_\phi - \cancel{\sin\theta\,\hat{\mathbf{e}}_\theta}) + \sin\theta(\sin\phi\,\sin\theta\,\hat{\mathbf{e}}_\rho + \cos\phi\,\sin\theta\,\hat{\mathbf{e}}_\phi + \cancel{\cos\theta\,\hat{\mathbf{e}}_\theta})$$

$$= \sin\phi(\cos^2\theta + \sin^2\theta)\hat{\mathbf{e}}_\rho + \cos\phi(\cos^2\theta + \sin^2\theta)\hat{\mathbf{e}}_\phi = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi;$$

$$\hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}} = \hat{\mathbf{e}}_\theta; \quad \text{and} \quad \hat{\mathbf{e}}_z = \hat{\boldsymbol{k}} = \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi.$$

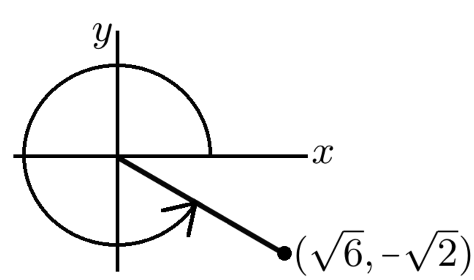


Figure 1: Answer for problem 6.2.6.1

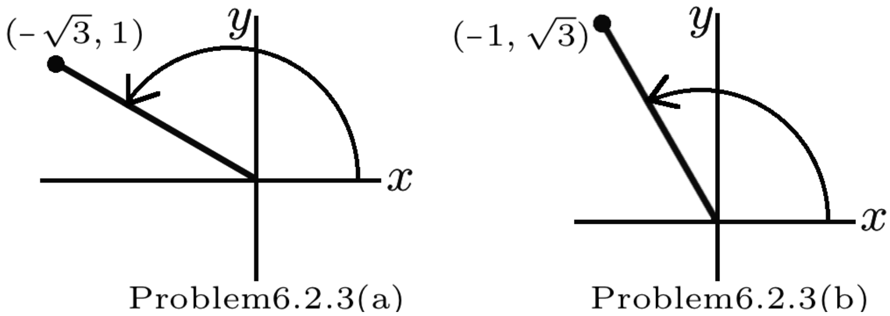


Figure 2: Problem 6.2.6.3(a) and Problem 6.2.6.3(b)

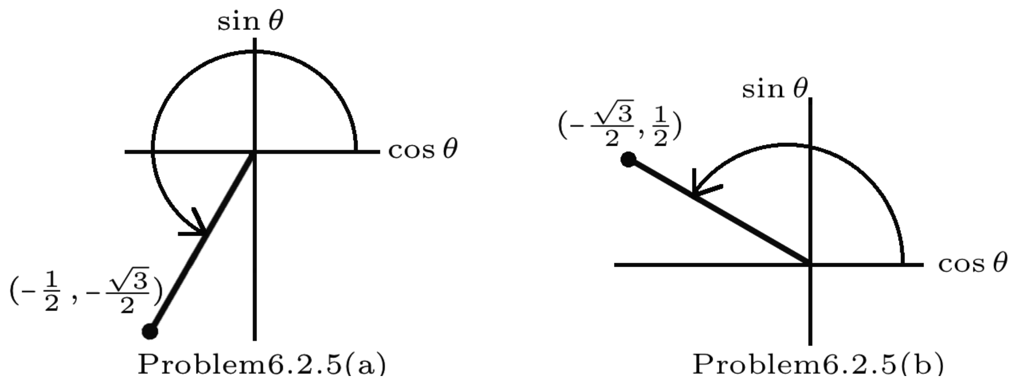


Figure 3: Problem 6.2.6.5(a) and Problem 6.2.6.5(b)

Section 6.3

6.3.4.1. $\mathbf{v} = (-1-1)\hat{\boldsymbol{\imath}} + (0-2)\,\hat{\boldsymbol{\jmath}} + (3-4)\hat{\boldsymbol{k}}$ is the direction vector, so the line is $\mathcal{C}:\ \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (1-2t)\hat{\boldsymbol{\imath}} + (2-2t)\,\hat{\boldsymbol{\jmath}} + (4-t)\hat{\boldsymbol{k}},\ 0 \le t \le 1$

6.3.4.3. Circle is $0 = x^2 + y^2 - 8y = x^2 + (y-4)^2 - 16$, that is, $x^2 + (y-4)^2 = 16$, that is, circle with center at $(0,4)$ and radius 4.

One parametrization is $\mathcal{C}_1:\ \mathbf{r} = 4\cos t\,\hat{\boldsymbol{\imath}} + (4+4\sin t)\hat{\boldsymbol{\jmath}}$, $0 \le t \le 2\pi$.

Alternatively, use polar coordinates: $r^2 = x^2 + y^2 = 8y = 8r\sin\theta$, hence $r = 4\sin\theta$, so another parametrization is $\mathcal{C}_2:\ \mathbf{r} = 8\sin\theta(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) = 4\sin 2\theta\,\hat{\boldsymbol{\imath}} + 4(1-\cos 2\theta)\,\hat{\boldsymbol{\jmath}}, 0 \le \theta \le \pi$.

Note that $\sin 2\theta$ and $\cos 2\theta$ are periodic with period π, so if we had used $0 \le \theta \le 2\pi$ then we would have traversed the circle twice in the parametrization $\mathcal{C}_2$.

6.3.4.5. ellipse with center at $(0,0)$, major axis of length 12 on the $x-$axis and minor axis of length 4 on the $y-$axis...see sketched figure

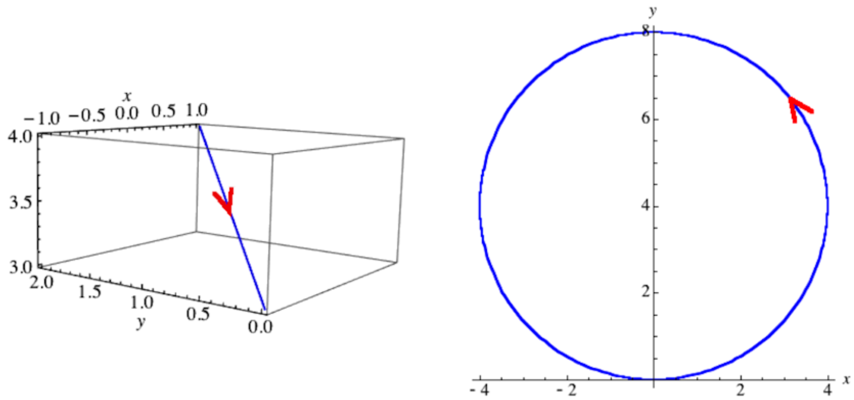
6.3.4.7. sketched figure is $\mathcal{C}:\ \mathbf{r} = \sin 3\theta(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}), 0 \le \theta \le 2\pi$ is called a "three-leaved rose"

6.3.4.9. see sketched figure

6.3.4.11. $\mathcal{C}:\ \mathbf{r} = 3\cos t\,\hat{\boldsymbol{\imath}} + 2\sin t\,\hat{\boldsymbol{\jmath}}, 0 \le t \le 2\pi$ gives a simple, closed curve

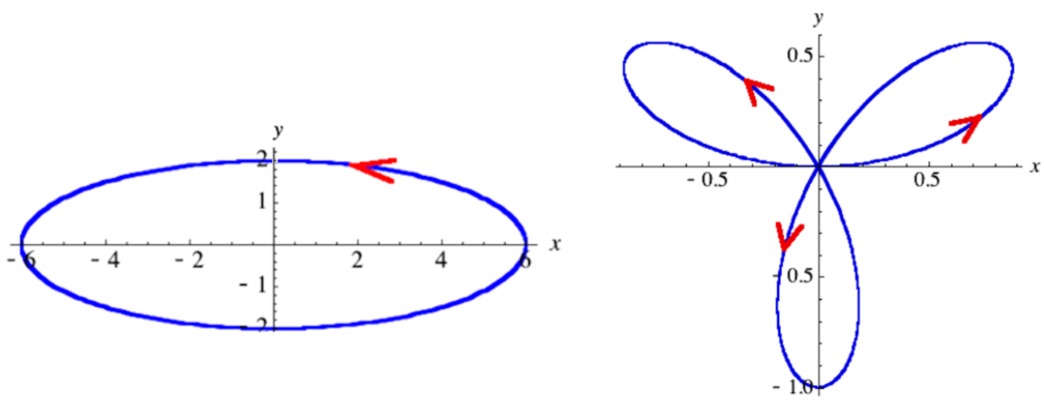
6.3.4.13. First rewrite the equation of the curve as $x^2 - 12x + 36 + 4y^2 = 36$ and then as $\dfrac{(x-6)^2}{36} + \dfrac{y^2}{4} = 1$. Its parametrization $\mathcal{C}:\ \mathbf{r} = (6+6\cos t)\hat{\boldsymbol{\imath}} + 2\sin t\,\hat{\boldsymbol{\jmath}}, 0 \le t \le 2\pi$ gives a simple, closed curve.

6.3.4.15. Equation (6.26) $\mathcal{C}_1:\ \mathbf{r}(\theta) = 4\cos\theta\cdot(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}})$, $0 \le \theta \le \pi$, is the circle of radius 4, center at $(2,0)$, traversed once counter-clockwise.



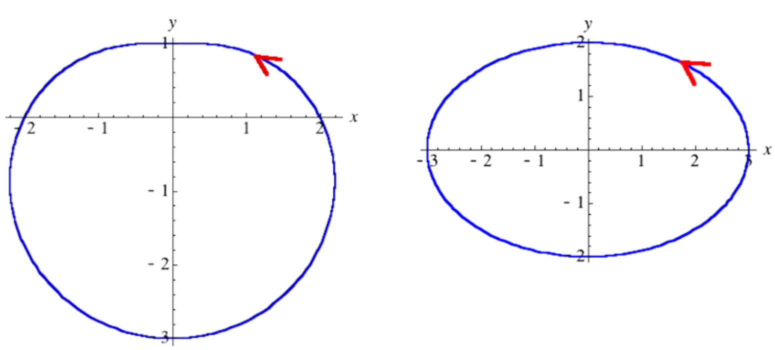
Problem 6.3.1 Problem 6.3.3

Figure 4: (a) Answer for problem 6.3.4.1 and (b) Answer for problem 6.3.4.3



Problem 6.3.5 Problem 6.3.7

Figure 5: (a) Answer for problem 6.3.4.5 and (b) Answer for problem 6.3.4.7



Problem 6.3.9 Problem 6.3.11

Figure 6: (a) Answer for problem 6.3.4.9 and (b) Answer for problem 6.3.4.11

$\mathcal{C}_2 : \mathbf{r}(\theta) = 4\cos\theta \cdot (\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}),\ 0 \le \theta \le 2\pi$ is the circle of radius 4, center at $(2,0)$, traversed counter-clockwise twice because $\mathcal{C}_2 : \mathbf{r}(\theta) = 2(1+\cos 2\theta)\hat{\boldsymbol{\imath}} + 2\sin 2\theta\,\hat{\boldsymbol{\jmath}},\ 0 \le \theta \le 2\pi$.

6.3.4.17. $\dot{\mathbf{r}} = \cos t\,\hat{\boldsymbol{\imath}} + \frac{6}{\pi}\,\hat{\boldsymbol{\jmath}}$. At the point $(\frac{\sqrt{3}}{2}, 1)$, $\sin t = x = \frac{\sqrt{3}}{2}$ and $1 = y = \frac{6}{\pi}t - 1$. The latter equation implies $t = \frac{\pi}{3}$, which agrees with the former equation. So, at the point $(\frac{\sqrt{3}}{2}, 1)$, $\dot{\mathbf{r}} = \cos\frac{\pi}{3}\,\hat{\boldsymbol{\imath}} + \frac{6}{\pi}\,\hat{\boldsymbol{\jmath}} = \frac{1}{2}\,\hat{\boldsymbol{\imath}} + \frac{6}{\pi}\,\hat{\boldsymbol{\jmath}}$.

$$x = \frac{\sqrt{3}}{2} + \frac{1}{2}t,\ y = 1 + \frac{6}{\pi}t,\ -\infty < t < \infty$$

are parametric equations of the desired tangent line. Other choices are possible.

6.3.4.19. $\dot{\mathbf{r}} = \cos t\,\hat{\boldsymbol{\imath}} + (1 - 2\sin t)\hat{\boldsymbol{\jmath}}$ is parallel to the $y-$axis whenever $1 - 2\sin t \ne 0$ *and* $\cos t = 0$, that is, when $t = \left(n + \frac{1}{2}\right)\pi$ for some integer n. This happens at the points

$$\mathbf{r}(t) = \sin\left(n + \frac{1}{2}\right)\pi\,\hat{\boldsymbol{\imath}} + \left(\left(n + \frac{1}{2}\right)\pi + 2\cos\left(n + \frac{1}{2}\right)\pi\right)\hat{\boldsymbol{\jmath}},$$

that is, at the points

$$\mathbf{r} = (-1)^n\hat{\boldsymbol{\imath}} + \left(n + \frac{1}{2}\right)\pi\,\hat{\boldsymbol{\jmath}}.$$

6.3.4.21. $\rho = 4\cos\phi \Longrightarrow \rho = 4\cos\phi$, that is, $x^2 + y^2 + z^2 = 4z$, that is, $x^2 + y^2 + (z-2)^2 = 4$, the sphere of radius 2 and center at $(0,0,2)$; see the figure

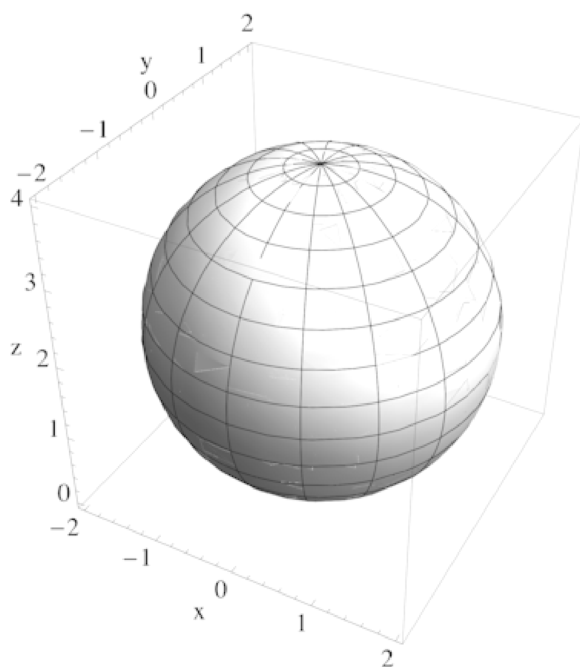


Figure 7: Answer for problem 6.3.4.21

6.3.4.23. $r = a\cos\theta \Longrightarrow r^2 = a\,r\cos\theta$, that is, $x^2 + y^2 = ax$, that is, $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$, circular cylindrical surface of radius $\frac{a}{2}$ and axis of symmetry being the line satisfying both $x = \frac{a}{2}$ and $y = 0$; see the figure

6.3.4.25. $x^2 + y^2 + z^2 + z = 0$, that is, $x^2 + y^2 + \left(z + \frac{1}{2}\right)^2 = \frac{1}{4}$, the sphere of radius $\frac{1}{2}$ and center at $(0,0,-\frac{1}{2})$

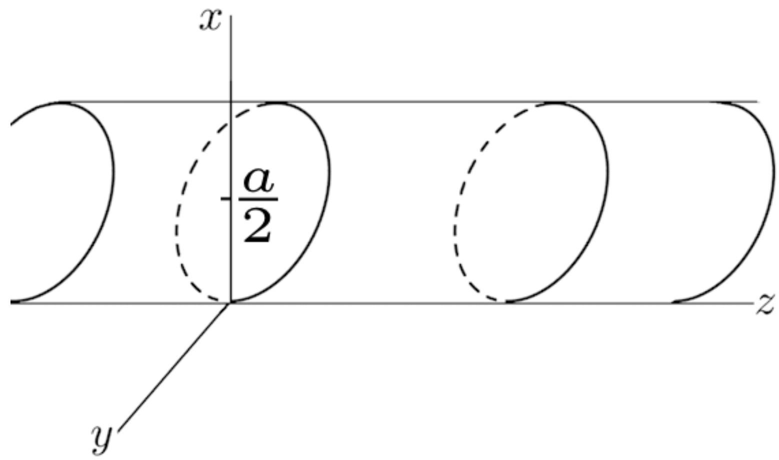


Figure 8: Answer for problem 6.3.4.23

One parametrization is

$$\mathcal{S}_1 : \mathbf{r}(\phi,\theta)= \frac{1}{2}\,\sin\phi\,\cos\theta\,\hat{\boldsymbol{\imath}} + \frac{1}{2}\,\sin\phi\,\sin\theta\,\hat{\boldsymbol{\jmath}} + \frac{1}{2}(-1+\cos\phi)\,\hat{\boldsymbol{k}},\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi.$$

Alternatively, use spherical coordinates: $\rho^2=-z=-\rho\cos\phi \Longrightarrow \rho=-\cos\phi$, so another parametrization is

$$\mathcal{S}_2 : \mathbf{r}(\phi,\theta)=-\cos\phi\left(\sin\phi\cos\theta\,\hat{\boldsymbol{\imath}}+\sin\phi\sin\theta\,\hat{\boldsymbol{\jmath}}+\cos\phi\,\hat{\boldsymbol{k}}\right)=-\tfrac{1}{2}\left(\sin 2\phi\,\cos\theta\,\hat{\boldsymbol{\imath}}+\sin 2\phi\,\sin\theta\,\hat{\boldsymbol{\jmath}}+(1+\cos 2\phi)\hat{\boldsymbol{k}}\right),$$

$$0\le\phi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi.$$

In this parametrization we only needed $0\le\phi\le\frac{\pi}{2}$ because all the terms are functions of 2ϕ instead of ϕ.

The preceding sentence explains why

$$\mathcal{S}_3 \colon \mathbf{r}(\phi,\theta)=-\frac{1}{2}\left(\sin 2\phi\cos\theta\,\hat{\boldsymbol{\imath}}+\sin 2\phi\sin\theta\,\hat{\boldsymbol{\jmath}}+(1+\cos 2\phi)\hat{\boldsymbol{k}}\right),\ \frac{\pi}{2}\le\phi\le\pi,\ 0\le\theta\le 2\pi$$

gives another parametrization.

6.3.4.27. $z=x^2+y^2$ is a circular paraboloid with axis being the positive $z-$ axis.

One parametrization is

$$\mathcal{S}_1 : \mathbf{r}(x,y)= x\,\hat{\boldsymbol{\imath}}+y\,\hat{\boldsymbol{\jmath}}+(x^2+y^2)\hat{\boldsymbol{k}}, -\infty<x<\infty,\ -\infty<y<\infty.$$

Another parametrization is to use polar coordinates:

$$\mathcal{S}_2 : \mathbf{r}(r,\theta)= r\cos\theta\,\hat{\boldsymbol{\imath}}+r\sin\theta\,\hat{\boldsymbol{\jmath}}+r^2\,\hat{\boldsymbol{k}},\ 0\le r<\infty,\ 0\le\theta\le 2\pi.$$

6.3.4.29. $z=\sqrt{x^2+\frac{1}{2}\,y^2}$ is an elliptic cone with axis being the positive $z-$ axis.

One parametrization is

$$\mathcal{S}_1 : \mathbf{r}(x,y)= x\,\hat{\boldsymbol{\imath}}+y\,\hat{\boldsymbol{\jmath}}+\sqrt{x^2+\frac{1}{2}\,y^2}\,\hat{\boldsymbol{k}}, -\infty<x<\infty,\ -\infty<y<\infty.$$

Another parametrization is to replace z by u and then parametrize the points on or inside ellipses $u^2=x^2+\frac{1}{2}\,y^2$, that is, $1=\dfrac{x^2}{u^2}+\dfrac{y^2}{2u^2}$:

$$\mathcal{S}_2 : \mathbf{r}(r,\theta)= u\cos v\,\hat{\boldsymbol{\imath}}+\sqrt{2}\,u\sin v\,\hat{\boldsymbol{\jmath}}+u\,\hat{\boldsymbol{k}},\ 0\le u<\infty,\ 0\le v\le 2\pi.$$

Another parametrization is to use polar coordinates:

$$\mathcal{S}_3 : \mathbf{r}(r,\theta) = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + r\cdot\sqrt{\cos^2\theta + \frac{1}{2}\sin^2\theta}\,\hat{\boldsymbol{k}},\ 0 \le r < \infty,\ 0 \le \theta \le 2\pi.$$

6.3.4.31. Equation (6.29) says the solutions of the ODE system, which is in companion form, can be written in the form

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} A\cos(\omega t - \delta) \\ -\omega A \sin(\omega t - \delta) \end{bmatrix},$$

where A, ω, δ are scalar constants and A, ω are positive.

The first data point tells us that for some "time" t_1 we get $(2,0) = \big(A\cos(\omega t_1 - \delta), -\omega A \sin(\omega t_1 - \delta)\big)$. Let $\theta \triangleq \omega t_1 - \delta$, so this becomes $2 = A\cos\theta$ and $0 = A\sin\theta$. Because A is positive, it follows that $\theta = 2k\pi$ for some integer k and $A = 2$.

Now, plug $A = 2$ into the form of the solution and the second data point to get $(0, -\sqrt{3}) = (2\cos(\omega t_2 - \delta), -2\omega\sin(\omega t_2 - \delta)$. From $\cos(\omega t_2 - \delta) = 0$ and the solution curve shown in the figure, it follows that $\omega t_2 - \delta = \omega t_1 - \delta + \frac{\pi}{2}$, hence

$$-\sqrt{3} = -2\omega\sin(\omega t_2 - \delta) = -2\omega\sin\Big(\omega t_1 - \delta + \frac{\pi}{2}\Big) = -2\omega\sin\Big(2k\pi + \frac{\pi}{2}\Big) = -2\omega.$$

It follows that the frequency of vibration is $\omega = \frac{\sqrt{3}}{2}$.

6.3.4.33. For $0 \le \theta \le 2\pi$, $-\infty < z < \infty$, $\mathcal{S} : \mathbf{r} = \mathbf{r}(x,y) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$

6.3.4.35. Multiply both sides of $\rho = 2\cos\phi$ by ρ to get $x^2 + y^2 + z^2 = \rho^2 = 2\rho\cos\phi = 2z$. So the surface is $x^2 + y^2 + z^2 - 2z + 1 = 1$, that is, $x^2 + y^2 + (z-1)^2 = 1$. The surface is the sphere with radius 1, center at $(x,y,z) = (0,0,1)$. It lies in the half-space $z \ge 0$, because $2z = x^2 + y^2 + z^2 \ge 0$.

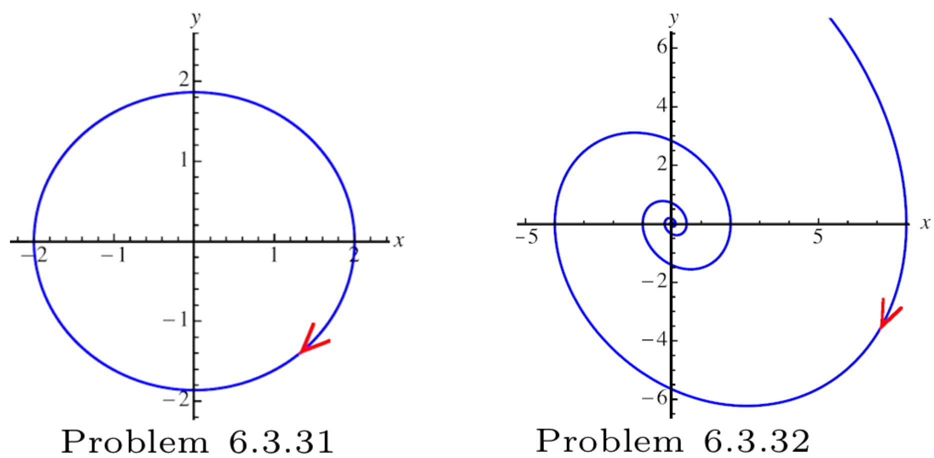


Figure 9: Figures useful for problems 6.3.4.31 and 6.3.4.32

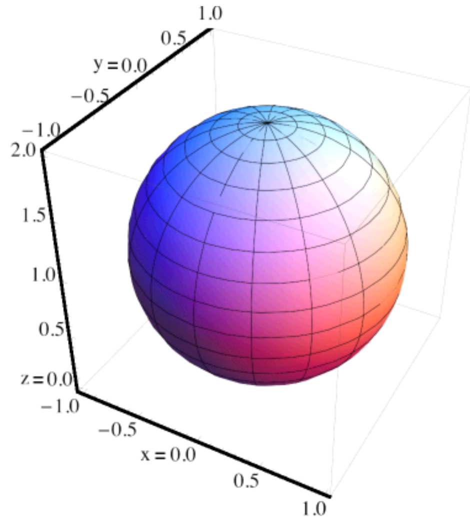


Figure 10: Answer 6.3.4.35

Section 6.4

6.4.5.1. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\Big[\dfrac{\partial}{\partial y}\Big[\sin(xy^2+3))\Big]\Big] = \dfrac{\partial}{\partial x}\Big[2xy\cos(xy^2+3)\Big] = 2y\cos(xy^2+3) - 2xy^3\sin(xy^2+3)$

6.4.5.3. That hyperbola is $z^2 = 4x^2 + (-2)^2$, on which implicit differentiation (Calculus I) gives $2z\dfrac{dz}{dx} = 8x$. At the point $(x,y,z) = (-1,-2,2\sqrt{2})$, $\dfrac{dz}{dx} = \dfrac{8x}{2z} = -\sqrt{2}$, so parametric equations of the tangent line to this hyperbola at that point are

$$\left\{\begin{array}{l} x = -1+t \\ y = -2 \\ z = 2\sqrt{2} - \sqrt{2}\,t \end{array}\right\},\ -\infty < t < \infty.$$

6.4.5.5. $\nabla f(x,y) = \dfrac{\partial}{\partial x}\Big[e^{-x^2+y}\Big]\hat{\imath} + \dfrac{\partial}{\partial y}\Big[e^{-x^2+y}\Big]\hat{\jmath} + e^{-x^2+y}\,\hat{\jmath}$

The directional derivative of $f(x,y)$ at the point $(1,2)$ in the direction of the vector $-\hat{\imath} + 2\,\hat{\jmath}$ is

$$(D_{\hat{\mathbf{u}}}f)(1,2) \triangleq \nabla f(1,2) \bullet \frac{1}{\sqrt{5}}(-\hat{\imath} + 2\,\hat{\jmath}) = \frac{1}{\sqrt{5}}\,(-2e^1\,\hat{\imath} + e^1\,\hat{\jmath}) \bullet (-\hat{\imath} + 2\,\hat{\jmath}) = \frac{4e}{\sqrt{5}}.$$

6.4.5.7. (a) $\nabla f(x,y,z) = \dfrac{\partial}{\partial x}\Big[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\Big]\hat{\imath} + \dfrac{\partial}{\partial y}\Big[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\Big]\hat{\jmath} + \dfrac{\partial}{\partial z}\Big[\dfrac{1}{\sqrt{x^2+y^2+z^2}}\Big]\hat{\boldsymbol{k}}$

$$= -\frac{1}{(x^2+y^2+z^2)^{3/2}}\left(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\boldsymbol{k}}\right).$$

(b) The directional derivative in the radial direction is

$$(D_{\hat{\mathbf{u}}}f)(x,y,z) \triangleq \nabla f(x,y,z) \bullet \frac{1}{\sqrt{x^2+y^2+z^2}}(x\hat{\imath}+y\hat{\jmath}+z\hat{\boldsymbol{k}}) = -\frac{1}{(x^2+y^2+z^2)^2}\,(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{\boldsymbol{k}}) \bullet (x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{\boldsymbol{k}}) = -1.$$

6.4.5.9. (a) $z = 3 - y^2 - 1^2$, that is, $z = 2 - y^2$.

(b) Differentiation (Calculus I) gives $\frac{dz}{dy} = -2y$. At the point $(x, y, z) = (1, 0.8, 1.36)$, $\frac{dz}{dy} = -1.6$, so parametric equations of the tangent line to this hyperbola at that point are

$$\left\{ \begin{array}{l} x = 1 \\ y = 0.8 + t \\ z = 1.36 - 1.6\,t \end{array} \right\}, \ -\infty < t < \infty.$$

(c) Pictorially, $\frac{\partial f}{\partial y}(1, 0.8)$ is the slope of part (b)'s tangent line to the curve $z = 2 - y^2$ in the plane $x = 1$ at the point $(x, y, z) = (1, 0.8, 1.36)$; see figure.

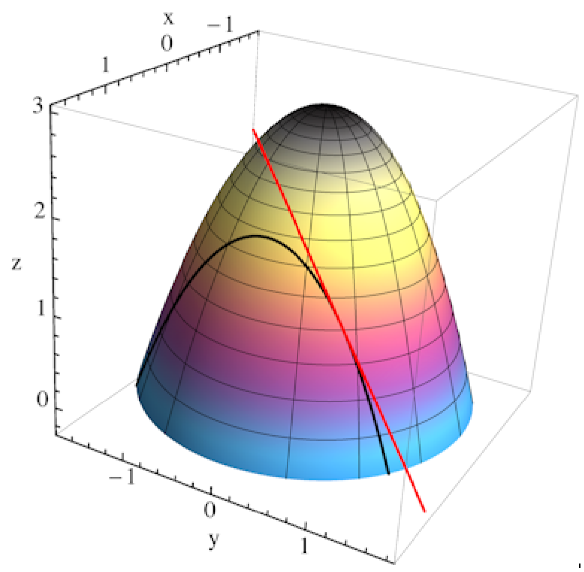


Figure 11: Answer for problem 6.4.5.9

6.4.5.11. $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\, x^{2y} \,\right] = \frac{\partial}{\partial y}\left[\, e^{\ln(x^{2y})} \,\right] = \frac{\partial}{\partial y}\left[\, e^{2y\ln x} \,\right] = 2\ln x\, e^{2y\ln x} = 2x^{2y}\ln x$, so
$\frac{dz}{dt}(2) = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\Big|_{t=2} = 2yx^{2y-1}\cdot 3 + 2x^{2y}\ln x\cdot(-2t)\Big|_{t=2} = 2\cdot 0\cdot 3^{-1}\cdot 3 + 2\cdot 3^0\ln 3\cdot(-4) = -8\ln 3.$

6.4.5.13. At $(u, v) = (-2, 1)$, $x = -2e^{-1}$ and $y = e^{-2}$.

(a) $\frac{\partial g}{\partial v}(-2, 1) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}\Big|_{u=-2,v=1}$

$$= \Big(\frac{1}{y} - 2xy\Big)\cdot e^{-v} + \Big(-\frac{x}{y^2} - x^2\Big)\cdot ve^u\Big|_{u=-2,v=1} = \Big(e^2 - 2(-2e^{-3})\Big)\cdot e^{-1} + \Big(-\frac{-2e^{-1}}{e^{-4}} - 4e^{-2}\Big)\cdot e^{-2}$$

$$= e + 4e^{-4} + 2e - 4e^{-4} = 3e.$$

(b) $\frac{\partial g}{\partial v}(-2, 1) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}\Big|_{u=-2,v=1}$

$$= \Big(\frac{1}{y} - 2xy\Big)\cdot\big(-ue^{-v} + \Big(-\frac{x}{y^2} - x^2\Big)\cdot e^u\Big|_{u=-2,v=1} = \Big(e^2 - 2(-2e^{-3}\Big)\cdot 2e^{-1} + \Big(-\frac{-2e^{-1}}{e^{-4}} - 4e^{-2}\Big)\cdot e^{-2}$$

$$= 2e + 8e^{-4} + 2e - 4e^{-4} = 4e + 4e^{-4}.$$

6.4.5.15. Note that $x = r\cos\theta$ and $y = r\sin\theta$. Use the multivariable chain rule to find

$$\frac{\partial z}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial z}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial z}{\partial y} = \cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial z}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial z}{\partial y} = -r\sin\theta\,\frac{\partial z}{\partial x} + r\cos\theta\,\frac{\partial z}{\partial y},$$

so the right hand side is

$$\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{1}{r}\cdot\frac{\partial z}{\partial \theta}\right)^2 = \left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)^2 + \left(-\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\,\frac{\partial z}{\partial y}\right)$$

$$= \cos^2\theta\cdot\left(\frac{\partial z}{\partial x}\right)^2 + 2\cos\theta\sin\theta\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + \sin^2\theta\cdot\left(\frac{\partial z}{\partial y}\right)^2 + \sin^2\theta\cdot\left(\frac{\partial z}{\partial x}\right)^2 - 2\cos\theta\sin\theta\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + \cos^2\theta\cdot\left(\frac{\partial z}{\partial y}\right)^2$$

$$= \left(\cos^2\theta + \sin^2\theta\right)\left(\frac{\partial z}{\partial x}\right)^2 + \left(\cos^2\theta + \sin^2\theta\right)\left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2,$$

which is the left hand side.

6.4.5.17. Define $f(x,y) = \ln\left(x^2 + y^3\right)$.

$$\ln\left((1.02)^2 + (2.99)^3\right) = f(1.02, 2.99) \approx f(1,3) + (1.02-1)\frac{\partial f}{\partial x}(1,3) + (2.99-3)\frac{\partial f}{\partial y}(1,3)$$

$$= \ln 28 + (0.02)\cdot\frac{2x}{x^2+y^3} - (0.01)\cdot\frac{3y^2}{x^2+y^3}\Big|_{x=1,y=3} = \ln 28 + (0.02)\cdot\frac{2}{28} - (0.01)\cdot\frac{27}{28} = \ln 28 - \frac{0.23}{28}$$

6.4.5.19. To find a potential function,

$$f = \int \frac{\partial f}{\partial x}\,\partial x = \int F_x\,\partial x = \int (yz - 2x)\partial x = xyz - x^2 + g(y,z),$$

where g is an arbitrary function of only y and z. Substitute this into $F_y = \dfrac{\partial f}{\partial y}$ to get

$$xz + \cos z = F_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\Big[xyz - x^2 + g(y,z)\Big] = xz + \frac{\partial g}{\partial y},$$

hence $\dfrac{\partial g}{\partial y} \equiv \cos z$, hence $g(y,z) = y\cos z + h(z)$, where $h(z)$ is an arbitrary function of z alone. Substitute $f = xyz - x^2 + y\cos z + h(z)$ into $F_z = \dfrac{\partial f}{\partial z}$ to get

$$xy - y\sin z = F_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\big[xyz - x^2 + y\cos z + h(z)\big] = xy - y\sin z + h'(z),$$

hence $h'(z) \equiv 0$, hence $h(z) = c$, where c is an arbitrary constant. So, a potential function for $\mathbf{F}$ is given by $f(x,y,z) = xyz - x^2 + y\cos z + c$, $c =$arb. const.

6.4.5.21. If we follow the process for finding a potential function, we will discover if the vector field is exact.

$$f = \int \frac{\partial f}{\partial x}\,\partial x = \int F_x\,\partial x = \int \cos y\,\partial x = x\cos y + g(y,z),$$

where g is an arbitrary function of only y and z. Substitute this into $F_y = \dfrac{\partial f}{\partial y}$ to get

$$-x\sin y + z = F_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big[x\cos y + g(y,z)\big] = -x\sin y + \frac{\partial g}{\partial y},$$

hence $\dfrac{\partial g}{\partial y} \equiv z$, hence $g(y,z) = yz + h(z)$, where $h(z)$ is an arbitrary function of z alone. Substitute $f = x\cos y + yz + h(z)$ into $F_z = \dfrac{\partial f}{\partial z}$ to get

$$y - 1 = F_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\big[\, x\cos y + yz + h(z)\,\big] = y + h'(z),$$

hence $h'(z) = -1$, hence $h(z) = -z + c$, where c is an arbitrary constant. So, a potential function for $\mathbf{F}$ is given by $f(x,y,z) = x\cos y + yz - z + c$, $c =$arb. const.

6.4.5.23. As usual, denote the velocity by $\mathbf{v} = \dot{\mathbf{r}}$. We calculate

$$\rho(t) = ||\mathbf{r}(t)|| \Longrightarrow \rho^2 = ||\mathbf{r}||^2 = \mathbf{r} \bullet \mathbf{r} \Longrightarrow 2\rho\frac{d\rho}{dt} = 2\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} \Longrightarrow \frac{d\rho}{dt} = \frac{1}{\rho}(\mathbf{r} \bullet \mathbf{v}) = \rho^{-1}(\mathbf{r} \bullet \mathbf{v}).$$

Concerning the desired identity, the product rule gives

$$LHS = \frac{d}{dt}\Big[\rho^{-3}\mathbf{r}\Big] = -3\rho^{-4}\frac{d\rho}{dt}\mathbf{r} + \rho^{-3}\mathbf{v} = -3\rho^{-4}\rho^{-1}(\mathbf{r} \bullet \mathbf{v})\mathbf{r} + \rho^{-3}\mathbf{v} = -3\rho^{-5}(\mathbf{r} \bullet \mathbf{v})\mathbf{r} + \rho^{-3}\mathbf{v}.$$

On the other hand, for some constants α and β,

$$RHS = \rho^{-5}\left(\alpha\,\mathbf{r} \times (\mathbf{v} \times \mathbf{r}) + \beta\,\rho^2\mathbf{v}\right) = \beta\rho^{-3}\mathbf{v} + \alpha\,\rho^{-5}\mathbf{r} \times (\mathbf{v} \times \mathbf{r}).$$

The vector triple product satisfies

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\,\mathbf{B} - (\mathbf{A} \bullet \mathbf{B})\,\mathbf{C},$$

so

$$RHS = \beta\rho^{-3}\mathbf{v} + \alpha\,\rho^{-5}\left((\mathbf{r} \bullet \mathbf{r})\mathbf{v} - (\mathbf{r} \bullet \mathbf{v})\mathbf{r}\right) = \beta\rho^{-3}\mathbf{v} + \alpha\,\rho^{-5}\left(\rho^2\right)\mathbf{v} - \alpha\,\rho^{-5}(\mathbf{r} \bullet \mathbf{v})\mathbf{r}$$

$$= -\alpha\,\rho^{-5}(\mathbf{r} \bullet \mathbf{v})\mathbf{r} + (\beta + \alpha)\rho^{-3}\mathbf{v}.$$

$LHS = RHS$ is an identity if, and only if, both $-3 = -\alpha$ and $1 = \beta + \alpha$ are true. If we choose $\alpha = 3$ and $\beta = -2$ then $LHS = RHS$ is an identity, that is, the desired identity is

$$\frac{d}{dt}\Big[\rho^{-3}\mathbf{r}\Big] = \rho^{-5}\left(\underline{\ 3\ }\,\mathbf{r} \times (\mathbf{v} \times \mathbf{r}) + \underline{\ -2\ }\,\rho^2\mathbf{v}\right).$$

Section 6.5

6.5.1.1. Let $f(x,y,z) \triangleq 3x^2 - y^2 + xz$. A normal vector to the surface is given by

$$\mathbf{n} = \boldsymbol{\nabla} f(1,2,8) = (6x+z)\,\hat{\boldsymbol{\imath}} - 2y\,\hat{\boldsymbol{\jmath}} + x\,\hat{\boldsymbol{k}}\Big|_{(x,y,z)=(1,2,8)} = 14\,\hat{\boldsymbol{\imath}} - 4\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}},$$

so an equation of the tangent plane is given by

$$14(x-1) - 4(y-2) + (z-8) = 0,$$

that is, $14x - 4y + z = 14$.

6.5.1.3. <u>Method 1</u>: Let $f(x,y,z) \triangleq x^2 - 3y^2 + xy - z$.
A normal vector to the surface is given by

$$\mathbf{n} {=} \boldsymbol{\nabla} f(1,-1,-3) {=} (2x+y)\,\hat{\boldsymbol{\imath}} + (-6y+x)\,\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}\Big|_{(x,y,z)=(1,-1,-3)} {=}\, \hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}},$$

so an equation of the tangent plane is given by

$$(x-1) + 7(y+1) - (z+3) = 0,$$

that is, $x + 7y - z = -3$.
<u>Method 2</u>: Define $g(x,y) \triangleq x^2 - 3y^2 + xy$. A normal vector to the surface is given by

$$\mathbf{n} = -\frac{\partial g}{\partial x}\,\hat{\boldsymbol{\imath}} - \frac{\partial g}{\partial y}\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\Big|_{(x,y)=(1,-1)} = -(2x+y)\hat{\boldsymbol{\imath}} - (-6y+x)\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\Big|_{(x,y)=(1,-1)} = -\hat{\boldsymbol{\imath}} - 7\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}},$$

so an equation of the tangent plane is given by

$$-(x-1) - 7(y+1) + (z+3) = 0,$$

that is, $x + 7y - z = -3$.

6.5.1.5. A normal vector to the surface is given by

$$\mathbf{n} = \left(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\,\hat{\boldsymbol{k}}\right) \times \left(3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right) = -3\,\hat{\boldsymbol{\imath}} - 7\,\hat{\boldsymbol{\jmath}} - 5\,\hat{\boldsymbol{k}}.$$

$\mathbf{n} = \dfrac{1}{\sqrt{83}}\,(3\,\hat{\boldsymbol{\imath}} + 7\,\hat{\boldsymbol{\jmath}} + 5\,\hat{\boldsymbol{k}})$ is an upward pointing unit vector that is perpendicular to the plane.

6.5.1.7. (a) A vector normal to the surface $0 = f(x,y,z) \triangleq z - g(x,y)$ is given by

$$\mathbf{n} {=} \boldsymbol{\nabla} f\Big|_{\text{at } P_0} = -\frac{\partial g}{\partial x}(x_0,y_0)\,\hat{\boldsymbol{\imath}} - \frac{\partial g}{\partial y}(x_0,y_0)\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\,.$$

(b) $z^2 = 2^2\left(\dfrac{x^2}{5^2} + \dfrac{y^2}{3^2} - 1\right)$ so $z = \pm 2 \cdot \sqrt{\dfrac{x^2}{5^2} + \dfrac{y^2}{3^2} - 1}$.
Near the point $P_0 = (-\frac{15}{\sqrt{17}}, \frac{15}{\sqrt{17}}, 2)$, the surface is given by

$$z = g(x,y) \triangleq 2 \cdot \sqrt{\frac{x^2}{5^2} + \frac{y^2}{3^2} - 1}.$$

Define

$$f(x,y,z) \triangleq z - 2 \cdot \sqrt{\frac{x^2}{5^2} + \frac{y^2}{3^2} - 1},$$

so our surface is $f(x,y,z) = 0$ near the point P_0.

(a) A normal vector to the surface at $P_0=(-\frac{15}{\sqrt{17}},\frac{15}{\sqrt{17}},2)$ is

$$\mathbf{n}=\boldsymbol{\nabla}f(-\frac{15}{\sqrt{17}},\frac{15}{\sqrt{17}},2)=-2\cdot\frac{1}{2}\left(\frac{x^2}{5^2}+\frac{y^2}{3^2}-1\right)^{-1/2}\cdot\frac{2x}{25}\,\hat{\boldsymbol{\imath}}\left(\frac{x^2}{5^2}+\frac{y^2}{3^2}-1\right)^{-1/2}\cdot\frac{2y}{9}\,\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}\Big|_{(x,y)=(-\frac{15}{\sqrt{17}},\frac{15}{\sqrt{17}})}$$

$$=...=\frac{6}{5\sqrt{17}}\,\hat{\boldsymbol{\imath}}-\frac{10}{3\sqrt{17}}\,\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}.$$

(b) Yes, this method produces a normal vector which agrees with (6.53) except for a factor of -1.

6.5.1.9. At any point (x,y,z) on the surface $0=\phi(x,y,z)\triangleq x^2-y^2+z^2-4$, a normal vector is given by

$$\widehat{\mathbf{n}}_1\triangleq\boldsymbol{\nabla}\phi(x,y,z)=2x\,\hat{\boldsymbol{\imath}}-2y\,\hat{\boldsymbol{\jmath}}+2z\,\hat{\boldsymbol{k}};$$

note that $\boldsymbol{\nabla}\phi(x,y,z)\neq\mathbf{0}$ because $(x,y,z)=(0,0,0)$ cannot be on the surface $x^2-y^2+z^2-4=0$.

At any point (x,y,z) on the surface $0=\psi(x,y,z)\triangleq z-\dfrac{1}{xy^2}$, a normal vector is given by

$$\widehat{\mathbf{n}}_2\triangleq\boldsymbol{\nabla}\phi(x,y,z)=\frac{1}{x^2y^2}\,\hat{\boldsymbol{\imath}}+\frac{2}{xy^3}\,\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}.$$

Because

$$\widehat{\mathbf{n}}_1\bullet\widehat{\mathbf{n}}_2=2x\cdot\left(\frac{1}{x^2y^2}\right)-2y\cdot\left(\frac{2}{xy^3}\right)+2z\cdot 1=\frac{2}{xy^2}-\frac{4}{xy^2}+2z=-\frac{2}{xy^2}+2z=2\psi(x,y,z)=0,$$

we see that the surfaces $0=\phi(x,y,z)\triangleq x^2-y^2+z^2-4$ and $0=\psi(x,y,z)=z-\dfrac{1}{xy^2}$ are perpendicular wherever they intersect.

6.5.1.11. The level sets are given by $k=f(x,y)=2x^2+y^2$. For $k<0$, the level set is empty. For $k=0$, the level set is the single point $(0,0$. For $k>0$, the level set is the ellipse

$$\frac{x^2}{k/2}+\frac{y^2}{k}=1.$$

At a point $\mathbf{r}_0=x_0\,\hat{\boldsymbol{\imath}}+y_0\,\hat{\boldsymbol{\jmath}}$ on a level set, a normal vector is given by

$$\boldsymbol{\nabla}f(\mathbf{r}_0)=\frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}}+\frac{\partial f}{\partial x}\,\hat{\boldsymbol{\jmath}}\Big|_{at\,\mathbf{r}_0}=4x_0\,\hat{\boldsymbol{\imath}}+2y_0\,\hat{\boldsymbol{\jmath}},$$

assuming the latter is non-zero.

Examples of normal vectors to level sets:

(1) for $k=1$, at the point $(\dfrac{1}{\sqrt{2}},0)$, $2\sqrt{2}\,\hat{\boldsymbol{\imath}}$ is a normal vector,

(2) for $k=4$, at the point $(1,-\sqrt{2})$, $4\,\hat{\boldsymbol{\imath}}-2\sqrt{2}\,\hat{\boldsymbol{\jmath}}$ is a normal vector,

(3) for $k=16$, at the point $(-\sqrt{2},2\sqrt{3})$, $-4\sqrt{2}\,\hat{\boldsymbol{\imath}}+4\sqrt{3}\,\hat{\boldsymbol{\jmath}}$ is a normal vector.

6.5.1.13. The level sets are given by $k=f(x,y)=\dfrac{x^2+y^2}{4x}$. For $k=0$, the level set is empty, because when the numerator, x^2+y^2, is zero, so is the denominator, making $f(0,0)$ undefined. For $k\neq 0$, the level set is where $4kx=x^2+y^2$, that is, $(x-2k)^2+y^2=(2k)^2$, that is, the level set is the ellipse

$$\frac{(x-2k)^2}{(2k)^2}+\frac{y^2}{(2k)^2}=1.$$

At a point $\mathbf{r}_0=x_0\,\hat{\boldsymbol{\imath}}+y_0\,\hat{\boldsymbol{\jmath}}$ on a level sets, a normal vector is given by

$$\boldsymbol{\nabla}f(\mathbf{r}_0)=\frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}}+\frac{\partial f}{\partial x}\,\hat{\boldsymbol{\jmath}}\Big|_{at\,\mathbf{r}_0}=\frac{x_0^2-y_0^2}{4x_0^2}\,\hat{\boldsymbol{\imath}}+\frac{y_0}{2x_0}\,\hat{\boldsymbol{\jmath}},$$

assuming the latter is non-zero.

Examples of normal vectors to level sets:

(1) for $k = \frac{1}{2}$, at the point $(1,1)$, $\dfrac{1}{2}\,\hat{\boldsymbol{\jmath}}$ is a normal vector, and $\widehat{\mathbf{n}} = \hat{\boldsymbol{\jmath}}$ is a unit normal vector,

(2) for $k = -1$, at the point $(-3, \sqrt{3})$, $\dfrac{1}{6}\left(\hat{\boldsymbol{\imath}} - \sqrt{3}\,\hat{\boldsymbol{\jmath}}\right)$ is a normal vector, and $\widehat{\mathbf{n}} = \dfrac{1}{2}\left(\hat{\boldsymbol{\imath}} - \sqrt{3}\,\hat{\boldsymbol{\jmath}}\right)$ is a unit normal vector

(3) for $k = 2$, at the point $(5, -\sqrt{15})$, $\dfrac{1}{10}\left(\hat{\boldsymbol{\imath}} - \sqrt{15}\,\hat{\boldsymbol{\jmath}}\right)$ is a normal vector, and $\widehat{\mathbf{n}} = \dfrac{1}{4}\left(\hat{\boldsymbol{\imath}} - \sqrt{15}\,\hat{\boldsymbol{\jmath}}\right)$ is a unit normal vector.

6.5.1.15. (a) The unit vector in the direction of $\boldsymbol{\nabla} f = 2x\,\hat{\boldsymbol{\imath}} + 4y\,\hat{\boldsymbol{\jmath}}$ is the direction of greatest increase of f. In particular, $\boldsymbol{\nabla} f(2,1) = 4\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}$, so $\dfrac{1}{\sqrt{2}}\,(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$ is the direction of greatest increase of f at the point $(x,y) = (2,1)$.

(b) The point $(x,y) = (2,1)$ is on the level curve $f(x,y) = f(2,1) = 6$, so $k = 6$. The level curve $6 = x^2 + 2y^2$ is the ellipse

$$\frac{x^2}{(\sqrt{6})^2} + \frac{y^2}{(\sqrt{3})^2} = 1.$$

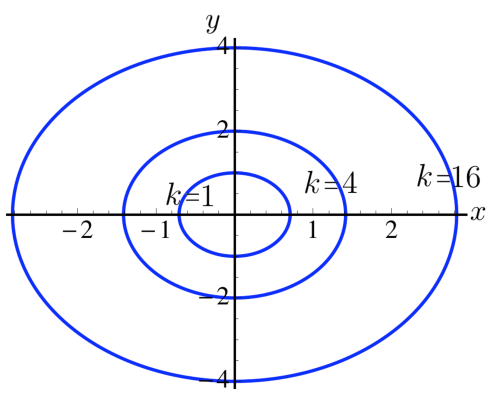


Figure 12: Answer 6.5.1.11

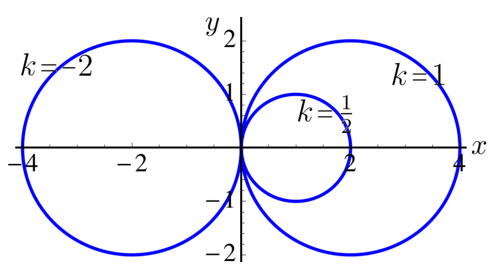


Figure 13: Answer 6.5.1.13

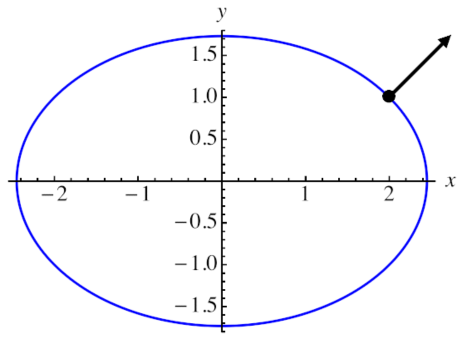


Figure 14: Answer 6.5.1.15(b)

Section 6.6

6.6.5.1. $\mathbf{A} = \hat{\boldsymbol{\imath}} - 2\,\hat{\boldsymbol{k}}$ and $\mathbf{B} = 2\,\hat{\boldsymbol{\imath}} + 3\,\hat{\boldsymbol{\jmath}}$ are sides determining the parallelogram, whose area is $||\mathbf{A}\times\mathbf{B}|| = ||6\,\hat{\boldsymbol{\imath}} - 4\,\hat{\boldsymbol{\jmath}} + 3\,\hat{\boldsymbol{k}}|| = \sqrt{61}$.

6.6.5.3. see sketch; $Volume = \left|\det\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2\end{bmatrix}\right| = 24$

6.6.5.5. see sketch of parallelopiped determined by sides $\mathbf{A} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$, $\mathbf{B} = 2\,\hat{\boldsymbol{\jmath}}$, and $\mathbf{C} = \hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 3\,\hat{\boldsymbol{k}}$;

$$Volume = \left|\det\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 3\end{bmatrix}\right| = 6$$

6.6.5.7. see sketch

6.6.5.9. see sketch

6.6.5.11. Let $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$. The picture has $D \mapsto D'$ and $B \mapsto B'$, so, respectively,

$$\begin{bmatrix} a \\ c\end{bmatrix} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} 1 \\ 0\end{bmatrix} = \begin{bmatrix} 2 \\ 1\end{bmatrix} \quad\text{and}\quad \begin{bmatrix} b \\ d\end{bmatrix} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} 0 \\ 1\end{bmatrix} = \begin{bmatrix} 0 \\ 1\end{bmatrix}$$

so $A = \begin{bmatrix} 2 & 0 \\ 1 & 1\end{bmatrix}$. [You can check your final conclusion by seeing if A has $C \mapsto C'$.]

6.6.5.13. $|\det(A)| = \left|\det\begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 5 \\ 1 & 0 & 3\end{bmatrix}\right| = ... = |16| = 16$ is the factor by which volume is multiplied.

6.6.5.15. (a) <u>Exs</u>. $A_1 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}$ implements $R_1 \leftrightarrow R_2$, $A_2 \triangleq \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$ implements $-2R_1 + R_2 \to R_2$, and $A_3 \triangleq \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$ implements $2R_1 \to R_1$.

(b) $|\det(A_1)| = 1$, $|\det(A_2)| = 1$, and $|\det(A_3)| = 2$, respectively, are the factors by which A_1, A_2, and A_3 multiply volume.

(c) <u>Ex</u>. Conjecture: The elementary row operation of adding a multiple of a row into another row or interchanging rows does not affect volume, but the operation of multiplying a row by k has the effect of multiplying volume by $|k|$.

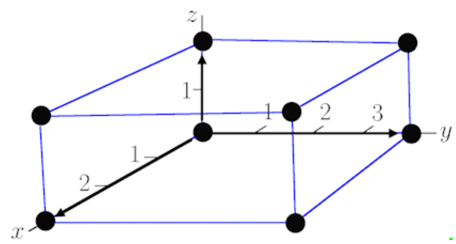


Figure 15: Answer for problem 6.6.5.3

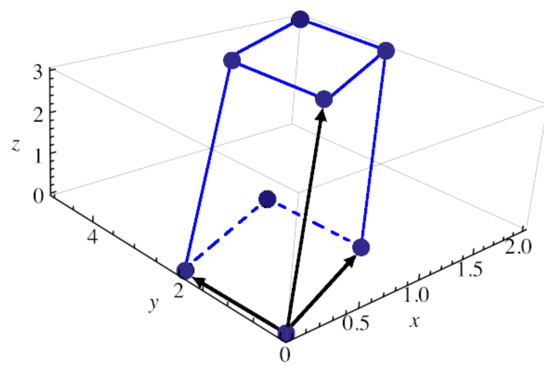


Figure 16: Answer for problem 6.6.5.5

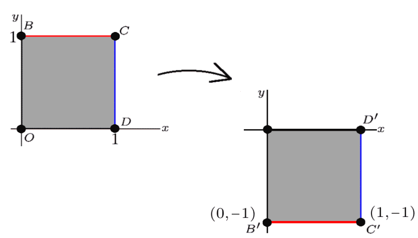


Figure 17: Problem 6.6.5.7

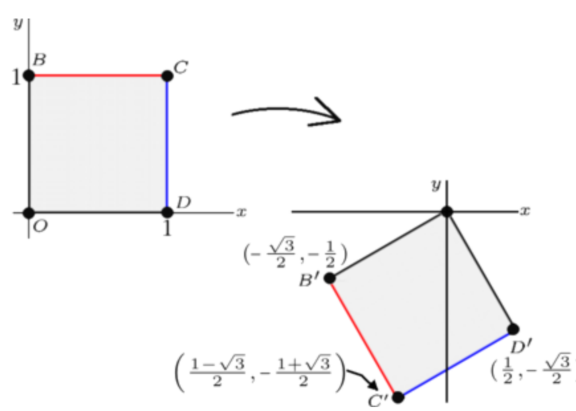


Figure 18: Problem 6.6.5.9

Section 6.7

6.7.6.1. $\boldsymbol{curl}\left(x^2y\,\hat{\boldsymbol{\imath}}+x^2z\,\hat{\boldsymbol{\jmath}}+x^3\,\hat{\boldsymbol{k}}\right)$

$$=\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & x^2z & x^3\end{bmatrix}=\left(0-x^2\right)\hat{\boldsymbol{\imath}}-\left(3x^2-0\right)\hat{\boldsymbol{\jmath}}-\left(2xz-x^2\right)\hat{\boldsymbol{k}}=-x^2\hat{\boldsymbol{\imath}}-3x^2\hat{\boldsymbol{\jmath}}+(2xz-x^2)\hat{\boldsymbol{k}}.$$

6.7.6.3. (a) $div(\mathbf{F})=\boldsymbol{\nabla}\bullet\mathbf{F}=\dfrac{\partial}{\partial x}\Big[\cos(xy^2)\Big]+\dfrac{\partial}{\partial y}\Big[\sin(y+z^2)\Big]+\dfrac{\partial}{\partial z}\Big[\ln|x-z|\Big]$

$$=-y^2\sin(xy^2)+\cos(y+z^2)-\frac{1}{x-z}.$$

(b) $curl(\mathbf{F})=\boldsymbol{\nabla}\times\mathbf{F}=\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \cos(xy^2) & \sin(y+z^2) & \ln|x-z|\end{bmatrix}$

$$=\left(0-2z\cos(y+z^2)\right)\hat{\boldsymbol{\imath}}-\left(\frac{1}{x-z}-0\right)\hat{\boldsymbol{\jmath}}+\left(0-\left(-2xy\sin(xy^2)\right)\right)\hat{\boldsymbol{k}}=-2z\cos(y+z^2)\,\hat{\boldsymbol{\imath}}-\frac{1}{x-z}\,\hat{\boldsymbol{\jmath}}+2xy\sin(xy^2)\,\hat{\boldsymbol{k}}.$$

6.7.6.5. $\mathbf{F}\triangleq\rho^{-2}(z\,\hat{\boldsymbol{\imath}}+x\,\hat{\boldsymbol{\jmath}}+y\,\hat{\boldsymbol{k}})=\dfrac{1}{x^2+y^2+z^2}\left(z\,\hat{\boldsymbol{\imath}}+x\,\hat{\boldsymbol{\jmath}}+y\,\hat{\boldsymbol{k}}\right).$

(a) $div(\mathbf{F})=\dfrac{\partial}{\partial x}\Big[\dfrac{z}{x^2+y^2+z^2}\Big]+\dfrac{\partial}{\partial y}\Big[\dfrac{x}{x^2+y^2+z^2}\Big]+\dfrac{\partial}{\partial z}\Big[\dfrac{y}{x^2+y^2+z^2}\Big]=\dfrac{-2\,(xz+xy+yz)}{\left(x^2+y^2+z^2\right)^2}.$

(b) $curl(\mathbf{F})=\boldsymbol{\nabla}\times\mathbf{F}=\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{z}{x^2+y^2+z^2} & \dfrac{x}{x^2+y^2+z^2} & \dfrac{y}{x^2+y^2+z^2}\end{bmatrix}$

$$=\left(\frac{x^2-y^2+z^2}{\left(x^2+y^2+z^2\right)^2}-\frac{-2xz}{\left(x^2+y^2+z^2\right)^2}\right)\hat{\boldsymbol{\imath}}-\left(\frac{-2xy}{\left(x^2+y^2+z^2\right)^2}-\frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^2}\right)\hat{\boldsymbol{\jmath}}$$

$$+\left(\frac{-x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}-\frac{-2yz}{\left(x^2+y^2+z^2\right)^2}\right)\hat{\boldsymbol{k}}$$

$$=\frac{1}{\left(x^2+y^2+z^2\right)^2}\left((x^2-y^2+z^2+2xz)\hat{\boldsymbol{\imath}}+(x^2+y^2-z^2+2xy)\hat{\boldsymbol{\jmath}}+\left(-x^2+y^2+z^2+2yz\right)\hat{\boldsymbol{k}}\right)$$

I think it's more difficult to do the problem by first rewriting $x,y,z,\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ in terms of spherical coordinates.

6.7.6.7. $\mathbf{v} = \frac{\partial}{\partial r}\Big[\Big(U\Big(r+\frac{a^2}{r}\Big)\cos(\theta-\alpha)\Big)\Big]\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\Big[U\Big(r+\frac{a^2}{r}\Big)\cos(\theta-\alpha)\Big]\hat{\mathbf{e}}_\theta$

$$= U\left(\Big(1-\frac{a^2}{r^2}\Big)\cos(\theta-\alpha)\,\hat{\mathbf{e}}_r - \Big(1+\frac{a^2}{r^2}\Big)\sin(\theta-\alpha)\,\hat{\mathbf{e}}_\theta\right)$$

6.7.6.9. (i) (a) In rectangular coordinates,

$$div(\mathbf{F}) = \left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right) \bullet \left(x(x^2+y^2)\hat{\boldsymbol{\imath}} + y(x^2+y^2-2z)\hat{\boldsymbol{\jmath}} + (z^2-3x^2-3y^2)\hat{\boldsymbol{k}}\right)$$

$$= \frac{\partial}{\partial x}\Big[x(x^2+y^2)\Big] + \frac{\partial}{\partial y}\Big[y(x^2+y^2-2z)\Big] + \frac{\partial}{\partial z}\Big[z^2-3x^2-3y^2\Big] = \big(3x^2+y^2\big) + \big(x^2+3y^2-2z\big) + \big(2z\big)$$

$$= 4(x^2+y^2).$$

(b) In cylindrical coordinates, $div(\mathbf{F}) = 4(x^2+y^2) = 4r^2$. Note that because $\mathbf{F}$ is given in terms of rectangular coordinates and rectangular coordinates basis vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, it makes sense to take the divergence in rectangular coordinates and then convert it to be in cylindrical coordinates.

(ii) (a) In rectangular coordinates,

$$curl(\mathbf{F}) = \left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right) \times \left(x(x^2+y^2)\hat{\boldsymbol{\imath}} + y(x^2+y^2-2z)\hat{\boldsymbol{\jmath}} + (z^2-3x^2-3y^2)\hat{\boldsymbol{k}}\right)$$

$$= \left(\frac{\partial}{\partial y}\Big[z^2-3x^2-3y^2\Big] - \frac{\partial}{\partial z}\Big[y(x^2+y^2-2z)\Big]\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial}{\partial z}\Big[x(x^2+y^2)\Big] - \frac{\partial}{\partial x}\Big[z^2-3x^2-3y^2\Big]\right)\hat{\boldsymbol{\jmath}}$$

$$+ \left(\frac{\partial}{\partial x}\Big[y(x^2+y^2-2z)\Big] - \frac{\partial}{\partial y}\Big[x(x^2+y^2)\Big]\right)\hat{\boldsymbol{k}}$$

$$= (-6y-(-2y))\,\hat{\boldsymbol{\imath}} + (0-(-6x))\,\hat{\boldsymbol{\jmath}} + (2xy-2xy)\,\hat{\boldsymbol{k}} = -4y\hat{\boldsymbol{\imath}} + 6x\hat{\boldsymbol{\jmath}}.$$

(b) In cylindrical coordinates, using Table 6.1 on p. 407,

$$curl(\mathbf{F}) = -4y\hat{\boldsymbol{\imath}} + 6x\hat{\boldsymbol{\jmath}} = -4r\sin\theta\,(\cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta) + 6r\cos\theta\,(\sin\theta\,\hat{\mathbf{e}}_r + \cos\theta\,\hat{\mathbf{e}}_\theta)$$

$$= 2r\cos\theta\sin\theta\,\hat{\mathbf{e}}_r + 2r\left(2\sin^2\theta + 3\cos^2\theta\right)\hat{\mathbf{e}}_\theta = r\sin 2\theta\,\hat{\mathbf{e}}_r + 2r(2+\cos^2\theta)\hat{\mathbf{e}}_\theta = r\sin 2\theta\,\hat{\mathbf{e}}_r + r(5+\cos 2\theta)\hat{\mathbf{e}}_\theta.$$

Again, it makes sense to take the curl in rectangular coordinates and then convert it to be in cylindrical coordinates.

6.7.6.11. (i) (a) In rectangular coordinates,

$$div(\mathbf{F}) = \left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right) \bullet$$

$$\left(x(x^2+y^2+z^2)\hat{\boldsymbol{\imath}} + \big(y(x^2+y^2+z^2) - 2(x^2+y^2)\big)\hat{\boldsymbol{\jmath}} + \big(z^2 - z(x^2+y^2+z^2)\big)\,\hat{\boldsymbol{k}}\right)$$

$$= \frac{\partial}{\partial x}\Big[x(x^2+y^2+z^2)\Big] + \frac{\partial}{\partial y}\Big[y(x^2+y^2+z^2) - 2(x^2+y^2)\Big] + \frac{\partial}{\partial z}\Big[z^2 - z(x^2+y^2+z^2)\Big]$$

$$= 3x^2 + \cancel{y^2} + z^2 + \cancel{x^2} + 3y^2 + z^2 - 4y + 2z - \cancel{x^2} - \cancel{y^2} - 3z^2 = 3x^2 + 3y^2 - z^2 - 4y + 2z$$

(b) In spherical coordinates,

$$div(\mathbf{F}) = 3x^2+3y^2-z^2-4y+2z = 3\left(x^2+y^2+z^2\right)-4z^2-4y+2z = \rho^2(3-4\,\cos^2\phi)+\rho\,(-4\sin\phi\,\sin\theta+2\cos\phi).$$

Note that because $\mathbf{F}$ is given in terms of rectangular coordinates and rectangular coordinates basis vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, it makes sense to take the divergence in rectangular coordinates and then convert it to be in spherical coordinates.

(ii) (a) In rectangular coordinates,

$$curl(\mathbf{F}) = \left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right) \times$$

$$\Big(x(x^2+y^2+z^2)\hat{\boldsymbol{\imath}} + \big(y(x^2+y^2+z^2) - 2(x^2+y^2)\big)\hat{\boldsymbol{\jmath}} + \big(z^2 - z(x^2+y^2+z^2)\big)\,\hat{\boldsymbol{k}}\Big)$$

$$= \left(\frac{\partial}{\partial y}\Big[\,z^2 - z(x^2+y^2+z^2)\Big] - \frac{\partial}{\partial z}\Big[\,y(x^2+y^2+z^2) - 2(x^2+y^2)\Big]\right)\hat{\boldsymbol{\imath}}$$

$$+\left(\frac{\partial}{\partial z}\Big[\,x(x^2+y^2+z^2)\Big] - \frac{\partial}{\partial x}\Big[\,z^2 - z(x^2+y^2+z^2)\Big]\right)\hat{\boldsymbol{\jmath}}$$

$$+\left(\frac{\partial}{\partial x}\Big[\,y(x^2+y^2+z^2) - 2(x^2+y^2)\Big] - \frac{\partial}{\partial y}\Big[\,x(x^2+y^2+z^2)\Big]\right)\hat{\boldsymbol{k}}$$

$$= (-2yz - 2yz)\,\hat{\boldsymbol{\imath}} + (2xz + 2xz)\,\hat{\boldsymbol{\jmath}} + (2xy - 4x - 2xy))\,\hat{\boldsymbol{k}} = -4yz\,\hat{\boldsymbol{\imath}} + 4xz\,\hat{\boldsymbol{\jmath}} - 4x\,\hat{\boldsymbol{k}}.$$

(b) In spherical coordinates, using Table 6.2 on p. 407,

$$curl(\mathbf{F}) = -4yz\,\hat{\boldsymbol{\jmath}} + 4xz\,\hat{\boldsymbol{\jmath}} - 4x\,\hat{\boldsymbol{k}}$$

$$= -4(\rho\sin\phi\,\sin\theta)(\rho\cos\phi)\,(\sin\phi\,\cos\theta\,\hat{\mathbf{e}}_\rho + \cos\phi\,\cos\theta\,\hat{\mathbf{e}}_\phi - \sin\theta\,\hat{\mathbf{e}}_\theta)$$

$$+4(\rho\sin\phi\,\cos\theta)(\rho\cos\phi)\,(\sin\phi\,\sin\theta\,\hat{\mathbf{e}}_\rho + \cos\phi\,\sin\theta\,\hat{\mathbf{e}}_\phi + \cos\theta\,\hat{\mathbf{e}}_\theta) - 4(\rho\sin\phi\,\cos\theta)\,(\cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi)$$

$$= \rho^2\Big(\big(-\sin^2\phi\,\cos\phi\,\cos\theta\,\sin\theta + \sin^2\phi\,\cos\phi\,\cos\theta\,\sin\theta\big)\,\hat{\mathbf{e}}_\rho$$

$$+\big(-\sin\phi\,\cos^2\phi\,\cos\theta\,\sin\theta + \sin\phi\,\cos^2\phi\,\cos\theta\,\sin\theta\big)\,\hat{\mathbf{e}}_\phi + \big(4\sin\phi\,\cos\phi\sin^2\theta + 4\sin\phi\,\cos\phi\cos^2\theta\big)\,\hat{\mathbf{e}}_\theta\Big)$$

$$+\rho\,\big(-4\sin\phi\,\cos\phi\,\cos\theta\,\hat{\mathbf{e}}_\rho + 4\sin^2\phi\,\cos\theta\,\hat{\mathbf{e}}_\phi\ \big)\,.$$

$$= 4\rho^2\sin\phi\,\cos\phi\,\hat{\mathbf{e}}_\theta - 4\rho\sin\phi\cos\theta\,(\cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi\ )\,.$$

Again, it makes sense to take the curl in rectangular coordinates and then convert it to be in spherical coordinates.

6.7.6.13. $\mathbf{F} = \rho^{-2}\mathbf{r} = \rho^{-2}(\rho\,\hat{\mathbf{e}}_\rho) = \rho^{-1}\hat{\mathbf{e}}_\rho$, so $div(\mathbf{F}) = \dfrac{1}{\rho^2}\,\dfrac{\partial}{\partial\rho}\Big[\,\rho^2\rho^{-1}\Big] + \dfrac{1}{\rho\sin\phi}\,\dfrac{\partial}{\partial\phi}\Big[\,0\Big] + \dfrac{1}{\rho\sin\phi}\,\dfrac{\partial}{\partial\theta}\Big[\,0\Big] = \rho^{-2}.$

6.7.6.15. <u>Method 1</u>: A normal to the surface $z = g(x,y) \triangleq 2r = 2\sqrt{x^2+y^2}$ is given by

$$\mathbf{n} = -\frac{\partial g}{\partial x}\,\hat{\boldsymbol{\imath}} - \frac{\partial g}{\partial x}\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}} = -\frac{2x}{r}\,\hat{\boldsymbol{\imath}} - \frac{2y}{r}\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}.$$

At the point $(x,y,z) = (-\sqrt{3},1,4)$, $\mathbf{n} = \sqrt{3}\,\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$, so an equation of the tangent plane is given by $\sqrt{3}(x+\sqrt{3}) - (y-1) + (z-4) = 0$, that is, $\sqrt{3}\,x - y + z = 0$.

<u>Method 2</u>: A normal to the surface $f(r,\theta,z) \triangleq 2r - z = 0$ is given by

$$\mathbf{n} = \boldsymbol{\nabla} f = \left(\frac{\partial}{\partial r}\Big[\,2r - z\Big]\right)\hat{\mathbf{e}}_r + \left(\frac{1}{r}\,\frac{\partial}{\partial\theta}\Big[\,2r - z\Big]\right)\hat{\mathbf{e}}_\theta + \left(\frac{\partial}{\partial z}\Big[\,2r - z\Big]\right)\hat{\mathbf{e}}_z = 2\hat{\mathbf{e}}_r - \hat{\mathbf{e}}_z$$

$$= 2(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) - \hat{\boldsymbol{k}} = \frac{2x}{r}\,\hat{\boldsymbol{\imath}} + \frac{2y}{r}\,\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}.$$

At the point $(x,y,z) = (-\sqrt{3},1,4)$, $\mathbf{n} = -\sqrt{3}\,\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$, so an equation of the tangent plane is given by $-\sqrt{3}(x+\sqrt{3}) + (y-1) - (z-4) = 0$, that is, $\sqrt{3}(x+\sqrt{3}) - (y-1) + (z-4) = 0$, that is, $\sqrt{3}\,x - y + z = 0$.

6.7.6.17. Using the product rule, the left hand side of (6.66)(6) is

$$\begin{aligned}\nabla \bullet (\mathbf{F}\times\mathbf{G}) =& \nabla \bullet \big((F_yG_z - F_zG_y)\hat{\boldsymbol{\imath}} - (F_xG_z - F_zG_x)\hat{\boldsymbol{\jmath}} + (F_xG_y - F_yG_x)\hat{\boldsymbol{k}}\big)\\ =& \frac{\partial(F_yG_z - F_zG_y)}{\partial x} - \frac{\partial(F_xG_z - F_zG_x)}{\partial y} + \frac{\partial(F_xG_y - F_yG_x)}{\partial z}\end{aligned}$$

$$= F_y\frac{\partial G_z}{\partial x} + G_z\frac{\partial F_y}{\partial x} - F_z\frac{\partial G_y}{\partial x} - G_y\frac{\partial F_z}{\partial x} - F_x\frac{\partial G_z}{\partial y} - G_z\frac{\partial F_x}{\partial y} + F_z\frac{\partial G_x}{\partial y} + G_x\frac{\partial F_z}{\partial y} + F_x\frac{\partial G_y}{\partial z} + G_y\frac{\partial F_x}{\partial z} - F_y\frac{\partial G_x}{\partial z} - G_x\frac{\partial F_y}{\partial z}$$

$$= G_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + G_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + G_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) - F_x\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right) - F_y\left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right) - F_z\left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)$$

$$= \mathbf{G}\bullet(\nabla\times\mathbf{F}) - \mathbf{F}\bullet(\nabla\times\mathbf{G}).$$

6.7.6.19. (a) If ∇f is continuously differentiable, then

$$\nabla\times(\nabla f) = \nabla\times\left(\frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\,\hat{\boldsymbol{k}}\right)$$

$$= \left(\frac{\partial}{\partial y}\Big[\frac{\partial f}{\partial z}\Big] - (\frac{\partial}{\partial z}\Big[\frac{\partial f}{\partial y}\Big]\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial}{\partial z}\Big[\frac{\partial f}{\partial x}\Big] - (\frac{\partial}{\partial x}\Big[\frac{\partial f}{\partial z}\Big]\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial}{\partial x}\Big[\frac{\partial f}{\partial y}\Big] - (\frac{\partial}{\partial y}\Big[\frac{\partial f}{\partial x}\Big]\right)\hat{\boldsymbol{k}} = \mathbf{0}.$$

(b) Using the result of part (a), along with Theorem 6.6(5), we have

$$\nabla\times(f\nabla g) = f\nabla\times(\nabla g) + (\nabla f)\times(\nabla g) = f\cdot\mathbf{0} + (\nabla f)\times(\nabla g) = (\nabla f)\times(\nabla g),$$

as was desired.

6.7.6.21. We will find a potential function, $u = u(\rho,\phi,\theta)$ satisfying

$$\frac{\partial u}{\partial\rho}\,\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\,\frac{\partial u}{\partial\phi}\,\hat{\mathbf{e}}_\phi + \frac{1}{\rho\sin\phi}\,\frac{\partial u}{\partial\theta}\,\hat{\mathbf{e}}_\theta = \nabla u = -\mathbf{H} = H_0\left(1 - 2\alpha\Big(\frac{a}{\rho}\Big)^3\right)\cos\phi\,\hat{\mathbf{e}}_\rho - H_0\left(1 + \alpha\Big(\frac{a}{\rho}\Big)^3\right)\sin\phi\,\mathbf{e}_\phi.$$

So,

$$u = \int\frac{\partial u}{\partial\rho}\,\partial\rho = \int -H_\rho\,\partial\rho = \int H_0\left(1 - 2\alpha\Big(\frac{a}{\rho}\Big)^3\right)\cos\phi\,\partial\rho = H_0\left(\rho + a\alpha\Big(\frac{a}{\rho}\Big)^2\right)\cos\phi + g(\phi,\theta),$$

where g is an arbitrary function of only ϕ and θ. Substitute this into $-H_\phi = \dfrac{1}{\rho}\,\dfrac{\partial u}{\partial\phi}$ to get

$$-H_0\left(1 + \alpha\Big(\frac{a}{\rho}\Big)^3\right)\sin\phi = -H_\phi = \frac{1}{\rho}\,\frac{\partial u}{\partial\phi} = \frac{1}{\rho}\,\frac{\partial}{\partial\phi}\Big[\,H_0\left(\rho + a\alpha\Big(\frac{a}{\rho}\Big)^2\right)\cos\phi + g(\phi,\theta)\,\Big]$$

$$= -H_0\left(1 + \alpha\Big(\frac{a}{\rho}\Big)^3\right)\sin\phi + \frac{1}{\rho}\,\frac{\partial g}{\partial\phi},$$

hence $\dfrac{\partial g}{\partial\phi} \equiv 0$, hence $g(\phi,\theta) = h(\theta)$, where $h(\theta)$ is an arbitrary function of θ alone. Substitute

$u = H_0\left(\rho + a\alpha\Big(\dfrac{a}{\rho}\Big)^2\right)\cos\phi + h(\theta)$ into $-H_\theta = \dfrac{1}{\rho\,\sin\phi}\,\dfrac{\partial u}{\partial\theta}$ to get

$$0 = -H_\theta = \frac{1}{\rho\,\sin\phi}\,\frac{\partial f}{\partial\theta} = \frac{1}{\rho\,\sin\phi}\,\frac{\partial}{\partial\theta}\Big[\,H_0\left(\rho + a\alpha\Big(\frac{a}{\rho}\Big)^2\right)\cos\phi + h(\theta)\Big] = \frac{1}{\rho\,\sin\phi}\,h'(\theta),$$

hence $h'(\theta) \equiv 0$, hence $h(\theta) = c$, where c is an arbitrary constant. So, a potential function for $\mathbf{H}$ is given by $u(r,\phi,\theta) = H_0\left(\rho + \alpha\,\left(\dfrac{a^3}{\rho^2}\right)\right)\cos\phi + c$, $c =$arb. const.

6.7.6.23. We will find a potential function, $f = f(r,\theta,z)$ satisfying

$$\frac{\partial f}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\,\frac{\partial f}{\partial\theta}\,\hat{\mathbf{e}}_\theta + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_z = \boldsymbol{\nabla} f = \mathbf{F} \triangleq (2Ar+1)(2\pi\theta-\theta^2)\,\hat{\mathbf{e}}_r + 2(Ar+1)(\pi-\theta)\,\hat{\mathbf{e}}_\theta + \cos 3z\,\mathbf{e}_z.$$

So,

$$f = \int \frac{\partial f}{\partial r}\,\partial r = \int F_r\,\partial r = \int (2Ar+1)(2\pi\theta-\theta^2)\partial r = (Ar^2+r)(2\pi\theta-\theta^2) + g(\theta,z),$$

where g is an arbitrary function of only θ and z. Substitute this into $F_\theta = \dfrac{\partial f}{\partial\theta}$ to get

$$2(Ar+1)(\pi-\theta) = F_\theta = \frac{1}{r}\,\frac{\partial f}{\partial\theta} = \frac{1}{r}\,\frac{\partial}{\partial\theta}\Big[\,(Ar^2+r)(2\pi\theta-\theta^2) + g(\theta,z)\,\Big] = (Ar+1)(2\pi-2\theta) + \frac{1}{r}\,\frac{\partial g}{\partial\theta},$$

hence $\dfrac{\partial g}{\partial\theta} \equiv 0$, hence $g(\theta,z) = h(z)$, where $h(z)$ is an arbitrary function of only z. Into $F_z = \dfrac{\partial f}{\partial z}$, substitute $f = (Ar^2+r)(2\pi\theta-\theta^2) + h(z)$ to get

$$\cos 3z = F_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\Big[\,(Ar^2+r)(2\pi\theta-\theta^2) + h(z)\,\Big] = h'(z),$$

hence $h'(z) = \cos 3z$, hence $h(z) = \dfrac{1}{3}\sin 3z + c$, where c is an arbitrary constant. So, a potential function for $\mathbf{F}$ is given by $f(r,\theta,z) = (Ar^2+r)(2\pi\theta-\theta^2) + \dfrac{1}{3}\sin(3z) + c$, arb. const. c

6.7.6.25. We will find a potential function, $f = f(r,\theta,z)$ satisfying

$$\frac{\partial f}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\,\frac{\partial f}{\partial\theta}\,\hat{\mathbf{e}}_\theta + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_z = \boldsymbol{\nabla} f = \mathbf{F} \triangleq \left(\frac{z^2}{2} + A^2\cos\theta + 2Ar\sin\theta\right)\hat{\mathbf{e}}_r + (-A^2\sin\theta + Ar\cos\theta)\hat{\mathbf{e}}_\theta + rz\mathbf{e}_z.$$

So,

$$f = \int \frac{\partial f}{\partial r}\,\partial r = \int F_r\,\partial r = \int\left(\frac{z^2}{2} + A^2\cos\theta + 2Ar\sin\theta\right)\partial r = \frac{rz^2}{2} + A^2r\cos\theta + Ar^2\sin\theta + g(\theta,z),$$

where g is an arbitrary function of only θ and z. Substitute this into $F_\theta = \dfrac{\partial f}{\partial\theta}$ to get

$$-A^2\sin\theta + Ar\cos\theta = F_\theta = \frac{1}{r}\,\frac{\partial f}{\partial\theta} = \frac{1}{r}\,\frac{\partial}{\partial\theta}\Big[\,\frac{rz^2}{2} + A^2r\cos\theta + Ar^2\sin\theta + g(\theta,z)\,\Big]$$

$$= -A^2\sin\theta + Ar\cos\theta + \frac{1}{r}\,\frac{\partial g}{\partial\theta},$$

hence $\dfrac{\partial g}{\partial\theta} \equiv 0$, hence $g(\theta,z) = h(z)$, where $h(z)$ is an arbitrary function of z alone. Substitute $f = \dfrac{rz^2}{2} + A^2r\cos\theta + Ar^2\sin\theta + h(z)$ into $F_z = \dfrac{\partial f}{\partial z}$ to get

$$rz = F_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\Big[\,\frac{rz^2}{2} + A^2r\cos\theta + Ar^2\sin\theta + h(z)\,\Big] = rz + h'(z),$$

hence $h'(z)=0$, hence $h(z)=c$, where c is an arbitrary constant. So, a potential function for $\mathbf{F}$ is given by $f(r,\theta,z)=\dfrac{rz^2}{2}+A^2 r\cos\theta+Ar^2\sin\theta+c$, $c=$arb. const.

6.7.6.27. (a) $\mathbf{F}\triangleq -\boldsymbol{\nabla}V(\mathbf{r})=-\dfrac{\partial V}{\partial\rho}\,\hat{\mathbf{e}}_\rho-\dfrac{1}{\rho}\,\dfrac{\partial V}{\partial\phi}\,\hat{\mathbf{e}}_\phi-\dfrac{1}{\rho\sin\phi}\,\dfrac{\partial V}{\partial\theta}\,\hat{\mathbf{e}}_\theta$

$$=-\frac{\partial}{\partial\rho}\Big[-k\,\frac{e^{-\mu\rho}}{\rho}\Big]\,\hat{\mathbf{e}}_\rho-\frac{1}{\rho}\,\frac{\partial}{\partial\theta}\Big[-k\,\frac{e^{-\mu\rho}}{\rho}\Big]\,\hat{\mathbf{e}}_\phi-\frac{1}{\rho\sin\phi}\,\frac{\partial}{\partial\theta}\Big[-k\,\frac{e^{-\mu\rho}}{\rho}\Big]\,\hat{\mathbf{e}}_\theta=-k\,\frac{\mu\rho+1}{\rho^2}\,e^{-\mu\rho}\,\hat{\mathbf{e}}_\rho$$

(b) $\nabla^2 V(\mathbf{r})=\dfrac{1}{\rho^2}\,\dfrac{\partial}{\partial\rho}\Big[\rho^2\dfrac{\partial V}{\partial\rho}\Big]+\dfrac{1}{\rho^2\sin\phi}\,\dfrac{\partial}{\partial\phi}\Big[\sin\phi\dfrac{\partial V}{\partial\phi}\Big]+\dfrac{1}{\rho^2\sin^2\phi}\,\dfrac{\partial^2 V}{\partial\theta^2}$

$$=\frac{1}{\rho^2}\,\frac{\partial}{\partial\rho}\left[\rho^2\frac{\partial}{\partial\rho}\Big[-k\,\frac{e^{-\mu\rho}}{\rho}\Big]\right]+\frac{1}{\rho^2\sin\phi}\,\frac{\partial}{\partial\phi}\Big[\sin\phi\cdot 0\Big]+\frac{1}{\rho^2\sin^2\phi}\cdot 0=\frac{1}{\rho^2}\,\frac{\partial}{\partial\rho}\left[\rho^2\cdot\left(k\,\frac{\mu\rho+1}{\rho^2}\,e^{-\mu\rho}\right)\right]$$

$$=k\frac{1}{\rho^2}\,\frac{\partial}{\partial\rho}\left[(\mu\rho+1)\,e^{-\mu\rho}\right]=k\frac{1}{\rho^2}\left(-\mu(\mu\rho+1)\,e^{-\mu\rho}+\mu\,e^{-\mu\rho}\right)=-\mu^2k\rho^{-1}e^{-\mu\rho}=\mu^2V(\mathbf{r})$$

6.7.6.29. $\boldsymbol{\nabla}\bullet\mathbf{F}\triangleq\boldsymbol{\nabla}\bullet(F_u\hat{\mathbf{e}}_u+F_v\hat{\mathbf{e}}_v+F_w\hat{\mathbf{e}}_w)$

$$=\boldsymbol{\nabla}\bullet\left(h_vh_wF_u\cdot\Big(\frac{1}{h_vh_w}\,\hat{\mathbf{e}}_u\Big)\right)+\boldsymbol{\nabla}\bullet\left(h_wh_uF_v\cdot\Big(\frac{1}{h_wh_u}\,\hat{\mathbf{e}}_v\Big)\right)+\boldsymbol{\nabla}\bullet\left(h_uh_vF_w\cdot\Big(\frac{1}{h_uh_v}\,\hat{\mathbf{e}}_w\Big)\right)$$

$$=h_vh_wF_u\cdot\boldsymbol{\nabla}\bullet\Big(\frac{1}{h_vh_w}\,\hat{\mathbf{e}}_u\Big)+h_wh_uF_v\cdot\boldsymbol{\nabla}\bullet\Big(\frac{1}{h_wh_u}\,\hat{\mathbf{e}}_v\Big)+h_uh_vF_w\cdot\boldsymbol{\nabla}\bullet\Big(\frac{1}{h_uh_v}\,\hat{\mathbf{e}}_w\Big)$$

$$+\Big(\boldsymbol{\nabla}(h_vh_wF_u)\Big)\bullet\Big(\frac{1}{h_vh_w}\,\hat{\mathbf{e}}_u\Big)+\Big(\boldsymbol{\nabla}(h_wh_uF_v)\Big)\bullet\Big(\frac{1}{h_wh_u}\,\hat{\mathbf{e}}_v\Big)+\Big(\boldsymbol{\nabla}(h_uh_vF_w)\Big)\bullet\Big(\frac{1}{h_uh_v}\,\hat{\mathbf{e}}_w\Big)$$

Using $\boldsymbol{\nabla}\bullet\Big(\dfrac{1}{h_vh_w}\,\hat{\mathbf{e}}_u\Big)=\mathbf{0}$, etc.,

$$\boldsymbol{\nabla}\bullet\mathbf{F}=h_vh_wF_u\cdot\mathbf{0}+h_wh_uF_v\cdot\mathbf{0}+h_uh_vF_w\cdot\mathbf{0}$$

$$+\Big(\boldsymbol{\nabla}(h_vh_wF_u)\Big)\bullet\Big(\frac{1}{h_vh_w}\,\hat{\mathbf{e}}_u\Big)+\Big(\boldsymbol{\nabla}(h_wh_uF_v)\Big)\bullet\Big(\frac{1}{h_wh_u}\,\hat{\mathbf{e}}_v\Big)+\Big(\boldsymbol{\nabla}(h_uh_vF_w)\Big)\bullet\Big(\frac{1}{h_uh_v}\,\hat{\mathbf{e}}_w\Big)$$

$$=\left(\frac{1}{h_u}\,\frac{\partial}{\partial u}\Big[h_vh_wF_u\Big]\hat{\mathbf{e}}_u+\frac{1}{h_v}\,\frac{\partial}{\partial v}\Big[h_vh_wF_u\Big]\hat{\mathbf{e}}_v+\frac{1}{h_w}\,\frac{\partial}{\partial w}\Big[h_vh_wF_u\Big]\hat{\mathbf{e}}_w\right)\bullet\Big(\frac{1}{h_vh_w}\,\hat{\mathbf{e}}_u\Big)$$

$$+\left(\frac{1}{h_u}\,\frac{\partial}{\partial u}\Big[h_wh_uF_v\Big]\hat{\mathbf{e}}_u+\frac{1}{h_v}\,\frac{\partial}{\partial v}\Big[h_wh_uF_v\Big]\hat{\mathbf{e}}_v+\frac{1}{h_w}\,\frac{\partial}{\partial w}\Big[h_wh_uF_v\Big]\hat{\mathbf{e}}_w\right)\bullet\Big(\frac{1}{h_wh_u}\,\hat{\mathbf{e}}_v\Big)$$

$$+\left(\frac{1}{h_u}\,\frac{\partial}{\partial u}\Big[h_uh_vF_w\Big]\hat{\mathbf{e}}_u+\frac{1}{h_v}\,\frac{\partial}{\partial v}\Big[h_uh_vF_w\Big]\hat{\mathbf{e}}_v+\frac{1}{h_w}\,\frac{\partial}{\partial w}\Big[h_uh_vF_w\Big]\hat{\mathbf{e}}_w\right)\bullet\Big(\frac{1}{h_uh_v}\,\hat{\mathbf{e}}_w\Big)$$

$$=\left(\frac{1}{h_u}\,\frac{\partial}{\partial u}\Big[h_vh_wF_u\Big]\right)\cdot\Big(\frac{1}{h_vh_w}\Big)+\left(\frac{1}{h_v}\,\frac{\partial}{\partial v}\Big[h_wh_uF_v\Big]\right)\cdot\Big(\frac{1}{h_wh_u}\Big)+\left(\frac{1}{h_w}\,\frac{\partial}{\partial w}\Big[h_uh_vF_w\Big]\right)\cdot\Big(\frac{1}{h_uh_v}\Big)$$

$$=\frac{1}{h_uh_vh_w}\cdot\left(\frac{\partial}{\partial u}\Big[h_vh_wF_u\Big]+\frac{\partial}{\partial v}\Big[h_wh_uF_v\Big]+\frac{\partial}{\partial w}\Big[h_uh_vF_w\Big]\right),$$

that is, (6.82) is true.

6.7.6.31. $\nabla^2 f = \boldsymbol{\nabla} \bullet (\boldsymbol{\nabla} f) = \boldsymbol{\nabla} \bullet \left(\frac{1}{h_u} \frac{\partial f}{\partial u} \hat{\mathbf{e}}_u + \frac{1}{h_v} \frac{\partial f}{\partial v} \hat{\mathbf{e}}_v + \frac{1}{h_w} \frac{\partial f}{\partial w} \hat{\mathbf{e}}_w \right)$

$$= \frac{1}{h_u h_v h_w} \left(\frac{\partial}{\partial u} \left[h_v h_w \cdot \frac{1}{h_u} \frac{\partial f}{\partial u} \right] + \frac{\partial}{\partial v} \left[h_w h_u \cdot \frac{1}{h_v} \frac{\partial f}{\partial v} \right] + \frac{\partial}{\partial w} \left[h_u h_v \cdot \frac{1}{h_w} \frac{\partial f}{\partial w} \right] \right)$$

$$= \frac{1}{h_u h_v h_w} \left(\frac{\partial}{\partial u} \left[\frac{h_v h_w}{h_u} \frac{\partial f}{\partial u} \right] + \frac{\partial}{\partial v} \left[\frac{h_w h_u}{h_v} \frac{\partial f}{\partial v} \right] + \frac{\partial}{\partial w} \left[\frac{h_u h_v}{h_w} \frac{\partial f}{\partial w} \right] \right),$$

that is, (6.84) is true.

6.7.6.33. (a) $A = \left[a_{ij} \right]$, so $f(x,y,z) \triangleq \mathbf{x}^T A \mathbf{x} = a_{11}x^2 + a_{12}xy + a_{13}xz + a_{21}xy + a_{22}y^2 + a_{23}yz + a_{31}xz + a_{32}yz + a_{33}z^2$. It follows that

$$\boldsymbol{\nabla} f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \\ \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2a_{11}x + a_{12}y + a_{13}z + a_{21}y + a_{31}z \\ \\ a_{12}x + a_{21}x + 2a_{22}y + a_{23}z + a_{32}z \\ \\ a_{13}x + a_{23}y + a_{31}x + a_{32}y + 2a_{33}z \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ \\ a_{21}x + a_{22}y + a_{23}z \\ \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix} + \begin{bmatrix} a_{11}x + a_{21}y + a_{31}z \\ \\ a_{12}x + a_{22}y + a_{32}z \\ \\ a_{13}x + a_{23}y + a_{33}z \end{bmatrix},$$

that is, $\boldsymbol{\nabla} f(x,y,z) = \left(A + A^T \right) \mathbf{x}$.

(b) If A is symmetric, then $\boldsymbol{\nabla} f(x,y,z) = 2A\mathbf{x}$.

Section 6.8

6.8.4.1. Using the value of $\boldsymbol{\omega}$ found in Example 6.34,

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = (7.29212\times10^{-5}/\mathrm{s})\hat{\boldsymbol{k}}\times\big((7.29212\times10^{-5}/\mathrm{s})\hat{\boldsymbol{k}}\times(6.4\times10^{6}\cos30^\circ\,\hat{\boldsymbol{\jmath}}+6.4\times10^{6}\sin30^\circ\,\hat{\boldsymbol{k}})\,\mathrm{m}\big)$$

$$= (7.29212\times10^{-5}/\mathrm{s})\hat{\boldsymbol{k}}\times\big((-7.29212\times10^{-5}/\mathrm{s})(6.4\times10^{6}\cos30^\circ)\hat{\boldsymbol{\imath}}\,\mathrm{m}\big)\approx -0.0294726\,\hat{\boldsymbol{\jmath}}\,\mathrm{m/s}^2$$

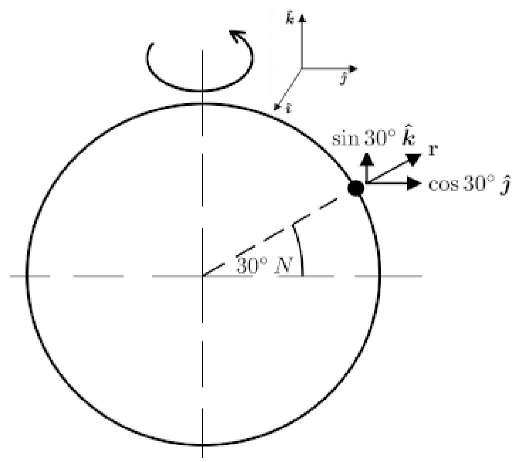


Figure 19: Problem 6.8.4.1

6.8.4.3. Note that $\rho(t)^2=||\mathbf{r}(t)||^2=\mathbf{r}(t)\bullet\mathbf{r}(t)$, so $2\big(\rho(t)\dot\rho(t)\big)=2\mathbf{r}(t)\bullet\dot{\mathbf{r}}(t)=2\mathbf{r}(t)\bullet\mathbf{v}(t)$.

Using the fact that $\mathbf{R}$ is a constant vector, the acceleration of the particle relative to the inertial reference frame is

$$\mathbf{a}(t)=\frac{d}{dt}\Big[\mathbf{v}(t)\Big]=\frac{d}{dt}\Big[\frac{1}{1+(\rho(t))^3}\,\mathbf{R}\Big]=-\,\frac{3\rho(t)\big(\rho(t)\dot\rho(t)\big)}{(1+(\rho(t))^3)^2}\,\mathbf{R}=-\,\frac{3\rho(t)\cdot2\mathbf{r}(t)\bullet\mathbf{v}(t)}{(1+(\rho(t))^3)^2}\,\mathbf{R}$$

$$=-\,\frac{6\rho(t)}{(1+(\rho(t))^3)^2}\,\big(\mathbf{r}(t)\bullet\mathbf{v}(t)\big)\mathbf{R},$$

which can also be rewritten as

$$\mathbf{a}(t)=-\,\frac{6\rho(t)}{(1+(\rho(t))^3)^2}\,\left(\mathbf{r}(t)\bullet\left(\frac{1}{1+(\rho(t))^3}\,\mathbf{R}\right)\right)\mathbf{R}=-\,\frac{6\rho(t)}{(1+(\rho(t))^3)}\,\left(\mathbf{r}(t)\bullet\mathbf{R}\right)\mathbf{R}\,.$$

6.8.4.5. Jupiter is rotating at $\omega=2\pi/(9.925\times3600)rad/s\approx1.75852\times10^{-4}$ rad/s, and Jupiter's radius is 7.1×10^{7} m. At latitude of $60^\circ\,N$, the position vector is $\mathbf{r}=(7.1\times10^{7}\cos60^\circ\,\hat{\boldsymbol{\jmath}}+7.1\times10^{7}\sin60^\circ\,\hat{\boldsymbol{k}})\,\mathrm{m}$.

We calculate that

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = (1.75852\times10^{-4}/\mathrm{s})\hat{\boldsymbol{k}}\times\big((1.75852\times10^{-4}\mathrm{s})\hat{\boldsymbol{k}}\times(7.1\times10^{7}\cos60^\circ\,\hat{\boldsymbol{\jmath}}+7.1\times10^{7}\sin60^\circ\,\hat{\boldsymbol{k}})\,\mathrm{m}\big)$$

$$= (1.75852\times10^{-4}/\mathrm{s})\hat{\boldsymbol{k}}\times\big((-1.75852\times10^{-4}/\mathrm{s})(7.1\times10^{7}\cos60^\circ)\hat{\boldsymbol{\imath}}\,\mathrm{m}\big)\approx 1.0980\,\hat{\boldsymbol{\jmath}}\,\mathrm{m/s}^2.$$

6.8.4.7. We want $9.81\,\mathrm{m/s}^2=||\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})||$. The latter is the greatest it can be when $(\boldsymbol{\omega}\times\mathbf{r})$ is perpendicular to $\boldsymbol{\omega}$, in which case

$$||\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})||=||\boldsymbol{\omega}||^2||\mathbf{r}||=\omega^2||\mathbf{r}||=\omega^2\cdot30\,\mathrm{m}.$$

So, we need

$$9.81\,\mathrm{m/s}^2=\omega^2\cdot30\,\mathrm{m},$$

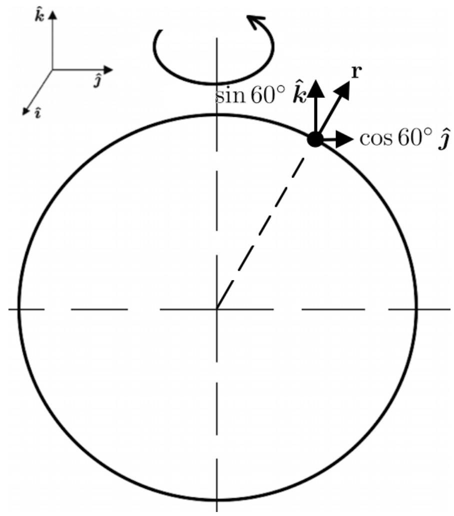


Figure 20: Problem 6.8.4.5

hence

$$\omega = \sqrt{\frac{9.81}{30}}\ /\mathrm{s} \approx 0.571839\ /\mathrm{s}$$

is the minimum angular speed. So, the maximum period of rotation should be

$$\frac{2\pi}{\omega} \approx 10.9877\,\mathrm{s},$$

or about 11 seconds.

6.8.4.9. At point $P = (-1, 2, 3)$, in m/s, using $\mathbf{r} = \overrightarrow{OP}$ and the given information that $\omega_x = 3$, we have

$12\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}} = \mathbf{V} = \boldsymbol{\omega} \times \mathbf{r} = (\omega_x\,\hat{\boldsymbol{\imath}} + \omega_y\,\hat{\boldsymbol{\jmath}} + \omega_z\,\hat{\boldsymbol{k}}) \times (-1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 3\,\hat{\boldsymbol{k}}) = (3\omega_y - 2\omega_z)\hat{\boldsymbol{\imath}} + (-\omega_z - 3\cdot 3)\hat{\boldsymbol{\jmath}} + (2\cdot 3 + \omega_y)\hat{\boldsymbol{k}})$.

This implies that

$$12\,\hat{\boldsymbol{\imath}} + 15\,\hat{\boldsymbol{\jmath}} + 0\cdot\hat{\boldsymbol{k}} = (3\omega_y - 2\omega_z)\hat{\boldsymbol{\imath}} + (-\omega_z)\hat{\boldsymbol{\jmath}} + (2\cdot 3 + \omega_y)\hat{\boldsymbol{k}}).$$

It follows that $\omega_z = -15$ and $\omega_y = -6$, so $\boldsymbol{\omega} = 3\,\hat{\boldsymbol{\imath}} - 6\,\hat{\boldsymbol{\jmath}} - 15\,\hat{\boldsymbol{k}}$.

6.8.4.11. Using formula (6.90), with $\nu_x = 0$, $\nu_y = 1$, $\nu_z = 0$, and $\omega = 1$, $e^{t\Omega} = \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix}$.

6.8.4.13. Using formula (6.90), with $\nu_x = \nu_y = \nu_z = \dfrac{1}{\sqrt{3}}$, and $\omega = \sqrt{3}$,

$$e^{t\Omega} = \frac{1}{3}\begin{bmatrix} 1+2\cos(\sqrt{3}\,t) & 1-\cos(\sqrt{3}\,t)-\sqrt{3}\sin(\sqrt{3}\,t) & 1-\cos(\sqrt{3}\,t)+\sqrt{3}\sin(\sqrt{3}\,t) \\ 1-\cos(\sqrt{3}\,t)+\sqrt{3}\sin(\sqrt{3}\,t) & 1+2\cos(\sqrt{3}\,t) & 1-\cos(\sqrt{3}\,t)-\sqrt{3}\sin(\sqrt{3}\,t) \\ 1-\cos(\sqrt{3}\,t)-\sqrt{3}\sin(\sqrt{3}\,t) & 1-\cos(\sqrt{3}\,t)+\sqrt{3}\sin(\sqrt{3}\,t) & 1+2\cos(\sqrt{3}\,t) \end{bmatrix}.$$

6.8.4.15. $LHS = \mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = (a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}) \bullet \Big((b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}) \times (c_1\hat{\boldsymbol{\imath}} + c_2\hat{\boldsymbol{\jmath}} + c_3\hat{\boldsymbol{k}})\Big)$

$$= (a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}) \bullet \Big((b_2c_3 - b_3c_2)\hat{\boldsymbol{\imath}} + (b_3c_1 - b_1c_3)\hat{\boldsymbol{\jmath}} + (b_1c_2 - b_2c_1)\hat{\boldsymbol{k}}\Big)$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \det[\,\mathbf{A} \,\vdots\, \mathbf{B} \,\vdots\, \mathbf{C}\,].$$

By this result, with the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ replaced by the vectors $\mathbf{C}, \mathbf{A}, \mathbf{B}$, respectively, we have

$$RHS = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B}) = \det[\,\mathbf{C} \,\vdots\, \mathbf{A} \,\vdots\, \mathbf{B}\,].$$

But, the column operation $C_1 \leftrightarrow C_2$ followed by $C_2 \leftrightarrow C_3$ explain why

$$RHS = \det[\,\mathbf{C} \,\vdots\, \mathbf{A} \,\vdots\, \mathbf{B}\,] = (-1)\det[\,\mathbf{A} \,\vdots\, \mathbf{C} \,\vdots\, \mathbf{B}\,] = (-1)(-1)\det[\,\mathbf{A} \,\vdots\, \mathbf{B} \,\vdots\, \mathbf{C}\,] = \det[\,\mathbf{A} \,\vdots\, \mathbf{B} \,\vdots\, \mathbf{C}\,].$$

Putting all of this together, we have that

$$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = LHS = \det[\,\mathbf{A} \,\vdots\, \mathbf{B} \,\vdots\, \mathbf{C}\,] = RHS = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B}),$$

as we were asked to explain.

6.8.4.17. Define a scalar function of t by $y(t) \triangleq \mathbf{e}_i(t) \bullet \mathbf{e}_j(t)$. We calculate that $y(0) = \mathbf{e}_i(0) \bullet \mathbf{e}_j(0) = \left\{ \begin{array}{ll} 0, & i \neq j \\ 1, & i = j \end{array} \right\}$, because we are assuming that initially we have a coordinate frame, whose three vectors must be an orthonormal set.

Using the result of problem 6.8.16, along with the given information about $\dot{\mathbf{e}}_\ell$, $\ell = 1, 2, 3$, we have

$$\dot{y}(t) = \dot{\mathbf{e}}_i(t) \bullet \mathbf{e}_j(t) + \mathbf{e}_i(t) \bullet \dot{\mathbf{e}}_j(t) = \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_i(t)\Big) \bullet \mathbf{e}_j(t) + \mathbf{e}_i(t) \bullet \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_i(t)\Big)$$

Applying symmetry of the dot product, along with the result of problem 6.8.15, to the first term gives

$$\dot{y}(t) = \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_i(t)\Big) \bullet \mathbf{e}_j(t) + \mathbf{e}_i(t) \bullet \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_j(t)\Big) = \mathbf{e}_j(t) \bullet \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_i(t)\Big) + \mathbf{e}_i(t) \bullet \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_j(t)\Big)$$

$$= \mathbf{e}_i(t) \bullet \Big(\mathbf{e}_j(t) \times \boldsymbol{\omega}(t)\Big) + \mathbf{e}_i(t) \bullet \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_j(t)\Big).$$

Now, anti-symmetry of the cross product applied to the first term implies t hat

$$\dot{y}(t) = \mathbf{e}_i(t) \bullet \Big(\mathbf{e}_j(t) \times \boldsymbol{\omega}(t)\Big) + \mathbf{e}_i(t) \bullet \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_j(t)\Big) = \mathbf{e}_i(t) \bullet \Big(-\boldsymbol{\omega}(t) \times \mathbf{e}_j(t)\Big) + \mathbf{e}_i(t) \bullet \Big(\boldsymbol{\omega}(t) \times \mathbf{e}_j(t)\Big) \equiv 0.$$

Because $\dot{y}(t) \equiv 0$, we conclude that $y(t) \equiv y(0)$. Because the set of three vectors was initially an orthonormal set, the set of three time dependent vectors remain an orthonormal set for $t \neq 0$.

6.8.4.19. $\dfrac{d}{dt}\Big[\mathbf{x}(t) \times \mathbf{z}(t)\Big] = \dfrac{d}{dt}\Big[\,(x_1\hat{\boldsymbol{\imath}} + x_2\hat{\boldsymbol{\jmath}} + x_3\hat{\boldsymbol{k}}) \times (z_1\hat{\boldsymbol{\imath}} + z_2\hat{\boldsymbol{\jmath}} + z_3\hat{\boldsymbol{k}})\,\Big]$

$$= \frac{d}{dt}\Big[\,(x_2z_3 - x_3z_2)\hat{\boldsymbol{\imath}} + (x_3z_1 - x_1z_3)\hat{\boldsymbol{\jmath}} + (x_1z_2 - x_2z_1)\hat{\boldsymbol{k}}\,\Big]$$

$$= (\dot{x}_2z_3 - \dot{x}_3z_2 + x_2\dot{z}_3 - x_3\dot{z}_2)\hat{\boldsymbol{\imath}} + (\dot{x}_3z_1 - \dot{x}_1z_3 + x_3\dot{z}_1 - x_1\dot{z}_3)\hat{\boldsymbol{\jmath}} + (\dot{x}_1z_2 - \dot{x}_2z_1 + x_1\dot{z}_2 - x_2\dot{z}_1)\hat{\boldsymbol{k}}$$

$$= (\dot{x}_2z_3 - \dot{x}_3z_2)\hat{\boldsymbol{\imath}} + (\dot{x}_3z_1 - \dot{x}_1z_3)\hat{\boldsymbol{\jmath}} + (\dot{x}_1z_2 - \dot{x}_2z_1)\hat{\boldsymbol{k}} + (x_2\dot{z}_3 - x_3\dot{z}_2)\hat{\boldsymbol{\imath}} + (x_3\dot{z}_1 - x_1\dot{z}_3)\hat{\boldsymbol{\jmath}} + (x_1\dot{z}_2 - x_2\dot{z}_1)\hat{\boldsymbol{k}}$$

$$\equiv \dot{\mathbf{x}}(t) \times \mathbf{z}(t) + \mathbf{x}(t) \times \dot{\mathbf{z}}(t).$$

Chapter Seven

Section 7.1

7.1.2.1. Using $w = 2t + \frac{\pi}{4}$, hence $dt = dw/2$,

$$\int_{-\pi}^{\frac{\pi}{2}} \cos(2t+\frac{\pi}{4})\, dt = \int_{-7\pi/4}^{5\pi/4} \cos(w)\, \frac{dw}{2} = \frac{1}{2}\Big[\ \sin(w)\Big]_{-7\pi/4}^{5\pi/4} = \frac{1}{2}\Big(\ \sin\big(\frac{7\pi}{4}\big) - \sin\big(-\frac{5\pi}{4}\big)\Big) = \frac{1}{2}\Big(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\Big) = -\frac{\sqrt{2}}{2}.$$

7.1.2.3. Using $w = 1 + x^2$, hence $dx = \dfrac{dw}{2x}$,

$$\int_0^5 \frac{3x}{1+x^2}\, dx = \int_1^{26} \frac{3x}{w} \cdot \frac{dw}{2x} = \frac{3}{2}\int_1^{26} \frac{1}{w}\, dw = \frac{3}{2}\Big[\ \ln|w|\Big]_1^{26} = \frac{3}{2}\Big(\ \ln(26) - \ln(1)\Big) = \frac{3}{2}\Big(\ \ln(26) - 0\Big) = \frac{3}{2}\ln 26.$$

7.1.2.5. Using $w = \sqrt{x}$, hence $dx = 2\sqrt{x}\, dw = 2w\, dw$,

$$\int_0^{\frac{\pi}{4}} \cos(\sqrt{x})\, dx = \int_0^{\sqrt{\pi/2}} \cos(w)\, 2w\, dw.$$

Using integration by parts with $u = w$ and $dv = \cos(w)\, dw$, hence $du = dw$ and $v = \displaystyle\int dv = \sin(w)$,

$$\int_0^{\frac{\pi}{4}} \cos(\sqrt{x})\, dx = 2w\sin(w)\Big|_0^{\sqrt{\pi}/2} - \int_0^{\sqrt{\pi}/2} 2\sin(w)\, dw = 2\,\frac{\sqrt{\pi}}{2}\sin\Big(\frac{\sqrt{\pi}}{2}\Big) - 0\cdot\sin(0) - \Big[\ -2\cos(w)\Big]_0^{\sqrt{\pi}/2}$$

$$= -2 + 2\cos\Big(\frac{\sqrt{\pi}}{2}\Big) + \sqrt{\pi}\sin\Big(\frac{\sqrt{\pi}}{2}\Big).$$

7.1.2.7. $\displaystyle\int_0^\infty \frac{x^2 dx}{(1+x^3)^2} = \lim_{b\to\infty}\Big(\int_0^b \frac{x^2 dx}{(1+x^3)^2}\Big) = \lim_{b\to\infty}\Big(\Big[-\frac{1}{3}(1+x^3)^{-1}\Big]_0^b\Big) = \lim_{b\to\infty}\Big(-\frac{1}{3}(1+b^3)^{-1} + \frac{1}{3}(1+0)^{-1}\Big).$

So, the improper integral is convergent and converges to $\dfrac{1}{3}$.

7.1.2.9. $\displaystyle\int_{\frac{4}{\pi}}^\infty x^{-2}\cos(x^{-1})\, dx = \lim_{b\to\infty}\Big(\int_{\frac{4}{\pi}}^b x^{-2}\cos(x^{-1})\, dx\Big) = \lim_{b\to\infty}\Big(\Big[-\sin(x^{-1})\Big]_{\frac{4}{\pi}}^b\Big) = \lim_{b\to\infty}\Big(-\sin(b^{-1}) + \sin\Big(\frac{\pi}{4}\Big)\Big)$

$$= 0 + \frac{\sqrt{2}}{2}.$$

So, the improper integral is convergent and converges to $\dfrac{1}{\sqrt{2}}$.

7.1.2.11. Using $x = \tan\theta$, hence $dx = \sec^2\theta\, d\theta$, we first find an indefinite integral:

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta = \ln\,|\,\sec\theta + \tan\theta\,| + c = \ln\,|\,x + \sqrt{x^2+1}\,| + c.$$

So,

$$\int_0^\infty \frac{dx}{\sqrt{1+x^2}} = \lim_{b\to\infty}\Big(\int_0^b \frac{dx}{\sqrt{1+x^2}}\Big) = \lim_{b\to\infty}\Big(\Big[\ln\,|\,x + \sqrt{x^2+1}\,|\Big]_0^b\Big) = \lim_{b\to\infty}\Big(\ln\,|\,\sqrt{b^2+1} + b\,| - \ln\,|\,0 + \sqrt{1}\,|\Big) = \infty.$$

So, the improper integral $\displaystyle\int_0^\infty \frac{dx}{\sqrt{1+x^2}}$ is divergent. This implies that the improper integral $\displaystyle\int_{-\infty}^\infty \frac{dx}{\sqrt{1+x^2}}$ is divergent.

7.1.2.13. For $p=1$, $\displaystyle\int \frac{1}{x^p}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$, so

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x}\,dx\right) = \lim_{b\to\infty}\left(\Big[\ln|x|\Big]_1^b\right) = \lim_{b\to\infty}(\ln(b)-\ln(1)) = \infty,$$

so the improper integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent. This implies that the improper integral $\displaystyle\int_0^\infty \frac{1}{x}\,dx$ is divergent.

For $p\neq 1$, $\displaystyle\int \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p} + c$.

For $p<1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x^p}\,dx\right) = \lim_{b\to\infty}\left(\left[\frac{x^{1-p}}{1-p}\right]_1^b\right) = \lim_{b\to\infty}\left(\left[\frac{b^{1-p}-1}{1-p}\right]\right) = \infty,$$

so the improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is divergent. This implies that the improper integral $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$ is divergent for $p<1$.

For $p>1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0+}\left(\int_a^1 \frac{1}{x^p}\,dx\right) = \lim_{a\to 0+}\left(\left[\frac{x^{1-p}}{1-p}\right]_a^1\right) = \lim_{a\to 0+}\left(\left[\frac{1-a^{1-p}}{1-p}\right]\right) = -\infty,$$

because $a^{1-p} = \dfrac{1}{a^{p-1}}$. So the improper integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ is divergent. This implies that the improper integral $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$ is divergent for $p>1$.

(b) So, for all real constants p, $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$ diverges.

(a) For no real constant p, does $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$ converge.

7.1.2.15. $\displaystyle\int_0^1 x\,f''(x)\,dx = \big[x\,f'(x)\big]_0^1 - \int_0^1 f'(x)\,dx = f'(1) - 0\cdot f'(0) - [f(x)]_0^1 = f'(1) - f(1) + f(0) = 0 - 0 + f(0) = f(0).$

Section 7.2

7.2.5.1. (a) Let $\mathcal{C}:\ \mathbf{r}=\mathbf{r}(t)=\mathbf{r}_0+t\mathbf{v}$, for $0\le t\le 1$, where constant vector $\mathbf{v}\triangleq\mathbf{r}_1-\mathbf{r}_0$. We have $\dot{\mathbf{r}}(t)\equiv\mathbf{v}$, so

$$\int_{\mathcal{C}} ds=\int_0^1 ||\dot{\mathbf{r}}(t)||\,dt=\int_0^1 ||\mathbf{v}||\,dt=||\mathbf{v}||\int_0^1 dt=||\mathbf{v}||\,\Big[\,t\,\Big]_0^1=||\mathbf{v}||\cdot 1=||\mathbf{r}_1-\mathbf{r}_0||,$$

which is the length of the line segment.

(b) Let $\mathcal{C}:\ \mathbf{r}=\mathbf{r}(t)=a\cos t\,\hat{\boldsymbol{\imath}}+a\sin t\,\hat{\boldsymbol{\jmath}}$, for $\theta_0\le t\le\theta_1$. We have $\dot{\mathbf{r}}(t)=-a\sin t\,\hat{\boldsymbol{\imath}}+a\cos t\,\hat{\boldsymbol{\jmath}}$, so

$$\int_{\mathcal{C}} ds=\int_0^1 ||\dot{\mathbf{r}}(t)||\,dt=\int_{\theta_0}^{\theta_1}\sqrt{a^2\cos^2+a^2\sin^2}\,dt=\int_{\theta_0}^{\theta_1}\sqrt{a^2}\,dt=\int_{\theta_0}^{\theta_1}|a|\,dt=|a|\int_{\theta_0}^{\theta_1}dt=|a|\,\Big[\,t\,\Big]_{\theta_0}^{\theta_1}$$

$$=|a|\cdot(\theta_1-\theta_0)=a\cdot(\theta_1-\theta_0),$$

which is the length of the arc of a circle of radius a from θ_0 to θ_1. [Implicitly we assumed the constant $a>0$.]

7.2.5.3. $\dot{\mathbf{r}}(t)=t\,\hat{\boldsymbol{\imath}}+\sqrt{6}\,\hat{\boldsymbol{\jmath}}+\dfrac{3}{t}\,\hat{\boldsymbol{k}}$, so $||\dot{\mathbf{r}}(t)||=\sqrt{t^2+6+(9/t^2)}=\sqrt{\big(t+(3/t)\big)^2}=\Big|t+\dfrac{3}{t}\Big|$. The exact arclength of the curve is

$$\int_{\mathcal{C}} ds=\int_1^e ||\dot{\mathbf{r}}(t)||\,dt=\int_1^e \Big|t+\frac{3}{t}\Big|\,dt=\int_1^e\Big(t+\frac{3}{t}\Big)dt=\Big[\,\frac{1}{2}t^2+3\ln t\,\Big]_1^e=\frac{1}{2}\,e^2-\frac{1}{2}\,1^2+3\ln e-3\ln 1=\frac{5+e^2}{2}.$$

7.2.5.5. $\dot{\mathbf{r}}(t)=-2\sin 2t\,\hat{\boldsymbol{\imath}}+2\cos 2t\,\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}$, so $||\dot{\mathbf{r}}(t)||=\sqrt{4\sin^2 2t+4\cos^2 2t+1}=\sqrt{4+1}=\sqrt{5}$. So, using integration by parts with $u=t$ and $dv=\cos 2t\,dt$, we calculate

$$\int_{\mathcal{C}} x\,z\,ds=\int_0^{\pi/4}(\cos 2t)\,t\,||\dot{\mathbf{r}}(t)||\,dt=\int_0^{\pi/4}\sqrt{5}\,t\,\cos 2t dt=\sqrt{5}\,\Big[\,\frac{1}{2}\,t\sin 2t+\frac{1}{4}\,\cos 2t\Big]_0^{\pi/4}$$

$$=\sqrt{5}\,\left(\Big(\frac{1}{2}\cdot\frac{\pi}{4}\sin\frac{\pi}{2}+\frac{1}{4}\,\cos\frac{\pi}{2}\Big)-\Big(0+\frac{1}{4}\cdot 1\Big)\right)=\sqrt{5}\,\Big(\frac{\pi}{8}-0-0-\frac{1}{4}\Big)=\frac{(\pi-2)\sqrt{5}}{8}.$$

7.2.5.7. $\mathcal{C}:\ \mathbf{r}=a\,\cos t\,\hat{\boldsymbol{\imath}}+a\,\sin t\,\hat{\boldsymbol{\jmath}},\ 0\le t\le\frac{\pi}{2}$, so the total mass, in grams, is

$$M=\int_{\mathcal{C}}\varrho\,ds=\int_0^{\pi/2}(1+x^2+y^2-xy)\,||\dot{\mathbf{r}}(t)||\,dt=\int_0^{\pi/2}(1+a^2-a^2\cos t\,\sin t)\,a\,dt=\Big[a\Big((1+a^2)t+\frac{1}{2}a^2\cos^2 t\Big)\Big]_0^{\pi/2}$$

$$=a\left((1+a^2)\frac{\pi}{2}+\frac{1}{2}a^2\cdot\Big(0-1\Big)\right)=\frac{a}{2}\Big((1+a^2)\pi-a^2\Big)=\frac{a}{2}\Big(\pi+a^2(\pi-1)\Big).$$

The moments in grams-m, are

$$M_y=\int_{\mathcal{C}}\varrho\,x\,ds=\int_0^{\pi/2}(1+x^2+y^2-xy)\,x\,||\dot{\mathbf{r}}(t)||\,dt=\int_0^{\pi/2}(1+a^2-a^2\cos t\,\sin t)a\cos t\,a\,dt$$

$$=\left[a^2\left((1+a^2)\sin t+\frac{1}{3}a^2\cos^3 t\right)\right]_0^{\pi/2}=a^2\left((1+a^2)(1-0)+\frac{1}{3}a^2\cdot(0-1)\right)=a^2\Big(1+\frac{2a^2}{3}\Big)$$

and

$$M_x=\int_{\mathcal{C}}\varrho\,y\,ds=\int_0^{\pi/2}(1+x^2+y^2-xy)\,y\,||\dot{\mathbf{r}}(t)||\,dt=\int_0^{\pi/2}(1+a^2-a^2\cos t\,\sin t)a\sin t\,a\,dt$$

$$=\left[a^2\left(-(1+a^2)\cos t-\frac{1}{3}a^2\sin^3 t\right)\right]_0^{\pi/2}=a^2\left(-(1+a^2)(0-1)-\frac{1}{3}a^2\cdot(1-0)\right)=a^2\Big(1+\frac{2a^2}{3}\Big).$$

The center of mass is at

$$(\overline{x},\overline{y})=\frac{2a}{\pi+a^2(\pi-1)}\cdot\Big(1+\frac{2a^2}{3},\,1+\frac{2a^2}{3}\Big).$$

7.2.5.9. $\mathcal{C}:\ \mathbf{r}=a\,\cos t\,\hat{\boldsymbol{\imath}}+a\,\sin t\,\hat{\boldsymbol{\jmath}},\ 0\le t\le\frac{\pi}{2}$, so the total mass, in grams, is

$$M=\int_{\mathcal{C}}\varrho\,ds=\int_0^{\pi/2}(1+0.2\,\theta)\,||\dot{\mathbf{r}}(t)||\,dt=\int_0^{\pi/2}(1+0.2t)\,a\,dt=a\Big[\,t+0.1t^2\,\Big]_0^{\pi/2}=a\left(\frac{\pi}{2}+0.1\Big(\frac{\pi}{2}\Big)^2\right).$$

The moments in grams-m, are

$$M_y=\int_{\mathcal{C}}\varrho\,x\,ds=\int_0^{\pi/2}(1+0.2\,\theta)\,x\,||\dot{\mathbf{r}}(t)||\,dt=\int_0^{\pi/2}(1+0.2t)a\cos t\,a\,dt=a^2\Big[\,\sin t+0.2t\sin t+0.2\cos t\Big]_0^{\pi/2}$$

$$=a^2\Big((1-0)+(0.1\pi-0)+(0-0.2)\Big)=a^2(0.8+0.1\pi)$$

and

$$M_y=\int_{\mathcal{C}}\varrho\,y\,ds=\int_0^{\pi/2}(1+0.2\,\theta)\,y\,||\dot{\mathbf{r}}(t)||\,dt=\int_0^{\pi/2}(1+0.2t)a\sin t\,a\,dt=a^2\Big[-\cos t-0.2t\cos t+0.2\sin t\Big]_0^{\pi/2}$$

$$=a^2\Big(-(0-1)+(0-0)+(0.2-0)\Big)=1.2\,a^2$$

The center of mass is at $(\overline{x},\overline{y})=\dfrac{4a}{\pi(2+0.1\pi)}\cdot(0.8+0.1\pi,1.2)\approx a\cdot(0.613005,0.6602324)$.

7.2.5.11. $\mathcal{C}:\ \mathbf{r}=a\,\cos t\,\hat{\boldsymbol{\imath}}+a\,\sin t\,\hat{\boldsymbol{\jmath}},\ \frac{\pi}{2}\le t\le\pi$

(a) $\displaystyle\int_{\mathcal{C}}xy^2\,ds=\int_{\pi/2}^{\pi}xy^2\,||\dot{\mathbf{r}}(t)||\,dt=\int_{\pi/2}^{\pi}a\cos t\,a^2\sin^2 t(a\,dt)=a^4\int_{\pi/2}^{\pi}\sin^2 t\,\cos t\,dt=a^4\Big[\,\frac{1}{3}\,\sin^3 t\,\Big]_{\pi/2}^{\pi}=\frac{a^4}{3}(0-1)$

$$=-\frac{a^4}{3}.$$

(b) $\displaystyle\int_{\mathcal{C}}xy^2\,dx=\int_{\pi/2}^{\pi}xy^2(-a\sin t\,dt)=\int_{\pi/2}^{\pi}a\cos t\,a^2\sin^2 t(-a\sin t\,dt)=-a^4\int_{\pi/2}^{\pi}\sin^3 t\,\cos t\,dt=-a^4\,\Big[\,\frac{1}{4}\sin^4 t\,\Big]_{\pi/2}^{\pi}$

$$=-\frac{a^4}{4}\Big(0-1\Big)=\frac{a^4}{4}.$$

(c) Using the Fundamental Theorem of Line Integrals with $f(x,y)=xy^2$,

$$\int_{\mathcal{C}}\boldsymbol{\nabla}(xy^2)\bullet\,d\mathbf{r}=xy^2\Big|_{(x,y)=(-a,0)}-xy^2\Big|_{(x,y)=(0,a)}=0-0=0.$$

7.2.5.13. The circulation of the velocity field $\mathbf{v}=y\,\hat{\boldsymbol{\imath}}-x\,\hat{\boldsymbol{\jmath}}+xy\,\hat{\boldsymbol{k}}$ around the half disk is

$$\Gamma=\oint_{\mathcal{C}}\mathbf{v}(\mathbf{r})\bullet d\mathbf{r}=\oint_{\mathcal{C}}(y\,\hat{\boldsymbol{\imath}}-x\,\hat{\boldsymbol{\jmath}}+xy\,\hat{\boldsymbol{k}})\bullet(dx\,\hat{\boldsymbol{\imath}}+dy\,\hat{\boldsymbol{\jmath}})=\oint_{\mathcal{C}}(y\,dx-x\,dy)=\int_{\mathcal{C}_1}(y\,dx-x\,dy)+\int_{\mathcal{C}_2}(y\,dx-x\,dy),$$

where $\mathcal{C}_1:\ \mathbf{r}=a\,\cos t\,\hat{\boldsymbol{\imath}}+a\,\sin t\,\hat{\boldsymbol{\jmath}},\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ and $\mathcal{C}_2:\ \mathbf{r}=-t\,\hat{\boldsymbol{\jmath}},\ -a\le t\le a$. So,

$$\Gamma=\int_{-\pi/2}^{\pi/2}\big(a\,\sin t\,(-a\sin t\,dt)-a\,\cos t\,(a\cos t\,dt)\big)+\int_{-a}^{a}\big((-t)\,(0\cdot dt)-(0)\,(-dt)\big)=-a^2\int_{-\pi/2}^{\pi/2}dt-a\int_{-a}^{a}0\cdot dt$$

$$=-a^2\cdot\pi=-\pi a^2.$$

7.2.5.15. (a) To find a potential function, begin with

$$f=\int\frac{\partial f}{\partial x}\,\partial x=\int F_x\,\partial x=\int y^2\,\partial x=xy^2+g(y,z),$$

where g is an arbitrary function of only y and z. Substitute this into $F_y=\dfrac{\partial f}{\partial y}$ to get

$$2xy+e^{3z}=F_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\Big[\,xy^2+g(y,z)\Big]=2xy+\frac{\partial g}{\partial y},$$

hence $\dfrac{\partial g}{\partial y} \equiv e^{3z}$, hence $g(y,z) = ye^{3z} + h(z)$, where $h(z)$ is an arbitrary function of *only* z. Substitute $f = xy^2 + ye^{3z} + h(z)$ into $F_z = \dfrac{\partial f}{\partial z}$ to get

$$3ye^{3z} + 1 = F_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\, xy^2 + ye^{3z} + h(z)\,\right] = 3ye^{3z} + h'(z),$$

hence $h'(z) \equiv 1$, hence $h(z) = z + c$, where c is an arbitrary constant. So, a potential function for $\mathbf{F}$ is given by $f(x,y,z) = xy^2 + ye^{3z} + z$.

(b) Using the Fundamental Theorem of Line Integrals with $f(x,y,z) = xy^2 + ye^{3z} + z$,

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathcal{C}} \boldsymbol{\nabla} f \bullet d\mathbf{r} = f(0,1,1) - f(1,0,1) = 0 + e^3 + 1 - (0+0+1) = e^3.$$

7.2.5.17. The force doing the work is $\mathbf{F} = +\frac{mMG}{||\mathbf{r}||^3}\,\mathbf{r} = \boldsymbol{\nabla}\Big[-\frac{mMG}{||\mathbf{r}||}\Big]$, so the work done against the gravitational field is

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathcal{C}} \boldsymbol{\nabla}\Big[-\frac{mMG}{||\mathbf{r}||}\Big] \bullet d\mathbf{r} = -\frac{mMG}{||\mathbf{r}||}\Big|_{(x,y,z)=(2,3,-1)} + \frac{mMG}{||\mathbf{r}||}\Big|_{(x,y,z)=(-2,1,1)} = \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{14}}\right) mMG.$$

7.2.5.19. Find a potential function: Substitute $w = x^2 + y^2$ and note that $\dfrac{\partial w}{\partial x} = 2x$ to get $f = \displaystyle\int \frac{x}{x^2+y^2}\,\partial x = \int \frac{1}{w}\,\frac{1}{2}\,\partial w = \frac{1}{2}\ln|w| + g(y) = \frac{1}{2}\ln\left(x^2+y^2\right) + g(y)$, where $g(y)$ is an arbitrary function of y alone. Match the rest of the vector field using $\dfrac{y}{x^2+y^2} = \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\Big[\dfrac{1}{2}\ln\left(x^2+y^2\right) + g(y)\Big] = \dfrac{y}{x^2+y^2} + g'(y)$, hence we may use $g(y) \equiv 0$ and $f = \frac{1}{2}\ln\left(x^2+y^2\right)$. By the Fundamental Theorem of Line Integrals, $\displaystyle\int_{\mathcal{C}} \left(\frac{x}{x^2+y^2}\,\hat{\boldsymbol{\imath}} + \frac{y}{x^2+y^2}\,\hat{\boldsymbol{\jmath}}\right) \bullet d\mathbf{r} = f(0,3) - f(2,0) = \frac{1}{2}(\ln 9 - \ln 4) = \ln\frac{3}{2}$. [Note that the equation $\mathbf{F} = \boldsymbol{\nabla} f$ holds in a domain in which $\boldsymbol{\nabla} f$ exists, that is, a domain not containing $(x,y) = (0,0)$.]

7.2.5.21. $\theta = \dfrac{3\pi}{2} + \varphi$ gives the angle in polar coordinates when the pendulum's bob has angle φ in Figure 7.17 in the textbook. Using the first column of Table 6.1 in Section 6.2, the force of gravity can be decomposed as

$$\mathbf{F} = -w\,\hat{\boldsymbol{\jmath}} = -w(\sin\theta\,\hat{\mathbf{e}}_r + \cos\theta\,\hat{\mathbf{e}}_\theta).$$

So, the work done by gravity on the bob as it travels from $\varphi = \alpha$ to $\varphi = 0$ along curve $-\mathcal{C}$, where $\mathcal{C}:\ \mathbf{r} = \ell(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}})$, $\dfrac{3\pi}{2} \le \theta \le \dfrac{3\pi}{2} + \alpha$, is

$$\begin{aligned} Work &= \int_{-\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = -\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = -\int_{3\pi/2}^{\alpha+(3\pi/2)} -w(\sin\theta\,\hat{\mathbf{e}}_r + \cos\theta\,\hat{\mathbf{e}}_\theta) \bullet \ell(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}})d\theta \\ &= w\ell\int_{3\pi/2}^{\alpha+(3\pi/2)} (\sin\theta\,\hat{\mathbf{e}}_r + \cos\theta\,\hat{\mathbf{e}}_\theta) \bullet (\hat{\mathbf{e}}_\theta)d\theta = w\ell\int_{3\pi/2}^{\alpha+(3\pi/2)} \cos\theta\,d\theta = w\ell\Big[\sin\theta\Big]_{3\pi/2}^{\alpha+(3\pi)/2)} \\ &= w\ell\left(\sin\left(\frac{3\pi}{2} + \alpha\right) - \sin\frac{3\pi}{2}\right) = w\ell\left(-\cos\alpha + 1\right). \end{aligned}$$

Section 7.3

7.3.7.1. Using rectangles with $\triangle x = 2$ and $\triangle y = 5$, $Volume \approx 2 \cdot 5 \cdot \big(f(1,12.5) + f(3,12.5) + f(1,17.5) + f(3,17.5)\big)$ $= 10(2+3+2+1) = 80$.

7.3.7.3. Using the substitution $w = xy$ and $\dfrac{\partial w}{\partial y} = x$,

$$\int_1^3 \int_2^4 x^2 e^{xy}\, dy\, dx = \int_1^3 \left(\Big[xe^{xy} \Big]_2^4 \right) dx.$$

Using integration by parts, this equals

$$= \int_1^3 \big(xe^{4x} - xe^{2x}\big) dx = \Big[\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} \Big]_1^3 = \frac{3}{4}e^{12} - \frac{1}{16}e^{12} - \frac{3}{2}e^6 + \frac{1}{4}e^6 - \Big(\frac{1}{4}e^4 - \frac{1}{16}e^4 - \frac{1}{2}e^2 + \frac{1}{4}e^2\Big)$$

$$= \frac{11}{16}e^{12} - \frac{5}{4}e^6 - \Big(\frac{3}{16}e^4 - \frac{1}{4}e^2\Big) = \frac{e^2}{16}\big(11e^{10} - 20e^4 - 3e^2 + 4\big).$$

7.3.7.5. Using the substitution $w = x + 2y$ and $\dfrac{\partial w}{\partial x} = 1$,

$$\int \frac{2}{(x+2y)^2}\, \partial x = \int \frac{2}{w^2}\, \partial w = -\frac{2}{w} = -\frac{2}{x+2y}.$$

So,

$$\iint_{\mathcal{D}} \frac{2}{(x+2y)^2}\, dA = \int_{-1}^0 \left(\int_4^5 \frac{2}{(x+2y)^2}\, dx \right) dy = \int_{-1}^0 \left(\Big[-\frac{2}{x+2y} \Big]_4^5 \right) dy = \int_{-1}^0 \left(-\frac{2}{5+2y} + \frac{2}{4+2y} \right) dy$$

$$= \Big[-\ln|5+2y| + \ln|4+2y| \Big]_{-1}^0 = -\ln 5 + \ln 3 + \ln 4 - \ln 2 = \ln \frac{3 \cdot 4}{5 \cdot 2} = \ln \frac{6}{5}.$$

7.3.7.7. $\mathcal{D} = \{(x,y) : 0 \le y \le 2,\ y \le x \le 3 - \frac{1}{2}\,y\}$, so

$$\iint_{\mathcal{D}} x\, dA = \int_0^2 \int_y^{3-\frac{1}{2}\,y} x\, dx\, dy = \int_0^2 \Big[\frac{1}{2}x^2 \Big]_y^{3-\frac{1}{2}\,y} dy = \frac{1}{2}\int_0^2 \big((3 - \frac{1}{2}\,y)^2 - y^2\big) dy = \frac{1}{2}\int_0^2 \big(9 - 3y - \frac{3}{4}\,y^2\big) dy$$

$$= \frac{1}{2}\Big[9y - \frac{3}{2}\,y^2 - \frac{1}{4}\,y^3 \Big]_0^2 = \frac{1}{2}\big(18 - 6 - 2 - (0 - 0 - 0)\big) = 5.$$

7.3.7.9. The curve $(x-2)^2 + y^2 = 4$, that is, $x^2 + y^2 = 4x$, can be rewritten in polar coordinates as $r = 4\cos\theta$. The curve $x^2 + (y-2)^2 = 4y$, that is, $x^2 + y^2 = 4y$, can be rewritten in polar coordinates as $r = 4\sin\theta$. These two curves intersect where $4y = x^2 + (y-2)^2 = 4x$, that is, where $y = x$. This occurs at the origin and one other point on the ray $\theta = \dfrac{\pi}{4}$.

We can write $\mathcal{D}$ as the union of two regions, as shown in the figure:

$$\mathcal{D} = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{4},\ 0 \le r \le 4\sin\theta\} \ \cup\ \{(r,\theta) : \frac{\pi}{4} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 4\cos\theta\}.$$

So,

$$\iint_{\mathcal{D}} x\, dA = \int_0^{\pi/4} \int_0^{4\sin\theta} (r\ \cos\theta) r dr\, d\theta + \int_{\pi/4}^{\pi/2} \int_0^{4\cos\theta} (r\ \cos\theta) r dr\, d\theta$$

$$= \int_0^{\pi/4} \Big[\frac{1}{3}r^3\ \cos\theta \Big]_0^{4\sin\theta} d\theta + \int_{\pi/4}^{\pi/2} \Big[\frac{1}{3}r^3\ \cos\theta \Big]_0^{4\cos\theta} d\theta = \int_0^{\pi/4} \frac{64}{3}\ \sin^3\theta\ \cos\theta\, d\theta + \int_{\pi/4}^{\pi/2} \frac{64}{3}\ \cos^4\theta\, d\theta$$

$$= \Big[\frac{16}{3}\ \sin^4\theta \Big]_0^{\pi/4} + \int_{\pi/4}^{\pi/2} \frac{16}{3}(1 + \cos 2\theta)^2 d\theta = \frac{16}{3}\Big(\frac{1}{4} - 0\Big) + \int_{\pi/4}^{\pi/2} \frac{16}{3}(1 + 2\cos 2\theta + \cos^2 2\theta) d\theta$$

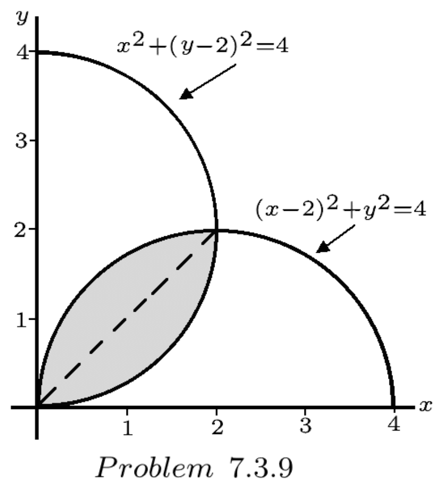


Figure 1: Problem 7.3.7.9

$$=\frac{4}{3}+\frac{16}{3}\Big[\,\frac{3}{2}\,\theta+\sin 2\theta+\frac{1}{4}\,\sin 4\theta\,\Big]_{\pi/4}^{\pi/2}=\frac{4}{3}+\frac{16}{3}\Big(\frac{3\pi}{8}+(0-1)+\frac{1}{4}(0-0)\Big)=\frac{4}{3}+2\pi-\frac{16}{3}=2\pi-4.$$

7.3.7.11. Points of intersection of the two bounding curves: $y^2=x=2y+3 \Longrightarrow 0=y^2-2y-3=(y-3)(y+1)$ $\Longrightarrow y=-1,3$, so

$$\mathcal{D}=\{(x,y):\,-1\le y\le 3,\,y^2\le x\le 2y+3\,\}$$

We calculate

$$\iint_{\mathcal{D}} x\,dA=\int_{-1}^{3}\int_{y^2}^{2y+3} x\,dx\,dy=\int_{-1}^{3}\Big[\,\frac{1}{2}\,x^2\,\Big]_{x=y^2}^{x=2y+3}dy=\frac{1}{2}\int_{-1}^{3}\Big((2y+3)^2-y^4\Big)dy=\frac{1}{2}\int_{-1}^{3}\Big(9+12y+4y^2-y^4\Big)dy$$

$$=\frac{1}{2}\Big[\,9y+6y^2+\frac{4}{3}\,y^3-\frac{1}{5}\,y^5\,\Big]_{-1}^{3}=\frac{1}{2}\Big(27+54+36-\frac{243}{5}+9-6+\frac{4}{3}+\frac{1}{5}\Big)=\frac{1}{2}\cdot\frac{1088}{15}=\frac{544}{15}.$$

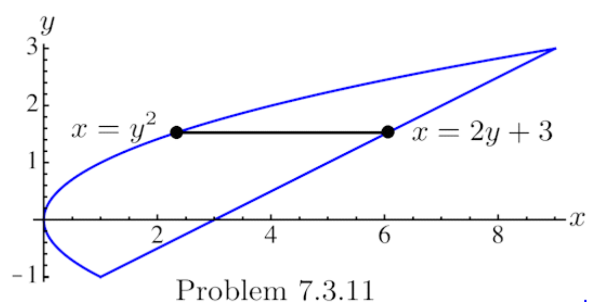


Figure 2: Problem 7.3.7.11

7.3.7.13. After drawing $\mathcal{D}$ we see that we can rewrite it:

$$\mathcal{D}=\{(x,y):\,0\le x\le 2,\,x\le y\le 2\,\}\;=\{(x,y):\,0\le y\le 2,\,0\le x\le y\,\}.$$

So,

$$\int_0^2\Big(\int_x^2 e^{-y^2}\,dy\Big)dx=\int_0^2\Big(\int_0^y e^{-y^2}\,dx\Big)dy=\int_0^2\Big[\,xe^{-y^2}\,\Big]_0^y dy=\int_0^2 y\,e^{-y^2}\,dy=\Big[-\frac{1}{2}\,e^{-y^2}\,\Big]_0^2=-\frac{1}{2}\big(e^{-4}-1\big)$$

$$=\frac{1}{2}\,(1-e^{-4}).$$

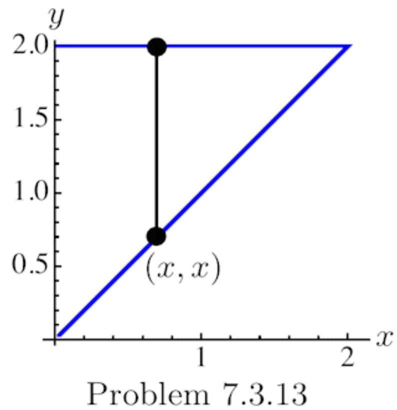


Figure 3: Problem 7.3.7.13

[Note that if we try to use vertical strips, then the first anti-partial-differentiation we would need to be able to do, $\int e^{-y^2}\,\partial y$, would have no closed form solution, making this approach a dead end.]

7.3.7.15. In polar coordinates,

$$\mathcal{D} = \{(r,\theta) : \frac{\pi}{6} \le \theta \le \frac{3\pi}{4},\ 0 \le r \le \sqrt{2}\},$$

so

$$M_y = \iint_{\mathcal{D}} \varrho\, x\, dA = \varrho_0 \int_{\pi/6}^{3\pi/4}\int_0^{\sqrt{2}} (r\,\cos\theta) r\, dr\, d\theta = \varrho_0 \left(\int_{\pi/6}^{3\pi/4} \cos\theta\, d\theta\right)\left(\int_0^{\sqrt{2}} r^2\, dr\right) = \varrho_0 \left(\Big[\ \sin\theta\ \Big]_{\pi/6}^{3\pi/4}\right)\left(\Big[\ \frac{1}{3}\, r^3\ \Big]_0^{\sqrt{2}}\right)$$

$$= \varrho_0 \Big(\frac{1}{\sqrt{2}} - \frac{1}{2}\Big)\Big(\frac{1}{3}\, 2\sqrt{2} - 0\Big) = \frac{2-\sqrt{2}}{3}\, \varrho_0.$$

7.3.7.17. The region between the circles $r = 1$ and $r = 2$ in the fourth quadrant is part of an annulus, specifically

$$\mathcal{D} = \{(r,\theta) : \frac{3\pi}{2} \le \theta \le 2\pi,\ 1 \le r \le 2\},$$

so

$$\iint_{\mathcal{D}} (x^2 - y^2) dA = \int_{3\pi/2}^{2\pi}\int_1^2 (x^2 - y^2) r dr\, d\theta = \int_0^{2\pi}\int_1^2 r^2(\cos^2\theta - \sin^2\theta) r dr d\theta = \int_{3\pi/2}^{2\pi}\int_1^2 \cos 2\theta\, r^3 dr\, d\theta$$

$$= \left(\Big[\ \frac{1}{2}\, \sin 2\theta\ \Big]_{3\pi/2}^{2\pi}\right)\left(\Big[\ \frac{1}{4}\, r^4\ \Big]_1^2\right) = 0 \cdot \frac{15}{4} = 0.$$

7.3.7.19. The solid consists of straws $0 \le z \le 1 - y$ above the domain $\mathcal{D} = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$ in the $xy-$plane. The solid has

$$Volume = \iint_{\mathcal{D}} (1-y)\, dA = \int_0^{2\pi}\int_0^a (1 - r\sin\theta) r\, dr\, d\theta = \int_0^{2\pi} \Big[\ \frac{1}{2}\, r^2 - \frac{1}{3}\, r^3 \sin\theta\ \Big]_0^a\, d\theta = \int_0^{2\pi} \Big(\frac{1}{2}\, a^2 - \frac{1}{3}\, a^3 \sin\theta\Big) d\theta$$

$$= \Big[\ \frac{1}{2}\, a^2\theta + \frac{1}{3}\, a^3 \cos\theta\ \Big]_0^{2\pi} = \frac{1}{2}\, a^2 \cdot 2\pi + 0 = \pi a^2.$$

7.3.7.21. The domain is

$$\mathcal{D} \triangleq \{(x,y) : \frac{\pi}{2} \le x \le \pi,\ \frac{\pi}{2} \le y \le \frac{5\pi}{2} - 2x\}.$$

Define $\mathbf{F} \triangleq \cos x\, \sin y\, \hat{\boldsymbol{\imath}} + \sin x\, \cos y\, \hat{\boldsymbol{\jmath}}$. We calculate

$$\oint_{\mathcal{C}} (\cos x\, \sin y\, dx + \sin x\, \cos y\, dy) = \oint_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial}{\partial x}\Big[\ \sin x\, \cos y\ \Big] - \frac{\partial}{\partial y}\Big[\ \cos x\, \sin y\ \Big]\right) dA$$

$$= \int_{\pi/2}^{\pi} \int_{\pi/2}^{(5\pi-4x)/2} (\cos x\, \cos y - \cos x\, \cos y) dy\, dx = \int_{\pi/2}^{\pi} \int_{\pi/2}^{(5\pi-4x)/2} (0) dy\, dx = 0.$$

7.3.7.23. The domain is

$$\mathcal{D} \triangleq \{(x, y) : 0 \le x \le 2,\, 0 \le y \le x(2-x)\}.$$

Define $\mathbf{F} \triangleq y^2\, \hat{\boldsymbol{\imath}} + 2x^2\, \hat{\boldsymbol{\jmath}}$. We calculate

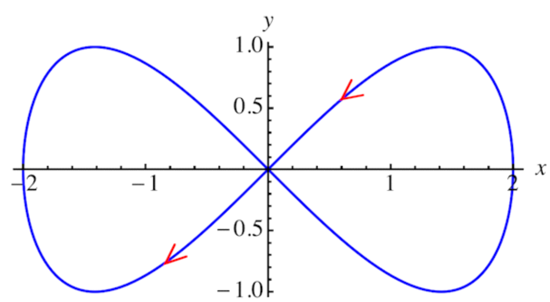
$$\oint_{\mathcal{C}} (y^2\, dx + 2x^2\, dy) = \oint_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial}{\partial x}\left[\, 2x^2 \,\right] - \frac{\partial}{\partial y}\left[\, y^2 \,\right] \right) dA = \int_0^2 \int_0^{x(2-x)} (4x - 2y) dy\, dx$$

$$= \int_0^2 \Big[\, 4xy - y^2 \,\Big]_0^{x(2-x)} dx = \int_0^2 \Big(4x \cdot x(2-x) - \big(x(2-x)\big)^2 \Big) dx$$

$$= \int_0^2 x^2(2-x)\big(4 - (2-x)\big) dx = \int_0^2 (4x^2 - x^4) dx = \left[\, \frac{4}{3}\, x^3 - \frac{1}{5}\, x^5 \,\right]_0^2 = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}.$$

7.3.7.25. Parametrize the ellipse by $\mathcal{C} :\ \mathbf{r} = a \cos t\, \hat{\boldsymbol{\imath}} + b \sin t\, \hat{\boldsymbol{\jmath}}$, $0 \le t \le 2\pi$. By Corollary 7.4, the area inside the ellipse is

$$\oint_{\mathcal{C}} x\, dy = \int_0^{2\pi} (a \cos t)(b \cos t\, dt) = ab \int_0^{2\pi} \cos^2 t\, dt = \frac{ab}{2} \int_0^{2\pi} (1 + \cos 2t) dt = \frac{ab}{2} \cdot \left[t + \frac{1}{2}\, \sin 2t \right]_0^{2\pi} = \pi ab.$$



Problem 7.3.25

Figure 4: Problem 7.3.7.25

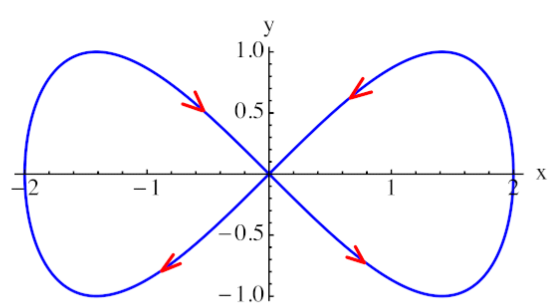


Figure 5: Problem 7.3.7.27

7.3.7.27. The curve $\mathcal{C}$ is not simple and positively oriented. But, symmetry implies the area enclosed by $\mathcal{C}$ is two times the area enclosed by the part of the curve given by the simple, positively oriented curve $\widetilde{\mathcal{C}} :\ \mathbf{r} = 2 \cos t\, \hat{\boldsymbol{\imath}} + \sin 2t\, \hat{\boldsymbol{\jmath}}$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

By Corollary 7.4, the area inside the curve $\mathcal{C}$ is

$$Area = 2 \oint_{\mathcal{C}} x\, dy = 2 \int_{-\pi/2}^{\pi/2} (2 \cos t)(2 \cos 2t\, dt) = 8 \int_{-\pi/2}^{\pi/2} \cos t \cdot (1 - 2 \sin^2 t)\, dt = 8 \left[\sin t - \frac{2}{3}\, \sin^3 t \right]_{-\pi/2}^{\pi/2}$$

$$= 8\Big(\Big(1-\frac{2}{3}\Big)-\Big(-1+\frac{2}{3}\Big)\Big) = \frac{16}{3}.$$

7.3.7.29. The curve $\mathcal{C}$ is the boundary of the triangle in the plane $z = 0$, traversed counter-clockwise. The circulation is

$$\Gamma = \int_{\mathcal{C}} \mathbf{v}\bullet d\mathbf{r} = \int_{\mathcal{C}} (z\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} - y\,\hat{\boldsymbol{k}})\bullet(dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}) = \int_{\mathcal{C}} (z\,dx + x\,dy).$$

But in the plane containing the triangle, $z \equiv 0$, so the circulation is

$$\Gamma = \int_{\mathcal{C}} (0\cdot dx + x\,dy) = \int_{\mathcal{C}} x\,dy,$$

so Green's theorem applied to the domain of the triangle

$$\mathcal{D} \triangleq \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 5-5x\}$$

gives that the circulation is

$$\Gamma = \iint_{\mathcal{D}} \Big(\frac{\partial}{\partial x}\big[\,x\,\big] - \frac{\partial}{\partial y}\big[\,0\,\big]\Big) dA = \iint_{\mathcal{D}} 1\,dA = Area(\mathcal{D}) = \frac{5}{2}.$$

7.3.7.31. Using rectangles with $\triangle x = 2$ and $\triangle y = 1$,

(a) $\Delta A = (2 miles)(1 mile)$ and $f(x,y)$ is measured in tenths of a mile, so the

$Volume \approx \Delta A\cdot(f(1,0.5) + f(1,1.5) + f(1,2.5) + f(3,0.5) + f(3,1.5) + f(3,2.5))$

$= (0.2\,\text{mile}^2)(1+2+3+4+5+5) = 4\,\text{mile}^3.$

(b) average height $= \dfrac{Volume}{Area(\mathcal{D})} \approx \dfrac{4\,\text{mile}^3}{12\,\text{mile}^2} = \dfrac{1}{3}$ mile.

7.3.7.33. <u>Method 1</u>: In polar coordinates, $\mathcal{D} = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 4\}$, so the average value of f over the quarter disk is

$$\bar{f} \triangleq \frac{1}{Area\ of\ \mathcal{D}} \iint_{\mathcal{D}} f(\mathbf{r})\,dA = \frac{1}{4\pi}\int_0^{\pi/2}\int_0^4 f(\mathbf{r})\,r\,dr\,d\theta.$$

In polar coordinates, $\mathcal{D}$ is a rectangle that we can break up into sub-rectangles, for example, the four sub-rectangles

$$\{(r,\theta) : 0\le\theta\le\frac{\pi}{4}, 0\le r\le 2\},\ \{(r,\theta) : 0\le\theta\le\frac{\pi}{4}, 2\le r\le 4\},\ \{(r,\theta) : \frac{\pi}{4}\le\theta\le\frac{\pi}{2}, 0\le r\le 2\},\ \{(r,\theta) : \frac{\pi}{4}\le\theta\le\frac{\pi}{2}, 2\le r\le 4\}.$$

The above four sub-rectangles have area $\Delta A = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$ in the $r\theta-$plane.

Sampling at the midpoints of those sub-rectangles for the integral of $\big(r\cdot f(r,\theta)\big)$ gives

$$\int_0^{\pi/2}\int_0^4 \big(rf(r,\theta)\big)dr\,d\theta \approx \Delta A\Big(r\cdot f(r,\theta)\Big|_{(r,\theta)=(1,\frac{\pi}{8})} + r\cdot f(r,\theta)\Big|_{(r,\theta)=(3,\frac{\pi}{8})} + r\cdot f(r,\theta)\Big|_{(r,\theta)=(1,\frac{3\pi}{8})} + r\cdot f(r,\theta)\Big|_{(r,\theta)=(3,\frac{3\pi}{8})}\Big).$$

Using the values given in the problem, we have

$$\int_0^{\pi/2}\int_0^4 \big(rf(r,\theta)\big)dr\,d\theta \approx \frac{\pi}{2}\big(1\cdot 2 + 3\cdot 3 + 1\cdot 1 + 3\cdot 5\big) = \frac{27\pi}{2}.$$

So, the average value of f on the disk is

$$\bar{f} \approx \frac{1}{4\pi}\cdot\frac{27\pi}{2} = \frac{27}{8}.$$

<u>Method 2</u>: In polar coordinates, $\mathcal{D} = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 4\}$, so the average value of f over the quarter disk is

$$\bar{f} \triangleq \frac{1}{Area\ of\ \mathcal{D}} \iint_{\mathcal{D}} f(\mathbf{r})\,dA = \frac{1}{4\pi}\int_0^{\pi/2}\int_0^4 f(\mathbf{r})\,r\,dr\,d\theta.$$

In Figure 7.36 we decomposed $\mathcal{D}$ into four sub-regions, each of which is either a wedge or a part of a wedge in the $xy-$plane:

$\{(r,\theta) : 0 \le \theta \le \frac{\pi}{4},\, 0 \le r \le 2\}$, which has area $\Delta A_1 = \frac{1}{2}\, \triangle\theta\, R^2 = \frac{1}{2} \cdot \dfrac{\pi}{4} \cdot 2^2 = \dfrac{\pi}{2}$,

$\{(r,\theta) : 0 \le \theta \le \frac{\pi}{4},\, 2 \le r \le 4\}$, which has area $\Delta A_2 = \frac{1}{2}\, \triangle\theta\big(R_2^2 - R_1^2\big) = \frac{1}{2} \cdot \dfrac{\pi}{4} \cdot (4^2 - 2^2) = \dfrac{3\pi}{2}$,

$\{(r,\theta) : \frac{\pi}{4} \le \theta \le \frac{\pi}{2},\, 0 \le r \le 2\}$, which has area $\Delta A_3 = \frac{1}{2}\, \triangle\theta\, R^2 = \frac{1}{2} \cdot \dfrac{\pi}{4} \cdot 2^2 = \dfrac{\pi}{2}$, and

$\{(r,\theta) : \frac{\pi}{4} \le \theta \le \frac{\pi}{2},\, 2 \le r \le 4\}$, which has area $\Delta A_4 = \frac{1}{2}\, \triangle\theta\big(R_2^2 - R_1^2\big) = \frac{1}{2} \cdot \dfrac{\pi}{4} \cdot (4^2 - 2^2) = \dfrac{3\pi}{2}$.

We approximate, using sampling at the polar coordinates version of midpoints:

$$\iint_{\mathcal{D}} f(\mathbf{r})\, dA \approx \Delta A_1 f(r,\theta)\Big|_{(r,\theta)=(1,\frac{\pi}{8})} + \Delta A_2 f(r,\theta)\Big|_{(r,\theta)=(3,\frac{\pi}{8})} + \Delta A_3 f(r,\theta)\Big|_{(r,\theta)=(1,\frac{\pi}{8})} + \Delta A_4 f(r,\theta)\Big|_{(r,\theta)=(3,\frac{\pi}{8})}$$

$$= \frac{\pi}{2} \cdot 2 + \frac{3\pi}{2} \cdot 3 + \frac{\pi}{2} \cdot 1 + \frac{3\pi}{2} \cdot 5 = \frac{27\pi}{2}.$$

So the average value of f over the quarter disk is

$$\bar{f} = \frac{1}{4\pi} \int_0^{\pi/2} \iint_{\mathcal{D}} f(\mathbf{r})\, dA \approx \frac{1}{4\pi} \cdot \frac{27\pi}{2} = \frac{27}{8}.$$

So, the two methods produced the same final conclusion.

7.3.7.35. $\mathcal{D} = \{(x,y) :\ 0 \le y \le 3,\, \frac{2}{3}\, y \le x \le \frac{4}{3}\, y\}$, so the lamina has mass

$$M = \iint_{\mathcal{D}} \varrho\, dA = \int_0^3 \int_{2y/3}^{4y/3} (10 - x + 0.1x^2 - y + 0.1y^2) dx\, dy = \frac{83}{4}\ g,$$

by using Mathematica.

The moments are

$$M_y = \iint_{\mathcal{D}} \varrho\, x\, dA = \int_0^3 \int_{2y/3}^{4y/3} (10 - x + 0.1x^2 - y + 0.1y^2) x\, dx\, dy = \frac{1967}{50}\ g - cm$$

and

$$M_x = \iint_{\mathcal{D}} \varrho\, y\, dA = \int_0^3 \int_{2y/3}^{4y/3} (10 - x + 0.1x^2 - y + 0.1y^2) y\, dx\, dy = \frac{198}{5}\ g - cm$$

again by using Mathematica.

The center of mass is at $(\overline{x}, \overline{y}) = \dfrac{1}{2075}\, (3934\ cm, 3960\ cm) \approx (1.8959\ cm, 1.90843\ cm)$.

The polar moment of inertia of the lamina is

$$I_0 = \iint_{\mathcal{D}} \varrho \cdot (x^2 + y^2)\, dA = \int_0^3 \int_{2y/3}^{4y/3} (10 - x + 0.1x^2 - y + 0.1y^2)(x^2 + y^2) dx\, dy = \frac{8771}{50}\ \ g - cm^2 = 175.42\ g - cm^2,$$

again by using Mathematica.

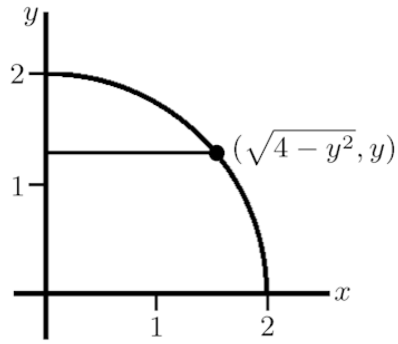
7.3.7.37. $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{4 - y^2}\, dy\, dx = \int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4 - y^2}\, dx dy = \int_0^2 \left(\Big[x \cdot \sqrt{4 - y^2} \Big]_0^{\sqrt{4-y^2}} \right) dy$

$$= \int_0^2 \Big(\sqrt{4 - y^2} \cdot \sqrt{4 - y^2} \Big) dy = \int_0^2 \big(4 - y^2\big) dy = \Big[4y - \frac{1}{3}\, y^3 \Big]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}.$$

7.3.7.39 Using Green's Theorem for the half disk $\mathcal{D} \triangleq \{(x,y) :\ 0 \le x^2 + y^2 \le a,\ x \ge 0\}$, the circulation of the velocity field $\mathbf{v} = y\, \hat{\boldsymbol{\imath}} - x\, \hat{\boldsymbol{\jmath}} + xy\, \hat{\boldsymbol{k}}$ around $\mathcal{D}$ is

$$\Gamma = \oint_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \bullet d\mathbf{r} = \oint_{\mathcal{C}} (y\, \hat{\boldsymbol{\imath}} - x\, \hat{\boldsymbol{\jmath}} + xy\, \hat{\boldsymbol{k}}) \bullet (dx\, \hat{\boldsymbol{\imath}} + dy\, \hat{\boldsymbol{\jmath}}) = \oint_{\mathcal{C}} (y\, dx - x\, dy) = \iint_{\mathcal{D}} \left(\frac{\partial}{\partial x} [\, -x\,] - \frac{\partial}{\partial y} [\, y\,] \right) dA$$

$$= \iint_{\mathcal{D}} (-1 - (1)) dA = -2 \cdot (Area\ of\ \mathcal{D}) = -2 \cdot (\frac{1}{2}\, \pi a^2) = -\pi a^2.$$



Problem 7.3.37

Figure 6: Problem 7.3.7.37

Section 7.4

7.4.3.1. The top of the tetrahedron passes through the points $(1,0,0)$, $(0,2,0)$, $(0,0,3)$, which are on the plane $Ax+By+Cz=D$ if

$$\left\{\begin{array}{c} A\cdot 1+B\cdot 0+C\cdot 0=D \\ A\cdot 0+B\cdot 2+C\cdot 0=D \\ A\cdot 0+B\cdot 0+C\cdot 3=D \end{array}\right\}.$$

One solution is $A=6, B=3, C=2, D=6$, so the plane is $6x+3y+2z=6$, that is, $z=3-3x-\dfrac{3y}{2}$.

The solid $\mathcal{V}$ sits above the planar region $\mathcal{D}$ as shown in the figure, so

$$\mathcal{V}=\left\{(x,y,z):(x,y) \text{ in } \mathcal{D},\ 0\le z\le 3-3x-\frac{3y}{2}\right\}=\left\{(x,y,z)\colon 0\le y\le 2, 0\le x\le 1-\frac{y}{2}, 0\le z\le 3-3x-\frac{3y}{2}\right\}.$$

<u>Example</u>: So, a final conclusion is that the integral can be re-written as

$$\int_{\underline{0}}^{\underline{2}}\int_{\underline{0}}^{\underline{1-\frac{y}{2}}}\int_{\underline{0}}^{\underline{3-3x-\frac{3y}{2}}}\underline{f(x,y,z)}\ \underline{dz}\ \underline{dx}\ \underline{dy}$$

Five other final conclusions can be correct.

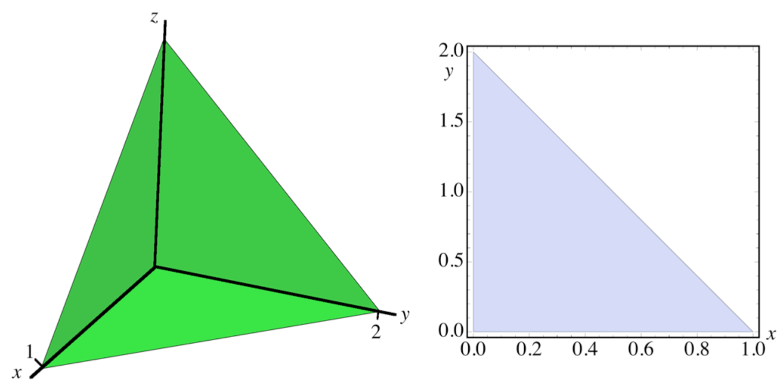


Figure 7: Problem 7.4.3.1

7.4.3.3. The top of the tetrahedron passes through the points $(1,0,0)$, $(0,3,0)$, $(0,0,2)$, which are on the plane $Ax+By+Cz=D$ if

$$\left\{\begin{array}{c} A\cdot 1+B\cdot 0+C\cdot 0=D \\ A\cdot 0+B\cdot 3+C\cdot 0=D \\ A\cdot 0+B\cdot 0+C\cdot 2=D \end{array}\right\}.$$

One solution is $A=6, B=2, C=3, D=6$, so the plane is $6x+2y+3z=6$, that is, $z=2-2x-\dfrac{2y}{3}$.

The solid $\mathcal{V}$ sits above the planar region $\mathcal{D}$ as shown in the figure, so

$$\mathcal{V}=\left\{(x,y,z):\ (x,y)\text{ in }\mathcal{D},\ 0\le z\le 2-2x-\frac{2y}{3}\right\}=\left\{(x,y,z)\colon 0\le x\le 1, 0\le y\le 3-3x, 0\le z\le 2\Big(1-x-\frac{y}{3}\Big)\right\}.$$

The problem concerns the centroid, so the mass density is a constant, ϱ_0. So, the mass is

$$M=\int_0^1\int_0^{3(1-x)}\int_0^{2(1-x-\frac{y}{3})}\varrho_0\,dz\,dy\,dx=\varrho_0\int_0^1\int_0^{3(1-x)}2\Big((1-x)-\frac{y}{3}\Big)dy\,dx=2\varrho_0\int_0^1\left(\Big[\,(1-x)-\frac{1}{6}\,y^2\,\Big]_0^{3(1-x)}\right)dx$$

$$=2\varrho_0\int_0^1\left((1-x)\cdot 3(1-x)-\frac{1}{6}\cdot 3^2(1-x)^2\right)dx=2\varrho_0\int_0^1(1-x)\Big(3-\frac{3}{2}\Big)(1-x)\,dx=3\varrho_0\int_0^1(1-x)^2dx=\varrho_0\Big[-(1-x)^3\Big]_0^1$$
$$=\varrho_0$$

and the moment $M_{y=0}$ is

$$M_{y=0}=\int_0^1\int_0^{3(1-x)}\int_0^{2(1-x-\frac{y}{3})}\varrho_0\,y\,dz\,dy\,dx=\varrho_0\int_0^1\int_0^{3(1-x)}2\Big((1-x)-\frac{y}{3}\Big)y\,dy\,dx$$

$$=\varrho_0\int_0^1\left(\Big[\,(1-x)y^2-\frac{2}{9}\,y^3\,\Big]_0^{3(1-x)}\right)dx=\varrho_0\int_0^1\left((1-x)\cdot 3^2(1-x)^2-\frac{2}{9}\cdot 3^3(1-x)^3\right)dx=\varrho_0\int_0^1(1-x)(9-6)(1-x)^2dx$$

$$=3\varrho_0\int_0^1(1-x)^3dx=\frac{3}{4}\,\varrho_0\Big[-(1-x)^4\Big]_0^1=\frac{3}{4}\,\varrho_0.$$

So, $y-$coordinate of the centroid is

$$\overline{y}=\frac{M_{y=0}}{M}=\frac{\frac{3}{4}\,\varrho_0}{\varrho_0}=\frac{3}{4}.$$

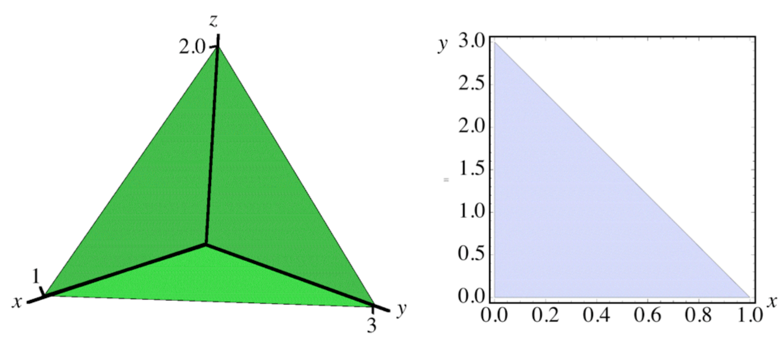


Figure 8: Problem 7.4.3

7.4.3.5. The solid $\mathcal{V}$ is given by

$$\mathcal{V}=\left\{(x,y,z)\colon 0\le z\le 4, 0\le x\le 2-\frac{z}{2}, 0\le y\le 1-\frac{x}{2}-\frac{z}{4}\right\}.$$

Graphing the region or working with the inequalities enables us to see that the region is a tetrahedron whose vertices are $(0,0,0)$, $(0,0,4)$, $(2,0,0)$, $(0,1,0)$. The top of the tetrahedron is part of the plane $y=1-\dfrac{x}{2}-\dfrac{z}{4}$, that is,

$$z=4-2x-4y,$$

which sits above the region $\mathcal{D}$ in the $xy-$plane which is the triangle with vertices $(0,0,0)$, $(2,0,0)$, and $(0,1,0)$. So,

$$\mathcal{V}=\left\{(x,y,z)\colon 0\le x\le 2, 0\le y\le 1-\frac{x}{2}, 0\le z\le 4-2x-4y\right\}.$$

The final conclusion is that the integral can be re-written as

$$\int_0^4\int_0^{2-\frac{z}{2}}\int_0^{1-\frac{x}{2}-\frac{z}{4}}f(x,y,z)\,dy\,dx\,dz=\int_{\underline{0}}^{\underline{2}}\int_{\underline{0}}^{\underline{1-\frac{x}{2}}}\int_{\underline{0}}^{\underline{4-2x-4y}}\underline{f(x,y,z)}\ \underline{dz}\ \underline{dy}\ \underline{dx}$$

7.4.3.7. The surfaces intersect on the curve $x^2+y^2+1=4$, and the upper surface is $z=4-(x^2+y^2)$, which lies above part of the disk $x^2+y^2\le 3$ in the $xy-$plane. Also, the solid is in the first octant, so $0\le\theta\le\frac{\pi}{2}$. In cylindrical coordinates,

$$\mathcal{V}=\Big\{(r,\theta,z)\colon 0\le\theta\le\frac{\pi}{2},\, 0\le r\le\sqrt{3},\, 1\le z\le 4-r^2\Big\}.$$

The problem concerns the centroid, so the mass density is a constant, ϱ_0. So, the mass is

$$M=\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{4-r^2}\varrho_0\, r\,dz\,dr\,d\theta=\varrho_0\int_0^{\pi/2}\int_0^{\sqrt{3}}(4-r^2-1)r\,dr\,d\theta=\varrho_0\int_0^{\pi/2}\bigg(\Big[\,\frac{3}{2}\,r^2-\frac{1}{4}\,r^4\,\Big]_0^{\sqrt{3}}\bigg)d\theta=\varrho_0\int_0^{\pi/2}\Big(\frac{9}{2}-\frac{9}{4}\Big)d\theta$$

$$=\varrho_0\,\frac{\pi}{2}\cdot\frac{9}{4}=\varrho_0\cdot\frac{9\pi}{8}.$$

The moments are

$$M_{xy}=\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{4-r^2}z\,\varrho_0\,r\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^{\sqrt{3}}\Big[\,\frac{1}{2}\,z^2\,\Big]_1^{4-r^2}\varrho_0 r\,dz\,dr\,d\theta=\frac{1}{2}\,\varrho_0\int_0^{\pi/2}\int_0^{\sqrt{3}}\Big((4-r^2)^2-1\Big)r\,dr\,d\theta$$

$$=\frac{1}{2}\,\varrho_0\int_0^{\pi/2}\int_0^{\sqrt{3}}\Big(15-8r^2+r^4\Big)r\,dr\,d\theta=\frac{1}{2}\,\varrho_0\int_0^{\pi/2}\bigg(\Big[\,\frac{15}{2}\,r^2-2r^4+\frac{1}{6}\,r^6\,\Big]_0^{\sqrt{3}}\bigg)\,d\theta$$

$$=\frac{1}{2}\,\varrho_0\int_0^{\pi/2}\Big(\frac{45}{2}-18+\frac{9}{2}\Big)d\theta=\frac{1}{2}\,\varrho_0\cdot\frac{\pi}{2}\cdot 9=\varrho_0\cdot\frac{9\pi}{4},$$

$$M_{yz}=\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{4-r^2}x\,\varrho_0\,r\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{4-r^2}r\cos\theta\,\varrho_0\,r\,dz\,dr\,d\theta=\varrho_0\int_0^{\pi/2}\int_0^{\sqrt{3}}(4-r^2-1)r^2\cos\theta\,dr\,d\theta$$

$$=\varrho_0\left(\int_0^{\pi/2}\cos\theta\,d\theta\right)\left(\int_0^{\sqrt{3}}(3r^2-r^4)\,dr\right).=\varrho_0\cdot 1\cdot\left(\Big[\,r^3-\frac{1}{5}\,r^5\,\Big]_0^{\sqrt{3}}\right)=\varrho_0\cdot\left(3\sqrt{3}-\frac{9\sqrt{3}}{5}\right)=\varrho_0\cdot\frac{6\sqrt{3}}{5}.$$

To find M_{zx}, either use symmetry regarding x versus y, or calculate

$$M_{zx}=\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{4-r^2}y\,\varrho_0\,r\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{4-r^2}r\sin\theta\,\varrho_0\,r\,dz\,dr\,d\theta=...=\varrho_0\left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\left(\int_0^{\sqrt{3}}(3r^2-r^4)\,dr\right)$$

$$=...=\varrho_0\cdot\frac{6\sqrt{3}}{5},$$

using results used in the calculation of M_{yz}.

The centroid is at

$$(\overline{x},\overline{y},\overline{z})=\frac{1}{M}(M_{yz},M_{zx},M_{xy})=\frac{1}{\varrho_0\cdot 9\pi/8}\left(\varrho_0\cdot\frac{6\sqrt{3}}{5},\ \varrho_0\cdot\frac{6\sqrt{3}}{5},\ \varrho_0\cdot\frac{9\pi}{4}\right)$$

that is,

$$(\overline{x},\overline{y},\overline{z})=\left(\frac{16\sqrt{3}}{15\pi},\ \frac{16\sqrt{3}}{15\pi},\ 2\right)\approx(0.5880841551,\ 0.5880841551,\ 2).$$

7.4.3.9. The solid is

$$\mathcal{V}=\{(r,\theta,z)\colon 0\le\theta\le 2\pi,\, 0\le r\le 2,\, -1\le z\le 2-y\}.$$

So,

$$\iiint_{\mathcal{V}}(x^2+y^2)dV=\int_0^{2\pi}\int_0^2\int_{-1}^{2-y}(x^2+y^2)r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2\int_{-1}^{2-r\sin\theta}r^2 r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 r^3\big((2-r\sin\theta)-(-1)\big)dr\,d\theta=\int_0^{2\pi}\int_0^2\big(3r^3-r^4\sin\theta\big)dr\,d\theta=\int_0^{2\pi}\Big[\,\frac{3}{4}\,r^4-\frac{1}{5}\,r^5\sin\theta\,\Big]_0^2 d\theta$$

$$=\int_0^{2\pi}\Big(12-\frac{32}{5}\,\sin\theta\Big)d\theta=12\cdot 2\pi-\frac{32}{5}\cdot 0=24\pi.$$

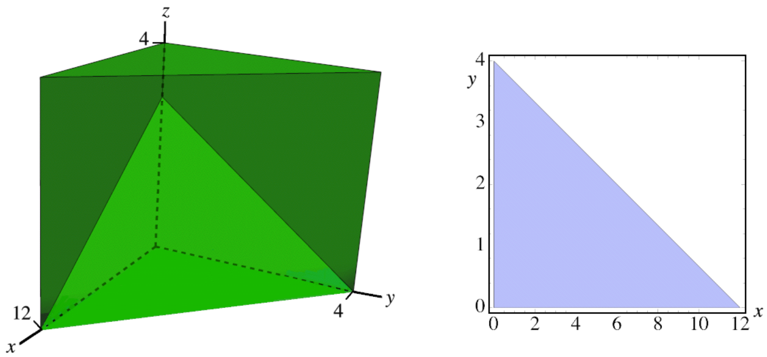


Figure 9: Problem 7.4.3.11

7.4.3.11. The solid $\mathcal{V}$ consists of vertical straws that run *from* the part of the plane $x + 3y + 4z = 12$ that lies in the first octant *to* the plane $z = 4$. We can rewrite the slanted plane as $z = 3 - \frac{x}{4} - \frac{3y}{4}$. So,

$$\mathcal{V} = \left\{ (x, y, z) : \ (x, y) \text{ in } \mathcal{D}, \ 3 - \frac{x}{4} - \frac{3y}{4} \le z \le 4 \right\}.$$

We can find $\mathcal{D}$ by thinking of it as the shadow cast by the slanted plane on the first quadrant of the $xy-$plane by a light at $z = \infty$. So,

$$\mathcal{D} = \left\{ (x, y) : \ 0 \le x \le 12, \ 0 \le y \le 4 - \frac{x}{3} \right\}.$$

Putting things together, this gives

$$\iiint_{\mathcal{V}} f(x, y, z) dV = \int_0^{12} \int_0^{4-\frac{x}{3}} \int_{3-\frac{x}{4}-\frac{3y}{4}}^{4} f(x, y, z) \, dz \, dy \, dx.$$

7.4.3.13. First, find the intersection of the two paraboloids that bound $\mathcal{V}$:

$$2\big(x^2 + y^2\big) = z = 9 - x^2 - y^2 \quad \iff \quad 3(x^2 + y^2) = 9 \quad \iff \quad r^2 = 3.$$

The solid $\mathcal{V}$ consists of vertical straws that run from the surface $z = 2\big(x^2 + y^2\big)$ to the surface $z = 9 - x^2 - y^2$. So,

$$\mathcal{V} = \big\{ (r, \theta, z) : \ (r, \theta) \text{ in } \mathcal{D}, \ 2\big(x^2 + y^2\big) \le z \le 9 - x^2 - y^2 \big\} = \left\{ (r, \theta, z) : \ 0 \le \theta \le 2\pi, \ 0 \le r \le \sqrt{3}, \ 2r^2 \le z \le 9 - r^2 \right\}.$$

The problem asks for the centroid so the mass density is a constant, ϱ_0. The total mass is

$$M = \iiint_{\mathcal{V}} \varrho_0 \, dV = \varrho_0 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{2r^2}^{9-r^2} r \, dz \, dr \, d\theta = \varrho_0 \int_0^{2\pi} \int_0^{\sqrt{3}} \big((9 - r^2) - 2r^2\big) r \, dr \, d\theta = \varrho_0 \int_0^{2\pi} \int_0^{\sqrt{3}} \big(9r - 3r^3\big) \, dr \, d\theta$$

$$= \varrho_0 \cdot 2\pi \cdot \Big[\frac{9}{2} r^2 - \frac{3}{4} r^4 \Big]_0^{\sqrt{3}} = \varrho_0 \cdot 2\pi \cdot \left(\frac{27}{2} - \frac{27}{4} \right) = \varrho_0 \cdot \frac{27\pi}{2}.$$

Regarding the $z-$coordinate of the centroid, we find

$$M_{xy} = \iiint_{\mathcal{V}} \varrho_0 z \, dV = \varrho_0 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{2r^2}^{9-r^2} r \, z \, dz \, dr \, d\theta = \varrho_0 \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{1}{2} \big((9 - r^2)^2 - (2r^2)^2\big) r \, dr \, d\theta$$

$$= \frac{1}{2} \varrho_0 \int_0^{2\pi} \int_0^{\sqrt{3}} \big(81r - 18r^3 - 3r^5\big) \, dr \, d\theta = \frac{1}{2} \varrho_0 \cdot 2\pi \cdot \Big[\frac{81}{2} r^2 - \frac{9}{2} r^4 - \frac{1}{2} r^6 \Big]_0^{\sqrt{3}} = \frac{1}{2} \varrho_0 \cdot 2\pi \cdot \left(\frac{243}{2} - \frac{81}{2} - \frac{27}{2} \right) = \varrho_0 \cdot \frac{135\pi}{2}.$$

The $z-$coordinate of the centroid is

$$\overline{z} = \frac{M_{xy}}{M} = \frac{\varrho_0 \cdot 135\pi/2}{\varrho_0 \cdot 27\pi/2} = 5.$$

7.4.3.15. First, find the intersection of the two surfaces that bound $\mathcal{V}$:

$$r = \sqrt{x^2 + y^2} = z = 2 - x^2 - y^2 = 2 - r^2 \iff 0 = r^2 + r - 2 = (r + 2)(r - 1),$$

because $r \ge 0$, we see that the surfaces intersect in the circle $r = 1$. The solid $\mathcal{V}$ consists of vertical straws that run from the surface $z = r$ to the surface $z = 2 - r^2$. So,

$$\mathcal{V} = \left\{(r, \theta, z) : \, 0 \le \theta \le 2\pi, \, 0 \le r \le 1, \, r \le z \le 2 - r^2\right\}.$$

The solid is homogeneous, so the mass density is a constant, ϱ_0. The polar moment of inertia for rotation about the $z-$axis is

$$I_0 = \iiint_{\mathcal{V}} \varrho_0 \left(x^2 + y^2\right) dV = \varrho_0 \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r^2 \cdot r \, dz \, dr \, d\theta = \varrho_0 \int_0^{2\pi} \int_0^1 \left((2 - r^2) - r\right) r^3 \, dr \, d\theta$$
$$= 2\pi\varrho_0 \cdot \left[\, \frac{1}{2} r^4 - \frac{1}{5} r^5 - \frac{1}{6} r^6 \,\right]_0^1 = 2\pi\varrho_0 \left(\frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right) = \frac{4\pi}{15}\, \varrho_0.$$

7.4.3.17. The mass density is $\varrho = k \cdot |x|$, where k is a constant; because $\mathcal{V}$ is in the half space $x \ge 0$, $\varrho = kx$. Again, because $x \ge 0$, the angle θ satisfies $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. The solid is

$$\mathcal{V} = \left\{(r, \phi, \theta) : \, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \, 0 \le \phi \le \pi, \, 2 \le \rho \le 3\right\}.$$

The solid has total mass

$$M = \iiint_{\mathcal{V}} \varrho \, dV = \iiint_{\mathcal{V}} kx \, dV = \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_2^3 (k\rho \sin\phi \, \cos\theta) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= k \left(\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta\right) \left(\int_0^{\pi} \sin^2\phi \, d\phi\right) \left(\int_2^3 \rho^3 \, d\rho\right) = k \left(\left[\, \sin\theta \,\right]_{-\pi/2}^{\pi/2}\right) \left(\int_0^{\pi} \frac{1}{2} (1 - \cos 2\phi) d\phi\right) \left(\left[\, \frac{1}{4} \rho^4 \,\right]_2^3\right)$$
$$= k \cdot 2 \cdot \left(\left[\, \frac{1}{2} \phi - \frac{1}{4} \sin 2\phi \,\right]_0^{\pi}\right) \cdot \frac{1}{4}\left(3^4 - 2^4\right) = k \cdot 2 \cdot \frac{\pi}{2} \cdot \frac{65}{4} = \frac{65\pi k}{4}.$$

7.4.3.19. In Example 7.25's alternate cylindrical coordinates (x, r, φ) in which the $x-$axis is the axis of revolution, the position vector is given by

$$\mathbf{r} = x\,\hat{\boldsymbol{\imath}} + \mathrm{r} \cos\varphi \,\hat{\boldsymbol{\jmath}} + \mathrm{r} \sin\varphi \,\hat{\boldsymbol{k}}.$$

In these coordinates, the "volume of revolution" solid is

$$\mathcal{V} = \{(x, \mathrm{r}, \varphi) : \, a \le x \le b, 0 \le \varphi \le 2\pi, \, g(x) \le \mathrm{r} \le f(x)\}.$$

The volume of the solid is

$$Volume = \iiint_{\mathcal{V}} dV = \int_a^b \int_0^{2\pi} \int_{g(x)}^{f(x)} \mathrm{r} \, d\mathrm{r} \, d\varphi \, dx = \int_a^b \int_0^{2\pi} \left(\left[\, \frac{1}{2} \mathrm{r}^2 \,\right]_{g(x)}^{f(x)}\right) d\varphi \, dx$$
$$= \int_a^b \int_0^{2\pi} \frac{1}{2} \left(\left(f(x)\right)^2 - \left(g(x)\right)^2\right) d\varphi \, dx = \pi \int_a^b \left(\left(f(x)\right)^2 - \left(g(x)\right)^2\right) dx,$$

as we have seen in a single variable calculus course.

7.4.3.21. Define another alternate cylindrical coordinates $(\mathrm{r}, y, \vartheta)$ in which the $y-$axis is the axis of revolution, so the position vector is given by

$$\mathbf{r} = \mathrm{r} \cos\vartheta \,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + \mathrm{r} \sin\vartheta \,\hat{\boldsymbol{k}}.$$

In these coordinates, the "volume of revolution" solid is

$$\mathcal{V} = \{(\mathrm{r}, y, \vartheta) : a \le \mathrm{r} \le b, 0 \le \varphi \le 2\pi, g(\mathrm{r}) \le y \le f(\mathrm{r})\}.$$

The volume of the solid is

$$Volume = \iiint_{\mathcal{V}} dV = \int_a^b \int_0^{2\pi} \int_{g(\mathrm{r})}^{f(\mathrm{r})} \mathrm{r} \, dy \, d\varphi \, d\mathrm{r} = \int_a^b \int_0^{2\pi} \left(\left[\, y \,\right]_{g(\mathrm{r})}^{f(\mathrm{r})}\right) \mathrm{r} \, d\varphi \, d\mathrm{r} = \int_a^b \int_0^{2\pi} \left(f(\mathrm{r}) - g(\mathrm{r})\right) \mathrm{r} \, d\varphi \, d\mathrm{r}$$
$$= \left(\int_0^{2\pi} d\varphi\right) \left(\int_a^b \left(f(\mathrm{r}) - g(\mathrm{r})\right) \mathrm{r} \, d\mathrm{r}\right) = 2\pi \int_a^b \left(f(\mathrm{r}) - g(\mathrm{r})\right) \mathrm{r} \, d\mathrm{r}.$$

The substitution $x = \mathrm{r}$ puts this into the usual format for the formula for volume using the method of cylindrical shells,

$$Volume = 2\pi \int_a^b x \left(f(x) - g(x)\right) dx,$$

that we see in a single variable calculus course.

Section 7.5

7.5.3.1. $dS = \Big|\Big|\dfrac{\partial \mathbf{r}}{\partial \theta} \times \dfrac{\partial \mathbf{r}}{\partial z}\Big|\Big|\, d\theta\, dz = \Big|\Big|\big(-\cos z\, \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}\big) \times \big(-a\sin\theta\, \hat{\boldsymbol{\imath}} + a\cos\theta\, \hat{\boldsymbol{\jmath}}\big)\Big|\Big| d\theta dz$

$= \Big|\Big| a\cos\theta\, \hat{\boldsymbol{\imath}} + a\sin\theta\, \hat{\boldsymbol{\jmath}} + a\cos\theta\cos z\, \hat{\boldsymbol{k}}\Big|\Big| d\theta dz = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta + a^2\cos^2\theta\cos^2 z}\, d\theta dz = a\sqrt{1+\cos^2\theta\cos^2 z}\, d\theta dz$

7.5.3.3. The cone $\phi = \dfrac{\pi}{6}$ and the sphere $\rho = 3$ intersect in a circle. A connection between cylindrical and spherical coordinates is that

$$\frac{1}{\sqrt{3}} = \tan\frac{\pi}{6} = \tan\phi = \frac{\rho\sin\phi}{\rho\cos\phi} = \frac{r}{z},$$

so the cone is given in cylindrical coordinates by

$$z = \sqrt{3}\, r,$$

hence $z^2 = 3r^2$.

The sphere $3 = \rho$ is given in cylindrical coordinates by

$$r^2 + z^2 = 3^2,$$

so the intersection of the cone and the sphere is where

$$9 - r^2 = z^2 = 3r^2,$$

that is, the circle $r = \dfrac{3}{2}$ at $z = \dfrac{3\sqrt{3}}{2}$.

The two parts of the surface $\mathcal{S}$ are (1) the side of the cone $z = \sqrt{3}\, r$ sitting above the disk $0 \le r \le \dfrac{3}{2}$ in the $xy-$plane and (2) the spherical cap which is the part of the sphere $\rho = 3$ sitting above the disk $0 \le r \le \dfrac{3}{2}$ in the $xy-$plane and, equivalently, having $0 \le \phi \le \dfrac{\pi}{6}$.

The spherical cap, $\mathcal{S}_1$ is part of the sphere $\rho = 3$ and can be parametrized by

$$\mathbf{r}_1 = 3\, \sin\phi\, \cos\theta\, \hat{\boldsymbol{\imath}} + 3\, \sin\phi\, \sin\theta\, \hat{\boldsymbol{\jmath}} + 3\, \cos\phi\, \hat{\boldsymbol{k}},$$

for $(\phi,\theta) = \mathcal{D}_1 = \{(\phi,\theta) : 0 \le \theta \le 2\pi,\, 0 \le \phi \le \dfrac{\pi}{6}\}$. On this spherical cap $\rho = 3$, so the element of surface area is $dS = 3^2 \sin\phi\, d\phi\, d\theta$. [Alternatively, we could use formula (7.33) in the textbook.] The surface area of the spherical cap $\mathcal{S}_1$ is

$$Area(\mathcal{S}_1) = \iint_{\mathcal{D}_1} dS = \int_0^{2\pi}\int_0^{\pi/6} 9\sin\phi\, d\phi d\theta = 18\pi\Big[-\cos\phi\Big]_0^{\pi/6} = 18\pi\Big(1 - \frac{\sqrt{3}}{2}\Big).$$

The surface $\mathcal{S}_2$, the side of the cone $z = \sqrt{3}\, r$, can be parametrized by

$$\mathbf{r}_2 = r\, \cos\theta\, \hat{\boldsymbol{\imath}} + r\, \sin\theta\, \hat{\boldsymbol{\jmath}} + \sqrt{3}\, r\, \hat{\boldsymbol{k}},$$

for $(r,\theta) = \mathcal{D}_2 = \{(r,\theta) : 0 \le \theta \le 2\pi,\, 0 \le r \le \frac{3}{2}\}$. The element of surface area is

$$\Big|\Big|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\Big|\Big|\, dr d\theta = \big|\big|\big(\cos\theta \hat{\boldsymbol{\imath}} + \sin\theta \hat{\boldsymbol{\jmath}} + \sqrt{3}\hat{\boldsymbol{k}}\big) \times \big(-r\sin\theta \hat{\boldsymbol{\imath}} + r\cos\theta \hat{\boldsymbol{\jmath}}\big)\big|\big|\, dr d\theta$$

$$= ... = \big|\big| -\sqrt{3}\, r\cos\theta\, \hat{\boldsymbol{\imath}} - \sqrt{3}\, r\sin\theta\, \hat{\boldsymbol{\imath}} + r\, \hat{\boldsymbol{k}}\big|\big|\, dr d\theta = ... = 2r\, dr d\theta.$$

We have

$$Area(\mathcal{S}_2) = \iint_{\mathcal{D}_2} 2r\, dr d\theta = 2 \cdot \int_0^{2\pi}\int_0^{3/2} r\, dr\, d\theta = 2 \cdot Area(\mathcal{D}_2) = 2 \cdot \pi \cdot \Big(\frac{3}{2}\Big)^2 = \frac{9}{2}\, \pi.$$

Alternatively, to find $Area(\mathcal{S}_2)$, one could use a classical formula for the area of the *side* of a right circular cone whose base radius is R and height is H:

$$Lateral\ surface\ area = \pi R\sqrt{R^2 + H^2} = \pi \cdot \frac{3}{2} \cdot \sqrt{\Big(\frac{3}{2}\Big)^2 + \Big(\sqrt{3} \cdot \frac{3}{2}\Big)^2} = \pi \cdot \frac{3}{2} \cdot \sqrt{9} = \frac{9}{2}\, \pi.$$

The total area of $\mathcal{S}$ is

$$Area(\mathcal{S}) = Area(\mathcal{S}_1) + Area(\mathcal{S}_2) = 18\pi\Big(1 - \frac{\sqrt{3}}{2}\Big) + \frac{9}{2}\,\pi = \frac{9\pi}{2}\,(5 - 2\sqrt{3}).$$

7.5.3.5. $\Big|\Big|\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}\Big|\Big| = ||\big(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}\big)\times\big(\hat{\boldsymbol{k}}\big)|| = || - \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}|| = \sqrt{2}$. The problem asks for the centroid so the mass density is a constant, ϱ_0. Define the domain of the parametrization by $\mathcal{D} = \{(u,v) : 0 \le u^2 + v^2 \le 1\} = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le 1\}$, where polar coordinates are $u = r\cos\theta$, $v = r\sin\theta$. The total mass is

$$M = \iint_{\mathcal{S}} \varrho_0\, dS = \iint_{\mathcal{D}} \varrho_0 \Big|\Big|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\Big|\Big| du dv = \iint_{\mathcal{D}} \varrho_0 \sqrt{2}\, du dv = \varrho_0\sqrt{2}\cdot Area(\mathcal{D}) = \varrho_0 \cdot \sqrt{2}\,\pi.$$

The moments are

$$M_{x=0} = \iint_{\mathcal{S}} \varrho_0\, x\, dS = \iint_{\mathcal{D}} \varrho_0 \Big|\Big|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\Big|\Big| u\, du dv = \iint_{\mathcal{D}} \varrho_0 \sqrt{2}\, u\, du dv = \varrho_0\sqrt{2}\int_0^{2\pi}\int_0^1 (r\,\cos\theta)\, r\, dr\, d\theta$$

$$= \varrho_0\cdot\sqrt{2}\Big(\int_0^{2\pi} \cos\theta\, d\theta\Big)\Big(\int_0^1 r^2\, dr\Big) = \varrho_0\cdot\sqrt{2}\cdot 0\cdot\frac{1}{3} = 0,$$

$$M_{y=0} = \iint_{\mathcal{S}} \varrho_0\, y\, dS = \iint_{\mathcal{D}} \varrho_0 \Big|\Big|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\Big|\Big| (2-u) du dv = \iint_{\mathcal{D}} \varrho_0 \sqrt{2}\, u\, du dv = \varrho_0\sqrt{2}\int_0^{2\pi}\int_0^1 (2 - r\,\cos\theta) r\, dr\, d\theta$$

$$= \varrho_0\cdot\sqrt{2}\int_0^{2\pi}\Big(\Big[r^2 - \frac{1}{3}\, r^3\cos\theta\Big]_0^1\Big)\, d\theta = \varrho_0\cdot\sqrt{2}\int_0^{2\pi}\Big(1 - \frac{1}{3}\,\cos\theta\Big) d\theta = \varrho_0\cdot 2\sqrt{2}\,\pi,$$

and

$$M_{z=0} = \iint_{\mathcal{S}} \varrho_0\, z\, dS = \iint_{\mathcal{D}} \varrho_0 \Big|\Big|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\Big|\Big| v\, du dv = \iint_{\mathcal{D}} \varrho_0 \sqrt{2}\, v\, du dv = \varrho_0\sqrt{2}\int_0^{2\pi}\int_0^1 (r\,\sin\theta)\, r\, dr\, d\theta$$

$$= \varrho_0\cdot\sqrt{2}\Big(\int_0^{2\pi} \sin\theta\, d\theta\Big)\Big(\int_0^1 r^2\, dr\Big) = \varrho_0\cdot\sqrt{2}\cdot 0\cdot\frac{1}{3} = 0.$$

So, the centroid is at

$$(\overline{x}, \overline{y}, \overline{z}) = \frac{1}{M}\,(M_{x=0}, M_{y=0}, M_{z=0}) = (0, 2, 0).$$

7.5.3.7. $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} = \big(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}\big)\times\big(\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{k}}\big) = -2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$, so the area of the surface is

$$Area(\mathcal{S}) = \iint_{\mathcal{D}} \sqrt{6}\, dA = \sqrt{6}\cdot Area(\mathcal{D}) = \sqrt{6}\cdot\pi\, 1^2 = \pi\,\sqrt{6}$$

and

$$\frac{1}{\varrho_0}\, M_{xy} = \frac{1}{\varrho_0}\, M_{z=0} = \iint_{\mathcal{S}} z\, dS = \iint_{\mathcal{D}} (2 - u - 2v)\sqrt{6}\, dA.$$

At this point we have a double integral in $\mathbb{R}^2$. We could write $dA = du\, dv$, but the domain $\mathcal{D}$ is best written in polar coordinates, $u = \mathrm{r}\cos\vartheta$ and $v = \mathrm{r}\sin\vartheta$. Then $dA = \mathrm{r}\, d\mathrm{r}\, d\vartheta$, so

$$\frac{1}{\varrho_0}\, M_{xy} = \iint_{\mathcal{D}} (2 - u - 2v)\sqrt{6}\, dA = \sqrt{6}\int_0^{2\pi}\int_0^1 (2 - \mathrm{r}\cos\vartheta - 2\mathrm{r}\sin\vartheta)\mathrm{r}\, d\mathrm{r}\, d\vartheta$$

$$= \sqrt{6}\int_0^{2\pi}\Big(\Big[\mathrm{r}^2 - \frac{1}{3}\,\mathrm{r}^3\cos\vartheta - \frac{2}{3}\,\mathrm{r}^3\sin\vartheta\Big]_0^1\Big) d\vartheta = \sqrt{6}\int_0^{2\pi}\Big(1 - \frac{1}{3}\,\cos\vartheta - \frac{2}{3}\,\sin\vartheta\Big) d\vartheta = \sqrt{6}\,\Big[\vartheta - \frac{1}{3}\,\sin\vartheta + \frac{2}{3}\,\cos\vartheta\Big]_0^{2\pi}$$

$$= 2\pi\sqrt{6}.$$

The z coordinate of the centroid is

$$\overline{z} = \frac{M_{xy}/\varrho_0}{Area(\mathcal{S})} = \frac{2\pi\sqrt{6}}{\pi\sqrt{6}} = 2.$$

7.5.3.9. Parametrize this half sphere, $\mathcal{S}$, using spherical coordinates, as usual: For $0 \le \phi \le \dfrac{\pi}{2}$, $0 \le \theta \le 2\pi$,

$$\mathbf{r} = a\,\sin\phi\,\cos\theta\,\hat{\boldsymbol{\imath}} + a\,\sin\phi\,\sin\theta\,\hat{\boldsymbol{\jmath}} + a\,\cos\phi\,\hat{\boldsymbol{k}}.$$

We know that the element of surface area is

$$dS = \left|\left|\frac{\partial\mathbf{r}}{\partial\phi} \times \frac{\partial\mathbf{r}}{\partial\theta}\right|\right| = a^2\sin\phi\,d\phi\,d\theta.$$

The problem asks for the centroid so the mass density is a constant, ϱ_0. The total mass is

$$M = \varrho_0 \cdot Area(\mathcal{S}) = \varrho_0 \cdot \frac{1}{2}\,4\pi a^2 = \varrho_0 \cdot 2\pi a^2.$$

The moment we need to find is $M_{z=0} = \varrho_0 \iint_{\mathcal{S}} z\,dS$, so we calculate

$$\iint_{\mathcal{S}} z\,dS = \varrho_0 \int_0^{2\pi}\int_0^{\pi/2} a\,\cos\phi(a^2\sin\phi\,d\phi\,d\theta) = 2\pi a^3\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi = 2\pi a^3\left[\frac{1}{2}\,\sin^2\phi\right]_0^{\pi/2} = \pi a^3.$$

The z coordinate of the centroid is

$$\overline{z} = \frac{M_{z=0}}{M} = \frac{\varrho_0\cdot\pi a^3}{\varrho_0\cdot 2\pi a^2} = \frac{a}{2}.$$

7.5.3.11. Parametrize the top, slanted surface $z = 5 - 2x$ using polar coordinates:

$$\mathcal{S} : \mathbf{r}(r,\theta) = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + (5 - 2r\cos\theta)\hat{\boldsymbol{k}}, (r,\theta) \text{ in } \mathcal{D} = \{(r,\theta) : 0 \le r < 2,\ 0 \le \theta \le 2\pi\}.$$

So, $\dfrac{\partial\mathbf{r}}{\partial r} \times \dfrac{\partial\mathbf{r}}{\partial\theta} = \big(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} - 2\cos\theta\,\hat{\boldsymbol{k}}\big)\times\big(-r\sin\theta\,\hat{\boldsymbol{\imath}} + r\cos\theta\,\hat{\boldsymbol{\jmath}} + 2r\sin\theta\,\hat{\boldsymbol{k}}\big) = \ldots = 2r\,\hat{\boldsymbol{\imath}} + r\,\hat{\boldsymbol{k}}$.

(a) The surface has

$$Area(\mathcal{S}) = \iint_{\mathcal{D}} \sqrt{5}\,r\,dA = \sqrt{5}\cdot\int_0^{2\pi}\int_0^2 r\,dr\,d\theta = \sqrt{5}\cdot\pi\,2^2 = 4\pi\sqrt{5}.$$

(b) $$\iint_{\mathcal{S}} z\,dS = \iint_{\mathcal{D}} (5 - 2x)\,\sqrt{5}\,(r\,dA) = \sqrt{5}\cdot\int_0^{2\pi}\int_0^2 (5 - 2r\,\cos\theta)(r\,dr\,d\theta) = \sqrt{5}\int_0^{2\pi}\left(\left[\frac{5}{2}\,r^2 - \frac{2}{3}r^3\cos\theta\right]_0^2\right)d\theta$$

$$= \sqrt{5}\int_0^{2\pi}\left(10 - \frac{16}{3}\,\cos\theta\right)d\theta = \sqrt{5}\left[10\,\theta - \frac{16}{3}\,\sin\theta\right]_0^{2\pi} = 20\pi\sqrt{5}.$$

7.5.3.13. $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1$ is the upper half sphere $x^2 + y^2 + z^2 = a^2, z \ge 0$ and $\mathcal{S}_2$ is the disk $x^2 + y^2 \le a^2, z = 0$ in the $xy-$plane.

We can parametrize $\mathcal{S}_1$ by

$$\mathbf{r}_1 = a\,\sin\phi\,\cos\theta\,\hat{\boldsymbol{\imath}} + a\,\sin\phi\,\sin\theta\,\hat{\boldsymbol{\jmath}} + a\,\cos\phi\,\hat{\boldsymbol{k}},$$

for $(\phi,\theta) = \mathcal{D}_1 = \{(\phi,\theta) : 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2}\}$. On this hemisphere the radius is a, so the element of surface area is $dS = a^2\sin\phi\,d\phi\,d\theta$, and the outward unit normal vector is

$$\widehat{\mathbf{n}} = \hat{\mathbf{e}}_\rho = \frac{1}{a}\,(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}).$$

So,

$$\iint_{\mathcal{S}_1}(y\,\hat{\boldsymbol{\imath}} + z^2\hat{\boldsymbol{k}})\bullet d\mathbf{S} = \iint_{\mathcal{D}_1}(y\,\hat{\boldsymbol{\imath}} + z^2\hat{\boldsymbol{k}})\bullet\widehat{\mathbf{n}}\,dS = \iint_{\mathcal{D}_1}(y\,\hat{\boldsymbol{\imath}} + z^2\hat{\boldsymbol{k}})\bullet\frac{1}{a}\,(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}})\,dS = \int_0^{2\pi}\int_0^{\pi/2}\frac{1}{a}\,(xy + z^3)a^2\sin\phi\,d\phi\,d\theta$$

$$= a\int_0^{2\pi}\int_0^{\pi/2}(a^2\sin^2\phi\,\cos\theta\,\sin\theta + a^3\cos^3\phi)\sin\phi\,d\phi\,d\theta$$

$$= a^3\left(\int_0^{2\pi}\cos\theta\,\sin\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^3\phi\,d\phi\right) + a^4\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi\right)$$

$$= a^3 \cdot 0 \cdot \left(\int_0^{\pi/2} \sin^3\phi \, d\phi \right) + 2\pi a^4 \Big[-\frac{1}{4}\cos^4\phi \Big]_0^{\pi/2} = \frac{\pi a^4}{2}.$$

[Alternatively, one can introduce spherical coordinates even earlier in the work, that is, one can find $\mathbf{F} \bullet \widehat{\mathbf{n}}$ using the basis $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta\}$.]

We can parametrize $\mathcal{S}_2$ by

$$\mathbf{r}_2 = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + 0 \cdot \hat{\boldsymbol{k}},$$

for $(r,\theta) = \mathcal{D}_2 = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le a\}$. On this flat disk the element of surface area is $dS = r\,dr\,d\theta$, and the outward unit normal vector is

$$\widehat{\mathbf{n}} = -\hat{\boldsymbol{k}}$$

and $z = 0$. So,

$$\iint_{\mathcal{S}_2} (y\,\hat{\boldsymbol{\imath}} + z^2\hat{\boldsymbol{k}}) \bullet d\mathbf{S} = \iint_{\mathcal{D}_2} (y\,\hat{\boldsymbol{\imath}} + z^2\hat{\boldsymbol{k}}) \bullet (-\hat{\boldsymbol{k}})dS = \int_0^{2\pi}\int_0^a (-z^2) r\,dr\,d\theta = \int_0^{2\pi}\int_0^a (-0^2) r\,dr\,d\theta = 0.$$

Putting things together,

$$\oiint_{\mathcal{S}} (y\,\hat{\boldsymbol{\imath}} + z^2\hat{\boldsymbol{k}}) \bullet d\mathbf{S} = \iint_{\mathcal{S}_1} (y\,\hat{\boldsymbol{\imath}} + z^2\hat{\boldsymbol{k}}) \bullet d\mathbf{S} + \iint_{\mathcal{S}_2} (y\,\hat{\boldsymbol{\imath}} + z^2\hat{\boldsymbol{k}}) \bullet d\mathbf{S} = \frac{\pi a^4}{2} + 0 = \frac{\pi a^4}{2}.$$

7.5.3.15. $\mathcal{S}$ consists of three parts, each with a corresponding unit normal vector $\widehat{\mathbf{n}}$ pointing *out* of the cylinder.
(1) The top, $\mathcal{S}_+$, that is, the surface $z = H,\ 0 \le x^2 + y^2 \le a^2$, can be parametrized by

$$\mathbf{r}_+ = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + H\,\hat{\boldsymbol{k}},$$

for $(r,\theta) = \mathcal{D}_+ = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le a\}$, so a normal vector is given by

$$\mathbf{n}_+ = \frac{\partial\mathbf{r}}{\partial r} \times \frac{\partial\mathbf{r}}{\partial\theta} = (\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} + 0 \cdot \hat{\boldsymbol{k}}) \times (-r\sin\theta\,\hat{\boldsymbol{\imath}} + r\cos\theta\,\hat{\boldsymbol{\jmath}} + 0 \cdot \hat{\boldsymbol{k}}) = r\,\hat{\boldsymbol{k}}.$$

An *outward* unit normal vector is given by $\widehat{\mathbf{n}} = \hat{\boldsymbol{k}}$.
(2) The bottom, $\mathcal{S}_-$, that is, the surface $z = -H,\ 0 \le x^2 + y^2 \le a^2$, can be parametrized by

$$\mathbf{r}_- = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} - H\,\hat{\boldsymbol{k}},$$

for $(r,\theta) = \mathcal{D}_- = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le a\}$, so a normal vector is given by

$$\mathbf{n}_- = \frac{\partial\mathbf{r}}{\partial r} \times \frac{\partial\mathbf{r}}{\partial\theta} = (\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} + 0 \cdot \hat{\boldsymbol{k}}) \times (-r\sin\theta\,\hat{\boldsymbol{\imath}} + r\cos\theta\,\hat{\boldsymbol{\jmath}} + 0 \cdot \hat{\boldsymbol{k}}) = r\,\hat{\boldsymbol{k}}.$$

An *outward* unit normal vector is given by $\widehat{\mathbf{n}} = -\hat{\boldsymbol{k}}$.
(3) The side, $\widetilde{\mathcal{S}}$, that is, the surface $-H \le z \le H,\ r = a$, can be parametrized by

$$\widetilde{\mathbf{r}} = a\cos\theta\,\hat{\boldsymbol{\imath}} + a\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}},$$

for $(\theta, z) = \widetilde{\mathcal{D}} = \{(\theta, z) : 0 \le \theta \le 2\pi,\ -H \le z \le H\}$, so a normal vector is given by

$$\widetilde{\mathbf{n}} = \frac{\partial\widetilde{\mathbf{r}}}{\partial\theta} \times \frac{\partial\widetilde{\mathbf{r}}}{\partial z} = (-a\sin\theta\,\hat{\boldsymbol{\imath}} + a\cos\theta\,\hat{\boldsymbol{\jmath}} + 0 \cdot \hat{\boldsymbol{k}}) \times (0 \cdot \hat{\boldsymbol{\imath}} + 0 \cdot \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) = a\cos\theta\,\hat{\boldsymbol{\imath}} + a\sin\theta\,\hat{\boldsymbol{\jmath}} + 0 \cdot \hat{\boldsymbol{k}}$$

$$= a(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) = a\,\hat{\mathbf{e}}_r.$$

An *outward* unit normal vector is given by $\widehat{\mathbf{n}} = \hat{\mathbf{e}}_r$ in cylindrical coordinates.

7.5.3.17. Formula (7.39) gives

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{S}} (x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}) \bullet \left(\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v} \right) dS = \iint_{\mathcal{D}} \mathbf{r}(u,v) \bullet \left(\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v} \right) dA$$

Define a scalar valued function by

$$f(u,v) \triangleq \mathbf{r}(u,v) \bullet \left(\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v} \right).$$

Break up $\mathcal{D} = \{(u,v):\ 0 \le u \le 6,\ 0 \le v \le 12\}$ into the six squares

$$\{(u,v): 0\le u\le 2, 0\le v\le 6\}, \{(u,v): 0\le u\le 2, 6\le v\le 12\}, \{(u,v): 2\le u\le 4, 0\le v\le 6\}, \{(u,v): 2\le u\le 4, 6\le v\le 12\},$$

$$\{(u,v): 4\le u\le 6, 0\le v\le 6\}, \quad \text{and} \quad \{(u,v): 4\le u\le 6, 6\le v\le 12\}.$$

Sampling $f(u,v)$ at midpoints $(1,3), (1,9), (3,3), (3,9), (5,3), (5,9)$ in the $uv-$plane, and defining $\Delta A = \triangle\, u \, \triangle\, v = 2{\cdot}6$, gives

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} \approx 2 \cdot 6 \cdot \big(\hat{\boldsymbol{\imath}} \bullet \hat{\boldsymbol{\jmath}} + (\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) \bullet \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}} \bullet (\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) + (-\hat{\boldsymbol{\jmath}}) \bullet \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\imath}} \bullet (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + \hat{\boldsymbol{\imath}} \bullet \hat{\boldsymbol{k}}\big) = 12(0+1+0+0+1+0) = 24.$$

7.5.3.19. (a) Rotation, by an angle φ, of the point $(x(t), y(t), 0)$ about the $x-$axis gives the point

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t)\cos\varphi \\ y(t)\sin\varphi \end{bmatrix}$$

So, rotating a plane curve $\mathcal{C}:\ \mathbf{r} = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}},\ \alpha \le t \le \beta$, about the $x-$axis gives a surface that we can parametrize as

$$\mathcal{S}:\ \mathbf{r}(t,\varphi) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\cos\varphi\,\hat{\boldsymbol{\jmath}} + y(t)\sin\varphi\,\hat{\boldsymbol{k}},$$

for (x,φ) in $\mathcal{D} = \{(t,\varphi):\ \alpha \le t \le \beta,\ 0 \le \varphi \le 2\pi\}$.

The element of surface area is

$$dS = \left|\left|\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \varphi}\right|\right| dy\, d\varphi = \left|\left|(\dot{x}(t)\hat{\boldsymbol{\imath}} + \dot{y}(t)\cos\varphi\,\hat{\boldsymbol{\jmath}} + \dot{y}(t)\sin\varphi\,\hat{\boldsymbol{k}}) \times (-y(t)\sin\varphi\,\hat{\boldsymbol{\jmath}} + y(t)\cos\varphi\,\hat{\boldsymbol{k}})\right|\right| dt\, d\varphi$$

$$= \left|\left|y(t)\dot{y}(t)\,\hat{\boldsymbol{\imath}} - y(t)\dot{x}(t)\cos\varphi\,\hat{\boldsymbol{\jmath}} - y(t)\dot{x}(t)\sin\varphi\,\hat{\boldsymbol{k}}\right|\right| dt\, d\varphi$$

$$= \sqrt{\big((y(t)\dot{y}(t)\big)^2 + (-y(t)\dot{x}(t)\cos\varphi)^2 + (-y(t)\dot{x}(t)\sin\varphi)^2} \cdot dt\, d\varphi = \sqrt{\big(y(t)\big)^2\Big(\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2\Big)}\ dt\, d\varphi$$

$$= |y(t)| \cdot \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\ dt\, d\varphi$$

So, $\mathcal{S}$ has surface area

$$Area(\mathcal{S}) = \iint_{\mathcal{S}} dS = \iint_{\mathcal{D}} |y(t)| \cdot \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\ dt\, d\varphi = \int_\alpha^\beta \int_0^{2\pi} |y(t)| \cdot \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\ dt\, d\varphi$$

$$= \left(\int_\alpha^\beta |y(t)| \cdot \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\ dt\right)\left(\int_0^{2\pi} d\varphi\right) = 2\pi \int_\alpha^\beta |y(t)| \cdot \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\ dt.$$

(b) $Area(\mathcal{S}) = 2\pi \displaystyle\int_2^3 \left|\frac{1}{3}t^3\right| \cdot \sqrt{\big(2t\big)^2 + \big(t^2\big)^2}\ dt = \frac{2\pi}{3}\int_2^3 t^4\sqrt{4+t^2}\ dt$

$$= \frac{4\pi}{3}\left(-\frac{28\sqrt{2}}{3} + 42\sqrt{13} - 4\ln\big(1+\sqrt{2}\big) + 4\ln\Big(\frac{3+\sqrt{13}}{2}\Big)\right),$$

with some help from Mathematica and the fact that $arcsinh(z) = \ln(z + \sqrt{1+z^2})$, for real $z > 0$.

Section 7.6

7.6.4.1. Using the Divergence Theorem, the total flux of $\boldsymbol{\nabla} f$ out of the sphere is

$$\oiint_{\mathcal{S}} \boldsymbol{\nabla} f \bullet \widehat{\mathbf{n}}\, dS = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \bullet (\boldsymbol{\nabla} f)\, dV = \iiint_{\mathcal{V}} \nabla^2 f\, dV.$$

But,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 8 - 6 - 4 = -2,$$

so the total flux out of the sphere is

$$\iiint_{\mathcal{V}} \nabla^2 f\, dV = \iiint_{\mathcal{V}} (-2)\, dV = -2 \cdot (Volume\ of\ the\ sphere) = -\frac{8\pi a^3}{3}.$$

7.6.4.3. The notation used in this problem is a little obscure, but we can rewrite

$$(xy\, dy\, dz - y^2\, dz\, dx + xy\, dx\, dy) = \left(xy\, \hat{\boldsymbol{\imath}} - y^2 \hat{\boldsymbol{\jmath}} + xy\, \hat{\boldsymbol{k}}\right) \bullet \left(dy\, dz\, \hat{\boldsymbol{\imath}} + dz\, dx\, \hat{\boldsymbol{\jmath}} + dx\, dy\, \hat{\boldsymbol{k}}\right) = \left(xy\, \hat{\boldsymbol{\imath}} - y^2 \hat{\boldsymbol{\jmath}} + xy\, \hat{\boldsymbol{k}}\right) \bullet d\mathbf{S}.$$

In general, $P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy$ is another notation for $\mathbf{F} \bullet d\mathbf{S}$. So,

$$\iint_{\mathcal{S}} (xy\, dy\, dz - y^2 dz\, dx + xy\, dx\, dy) = \iint_{\mathcal{S}} \left(xy\, \hat{\boldsymbol{\imath}} - y^2 \hat{\boldsymbol{\jmath}} + xy\, \hat{\boldsymbol{k}}\right) \bullet d\mathbf{S} = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \bullet \left(xy\, \hat{\boldsymbol{\imath}} - y^2 \hat{\boldsymbol{\jmath}} + xy\, \hat{\boldsymbol{k}}\right) dV$$

$$= \iiint_{\mathcal{V}} (y - 2y + 0)\, dV = -\iiint_{\mathcal{V}} y\, dV = -\int_0^{\pi} \int_0^3 \int_0^2 (r \sin\theta) r\, dz dr d\theta = -2 \int_0^{\pi} \int_0^3 (r \sin\theta) r\, dr d\theta$$

$$= -2 \left(\int_0^{\pi} \sin\theta\, d\theta \right) \left(\int_0^3 r^2\, dr \right) = -2 \cdot \Big[-\cos\theta \Big]_0^{\pi} \cdot \Big[\frac{1}{3} r^3 \Big]_0^3 = -2 \cdot 2 \cdot 9 = -36.$$

7.6.4.5. $\iiint_{\mathcal{V}} \psi\, dV \iiint_{\mathcal{V}} \nabla^2 \phi\, dV = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \bullet \boldsymbol{\nabla} \phi\, dV = \oiint_{\mathcal{S}} (\boldsymbol{\nabla} \phi) \bullet \widehat{\mathbf{n}}\, dS = \oiint_{\mathcal{S}} \widehat{\mathbf{n}} \bullet \boldsymbol{\nabla} \phi\ dS \triangleq \oiint_{\mathcal{S}} \frac{\partial \phi}{\partial n}\ dS.$

7.6.4.7. We can't use the Divergence Theorem to conclude that the total flux out of the sphere $\mathcal{S}_a = \{\mathbf{r} :\ ||\mathbf{r}|| = a\}$ is zero because $\mathbf{F}(\mathbf{r})$ is not continuously differentiable at $\mathbf{r} = \mathbf{0}$, which is in the solid ball $\mathcal{V}_a = \{\mathbf{r} :\ ||\mathbf{r}|| \le a\}$ enclosed by $\mathcal{S}_a = \{\mathbf{r} :\ ||\mathbf{r}|| = a\}$. So, this is a situation where the "mathematical technicalities" have a serious effect on the result, including its physical interpretation.

7.6.4.9. The paraboloid $z = 5 - 3(x^2 + y^2)$ intersects the plane $z = 2$ in the curve $2 = 5 - 3(x^2 + y^2)$, that is, $x^2 + y^2 = 1$. We will parametrize that curve, $\mathcal{C}$, so that it is positively oriented with respect to the oriented upward surface $\mathcal{S}$ that is the part of the paraboloid $z = 5 - 3(x^2 + y^2)$ that lies above the plane $z = 2$:

$$\mathcal{C} :\ \mathbf{r} = \cos t\, \hat{\boldsymbol{\imath}} + \sin t\, \hat{\boldsymbol{\jmath}} + 2\, \hat{\boldsymbol{k}},\ 0 \le t \le 2\pi.$$

Using Stokes's Theorem,

$$\iint_{\mathcal{S}} \boldsymbol{curl}(xy\, \hat{\boldsymbol{\imath}} - xz\, \hat{\boldsymbol{\jmath}}) \bullet d\mathbf{S} = \oint_{\mathcal{C}} (xy\, \hat{\boldsymbol{\imath}} - xz\, \hat{\boldsymbol{\jmath}}) \bullet d\mathbf{r} = \int_0^{2\pi} (\cos t\, \sin t\, \hat{\boldsymbol{\imath}} - \cos t \cdot 2\, \hat{\boldsymbol{\jmath}}) \bullet (-\sin t\, \hat{\boldsymbol{\imath}} + \cos t\, \hat{\boldsymbol{\jmath}} + 0\, \hat{\boldsymbol{k}}) dt$$

$$= \int_0^{2\pi} (-\cos t\, \sin^2 t - 2\cos^2 t) dt = \int (-\cos t\, \sin^2 t - 1 - \cos 2t) dt = \Big[-\frac{1}{3} \sin^3 t - t - \frac{1}{2} \sin 2t \Big]_0^{2\pi} = -2\pi.$$

7.6.4.11. Define the curve $\mathcal{C}$ to be the the circle $x^2 + y^2 = 1$ oriented positively in the $z = 0$ plane, and define $\mathcal{S}$ to be the northern hemisphere $x^2 + y^2 + z^2 = 1$ oriented upward. Then $\mathcal{C}$ is the boundary curve for $\mathcal{S}$. Using Stokes's Theorem, the circulation of $\mathbf{F}$ around $\mathcal{C}$ is

$$\Gamma = \oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{F}) \bullet d\mathbf{S} = \iint_{\mathcal{S}} \left((\boldsymbol{\nabla} \times (-\frac{y^3}{3}\, \hat{\boldsymbol{\imath}} + \frac{x^3}{3}\, \hat{\boldsymbol{\jmath}} + z\, \hat{\boldsymbol{k}}) \right) \bullet \widehat{\mathbf{n}}\, dS = \iint_{\mathcal{S}} \left((x^2 + y^2) \hat{\boldsymbol{k}} \right) \bullet \hat{\mathbf{e}}_\rho\, dS$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \left(1^2 \sin^2 \phi\, \hat{\boldsymbol{k}} \right) \bullet \hat{\mathbf{e}}_\rho\, (1^2 \sin\phi\, d\phi\, d\theta) = \int_0^{2\pi} \int_0^{\pi/2} \sin^2 \phi\, (\hat{\boldsymbol{k}} \bullet \hat{\mathbf{e}}_\rho) \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \sin^2 \phi\, (\cos\phi) \sin\phi\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi/2} \sin^3\phi \, \cos\phi \, d\phi = 2\pi \Big[\frac{1}{4} \sin^4\phi \Big]_0^{\pi/2} = \frac{\pi}{2}.$$

7.6.4.13. Lift$=\varrho_0\Gamma$ where Γ is the circulation around a given simple, closed piecewise smooth curve $\mathcal{C}$. Find a piecewise smooth, oriented surface $\mathcal{S}$ which has $\mathcal{C}$ as its boundary curve. Using the given information that $\nabla \times \mathbf{v} = \mathbf{0}$, by Stokes's Theorem,

$$\Gamma = \oint_{\mathcal{C}} \mathbf{v} \bullet d\mathbf{r} = \iint_{\mathcal{S}} (\nabla \times \mathbf{v}) \bullet d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{0} \bullet d\mathbf{S} = \iint_{\mathcal{S}} 0 \, dS = 0.$$

So, the lift is zero.

7.6.4.15. The homogeneous spherical shell of matter occupies the solid $\mathcal{V} = \{(\rho,\phi,\theta) : \, a < \rho < b, \, 0 \le \phi \le \pi, \, 0 \le \theta \le 2\pi\}$, for some constants a, b with $0 < a < b < \infty$. Let point P_i have coordinates (x_i, y_i, z_i), for $i = 0, 1$. Denote a general point in $\mathcal{V}$ by P, whose spherical coordinates are (ρ, ϕ, θ). Let ϱ_0 be the constant mass density of the matter in $\mathcal{V}$.

(a) If point P_0 is inside the shell $\mathcal{V}$, then $x_0^2 + y_0^2 + z_0^2 < a^2$. Without loss of generality, the spherical coordinates of P_0 are $(\rho, \phi, \theta) = (\rho_0, 0, 0)$, where $\rho_0 < a$. The object of mass m at a position $\tilde{P}_0$ experiences the element of gravitational force

$$\mathbf{dF} = -\frac{k\mathbf{r}}{||\mathbf{r}||^3},$$

where $k = mG\varrho_0 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, G is the universal gravitational constant, and

$$\mathbf{r} = \overrightarrow{P - P_0} = \rho \sin\phi \, \cos\theta \, \hat{\boldsymbol{\imath}} + \rho \sin\phi \, \sin\theta \, \hat{\boldsymbol{\jmath}} + (\rho\cos\phi - \rho_0)\hat{\boldsymbol{k}}.$$

So,

$$\begin{aligned} ||\mathbf{r}|| = \sqrt{(\rho\sin\phi\,\cos\theta)^2 + (\rho\sin\phi\,\sin\theta)^2 + (\rho\cos\phi - \rho_0)^2} &= \sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho_0\rho\cos\phi + \rho_0^2} \\ &= \sqrt{\rho^2 + \rho_0^2 - 2\rho_0\rho\cos\phi}. \end{aligned}$$

The total force exerted by the ball on the object is

$$\mathbf{F} = -mG\varrho_0 \int_0^{2\pi}\int_0^{\pi}\int_a^b (\rho\sin\phi\,\cos\theta\,\hat{\boldsymbol{\imath}} + \rho\sin\phi\,\sin\theta\,\hat{\boldsymbol{\jmath}} + (\rho\cos\phi - \rho_0)\hat{\boldsymbol{k}}) \frac{1}{(\rho^2 + \rho_0^2 - 2\rho_0\rho\cos\phi)^{3/2}} \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

The $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components of $\mathbf{F}$ are zero, because, for example,

$$\hat{\boldsymbol{\imath}} \bullet \mathbf{F} = -mG\varrho_0 \int_0^{2\pi}\int_0^{\pi}\int_a^b \rho\sin\phi\,\cos\theta \, \frac{1}{(\rho^2 + \rho_0^2 - 2\rho_0\rho\cos\phi)^{3/2}} \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} \cos\theta \, d\theta \right) \left(-mG\varrho_0 \int_0^{\pi}\int_a^b \rho\sin\phi \, \frac{1}{(\rho^2 + \rho_0^2 - 2\rho_0\rho\cos\phi)^{3/2}} \, \rho^2\sin\phi \, d\rho \, d\phi \right) = 0 \cdot \Big(... \Big) = 0.$$

Note that because $\rho_0 < a$, for $a < \rho < b$ we have

$$\rho^2 + \rho_0^2 - 2\rho_0\rho\cos\phi = ||\mathbf{r}||^2 = ||\overrightarrow{OP} - \overrightarrow{OP_0}||^2 \ge (||\overrightarrow{OP}|| - ||\overrightarrow{OP_0}||)^2 = (\rho - \rho_0)^2 > 0,$$

so the second factor is not an improper integral.

As for the $\hat{\boldsymbol{k}}$ component of $\mathbf{F}$, we calculate

$$\hat{\boldsymbol{k}} \bullet \mathbf{F} = -mG\varrho_0 \int_0^{2\pi}\int_0^{\pi}\int_a^b (\rho\cos\phi - \rho_0) \frac{1}{(\rho^2 + \rho_0^2 - 2\rho_0\rho\cos\phi)^{3/2}} \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Changing from spherical coordinates to rectangular coordinates, where the rectangular coordinates of (ρ, ϕ, θ) are $(x, y, z) = (x, y, z)$ and the rectangular coordinates of $(\rho_0, 0, 0)$ are $(x, y, z) = (0, 0, \rho_0)$, we have

$$\hat{\boldsymbol{k}} \bullet \mathbf{F} = -mG\varrho_0 \iiint_{\mathcal{V}} \frac{(z - \rho_0)}{x^2 + y^2 + z^2 + \rho_0^2 - 2\rho_0 z)^{3/2}} \, dV = -mG\varrho_0 \iiint_{\mathcal{V}} \frac{(z - \rho_0)}{x^2 + y^2 + (z - \rho_0)^2)^{3/2}} \, dV.$$

We can find a vector field $\mathbf{H}(x, y, z)$ so that the integrand can be expressed as a divergence:

$$\frac{(z - \rho_0)}{x^2 + y^2 + (z - \rho_0)^2)^{3/2}} = \nabla \bullet \mathbf{H}(x, y, z), \quad \text{where} \quad \mathbf{H}(x, y, z) = \frac{1}{(x^2 + y^2 + (z - \rho_0)^2)^{1/2}} \, \hat{\boldsymbol{k}}.$$

It follows from this and the Divergence Theorem that

$$\hat{\boldsymbol{k}} \bullet \mathbf{F} = -mG\varrho_0 \iiint_{\mathcal{V}} \frac{(z-\rho_0)}{(x^2+y^2+(z-\rho_0)^2)^{3/2}}\, dV = -mG\varrho_0 \iiint_{\mathcal{V}} \big(\boldsymbol{\nabla} \bullet \mathbf{H}(x,y,z)\big) dV = -mG\varrho_0 \oiint_{\partial\mathcal{V}} \mathbf{H}(x,y,z) \bullet d\mathbf{S}.$$

The positively oriented boundary of the spherical shell $\mathcal{V}$ consists of two parts: the outer sphere $\mathbf{S}_b : \rho = b$, where $\widehat{\mathbf{n}} = \mathbf{e}_\rho$, and the inner sphere $\mathbf{S}_a : \rho = a$, where $\widehat{\mathbf{n}} = -\mathbf{e}_\rho$, so

$$\hat{\boldsymbol{k}} \bullet \mathbf{F} = -mG\varrho_0 \left(\oiint_{\mathbf{S}_b} \mathbf{H}(x,y,z) \bullet d\mathbf{S} + \oiint_{\mathbf{S}_a} \mathbf{H}(x,y,z) \bullet d\mathbf{S} \right)$$

$$= -mG\varrho_0 \left(\int_0^{2\pi}\int_0^{\pi} \mathbf{H}\big|_{\rho=b} \bullet \mathbf{e}_\rho b^2 \sin\phi\, d\phi\, d\theta + \int_0^{2\pi}\int_0^{\pi} \mathbf{H}\big|_{\rho=a} \bullet (-\mathbf{e}_\rho) a^2 \sin\phi\, d\phi\, d\theta \right).$$

Note that on $\mathbf{S}_b$,

$$\mathbf{H}\big|_{\rho=b} \bullet \mathbf{e}_\rho = \frac{1}{(x^2+y^2+(z-\rho_0)^2)^{1/2}}\Big|_{\rho=b} \hat{\boldsymbol{k}} \bullet \mathbf{e}_\rho = \frac{1}{(b^2+\rho_0^2-2\rho_0 b\cos\phi)^{1/2}}\, \hat{\boldsymbol{k}} \bullet (\sin\phi\,\cos\theta\hat{\boldsymbol{\imath}} + \sin\phi\,\sin\theta\hat{\boldsymbol{\jmath}} + \cos\phi\,\hat{\boldsymbol{k}})$$

$$= \frac{\cos\phi}{(b^2+\rho_0^2-2\rho_0 b\cos\phi)^{1/2}},$$

so

$$\hat{\boldsymbol{k}} \bullet \mathbf{F} = -mG\varrho_0 \left(\int_0^{2\pi}\int_0^{\pi} \frac{\cos\phi}{(b^2+\rho_0^2-2\rho_0 b\cos\phi)^{1/2}} b^2 \sin\phi\, d\phi\, d\theta - \int_0^{2\pi}\int_0^{\pi} \frac{\cos\phi}{(a^2+\rho_0^2-2\rho_0 a\cos\phi)^{1/2}} a^2 \sin\phi\, d\phi\, d\theta \right).$$

Note that there is no θ dependence in the integrand, so integration with respect to θ just introduces a factor of 2π.

Mathematca calculates that

$$\int_0^{\pi} \frac{b^2\sin\phi\cos\phi}{(b^2+\rho_0^2-2\rho_0 b\cos\phi)^{1/2}}\, d\phi = \frac{|b+\rho_0|(b^2-b\rho_0+\rho_0^2) - |b-\rho_0|(b^2+b\rho_0+\rho_0^2)}{3\rho_0^2}$$

and

$$\int_0^{\pi} \frac{a^2\sin\phi\cos\phi}{(a^2+\rho_0^2-2\rho_0 a\cos\phi)^{1/2}}\, d\phi = \frac{|a+\rho_0|(a^2-a\rho_0+\rho_0^2) - |a-\rho_0|(a^2+a\rho_0+\rho_0^2)}{3\rho_0^2}.$$

Because $b > a > \rho_0$, $\hat{\boldsymbol{k}} \bullet \mathbf{F} =$

$$= -2\pi mG\varrho_0 \frac{1}{3\rho_0^2}\Big(\big((b+\rho_0)(b^2-b\rho_0+\rho_0^2) - (b-\rho_0)(b^2+b\rho_0+\rho_0^2)\big) - \big((a+\rho_0)(a^2-a\rho_0+\rho_0^2) - (a-\rho_0)(a^2+a\rho_0+\rho_0^2)\big)\Big)$$

$$= -2\pi mG\varrho_0 \frac{1}{3\rho_0^2}\Big(\big(2\rho_0^3\big) - \big(2\rho_0^3\big)\Big) = 0,$$

so $\hat{\boldsymbol{k}} \bullet \mathbf{F} = 0$.

In summary, if the point P_0, whose rectangular coordinates are $(x,y,z) = (0,0,\rho_1)$, is inside the spherical shell of matter then it experiences a gravitational force of $\mathbf{0}$.

(b) If point P_1 is outside the shell $\mathcal{V}$, then $\rho_1^2 = x_1^2 + y_1^2 + z_1^2 > b^2$. Almost all of the calculations we did in part (a) apply to this situation, so that we get that the spherical shell of matter exerts a gravitational force of

$$\mathbf{F} = 0 \cdot \hat{\boldsymbol{\imath}} + 0 \cdot \hat{\boldsymbol{\jmath}} - 2\pi mG\varrho_0 \frac{1}{3\rho_1^2}\Big(\big(|b+\rho_1|(b^2-b\rho_1+\rho_1^2) - |b-\rho_1|(b^2+b\rho_1+\rho_1^2)\big)$$

$$-\big(|a+\rho_1|(a^2-a\rho_1+\rho_1^2) - |a-\rho_1|(a^2+a\rho_1+\rho_1^2)\big)\Big)\hat{\boldsymbol{k}}.$$

Because $\rho_1 > b > a$,

$$\mathbf{F} = -2\pi mG\varrho_0 \frac{1}{3\rho_1^2}\Big(\big((b+\rho_1)(b^2-b\rho_1+\rho_1^2) - (\rho_1-b)(b^2+b\rho_1+\rho_1^2)\big) - \big((a+\rho_1)(a^2-a\rho_1+\rho_1^2) - (\rho_1-a)(a^2+a\rho_1+\rho_1^2)\big)\Big)\hat{\boldsymbol{k}}$$

$$= -2\pi mG\varrho_0 \frac{1}{3\rho_1^2}\Big(\big(2b^3\big) - \big(2a^3\big)\Big) = -\frac{4\pi\varrho_0}{3}(b^3-a^3) \cdot \frac{1}{\rho_1^2}\, mG\hat{\boldsymbol{k}} = -\frac{mMG}{\rho_1^2}\,\hat{\boldsymbol{k}},$$

where M is the mass of the spherical shell $\mathcal{V}$. It follows that if the point P_1 is inside the spherical shell of matter then it experiences a gravitational force of

$$\mathbf{F} = -\frac{mMG}{\rho_1^2}\,\hat{\boldsymbol{k}} = -\frac{GMm}{||\mathbf{r}_1||^3}\,\mathbf{r}_1,$$

where $\mathbf{r}_1 = \overrightarrow{OP_1}$.

7.6.4.17. Traveling along a curve $\mathcal{C}: \mathbf{r} = \mathbf{r}(t),\ a \le t \le b$, the non-zero tangent vectors are

$$\dot{\mathbf{r}}(t) = \dot{x}(t)\,\hat{\boldsymbol{\imath}} + \dot{y}(t)\,\hat{\boldsymbol{\jmath}}.$$

It follows that $\mathbf{n} \triangleq \dot{y}(t)\,\hat{\boldsymbol{\imath}} - \dot{x}(t)\,\hat{\boldsymbol{\jmath}}$ is normal to $\mathcal{C}$. Let us leave aside for the moment the question of why $\mathbf{n}$ points *out* of $\mathcal{D}$.

Note that $\widehat{\mathbf{n}} = \dfrac{1}{||\mathbf{n}||}\,\mathbf{n}$ and

$$||\mathbf{n}|| = \sqrt{\big(\dot{y}(t)\big)^2 + \big(-\dot{x}(t)\big)^2} = \sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2} = ||\dot{\mathbf{r}}||,$$

hence

$$\widehat{\mathbf{n}}\,ds = \frac{1}{||\mathbf{n}||}(\dot{y}(t)\,\hat{\boldsymbol{\imath}} - \dot{x}(t)\,\hat{\boldsymbol{\jmath}})\,||\dot{\mathbf{r}}||\,dt = (\dot{y}(t)\,\hat{\boldsymbol{\imath}} - \dot{x}(t)\,\hat{\boldsymbol{\jmath}})dt.$$

Next, rewrite

$$(P\,\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}) \bullet (\dot{y}(t)\,\hat{\boldsymbol{\imath}} - \dot{x}(t)\hat{\boldsymbol{\jmath}})dt = (-Q\,\hat{\boldsymbol{\imath}} + P\hat{\boldsymbol{\jmath}}) \bullet (dx\,\hat{\boldsymbol{\imath}} + dy\hat{\boldsymbol{\jmath}}) = -Q\,dx + P\,dy.$$

It follows that

$$\oint_{\mathcal{C}} (P\,\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}) \bullet \widehat{\mathbf{n}}\,ds = \oint_{\mathcal{C}} (-Q\,dx + P\,dy)$$

Green's Theorem implies that

$$\oint_{\mathcal{C}} (P\,\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}) \bullet \widehat{\mathbf{n}}\,ds = \oint_{\mathcal{C}} (-Q\,dx + P\,dy) = \iint_{\mathcal{D}} \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big) dA,$$

that is,

$$(\star) \qquad \oint_{\mathcal{C}} \mathbf{F} \bullet \widehat{\mathbf{n}}\,ds = \iint_{\mathcal{D}} \boldsymbol{\nabla} \bullet \mathbf{F}\,dA,$$

as was desired.

So, why does $\mathbf{n} \triangleq \dot{y}(t)\,\hat{\boldsymbol{\imath}} - \dot{x}(t)\,\hat{\boldsymbol{\jmath}}$ point *out* of $\mathcal{D}$? It will suffice to show that $\dot{\mathbf{r}}(t) = \dot{x}(t)\,\hat{\boldsymbol{\imath}} + \dot{y}(t)\,\hat{\boldsymbol{\jmath}}$ is always to the left of $\mathbf{n}$:

Case 1: If $\dot{\mathbf{r}}(t)$ is in the first quadrant, that is, QI, that is, $\dot{x}(t) > 0$ and $\dot{y}(t) > 0$, then $\dot{y}(t) > 0$ and $-\dot{x}(t) < 0$, so $\mathbf{n}$ is in QIV, hence $\dot{\mathbf{r}}(t)$ is to the left of $\mathbf{n}$.

Case 2: If $\dot{\mathbf{r}}(t)$ is in the second quadrant, that is, QII, that is, $\dot{x}(t) < 0$ and $\dot{y}(t) > 0$, then $\dot{y}(t) > 0$ and $-\dot{x}(t) > 0$, so $\mathbf{n}$ is in QI, hence $\dot{\mathbf{r}}(t)$ is to the left of $\mathbf{n}$.

Case 3: If $\dot{\mathbf{r}}(t)$ is in the third quadrant, that is, QIII, that is, $\dot{x}(t) < 0$ and $\dot{y}(t) < 0$, then $\dot{y}(t) < 0$ and $-\dot{x}(t) > 0$, so $\mathbf{n}$ is in QII, hence $\dot{\mathbf{r}}(t)$ is to the left of $\mathbf{n}$.

Case 4: If $\dot{\mathbf{r}}(t)$ is in the fourth quadrant, that is, QIV, that is, $\dot{x}(t) > 0$ and $\dot{y}(t) < 0$, then $\dot{y}(t) < 0$ and $-\dot{x}(t) < 0$, so $\mathbf{n}$ is in QIII, hence $\dot{\mathbf{r}}(t)$ is to the left of $\mathbf{n}$.

Section 7.7

7.7.2.1. Using the results of Example 7.50, the standard deviation of the fair die of Example 7.45 is

$$\sigma = \sqrt{E[\,X^2\,] - (E[\,X\,])^2} = \sqrt{\frac{91}{6} - \left(\frac{7}{2}\right)^2} = \sqrt{\frac{35}{12}}.$$

7.7.2.3. continuing from the hint,

$$E[(X-\mu)^2] = E[X^2] - 2E[\mu X] + E[\mu)^2] = E[X^2] - 2\mu E[X] + \mu^2 E[1] = E[X^2] - 2\mu\mu + \mu^2 \cdot 1 = E[X^2] - \mu^2,$$

as we wanted to show.

7.7.2.5. By Theorem 7.21, the random variable $Z \triangleq X + Y$ has PDF

$$k(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du,$$

where both X and Y have the same probability density function (PDF), namely $f(x) = \dfrac{1}{\sqrt{2\pi}\ \sigma}\ \exp\left(-\dfrac{x^2}{2\sigma^2}\right)$. So,

$$k(x) = \int_{-\infty}^{\infty} f(u)f(x-u)du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\ \sigma}\ \exp\left(-\frac{u^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi}\ \sigma}\ \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right) du$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2+(x-u)^2}{2\sigma^2}\right) du = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{2(u^2 - xu + \frac{1}{2}\,x^2)}{2\sigma^2}\right) du$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{(u-\frac{1}{2}\,x)^2 + \frac{1}{4}\,x^2}{\sigma^2}\right) du = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{(u-\frac{1}{2}\,x)^2}{\sigma^2}\right) \exp\left(-\frac{x^2}{4\sigma^2}\right) du$$

$$= \frac{1}{\sqrt{\pi}\,\sigma} \exp\left(-\frac{x^2}{4\sigma^2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,(\sigma/\sqrt{2})}\ \exp\left(-\frac{(u-\frac{1}{2}\,x)^2}{2(\sigma/\sqrt{2})^2}\right) du.$$

Note that $w(u) \triangleq \dfrac{1}{\sqrt{2\pi}\,(\sigma/\sqrt{2})} \exp\left(-\dfrac{(u-\frac{1}{2}\,x)^2}{2(\sigma/\sqrt{2})^2}\right)$ is the PDF of a normal random variable, W, whose mean is $\frac{1}{2}\,x$ and whose standard deviation is $\sigma/\sqrt{2}$. By definition of "normal random variable," we must have $\displaystyle\int_{-\infty}^{\infty} w(u)du = 1$. So,

$$k(x) = \frac{1}{\sqrt{\pi}\,\sigma} \exp\left(-\frac{x^2}{4\sigma^2}\right) = 2 \cdot \frac{1}{\sqrt{2\pi}\,\sqrt{2}\,\sigma} \exp\left(-\frac{x^2}{2(\sqrt{2}\,\sigma)^2}\right),$$

which is the PDF of two times a normal random variable whose mean is 0 and whose standard deviation is $\sqrt{2}\,\sigma$.

7.7.2.7. Equation (7.49) is that $\sigma^2 = E[\,X^2\,] - (E[\,X\,])^2$, where X is a random variable. If X is a discrete random variable taking on the values x_1, ... , x_m, each with probability $\frac{1}{m}$, then

$$E[\,X\,] = \sum_{n=1}^{m} x_n P(X = x_n) = \sum_{n=1}^{m} x_n \cdot \frac{1}{m} = \frac{1}{m}\sum_{i=1}^{m} x_i \triangleq \overline{x},$$

as we defined the latter notation in Section 2.5.

In Section 2.5, we found that the method of least squares solutions concludes that the regression line for observations $(x,y) = (x_n, y_n),\ n = 1,..,m$, is given by

$$y = \frac{1}{\overline{x^2} - \left(\overline{x}\right)^2} \left((\overline{xy} - \overline{x}\ \overline{y})x + (-\overline{x}\ \overline{xy} + \overline{x^2}\ \overline{y}) \right),$$

assuming at least two of the x_n's are distinct. Note that also in Section 2.5, we defined the notation

$$\overline{x^2} \triangleq \frac{1}{m}\sum_{i=n}^{m} x_n^2 = \sum_{n=1}^{m} x_n^2 \cdot \frac{1}{m} = \sum_{n=1}^{m} x_n^2 P(X = x_n) = E[\,X^2\,].$$

So, the formula for the regression line has in it the factor

$$\frac{1}{\overline{x^2} - \left(\overline{x}\right)^2} = \frac{1}{E[\,X^2\,] - (E[\,X\,])^2}.$$

Section 8.1.8

8.1.8.1. Let $f(x) = e^{-1.5x} - \cos 2x$. Newton's Method is $x_{k+1} = x_k - \dfrac{e^{-1.5x_k} - \cos 2x_k}{-1.5e^{-1.5x_k} + 2\sin 2x_k}$.

(a) In 8 steps, Newton's Method gave approximate solution $x_8 = 2.370476901$, with $f(x_8) = -3.677613769E-16$.

(b) In 26 steps, the Bisection Method gave approximate solution $x_{26} = 2.370476902$, with $f(x_{24}) = -1.7873903989E-09$.

Table 1: Problem 8.1.8.1(a) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$2.000000000E+00$	$7.034306892E-01$	$4.428867782E-01$
1	$2.442886778E+00$	$-1.468958327E-01$	$-7.313908598E-02$
2	$2.369747692E+00$	$1.489109163E-03$	$7.291856883E-04$
3	$2.370476878E+00$	$4.643574163E-08$	$2.274003987E-08$
4	$2.370476901E+00$	$-3.677613769E-16$	$-1.800963675E-16$
5	$2.370476901E+00$	$-3.677613769E-16$	$-1.800963675E-16$

Table 2: Problem 8.1.8.1(b) by the Bisection method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$1.000000000E+00$	$6.392769967E-01$	
1	$3.000000000E+00$	$-9.490612901E-01$	$-2.000000000E+00$
2	$2.000000000E+00$	$7.034306892E-01$	$1.000000000E+00$
3	$2.500000000E+00$	$-2.601444396E-01$	$-5.000000000E-01$
4	$2.250000000E+00$	$2.450139177E-01$	$2.500000000E-01$
5	$2.375000000E+00$	$-9.234336438E-03$	$-1.250000000E-01$
.	.	.	.
25	$2.3704768419E+00$	$1.1992681513E-07$	$1.192092896E-07$
26	$2.370476902E+00$	$-1.7873903989E-09$	$-5.960464478E-08$

8.1.8.3. Let $f(x) = \ln(x+3) - \cos 2x$. Newton's Method is $x_{k+1} = x_k - \dfrac{\ln(x_k+3) - \cos 2x_k}{(x_k+3)^{-1} + 2\sin 2x_k}$.

(a) In 5 steps, Newton's Method gave approximate solution $x_5 = -2.226394727$, with $f(x_5) = 0.000000000E+00$.

(b) In 25 steps, the Bisection Method gave approximate solution $x_{25} = -2.226394721$, with $f(x_{25}) = 1.841991815E-08$.

Table 3: Problem 8.1.8.3(a) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$-2.500000000E+00$	$-9.768093660E-01$	$2.493229010E-01$
1	$-2.250677099E+00$	$-7.911344164E-02$	$2.404541450E-02$
2	$-2.226631684E+00$	$-7.644150974E-04$	$2.369343572E-04$
3	$-2.226394750E+00$	$-7.570689420E-08$	$2.347038709E-08$
4	$-2.226394727E+00$	$-1.165734176E-15$	$3.613968474E-16$
5	$-2.226394727E+00$	$0.000000000E+00$	$0.000000000E+00$

8.1.8.5. Let $f(x) = \ln(x+3) - x^2 + 2$.

Table 4: Problem 8.1.8.3(b) by the Bisection method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$-2.999900000E+00$	$-1.017045476E+01$	
1	$-2.000000000E+00$	$6.536436209E-01$	$-9.999000000E-01$
2	$-2.499950000E+00$	$-9.766134772E-01$	$4.999500000E-01$
3	$-2.249975000E+00$	$-7.680406400E-02$	$-2.499750000E-01$
4	$-2.124987500E+00$	$3.125927575E-01$	$-1.249875000E-01$
5	$-2.187481250E+00$	$1.234435047E-01$	$6.249375000E-02$
.	.	.	.
24	$-2.226394781E+00$	$-1.738236637E-07$	$1.191973684E-07$
25	$-2.226394721E+00$	$1.841991815E-08$	$-5.959868421E-08$

(1) For initial guess $x_0 = 0.5$, in 7 steps Newton's method gives approximate solution 1.894215957. It appears to have quadratic convergence.
(2) For initial guess $x_0 = -2$, in 5 steps Newton's method gives approximate solution -1.541814739. It appears to have quadratic convergence.

Table 5: First try for problem 8.1.8.5 by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$5.000000000E-01$	$3.002762968E+00$	$4.203868156E+00$
1	$4.703868156E+00$	$-1.808465307E+01$	$-1.949211770E+00$
2	$2.754656386E+00$	$-3.838122471E+00$	$-7.193502714E-01$
3	$2.035306114E+00$	$-5.259966580E-01$	$-1.358457331E-01$
4	$1.899460381E+00$	$-1.882466764E-02$	$-5.236614118E-03$
5	$1.894223767E+00$	$-2.799371692E-05$	$-7.810474410E-06$
6	$1.894215957E+00$	$-6.227729443E-11$	$-1.737594833E-11$
7	$1.894215957E+00$	$0.000000000E+00$	$0.000000000E+00$

Table 6: Second try for problem 8.1.8.5 by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$-2.000000000E+00$	$-2.000000000E+00$	$4.000000000E-01$
1	$-1.600000000E+00$	$-2.235277634E-01$	$5.710563298E-02$
2	$-1.542894367E+00$	$-4.071002868E-03$	$1.079245941E-03$
3	$-1.541815121E+00$	$-1.438938307E-06$	$3.817405511E-07$
4	$-1.541814739E+00$	$-1.798561300E-13$	$4.771462058E-14$
5	$-1.541814739E+00$	$0.000000000E+00$	$0.000000000E+00$

8.1.8.7. Using $f(x) = x - \dfrac{1}{1+x^2} + e^{-1.5x}$ and initial guesses $x_0 = 0.5$ and $x_1 = 1$, in 9 steps the Secant method gives approximate solution 0.2591432818.

8.1.8.9. With initial guesses 4, 7, 10, 13, get approximate solutions 3.92660231, 7.06858275, 10.2101761, 13.3517688, respectively.

Table 7: Problem 8.1.8.7 by a Secant method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$5.000000000E-01$	$1.723665527E-01$	
1	$1.000000000E+00$	$7.231301601E-01$	$-6.564796134E-01$
2	$3.435203866E-01$	$4.640398033E-02$	$-4.501564735E-02$
3	$2.985047392E-01$	$1.937957800E-02$	$-3.228135218E-02$
4	$2.662233871E-01$	$3.173390946E-03$	$-6.321126025E-03$
5	$2.599022610E-01$	$3.334149856E-04$	$-7.421042191E-04$
6	$2.591601568E-01$	$7.395333631E-06$	$-1.683367324E-05$
7	$2.591433231E-01$	$1.812698136E-08$	$-4.136303827E-08$
8	$2.591432818E-01$	$9.910960941E-13$	$-2.261655505E-12$
9	$2.591432818E-01$	$0.000000000E+00$	$0.000000000E+00$

Table 8: Problem 8.1.8.9(i) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$4.00000000E+00$	$-1.58491983E-01$	$-6.77545149E-02$
1	$3.93224549E+00$	$-1.13329434E-02$	$-5.61118437E-03$
2	$3.92663430E+00$	$-6.38800963E-05$	$-3.19876740E-05$
3	$3.92660231E+00$	$-2.04500739E-09$	$-1.02409392E-09$
4	$3.92660231E+00$	$0.00000000E+00$	$0.00000000E+00$
5	$3.92660231E+00$	$0.00000000E+00$	$0.00000000E+00$

Table 9: Problem 8.1.8.9(ii) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$7.00000000E+00$	$1.28550354E-01$	$7.30641242E-02$
1	$7.07306412E+00$	$-9.00313785E-03$	$-4.46123588E-03$
2	$7.06860289E+00$	$-4.02859866E-05$	$-2.01422403E-05$
3	$7.06858275E+00$	$-8.11462564E-10$	$-4.05732458E-10$
4	$7.06858275E+00$	$0.00000000E+00$	$0.00000000E+00$
5	$7.06858275E+00$	$0.00000000E+00$	$0.00000000E+00$

Table 10: Problem 8.1.8.9(iii) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$1.00000000E+01$	$3.51639168E-01$	$2.47568404E-01$
1	$1.02475684E+01$	$-7.77271880E-02$	$-3.59599057E-02$
2	$1.02116085E+01$	$-2.86886243E-03$	$-1.43032194E-03$
3	$1.02101782E+01$	$-4.10732374E-06$	$-2.05365344E-06$
4	$1.02101761E+01$	$-8.43591863E-12$	$-4.21795934E-12$
5	$1.02101761E+01$	$-1.77635684E-15$	$-8.88178425E-16$
6	$1.02101761E+01$	$-1.77635684E-15$	$-8.88178425E-16$

Table 11: Problem 8.1.8.9(iv) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$1.30000000E+01$	$5.36978867E-01$	$4.42180436E-01$
1	$1.34421804E+01$	$-1.99394502E-01$	$-8.17677443E-02$
2	$1.33604127E+01$	$-1.74390033E-02$	$-8.56876791E-03$
3	$1.33518439E+01$	$-1.50303186E-04$	$-7.51402985E-05$
4	$1.33517688E+01$	$-1.12943909E-08$	$-5.64719537E-09$
5	$1.33517688E+01$	$-1.55431223E-15$	$-7.77156117E-16$
6	$1.33517688E+01$	$-1.55431223E-15$	$-7.77156117E-16$

8.1.8.11. Define $g(x) \triangleq \dfrac{1}{1+x^2} - e^{-1.5x}$. Try initial guess $x_0 = 1$.

(a) After 30 steps, the method of successive approximations gives approximate solution of about 0.259143282 and appears to converge.

(b) After 8 steps, Aitken's method gives approximate solution of about 0.259143282 but after that bounces around that value.

Table 12: Problem 8.1.8.11(a) by successive approximations

k	x_k	$g(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$1.000000000E+00$	$2.768698399E-01$	$-7.231301601E-01$
1	$2.768698399E-01$	$2.686619217E-01$	$-8.207918147E-03$
.	.	.	.
4	$2.620371258E-01$	$2.607571320E-01$	$-1.279993765E-03$
5	$2.607571320E-01$	$2.600462223E-01$	$-7.109096449E-04$
.	.	.	.
30	$2.591432826E-01$	$2.591432823E-01$	$-3.843582119E-10$
.	.	.	.
33	$2.591432819E-01$	$2.591432819E-01$	$-6.814626641E-11$
.	.	.	.
38	$2.591432818E-01$	$2.591432818E-01$	$-3.813283023E-12$

8.1.8.13. Define $f(x) \triangleq \sqrt{2}\cos(\sqrt{2}\,x)\sin(\sqrt{3}\,x) + \sqrt{3}\sin(\sqrt{2}\,x)\cos(\sqrt{3}\,x)$, where $x = \sqrt{\lambda}$. A graphing calculator suggests that there are solutions of $f(x) = 0$ close to every positive integer.

Newton's method for the equation $f(x) = 0$ gives approximate solutions 0.988590027, 2.01623831, 2.96951504, 4.02486196, 4.96041337, 6.02137429, 6.96387382, 8.00617912, 8.97750897, 9.98413271, 10.9947678, 11.9624699,...

This gives approximate solutions for $\lambda = x^2$ of 9.77310241, 4.06521693, 8.81801955, 16.1995138, 24.6057008, 36.2569483, 48.4955386, 64.0989040, 80.5956673, 99.6829060, 120.884919, 143.100686,...

$\lambda = 0$ is also a solution of the original equation but it is not allowed in Example 9.32 because the characteristic equation derived there assumed that $\lambda > 0$.

Table 13: Third try for problem 8.1.8.11(b) by Aitken's method

k	x_k	$g(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$1.000000000E+00$	$2.768698399E-01$	$-8.302152060E-03$
1	$2.685676878E-01$	$2.643125964E-01$	$-5.020291930E-03$
2	$2.592923045E-01$	$2.592269687E-01$	$-8.364653451E-05$
3	$2.591433221E-01$	$2.591433044E-01$	$-2.186316256E-08$
4	$2.591432826E-01$	$2.591432822E-01$	$-2.433423829E-08$
5	$2.591432579E-01$	$2.591432684E-01$	$1.299105579E-08$
6	$2.591432813E-01$	$2.591432815E-01$	$-1.313784110E-07$
7	$2.591431502E-01$	$2.591432078E-01$	$7.416190212E-08$
8	$2.591432820E-01$	$2.591432819E-01$	$-3.951235061E-09$
.	.	.	.
34	$2.591433121E-01$	$2.591432988E-01$	$-1.543237615E-08$
.	.	.	.
36	$2.591432668E-01$	$2.591432733E-01$	$9.633460751E-09$
37	$2.591432830E-01$	$2.591432824E-01$	$2.474014565E-08$

Table 14: Problem 8.1.8.13(i) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$1.00000000E+00$	$-5.70133144E-02$	$-1.14085586E-02$
1	$9.88591441E-01$	$-7.06520986E-06$	$-1.41442394E-06$
2	$9.88590027E-01$	$-4.81059637E-13$	$-9.63060367E-14$
3	$9.88590027E-01$	$0.00000000E+00$	$0.00000000E+00$
4	$9.88590027E-01$	$0.00000000E+00$	$0.00000000E+00$

Table 15: Problem 8.1.8.13(ii) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$2.00000000E+00$	$-7.96549861E-02$	$1.62275408E-02$
1	$2.01622754E+00$	$-5.27835765E-05$	$1.07722125E-05$
2	$2.01623831E+00$	$-5.39907008E-11$	$1.10185882E-11$
3	$2.01623831E+00$	$7.21644966E-16$	$-1.47275523E-16$
4	$2.01623831E+00$	$7.21644966E-16$	$-1.47275523E-16$

Table 16: Problem 8.1.8.13(iii) by Newton's method

k	x_k	$f(x_k)$	$\triangle x_{k+1} \triangleq x_{k+1} - x_k$
0	$3.00000000E+00$	$-1.51637660E-01$	$-3.04600934E-02$
1	$2.96953991E+00$	$-1.23422510E-04$	$-2.48707064E-05$
2	$2.96951504E+00$	$-3.89727806E-10$	$-7.85340297E-11$
3	$2.96951504E+00$	$1.11022302E-15$	$2.23720983E-16$
4	$2.96951504E+00$	$0.00000000E+00$	$0.00000000E+00$
5	$2.96951504E+00$	$0.00000000E+00$	$0.00000000E+00$

Section 8.2.5

8.2.5.1. (a) For the system of equations $\left\{\begin{array}{ll} 0 = xy^2 - y + 1 & \triangleq f(x,y) \\ 0 = x^2y + x - 0.5 & \triangleq g(x,y) \end{array}\right\}$, define

$$\mathbf{x} \triangleq \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} xy^2 - y + 1 \\ x^2y + x - 0.5 \end{bmatrix}.$$

The Jacobian matrix is

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) = \begin{bmatrix} y^2 & 2xy - 1 \\ 2xy + 1 & x^2 \end{bmatrix},$$

so

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x})\right)^{-1} = \frac{1}{1 - 3x^2y^2} \begin{bmatrix} x^2 & 1 - 2xy \\ -1 - 2xy & y^2 \end{bmatrix}.$$

Newton's method, as in (8.18), is $\mathbf{x}_{k+1} = \mathbf{x}_k - \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k)\right)^{-1} \mathbf{f}(\mathbf{x}_k)$, that is,

$$\begin{bmatrix} x_{k+1} \\ \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ \\ y_k \end{bmatrix} - \frac{1}{1 - 3x_k^2y_k^2} \begin{bmatrix} x_k^2 & 1 - 2x_ky_k \\ \\ -1 - 2x_ky_k & y_k^2 \end{bmatrix} \begin{bmatrix} x_ky_k^2 - y_k + 1 \\ \\ x_k^2y + x_k - 0.5 \end{bmatrix}.$$

(b) $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} \quad \Rightarrow \quad \mathbf{f}(\mathbf{x}_0) = \begin{bmatrix} 0 \cdot (-0.5)^2 - (-0.5) + 1 \\ 0^2 \cdot (-0.5) + 0 - 0.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$ and

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0)\right)^{-1} = \frac{1}{1 - 3 \cdot 0^2 \cdot (-0.5)^2} \begin{bmatrix} 0^2 & 1 - 2 \cdot 0 \cdot (-0.5) \\ -1 - 2 \cdot 0 \cdot (-0.5) & (-0.5)^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0.25 \end{bmatrix}.$$

So

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0.25 \end{bmatrix} \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.125 \end{bmatrix}.$$

8.2.5.3. For the system of equations

$$\left\{\begin{array}{ll} 0 = x + 2y - 9 & \triangleq f(x,y) \\ 0 = x^2 + y^2 - 4y - 21 & \triangleq g(x,y) \end{array}\right\},$$

define

$$\mathbf{x} \triangleq \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} x + 2y - 9 \\ x^2 + y^2 - 4y - 21 \end{bmatrix}.$$

The Jacobian matrix is

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 2x & 2y - 4 \end{bmatrix},$$

so

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x})\right)^{-1} = \frac{1}{-4+2y-4x} \begin{bmatrix} 2y - 4 & -2 \\ -2x & 1 \end{bmatrix}.$$

Newton's method, as in (8.18), is $\mathbf{x}_{k+1} = \mathbf{x}_k - \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k)\right)^{-1} \mathbf{f}(\mathbf{x}_k)$, that is,

$$\begin{bmatrix} x_{k+1} \\ \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ \\ y_k \end{bmatrix} - \frac{1}{-4+2y_k-4x_k} \begin{bmatrix} 2y_k - 4 & -2 \\ \\ -2x_k & 1 \end{bmatrix} \begin{bmatrix} x_k + 2y_k - 9 \\ \\ x_k^2 + y_k^2 - 4y_k - 21 \end{bmatrix}$$

Using an $\textsf{EXCEL}^{\text{TM}}$spreadsheet, we found approximate solutions of the system of equations:

$$(x,y) \approx (5.0000000,\ 2.0000000) \quad \text{and} \quad (x,y) \approx (-3.0000000,\ 6.0000000).$$

8.2.5.5. For the system of equations

$$\left\{\begin{array}{ll} 0 = xy^2 - 8x - y^2 + 8 & \triangleq f(x,y) \\ 0 = 2xy - x^2y & \triangleq g(x,y) \end{array}\right\},$$

Table 17: First Newton's Method's approximate solution of Problem 8.2.5.3

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$4.3000000E+00$	$-3.7000000E+00$	$-1.2100000E+01$	$2.5980000E+01$	
1	$7.3062937E+00$	$8.4685315E-01$	$0.0000000E+00$	$2.9711675E+01$	$5.4508417E+00$
2	$5.4217208E+00$	$1.7891396E+00$	$0.0000000E+00$	$4.4395187E+00$	$2.1070165E+00$
3	$5.0201108E+00$	$1.9899446E+00$	$0.0000000E+00$	$2.0161328E-01$	$4.4901368E-01$
4	$5.0000503E+00$	$1.9999748E+00$	$0.0000000E+00$	$5.0302809E-04$	$2.2428288E-02$
5	$5.0000000E+00$	$2.0000000E+00$	$0.0000000E+00$	$3.1628815E-09$	$5.6239543E-05$
6	$5.0000000E+00$	$2.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$3.5362090E-10$
7	$5.0000000E+00$	$2.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

Table 18: Second Newton's Method's approximate solution of Problem 8.2.5.3

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$-4.8000000E+00$	$-5.0000000E-01$	$-1.4800000E+01$	$4.2900000E+00$	
1	$-9.4070423E+00$	$9.2035211E+00$	$0.0000000E+00$	$1.1538316E+02$	$1.0741655E+01$
2	$-4.9722314E+00$	$6.9861157E+00$	$0.0000000E+00$	$2.4584434E+01$	$4.9582693E+00$
3	$-3.3256485E+00$	$6.1628243E+00$	$0.0000000E+00$	$3.3890439E+00$	$1.8409356E+00$
4	$-3.0122579E+00$	$6.0061290E+00$	$0.0000000E+00$	$1.2276708E-01$	$3.5038133E-01$
5	$-3.0000187E+00$	$6.0000094E+00$	$0.0000000E+00$	$1.8724755E-04$	$1.3683843E-02$
6	$-3.0000000E+00$	$6.0000000E+00$	$0.0000000E+00$	$4.3826986E-10$	$2.0934815E-05$
7	$-3.0000000E+00$	$6.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$4.9000061E-11$
8	$-3.0000000E+00$	$6.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

Table 19: Third Newton's Method's approximate solution of Problem 8.2.5.3

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$2.0000000E+00$	$3.0000000E+00$	$-1.0000000E+00$	$-2.0000000E+01$	
1	$8.3333333E+00$	$3.3333333E-01$	$0.0000000E+00$	$4.7222222E+01$	$6.8718427E+00$
2	$5.7575758E+00$	$1.6212121E+00$	$0.0000000E+00$	$8.2931589E+00$	$2.8797845E+00$
3	$5.0603165E+00$	$1.9698417E+00$	$0.0000000E+00$	$6.0771302E-01$	$7.7955950E-01$
4	$5.0004480E+00$	$1.9997760E+00$	$0.0000000E+00$	$4.4803020E-03$	$6.6935058E-02$
5	$5.0000000E+00$	$2.0000000E+00$	$0.0000000E+00$	$2.5085763E-07$	$5.0085690E-04$
6	$5.0000000E+00$	$2.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$2.8046735E-08$
7	$5.0000000E+00$	$2.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

define

$$\mathbf{x} \triangleq \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} xy^2 - 8x - y^2 + 8 \\ 2xy - x^2 y \end{bmatrix}.$$

The Jacobian matrix is

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) = \begin{bmatrix} y^2 - 8 & 2xy - 2y \\ 2y - 2xy & 2x - x^2 \end{bmatrix},$$

so

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x})\right)^{-1} = \frac{1}{(y^2-8)x(2-x) - 2(x-1)y \cdot 2y(1-x)} \begin{bmatrix} 2x - x^2 & 2y - 2xy \\ 2xy - 2y & y^2 - 8 \end{bmatrix}$$

$$= \frac{1}{3x^2y^2 + 8x^2 - 6xy^2 - 16x + 4y^2} \begin{bmatrix} 2x - x^2 & 2y - 2xy \\ 2xy - 2y & y^2 - 8 \end{bmatrix}.$$

Newton's method, as in (8.18), is $\mathbf{x}_{k+1} = \mathbf{x}_k - \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k)\right)^{-1} \mathbf{f}(\mathbf{x}_k)$, that is,

$$\begin{bmatrix} x_{k+1} \\ \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ \\ y_k \end{bmatrix} - \frac{1}{3x_k^2y_k^2 + 8x_k^2 - 6x_ky_k^2 - 16x_k + 4y_k^2} \begin{bmatrix} 2x_k - x_k^2 & 2y_k - 2x_ky_k \\ \\ 2x_ky_k - 2y_k & y_k^2 - 8 \end{bmatrix}.$$

$$\cdot \begin{bmatrix} x_ky_k^2 - 8x_k - y_k^2 + 8 \\ \\ 2x_ky_k - x_k^2 y_k \end{bmatrix}$$

Using an EXCEL™spreadsheet, we found approximate solutions of the system of equations:

$$(x, y) \approx (0.0000000,\ -2.8284271),\ (x, y) \approx (1.0000000,\ 0),\ \text{and}\ (x, y) \approx (0.0000000,\ 2.8284271).$$

These are good approximations of three of the exact solutions. [There are two other exact solutions that seemed to be impossible to find using Newton's method.]

Table 20: First Newton's Method's approximate solution of Problem 8.2.5.5

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$-5.0000000E+00$	$-3.0000000E+00$	$-6.0000000E+00$	$1.0500000E+02$	
1	$3.2910389E-01$	$-2.7692308E+00$	$2.2230877E-01$	$-1.5227956E+00$	$5.3340981E+00$
2	$1.0855160E-01$	$-2.8371547E+00$	$-4.4079210E-02$	$-5.8252389E-01$	$2.3077472E-01$
3	$7.8076387E-03$	$-2.8283794E+00$	$2.6772678E-04$	$-4.3993514E-02$	$1.0112542E-01$
4	$3.0827960E-05$	$-2.8284271E+00$	$-1.0665527E-08$	$-1.7438659E-04$	$7.7769571E-03$
5	$4.7523141E-10$	$-2.8284271E+00$	$0.0000000E+00$	$-2.6883148E-09$	$3.0827485E-05$
6	$1.1292235E-19$	$-2.8284271E+00$	$0.0000000E+00$	$-6.3878530E-19$	$4.7523141E-10$
7	$0.0000000E+00$	$-2.8284271E+00$	$0.0000000E+00$	$0.0000000E+00$	$1.1292235E-19$
8	$0.0000000E+00$	$-2.8284271E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

8.2.5.7. For the system of equations

$$\left\{ \begin{array}{ll} 0 = 4x - 6y - \pi & \triangleq f(x,y) \\ 0 = \cos x - 2\cos x \sin y & \triangleq g(x,y) \end{array} \right\},$$

define

$$\mathbf{x} \triangleq \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} 4x - 6y - \pi \\ \cos x - 2\cos x \sin y \end{bmatrix}.$$

The Jacobian matrix is

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) = \begin{bmatrix} 4 & -6 \\ -\sin x(1 - 2\sin y) & -2\cos x \cos y \end{bmatrix},$$

so

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x})\right)^{-1} = \frac{1}{-8\cos x \cos y - 6\sin x(1 - 2\sin y)} \begin{bmatrix} -2\cos x \cos y & 6 \\ \sin x(1 - 2\sin y) & 4 \end{bmatrix}.$$

Table 21: Second Newton's Method's approximate solution of Problem 8.2.5.5

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$4.3000000E+00$	$-3.7000000E+00$	$1.8777000E+01$	$3.6593000E+01$	
1	$8.3320406E-01$	$-2.1301731E+00$	$5.7750800E-01$	$-2.0709098E+00$	$3.8056577E+00$
2	$-1.0255886E+00$	$-1.9711434E-01$	$1.6126006E+01$	$6.1164757E-01$	$2.6817581E+00$
3	$9.7806186E-01$	$-7.7040791E-01$	$1.6248421E-01$	$-7.7003713E-01$	$2.0840539E+00$
4	$8.4673901E-01$	$-3.2221125E-02$	$1.2259288E+00$	$-3.1464285E-02$	$7.4977692E-01$
5	$9.9982190E-01$	$-7.0135574E-03$	$1.4248235E-03$	$-7.0135572E-03$	$1.5514442E-01$
6	$9.9998771E-01$	$-2.4973958E-06$	$9.8345537E-05$	$-2.4973958E-06$	$7.0130205E-03$
7	$1.0000000E+00$	$-6.1400423E-11$	$1.2473578E-11$	$-6.1400423E-11$	$1.2544290E-05$
8	$1.0000000E+00$	$-1.9146646E-22$	$0.0000000E+00$	$-1.9146646E-22$	$6.1420217E-11$
9	$1.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$1.9146646E-22$
10	$1.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

Table 22: Third Newton's Method's approximate solution of Problem 8.2.5.5

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$-1.0000000E+00$	$1.5000000E+00$	$1.1500000E+01$	$-4.5000000E+00$	
1	$4.0845070E-01$	$2.9577465E+00$	$-4.4263518E-01$	$1.9227406E+00$	$2.0270072E+00$
2	$2.6747728E-01$	$2.8070859E+00$	$8.8099434E-02$	$1.3008330E+00$	$2.0633009E-01$
3	$6.3579721E-02$	$2.8300119E+00$	$-8.3973680E-03$	$3.4842275E-01$	$2.0518240E-01$
4	$2.3441815E-03$	$2.8284240E+00$	$1.7753342E-05$	$1.3245135E-02$	$6.1256125E-02$
5	$2.7579042E-06$	$2.8284271E+00$	$-5.9067418E-11$	$1.5601040E-05$	$2.3414257E-03$
6	$3.8030489E-12$	$2.8284271E+00$	$0.0000000E+00$	$2.1513293E-11$	$2.7579004E-06$
7	$7.2313680E-24$	$2.8284271E+00$	$0.0000000E+00$	$4.0906795E-23$	$3.8030489E-12$
8	$0.0000000E+00$	$2.8284271E+00$	$0.0000000E+00$	$0.0000000E+00$	$7.2313680E-24$
9	$0.0000000E+00$	$2.8284271E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

Newton's method, as in (8.18), is $\mathbf{x}_{k+1} = \mathbf{x}_k - \left(\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k) \right)^{-1} \mathbf{f}(\mathbf{x}_k)$, that is,

$$\begin{bmatrix} x_{k+1} \\ \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ \\ y_k \end{bmatrix} - \frac{1}{-8\cos x_k \cos y_k - 6\sin x_k(1-2\sin y_k)} \begin{bmatrix} -2\cos x_k \cos y_k & 6 \\ \\ \sin x_k(1-2\sin y_k) & 4 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 4x_k - 6y_k - \pi \\ \\ \cos x_k - 2\cos x \sin y_k \end{bmatrix}.$$

Using an $\mathsf{EXCEL}^{\text{TM}}$spreadsheet, we found approximate solutions of the system of equations:

$$(x,y) \approx (-4.7123890, -3.6651914), (x,y) \approx (-42.411501, -28.797933), (x,y) \approx (29.845130, 19.373155).$$

Note that because the system involves trigonometric functions, which are periodic, it makes sense to use several choices of initial guesses that vary in magnitude.

Table 23: First Newton's Method's approximate solution of Problem 8.2.5.7

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$4.3000000E+00$	$-3.7000000E+00$	$3.6258407E+01$	$2.3916601E-02$	
1	$-3.7417972E+00$	$-3.0181302E+00$	$-3.5527137E-15$	$-1.0284702E+00$	$8.0706535E+00$
2	$-4.3144926E+00$	$-3.3999272E+00$	$0.0000000E+00$	$-1.8950037E-01$	$6.8829427E-01$
3	$-4.5138935E+00$	$-3.5328611E+00$	$0.0000000E+00$	$-4.6789791E-02$	$2.3965008E-01$
4	$-4.6122588E+00$	$-3.5984380E+00$	$0.0000000E+00$	$-1.1771806E-02$	$1.1822034E-01$
5	$-4.6619735E+00$	$-3.6315811E+00$	$3.5527137E-15$	$-2.9615935E-03$	$5.9749612E-02$
6	$-4.6870760E+00$	$-3.6483161E+00$	$0.0000000E+00$	$-7.4336323E-04$	$3.0169411E-02$
7	$-4.6997038E+00$	$-3.6567346E+00$	$0.0000000E+00$	$-1.8625352E-04$	$1.5176786E-02$
8	$-4.7060389E+00$	$-3.6609581E+00$	$3.5527137E-15$	$-4.6617695E-05$	$7.6138659E-03$
9	$-4.7092120E+00$	$-3.6630735E+00$	$0.0000000E+00$	$-1.1661386E-05$	$3.8136189E-03$
.	.	.	.	.	.
44	$-4.7123890E+00$	$-3.6651914E+00$	$0.0000000E+00$	$-9.8645628E-27$	$1.1111638E-13$
45	$-4.7123890E+00$	$-3.6651914E+00$	$0.0000000E+00$	$-2.4813313E-27$	$5.5542351E-14$
46	$-4.7123890E+00$	$-3.6651914E+00$	$0.0000000E+00$	$-6.2020618E-28$	$2.7856600E-14$

Table 24: Second Newton's Method's approximate solution of Problem 8.2.5.7

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$4.2000000E+00$	$3.0000000E+00$	$4.0584073E+00$	$-3.5188960E-01$	
1	$-4.2581583E+01$	$-2.7511321E+01$	$-8.4000000E+00$	$4.0322049E-01$	$5.5852101E+01$
2	$-4.2553809E+01$	$-2.8892805E+01$	$-3.9079850E-14$	$-2.2633025E-02$	$1.3817631E+00$
3	$-4.2481277E+01$	$-2.8844450E+01$	$1.7763568E-14$	$-5.5399795E-03$	$8.7172287E-02$
4	$-4.2446108E+01$	$-2.8821004E+01$	$-1.0658141E-14$	$-1.3733629E-03$	$4.2267930E-02$
5	$-4.2428741E+01$	$-2.8809426E+01$	$1.7763568E-14$	$-3.4205684E-04$	$2.0872594E-02$
6	$-4.2420106E+01$	$-2.8803670E+01$	$-1.0658141E-14$	$-8.5363708E-05$	$1.0378313E-02$
7	$-4.2415800E+01$	$-2.8800799E+01$	$1.7763568E-14$	$-2.1322720E-05$	$5.1755234E-03$
.	.	.	.	.	.
44	$-4.2411501E+01$	$-2.8797933E+01$	$1.7763568E-14$	$-1.2900012E-27$	$3.7326840E-14$
45	$-4.2411501E+01$	$-2.8797933E+01$	$-1.0658141E-14$	$-2.7801442E-28$	$1.9308356E-14$
46	$-4.2411501E+01$	$-2.8797933E+01$	$-1.0658141E-14$	$-1.0054434E-28$	$1.0180733E-14$

Table 25: Third Newton's Method's approximate solution of Problem 8.2.5.7

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$2.0000000E+01$	$2.0000000E+01$	$-4.3141593E+01$	$-3.3703110E-01$	
1	$1.6131654E+01$	$1.0230837E+01$	$3.5527137E-15$	$-2.2271020E+00$	$1.0507171E+01$
2	$2.9800522E+01$	$1.9343416E+01$	$1.7763568E-14$	$-2.3163668E-03$	$1.6427935E+01$
3	$2.9822742E+01$	$1.9358229E+01$	$-1.0658141E-14$	$-5.8119291E-04$	$2.6705426E-02$
4	$2.9833914E+01$	$1.9365677E+01$	$3.5527137E-15$	$-1.4558679E-04$	$1.3426417E-02$
5	$2.9839516E+01$	$1.9369412E+01$	$3.5527137E-15$	$-3.6434440E-05$	$6.7333426E-03$
6	$2.9842322E+01$	$1.9371282E+01$	$3.5527137E-15$	$-9.1134338E-06$	$3.3719252E-03$
7	$2.9843726E+01$	$1.9372218E+01$	$-1.0658141E-14$	$-2.2789681E-06$	$1.6873041E-03$
.	.	.	.	.	.
44	$2.9845130E+01$	$1.9373155E+01$	$-1.0658141E-14$	$-8.5873563E-29$	$1.2410859E-14$
45	$2.9845130E+01$	$1.9373155E+01$	$-1.0658141E-14$	$-7.1803639E-30$	$6.1606666E-15$
46	$2.9845130E+01$	$1.9373155E+01$	$3.5527137E-15$	$1.4971060E-30$	$2.7807670E-15$
47	$2.9845130E+01$	$1.9373155E+01$	$3.5527137E-15$	$1.4971060E-30$	$5.6337739E-16$

8.2.5.9. For the system of equations

$$\left\{\begin{array}{ll} 0 = xy^2 - y + 1 & \triangleq f(x,y) \\ 0 = x^2y + x - 0.5 & \triangleq g(x,y) \end{array}\right\},$$

define

$$\mathbf{x} \triangleq \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} xy^2 - y + 1 \\ x^2y + x - 0.5 \end{bmatrix}.$$

The approximate Jacobian matrix is

$$\left[\frac{\triangle f_{i,k}}{\triangle x_{j,k}}\right] = \frac{f_i(\mathbf{x}_{k-1} + (x_{j,k} - x_{j,k-1})\mathbf{e}^{(j)})}{x_{j,k} - x_{j,k-1}}$$

$$= \begin{bmatrix} y_{k-1}^2 \frac{x_k - x_{k-1}}{x_k - x_{k-1}} - 0 + 0 & x_{k-1} \frac{y_k^2 - y_{k-1}^2}{y_k - y_{k-1}} - \frac{y_k - y_{k-1}}{y_k - y_{k-1}} + 0 \\ y_{k-1} \frac{x_k^2 - x_{k-1}^2}{x_k - x_{k-1}} + \frac{x_k - x_{k-1}}{x_k - x_{k-1}} - 0 & x_{k-1}^2 \frac{y_k - y_{k-1}}{x_k - x_{k-1}} + 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} y_{k-1}^2 & x_{k-1}(y_k + y_{k-1}) - 1 \\ y_{k-1}(x_k + x_{k-1}) + 1 & x_{k-1}^2 \end{bmatrix}$$

so

$$\left[\frac{\triangle f_{i,k}}{\triangle x_{j,k}}\right]^{-1} = \frac{1}{x_{k-1}^2 y_{k-1}^2 - \big(1 + y_{k-1}(x_k + x_{k-1})\big)\big(-1 + x_{k-1}(y_k + y_{k-1})\big)} \cdot$$

$$\cdot \begin{bmatrix} x_{k-1}^2 & 1 - x_{k-1}(y_k + y_{k-1}) \\ -1 - y_{k-1}(x_k + x_{k-1}) & y_{k-1}^2 \end{bmatrix}.$$

A Secant method, as in (8.22), with relaxation parameter $w = 1$, is

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \left[\frac{\triangle f_{i,k}}{\triangle x_{j,k}}\right]^{-1} \mathbf{f}(\mathbf{x}_k).$$

Using an EXCEL$^{\text{TM}}$spreadsheet, we found approximate solutions of the system of equations:

$$(x,y) \approx (-4.7123890, -3.6651914), (x,y) \approx (-42.411501, -28.797933), (x,y) \approx (29.845130, 19.373155).$$

Table 26: First Secant Method's approximate solution of Problem 8.2.5.9

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$4.3000000E+00$	$-3.7000000E+00$	$3.6258407E+01$	$2.3916601E-02$	
1	$-3.7417972E+00$	$-3.0181302E+00$	$-3.5527137E-15$	$-1.0284702E+00$	$8.0706535E+00$
2	$-4.3144926E+00$	$-3.3999272E+00$	$0.0000000E+00$	$-1.8950037E-01$	$6.8829427E-01$
3	$-4.5138935E+00$	$-3.5328611E+00$	$0.0000000E+00$	$-4.6789791E-02$	$2.3965008E-01$
4	$-4.6122588E+00$	$-3.5984380E+00$	$0.0000000E+00$	$-1.1771806E-02$	$1.1822034E-01$
5	$-4.6619735E+00$	$-3.6315811E+00$	$3.5527137E-15$	$-2.9615935E-03$	$5.9749612E-02$
6	$-4.6870760E+00$	$-3.6483161E+00$	$0.0000000E+00$	$-7.4336323E-04$	$3.0169411E-02$
7	$-4.6997038E+00$	$-3.6567346E+00$	$0.0000000E+00$	$-1.8625352E-04$	$1.5176786E-02$
8	$-4.7060389E+00$	$-3.6609581E+00$	$3.5527137E-15$	$-4.6617695E-05$	$7.6138659E-03$
9	$-4.7092120E+00$	$-3.6630735E+00$	$0.0000000E+00$	$-1.1661386E-05$	$3.8136189E-03$
.	.	.	.	.	.
44	$-4.7123890E+00$	$-3.6651914E+00$	$0.0000000E+00$	$-9.8645628E-27$	$1.1111638E-13$
45	$-4.7123890E+00$	$-3.6651914E+00$	$0.0000000E+00$	$-2.4813313E-27$	$5.5542351E-14$
46	$-4.7123890E+00$	$-3.6651914E+00$	$0.0000000E+00$	$-6.2020618E-28$	$2.7856600E-14$

Table 27: Second Secant Method's approximate solution of Problem 8.2.5.9

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$4.2000000E+00$	$3.0000000E+00$	$4.0584073E+00$	$-3.5188960E-01$	
1	$-4.2581583E+01$	$-2.7511321E+01$	$-8.4000000E+00$	$4.0322049E-01$	$5.5852101E+01$
2	$-4.2553809E+01$	$-2.8892805E+01$	$-3.9079850E-14$	$-2.2633025E-02$	$1.3817631E+00$
3	$-4.2481277E+01$	$-2.8844450E+01$	$1.7763568E-14$	$-5.5399795E-03$	$8.7172287E-02$
4	$-4.2446108E+01$	$-2.8821004E+01$	$-1.0658141E-14$	$-1.3733629E-03$	$4.2267930E-02$
5	$-4.2428741E+01$	$-2.8809426E+01$	$1.7763568E-14$	$-3.4205684E-04$	$2.0872594E-02$
6	$-4.2420106E+01$	$-2.8803670E+01$	$-1.0658141E-14$	$-8.5363708E-05$	$1.0378313E-02$
7	$-4.2415800E+01$	$-2.8800799E+01$	$1.7763568E-14$	$-2.1322720E-05$	$5.1755234E-03$
.	.	.	.	.	.
44	$-4.2411501E+01$	$-2.8797933E+01$	$1.7763568E-14$	$-1.2900012E-27$	$3.7326840E-14$
45	$-4.2411501E+01$	$-2.8797933E+01$	$-1.0658141E-14$	$-2.7801442E-28$	$1.9308356E-14$
46	$-4.2411501E+01$	$-2.8797933E+01$	$-1.0658141E-14$	$-1.0054434E-28$	$1.0180733E-14$

Table 28: Third Secant Method's approximate solution of Problem 8.2.5.9

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$2.0000000E+01$	$2.0000000E+01$	$-4.3141593E+01$	$-3.3703110E-01$	
1	$1.6131654E+01$	$1.0230837E+01$	$3.5527137E-15$	$-2.2271020E+00$	$1.0507171E+01$
2	$2.9800522E+01$	$1.9343416E+01$	$1.7763568E-14$	$-2.3163668E-03$	$1.6427935E+01$
3	$2.9822742E+01$	$1.9358229E+01$	$-1.0658141E-14$	$-5.8119291E-04$	$2.6705426E-02$
4	$2.9833914E+01$	$1.9365677E+01$	$3.5527137E-15$	$-1.4558679E-04$	$1.3426417E-02$
5	$2.9839516E+01$	$1.9369412E+01$	$3.5527137E-15$	$-3.6434440E-05$	$6.7333426E-03$
6	$2.9842322E+01$	$1.9371282E+01$	$3.5527137E-15$	$-9.1134338E-06$	$3.3719252E-03$
7	$2.9843726E+01$	$1.9372218E+01$	$-1.0658141E-14$	$-2.2789681E-06$	$1.6873041E-03$
.	.	.	.	.	.
44	$2.9845130E+01$	$1.9373155E+01$	$-1.0658141E-14$	$-8.5873563E-29$	$1.2410859E-14$
45	$2.9845130E+01$	$1.9373155E+01$	$-1.0658141E-14$	$-7.1803639E-30$	$6.1606666E-15$
46	$2.9845130E+01$	$1.9373155E+01$	$3.5527137E-15$	$1.4971060E-30$	$2.7807670E-15$
47	$2.9845130E+01$	$1.9373155E+01$	$3.5527137E-15$	$1.4971060E-30$	$5.6337739E-16$

8.2.5.11. For the system of equations

$$\left\{\begin{array}{ll} 0 = x + 2y - 6 & \triangleq f(x,y) \\ 0 = x^2 + 2x + y^2 - 2y - 23 & \triangleq g(x,y) \end{array}\right\},$$

define

$$\mathbf{x} \triangleq \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} x + 2y - 6 \\ x^2 + 2x + y^2 - 2y - 23 \end{bmatrix}.$$

The approximate Jacobian matrix is

$$\left[\frac{\triangle f_{i,k}}{\triangle x_{j,k}}\right] = \frac{f_i(\mathbf{x}_{k-1} + (x_{j,k} - x_{j,k-1})\mathbf{e}^{(j)})}{x_{j,k} - x_{j,k-1}}$$

$$= \begin{bmatrix} \frac{x_k - x_{k-1}}{x_k - x_{k-1}} + 0 + 0 & 0 + 2\,\frac{y_k - y_{k-1}}{y_k - y_{k-1}} + 0 \\ \\ \frac{x_k^2 - x_{k-1}^2}{x_k - x_{k-1}} + 2\,\frac{x_k - x_{k-1}}{x_k - x_{k-1}} + 0 - 0 & 0 - 0 + \frac{y_k^2 - y_{k-1}^2}{y_k - y_{k-1}} - 2\,\frac{y_k - y_{k-1}}{y_k - y_{k-1}} - 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ \\ x_k + x_{k-1} + 2 & y_k + y_{k-1} - 2 \end{bmatrix}$$

so

$$\left[\frac{\triangle f_{i,k}}{\triangle x_{j,k}}\right]^{-1} = \frac{1}{(y_k + y_{k-1} - 2) - 2(x_k + x_{k-1} + 2)} \begin{bmatrix} y_k + y_{k-1} - 2 & -2 \\ \\ -2 - x_k - x_{k-1} & 1 \end{bmatrix}.$$

A Secant method, as in (8.22), with relaxation parameter $w = 1$, is

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \left[\frac{\triangle f_{i,k}}{\triangle x_{j,k}}\right]^{-1} \mathbf{f}(\mathbf{x}_k).$$

Using an $\mathsf{EXCEL}^{\mathrm{TM}}$spreadsheet, we found approximate solutions of the system of equations:

$$(x,y) \approx (4.0000000, 1.0000000), (x,y) \approx (5.8987177, 0.050641131), \quad \text{and}$$

$$(x,y) \approx (-4.0000000, 5.0000000).$$

We see that for one of the approximate solutions, a Secant Method gives comparable conclusions as did Newton's Method in problem 8.2.5.4, and a Secant Method only needed roughly the same number of steps to satisfy a comparable stopping rule.

A Secant Method found to other approximate solutions, although for one of those two it took many steps to do so.

Table 29: First Secant Method's approximate solution of Problem 8.2.5.11

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$-3.0000000E-01$	$7.0000000E-01$	$-4.9000000E+00$	$-2.4420000E+01$	
1	$-2.0000000E-01$	$8.0000000E-01$	$-4.6000000E+00$	$-2.4320000E+01$	
2	$1.4354286E+01$	$-4.1771429E+00$	$0.0000000E+00$	$2.3755690E+02$	$1.5381781E+01$
3	$1.7470205E+00$	$2.1264898E+00$	$0.0000000E+00$	$-1.6184899E+01$	$1.4095351E+01$
4	$2.5511738E+00$	$1.7244131E+00$	$0.0000000E+00$	$-1.1864390E+01$	$8.9907079E-01$
5	$4.7594295E+00$	$6.2028523E-01$	$0.0000000E+00$	$8.3152119E+00$	$2.4689049E+00$
6	$3.8494951E+00$	$1.0752524E+00$	$0.0000000E+00$	$-1.4767340E+00$	$1.0173376E+00$
7	$3.9867233E+00$	$1.0066383E+00$	$0.0000000E+00$	$-1.3254635E-01$	$1.5342579E-01$
8	$4.0002550E+00$	$9.9987250E-01$	$0.0000000E+00$	$2.5500399E-03$	$1.5128861E-02$
9	$3.9999996E+00$	$1.0000002E+00$	$0.0000000E+00$	$-4.2387687E-06$	$2.8556795E-04$
10	$4.0000000E+00$	$1.0000000E+00$	$0.0000000E+00$	$-1.3510615E-10$	$4.7389367E-07$
11	$4.0000000E+00$	$1.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$1.5105327E-11$
12	$4.0000000E+00$	$1.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

Table 30: Second Secant Method's approximate solution of Problem 8.2.5.11

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$2.0000000E+00$	$-2.0000000E+00$	$2.0000000E+00$	$-1.9000000E+01$	
1	$-2.0000000E-01$	$8.0000000E-01$	$5.4000000E+00$	$-2.1920000E+01$	
2	$2.2592593E+00$	$-3.1296296E+00$	$-1.0000000E+01$	$-1.2619684E+01$	$4.6357249E+00$
3	$7.7649509E+00$	$-8.8247545E-01$	$0.0000000E+00$	$2.2543324E+01$	$5.9466245E+00$
4	$6.2650887E+00$	$-1.3254433E-01$	$0.0000000E+00$	$3.7387265E+00$	$1.6768970E+00$
5	$6.0519046E+00$	$-2.5952282E-02$	$0.0000000E+00$	$1.5224132E+00$	$2.3834706E-01$
6	$5.9530224E+00$	$2.3488786E-02$	$0.0000000E+00$	$5.3298289E-01$	$1.1055359E-01$
.	.	.	.	.	.
12	$5.8988029E+00$	$5.0598569E-02$	$0.0000000E+00$	$8.2970312E-04$	$1.8549688E-04$
.	.	.	.	.	.
21	$5.8987177E+00$	$5.0641129E-02$	$0.0000000E+00$	$4.9135103E-08$	$1.0986944E-08$
22	$5.8987177E+00$	$5.0641130E-02$	$0.0000000E+00$	$1.6659591E-08$	$3.7251971E-09$
23	$5.8987177E+00$	$5.0641131E-02$	$0.0000000E+00$	$5.6485412E-09$	$1.2630534E-09$
.	.	.	.	.	.
34	$5.8987177E+00$	$5.0641131E-02$	$0.0000000E+00$	$3.1974423E-14$	$8.8885625E-15$
35	$5.8987177E+00$	$5.0641131E-02$	$0.0000000E+00$	$0.0000000E+00$	$2.4241534E-15$
36	$5.8987177E+00$	$5.0641131E-02$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

Table 31: Third Secant Method's approximate solution of Problem 8.2.5.11

k	x_k	y_k	$f(x_k, y_k)$	$g(x_k, y_k)$	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$-3.0000000E+00$	$2.0000000E+00$	$-5.0000000E+00$	$-2.0000000E+01$	
1	$-2.0000000E-01$	$8.0000000E-01$	$-4.6000000E+00$	$-2.4320000E+01$	
2	$-1.4250000E+01$	$1.0125000E+01$	$0.0000000E+00$	$2.3382813E+02$	$1.6862922E+01$
3	$-4.2424242E-01$	$3.2121212E+00$	$0.0000000E+00$	$-1.9775023E+01$	$1.5457667E+01$
4	$-1.5023232E+00$	$3.7511616E+00$	$0.0000000E+00$	$-1.7178781E+01$	$1.2053309E+00$
5	$-8.6357554E+00$	$7.3178777E+00$	$0.0000000E+00$	$7.3220338E+01$	$7.9754196E+00$
6	$-2.8579080E+00$	$4.4289540E+00$	$0.0000000E+00$	$-9.7904525E+00$	$6.4598298E+00$
7	$-3.5393584E+00$	$4.7696792E+00$	$0.0000000E+00$	$-4.3411775E+00$	$7.6188476E-01$
8	$-4.0822375E+00$	$5.0411187E+00$	$0.0000000E+00$	$8.3082857E-01$	$6.0695724E-01$
9	$-3.9950296E+00$	$4.9975148E+00$	$0.0000000E+00$	$-4.9672625E-02$	$9.7501320E-02$
10	$-3.9999494E+00$	$4.9999747E+00$	$0.0000000E+00$	$-5.0604557E-04$	$5.5004429E-03$
11	$-4.0000000E+00$	$5.0000000E+00$	$0.0000000E+00$	$3.1460242E-07$	$5.6613147E-05$
12	$-4.0000000E+00$	$5.0000000E+00$	$0.0000000E+00$	$-1.9895197E-12$	$3.5173843E-08$
13	$-4.0000000E+00$	$5.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$2.2243506E-13$
14	$-4.0000000E+00$	$5.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$	$0.0000000E+00$

Section 8.3.4

8.3.4.1. $L=\int_0^{2\pi}||\dot{\mathbf{r}}(t)||\,dt=\int_0^{2\pi}\sqrt{(-2\sin 2t)^2+(\cos t)^2+1^2}\;dt=\int_0^{2\pi}\sqrt{4\sin^2 2t+\cos^2 t+1}\;dt.$

Define $f(t)\triangleq\sqrt{4\sin^2 2t+\cos^2 t+1}$. Simpson's Rule with $h=\dfrac{2\pi}{8}$ gives

$$L\approx S_8=\frac{h}{3}\Big(f(0)+4f\big(\frac{\pi}{4}\big)+2f\big(\frac{2\pi}{4}\big)+4f\big(\frac{3\pi}{4}\big)+2f\big(\frac{4\pi}{4}\big)+4f\big(\frac{5\pi}{4}\big)+2f\big(\frac{6\pi}{4}\big)+4f\big(\frac{7\pi}{4}\big)+f\big(\frac{8\pi}{4}\big)\Big)$$
$$=\frac{\pi}{3}\big(1+\sqrt{2}+2\sqrt{22}\big)\approx 12.35174232614476.$$

Mathematica$^{\mathrm{TM}}$helped when doing the evaluations and simplifications, but the explanatory work preceding that was done by hand.

8.3.4.3. $T_{12}=\frac{h}{2}\left(f_0+2f_1+2f_2+...+2f_{11}+f_{12}\right)$

$$S_{12}=\tfrac{h}{3}\big(f_0+4f_1+2f_2+4f_3+2f_4+4f_5+2f_6+4f_7+2f_8+4f_9+2f_{10}+4f_{11}+f_{12}\big)$$

$$U_{12}=\tfrac{2h}{45}\big(7f_0+32f_1+12f_2+32f_3+14f_4+32f_5+12f_6+32f_7+14f_8+32f_9+12f_{10}+32f_{11}+7f_{12}\big)$$

8.3.4.5. (a) Partition $[-\pi/4,\pi/3]$ into four subintervals of equal length: $h=\frac{7\pi}{48}$, $\theta_k=-\frac{\pi}{4}+\frac{7\pi k}{48}$, $k=0,...,4$, so $I=\displaystyle\int_{-\pi/4}^{\pi/3}\tan\theta\,d\theta\approx T_4=$

$$=\frac{\frac{7\pi}{48}}{2}\left(\tan\left(-\frac{\pi}{4}\right)+2\tan\left(-\frac{5\pi}{48}\right)+2\tan\left(\frac{2\pi}{48}\right)+2\tan\left(\frac{9\pi}{48}\right)+\tan\left(\frac{\pi}{3}\right)\right)\approx 0.3786153194106639.$$

(b) To find an error bound, first we need to calculate $f''(x)$, where $f(\theta)=\tan\theta$: We get $f'(x)=\sec^2\theta$ and then $f''(\theta)=2\sec\theta\cdot\sec\theta\,\tan\theta=2\sec^2\theta\,\tan\theta$.

We graphed $|f''(\theta)|$ versus θ over the interval $\Big[-\frac{\pi}{4},\frac{\pi}{3}]$ and found that $|f''(\theta)|\le 14\triangleq k_2$ for $-\frac{\pi}{4}\le x\le\frac{\pi}{3}$.

The error bound is

$$\left|\int_{-\pi/4}^{\pi/3}\tan\theta\,d\theta-T_4\right|\le\frac{\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)}{12}\;k_2\;\left(\frac{\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)}{4}\right)^2=\frac{4802\pi^3}{331776}\approx 0.4487730897.$$

(c) $I=-\ln|\cos\frac{\pi}{3}|+\ln|\cos\left(-\frac{\pi}{4}\right)|=-\ln\frac{1}{2}+\ln\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2}\,\ln 2\approx 0.3465735903.$

(d) Yes, the absolute error $|\,I-T_4\,|\approx|0.3465735903-0.3786153194106639|\approx 0.032041729130691254$ is well less than the error bound of 0.4487730897.

8.3.4.7. The change of the position is
(a) $x(4)-x(0)\approx S_8=\frac{0.5}{3}(1+4\cdot 4+2\cdot 8+4\cdot 13+2\cdot 16+4\cdot 20+2\cdot 24+4\cdot 27+30)=63.833...,$

(b) $x(4)-x(0)\approx U_8=\frac{2(0.5)}{45}(7\cdot 1+32\cdot 4+12\cdot 8+32\cdot 13+14\cdot 16+32\cdot 20+12\cdot 24+32\cdot 27+7\cdot 30)=63.844...$

Section 8.4.3

8.4.3.1. $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -5 \\ -2 & 7 & -4 \end{bmatrix}$, so $\kappa(A) = ||\,A\,||_F \;||A^{-1}||_F = \sqrt{29} \cdot \sqrt{198} \approx 75.77598564.$

So, 75.8 is an upper bound on the condition number.

8.4.3.3. (a) The Gauss-Jordan algorithm begins with

$$\left[\begin{array}{cc|c} 5.000 \times 10^{-5} & 1.000 & 1.000 \\ 1.000 & 1.000 & 2.000 \end{array}\right] \underset{\substack{2 \times 10^4 R_1 \to R_1 \\ -R_1 + R_2 \to R_2}}{\sim} \left[\begin{array}{cc|c} 1.000 & 2.000 \times 10^4 & 2.000 \times 10^4 \\ 0.000 & -2.000 \times 10^4 & -2.000 \times 10^4 \end{array}\right].$$

The $(2,2)$ entry in the latter matrix is -2.000×10^4 because $1.000 - 20,000 \approx -20,000$, rounded off to four digits of precision. The $(2,3)$ entry in the latter matrix is -2.000×10^4 because $2.000 - 20,000 \approx -20,000$, rounded off to four digits of precision.

Continuing further with the Gauss-Jordan algorithm gives

$$\underset{-5 \times 10^{-5} R_2 \to R_2}{\sim} \left[\begin{array}{cc|c} 1.000 & 2.000 \times 10^4 & 2.000 \times 10^4 \\ 0.000 & 1.000 & 1.000 \end{array}\right] \underset{-2 \times 10^4 R_2 + R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 1.000 \end{array}\right].$$

so, in four digit arithmetic the solution to the system seems to be $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.000 \\ 1.000 \end{bmatrix}$, without partial pivoting.

(b) The Gauss-Jordan algorithm with partial pivoting begins with

$$\left[\begin{array}{cc|c} 5.000 \times 10^{-5} & 1.000 & 1.000 \\ 1.000 & 1.000 & 2.000 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1.000 & 1.000 & 2.000 \\ 5.000 \times 10^{-5} & 1.000 & 1.000 \end{array}\right]$$

followed by

$$\underset{-5 \times 10^{-5} R_1 + R_2 \to R_2}{\sim} \left[\begin{array}{cc|c} 1.000 & 1.000 & 2.000 \\ 0.000 & 1.000 & 9.999 \times 10^{-1} \end{array}\right].$$

The $(2,2)$ entry in the latter matrix is 1.000 because $1.000 - 0.00005 = 0.99995 \approx 1.000$, rounded off to four digits of precision.

Continuing further with the Gauss-Jordan algorithm with partial pivoting gives

$$\underset{-R_2 + R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 1.000 & 0.000 & 1.000 \\ 0.000 & 1.000 & 9.999 \times 10^{-1} \end{array}\right]$$

so, in four digit arithmetic the solution to the system seems to be $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.000 \\ 0.9999 \end{bmatrix}$.

To find the true solution we can use the inverse of a 2×2 matrix to get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5.000 \times 10^{-5} & 1.000 \\ 1.000 & 1.000 \end{bmatrix}^{-1} \begin{bmatrix} 1.000 \\ 2.000 \end{bmatrix}$$

$$= \frac{1}{5.000 \times 10^{-5} - 1.000} \begin{bmatrix} 1.000 & -1.000 \\ -1.000 & 5.000 \times 10^{-5} \end{bmatrix} \begin{bmatrix} 1.000 \\ 2.000 \end{bmatrix} = \frac{1}{-0.99995} \begin{bmatrix} -1 \\ -0.9999 \end{bmatrix} \approx \begin{bmatrix} 1.0000500025 \\ 0.9999499975 \end{bmatrix}.$$

In this example, the Gauss-Jordan algorithm without partial pivoting gave a very inaccurate solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.000 \\ 1.000 \end{bmatrix}$.

In this example, the Gauss-Jordan algorithm with partial pivoting gave a quite accurate solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.000 \\ 0.9999 \end{bmatrix}$.

8.4.3.5. (a) The Gauss-Jordan algorithm with partial pivoting begins with

$$\left[\begin{array}{ll|l} 1.000 & 1.000 & 2.000 \\ 5.000 & 1.000\times 10^5 & 1.000\times 10^5 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ll|l} 5.000 & 1.000\times 10^5 & 1.000\times 10^5 \\ 1.000 & 1.000 & 2.000 \end{array}\right]$$

followed by

$$\underset{\substack{0.2R_1 \to R_1 \\ -R_1+R_2 \to R_2}}{\sim} \left[\begin{array}{rr|r} 1.000 & 2.000\times 10^4 & 2.000\times 10^4 \\ 0.000 & -2.000\times 10^4 & -2.000\times 10^4 \end{array}\right].$$

The $(2,2)$ entry in the latter matrix is -2.000×10^4 because $1.000 - 20,000 = -19,9999 \approx -2.000\times 10^4$, rounded off to four digits of precision. The $(2,3)$ entry in the latter matrix is -2.000×10^4 because $2.000 - 20,000 = -19,9998 \approx -2.000\times 10^4$, rounded off to four digits of precision.

Continuing further with the Gauss-Jordan algorithm with partial pivoting gives

$$\underset{\substack{-5\times 10^{-5}R_2 \to R_2 \\ -2.000\times 10^4 R_2 + R_1 \to R_1}}{\sim} \left[\begin{array}{ll|l} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 1.000 \end{array}\right]$$

so, in four digit arithmetic the solution to the system seems to be $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.000 \\ 1.000 \end{bmatrix}$.

(b) The Gauss-Jordan algorithm with implicit partial pivoting begins by separately storing each non-zero row (of the coefficient matrix), normalized so that its entry of largest magnitude is 1. Let us denote by $\widetilde{R}_j$ those normalized rows. Here we have

$$R_1 = [\,1.000 \quad 1.000\,] \Rightarrow \widetilde{R}_1 = [\,1.000 \quad 1.000\,]$$

and

$$R_2 = [\,5.000 \quad 1.000\times 10^5\,] \Rightarrow \widetilde{R}_2 = 10^{-5}R_2 = [\,5.000\times 10^{-5} \quad 1.000\,],$$

so it is the first row that has (implicitly) the largest entry in the first column. So, the method of implicit partial pivoting says to *not* to exchange the first and second rows, as we would do using the method of partial pivoting.

$$\left[\begin{array}{ll|l} 1.000 & 1.000 & 2.000 \\ 5.000 & 1.000\times 10^5 & 1.000\times 10^5 \end{array}\right] \underset{-5R_1+R_2 \to R_2}{\sim} \left[\begin{array}{ll|l} 1.000 & 1.000 & 2.000 \\ 0.000 & 1.000\times 10^5 & 1.000\times 10^5 \end{array}\right],$$

using the facts that to four digits of precision $1.000\times 10^5 - 5.000 = 99,995 \approx 1.000\times 10^5$ and $1.000\times 10^5 - 10.00 = 99,990 \approx 1.000\times 10^5$, followed by

$$\underset{\substack{10^{-5}R_2 \to R_2 \\ -R_2+R_1 \to R_1}}{\sim} \left[\begin{array}{ll|l} 1.000 & 0.000 & 1.000 \\ 0.000 & 1.000 & 1.000 \end{array}\right].$$

So, in four digit arithmetic the solution to the system seems to be $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$.

To find the true solution we can use the inverse of a 2×2 matrix to get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.000 & 1.000 \\ 5.000 & 1.000\times 10^5 \end{bmatrix}^{-1} \begin{bmatrix} 2.000 \\ 1.000\times 10^5 \end{bmatrix}$$

$$= \frac{1}{1.000\times 10^5 - 5.000}\begin{bmatrix} 1.000\times 10^5 & -1.000 \\ -5.000 & 1.000 \end{bmatrix}\begin{bmatrix} 2.000 \\ 1.000\times 10^5 \end{bmatrix} = \frac{1}{9.9995\times 10^4}\begin{bmatrix} 1.000\times 10^5 \\ 9.999\times 10^4 \end{bmatrix} \approx \begin{bmatrix} 1.0000500025 \\ 0.9999499975 \end{bmatrix}.$$

In this example, the Gauss-Jordan algorithm with partial pivoting gave a very inaccurate solution: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.000 \\ 1.000 \end{bmatrix}$.

The Gauss-Jordan algorithm with implicit partial pivoting gave quite an accurate solution: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.000 \\ 1.000 \end{bmatrix}$.

8.4.3.7. The system of Example 8.13 is the same as the system in Examples 8.11 and 8.12, so the true solution is $x = 1.0000100001...,\, y = 0.99998999989...$

We tried the SOR method using initial guess $x = 3$, $y = 2$. (i) With relaxation parameter $\omega = 1$, that is, using the Gauss-Seidel iteration, after 10 steps the method converged to $x = 1.000010000100001$, $y = 0.9999899998999990$. (ii) Instead, with relaxation parameter $\omega = 0.9$, after 38 steps the method again converged to $x = 1.000010000100001$, $y = 0.9999899998999990$.

Table 32: Approximate solution of Problem 8.4.3.7(i): SOR with $\omega = 1.0$

k	x_k	y_k	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$3.000000000E+00$	$2.000000000E+00$	
1	$2.000000000E+00$	$9.999800000E-01$	$1.414227705E+00$
2	$0.000000000E+00$	$1.000000000E+00$	$2.000000000E+00$
3	$1.000020000E+00$	$9.999899998E-01$	$1.000020000E+00$
4	$1.000000000E+00$	$9.999900000E-01$	$2.000000000E-05$
5	$1.000010000E+00$	$9.999899999E-01$	$1.000020000E-05$
6	$1.000010000E+00$	$9.999899999E-01$	$2.000000166E-10$
7	$1.000010000E+00$	$9.999899999E-01$	$1.000017846E-10$
8	$1.000010000E+00$	$9.999899999E-01$	$1.776356839E-15$
9	$1.000010000E+00$	$9.999899999E-01$	$8.881784197E-16$
10	$1.000010000E+00$	$9.999899999E-01$	$0.000000000E+00$

8.4.3.9. From problem 2.4.4.12, if Q is a real, orthogonal matrix then $||\,Q\mathbf{x}\,|| = ||\,\mathbf{x}\,||$ for all $\mathbf{x}$. It follows that $||\,Q\,|| = 1$. Also, because Q is a real, orthogonal matrix, it is square and satisfies $Q^TQ = I = QQ^T$, hence there exists $Q^{-1} = Q^T$, which is also a real, orthogonal matrix. From problem 2.4.12, it follows that $||Q^{-1}\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x}$, hence $||Q^{-1}|| = 1$. So, the standard condition number of Q is $\kappa(Q) = ||\,Q\,||\;||Q^{-1}|| = 1\cdot 1 = 1$. This says that Q is perfectly conditioned.

8.4.3.11. By the remark after Definition 8.1, for all $\mathbf{y}$, $||A\mathbf{y}|| \le ||A||\;||\mathbf{y}||$. Replacing $\mathbf{y}$ by $B\mathbf{x}$, we have that for all $\mathbf{x}$,

$$||(AB)\mathbf{x}|| = ||A(B\mathbf{x})|| \le ||A||\;||B\mathbf{x}||.$$

By the same remark, for all $\mathbf{x}$, $||B\mathbf{x}|| \le ||B||\;||\mathbf{x}||$. So, for all $\mathbf{x}$,

$$||(AB)\mathbf{x}|| \le ||A||\;||B\mathbf{x}|| \le ||A||\;||B||\;||\mathbf{x}||.$$

So, for all $\mathbf{x}$, we have $||(AB)\mathbf{x}|| \le M\,||\mathbf{x}||$, where $M = ||A||\,||B||$. It follows from Definition 8.1 that $||AB|| \le ||A||\;||B||$.

8.4.3.13. The Gauss-Jordan algorithm with implicit partial pivoting begins by separately storing each non-zero row (of the coefficient matrix), normalized so that its entry of largest magnitude is 1. Let us denote by $\widetilde{R}_j$ those normalized rows. Here we have

$$R_1 = [\,1.000 \qquad 1.000\,] \;\Rightarrow \widetilde{R}_1 = [\,1.000 \qquad 1.000\,]$$

and

$$R_2 = [\,10.00 \qquad 1.000\times 10^6\,] \;\Rightarrow \widetilde{R}_2 = 10^{-6}R_2 = [\,1.000\times 10^{-5} \qquad 1.000\,],$$

Table 33: Approximate solution of Problem 8.4.3.7(ii): SOR with $\omega = 0.9$

k	x_k	y_k	$\|\|(\triangle x_k, \triangle y_k)\|\|$
0	$3.000000000E+00$	$2.000000000E+00$	
1	$1.800000000E+00$	$8.999838000E-01$	$1.627893006E+00$
2	$3.000000000E-01$	$1.099997300E+00$	$1.513276379E+00$
3	$1.170014580E+00$	$9.899878499E-01$	$8.769421010E-01$
4	$8.400024300E-01$	$1.009992170E+00$	$3.306178942E-01$
5	$1.026012393E+00$	$9.989895509E-01$	$1.863350853E-01$
6	$9.750072900E-01$	$1.000990442E+00$	$5.104433472E-02$
7	$1.003510644E+00$	$9.998899235E-01$	$2.852459118E-02$
8	$9.966093313E-01$	$1.000090075E+00$	$6.904214034E-03$
9	$1.000450133E+00$	$9.999799883E-01$	$3.842379291E-03$
10	$9.995798659E-01$	$1.000000011E+00$	$8.704976336E-04$
.	.	.	.
33	$1.000010000E+00$	$9.999899999E-01$	$1.576555787E-14$
34	$1.000010000E+00$	$9.999899999E-01$	$2.886579864E-15$
35	$1.000010000E+00$	$9.999899999E-01$	$1.332267630E-15$
36	$1.000010000E+00$	$9.999899999E-01$	$2.220446049E-16$
37	$1.000010000E+00$	$9.999899999E-01$	$2.220446049E-16$
38	$1.000010000E+00$	$9.999899999E-01$	$0.000000000E+00$

so it is the first row that has (implicitly) the largest entry in the first column. So, the method of implicit partial pivoting says to *not* to exchange the first and second rows, as we would do using the method of partial pivoting.

$$\left[\begin{array}{cc|c} 1.000 & 1.000 & 2.000 \\ 10.00 & 1.000\times 10^6 & 1.000\times 10^6 \end{array}\right] \underset{-10R_1+R_2\to R_2}{\sim} \left[\begin{array}{cc|c} 1.000 & 1.000 & 2.000 \\ 0.000 & 1.000\times 10^6 & 1.000\times 10^6 \end{array}\right],$$

using the facts that to four digits of precision $1.000\times 10^6 - 10.00 = 999,990 \approx 1.000\times 10^6$ and $1.000\times 10^6 - 20.00 = 999,980 \approx 1.000\times 10^6$, followed by

$$\underset{\substack{10^{-6}R_2\to R_2 \\ -R_2+R_1\to R_1}}{\sim} \left[\begin{array}{cc|c} 1.000 & 0.000 & 1.000 \\ 0.000 & 1.000 & 1.000 \end{array}\right].$$

So, in four digit arithmetic the solution to the system seems to be $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$.

The Gauss-Jordan algorithm with implicit partial pivoting gave a solution that is quite accurate compared to the exact solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.0000100001... \\ 0.99998999989... \end{bmatrix}$.

Section 8.5.7

8.5.7.1. Write

$$A=\begin{bmatrix}1&1&1&1\\1&2&0&1\\1&0&-1&1\\0&-1&2&1\end{bmatrix}=\left[\begin{array}{c|c}a_{11}&\mathbf{y}^T\\--&--\\ & \\ \mathbf{x}&A^{(3)}\end{array}\right],\quad\text{where}\quad \mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

The first step is to do a similarity transformation of A using a matrix of the form

$$Q_1=\left[\begin{array}{c|c}1&\mathbf{0}^T\\--&--\\ & \\ \mathbf{0}&{}^{(3)}P\end{array}\right]=\left[\begin{array}{c|c}1&\mathbf{0}^T\\---&---\\ & \\ \mathbf{0}&I_3-\beta_{\mathbf{u}}\mathbf{u}\mathbf{u}^T\end{array}\right]=I_4-\beta_{\mathbf{u}}\mathbf{v}\mathbf{v}^T,$$

where $\beta_{\mathbf{u}}=\dfrac{2}{||\mathbf{u}||^2}$. This step should produce

$$A_1\triangleq Q_1AQ_1=\begin{bmatrix}\star&\star&\star&\star\\ \blacksquare&\star&\star&\star\\0&\star&\star&\star\\0&\star&\star&\star\end{bmatrix}.$$

In this problem,

$$\mathbf{u}=\mathbf{x}+(Sign(x_1)||\,\mathbf{x}\,||)\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}+\sqrt{2}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1+\sqrt{2}\\1\\0\end{bmatrix}\quad\text{and}\quad\beta_{\mathbf{u}}=...=1-\frac{1}{\sqrt{2}}.$$

We have

$$\begin{aligned}{}^{(3)}P&=I_3-\frac{2}{||\mathbf{u}||^2}\mathbf{u}\mathbf{u}^T=I_3-\left(1-\frac{1}{\sqrt{2}}\right)\begin{bmatrix}1+\sqrt{2}\\1\\0\end{bmatrix}\begin{bmatrix}1+\sqrt{2}&1&0\end{bmatrix}\\
&=\begin{bmatrix}1-\left(1-\frac{1}{\sqrt{2}}\right)(1+\sqrt{2})^2&-(1-\frac{1}{\sqrt{2}})(1+\sqrt{2})&0\\-\left(1-\frac{1}{\sqrt{2}}\right)(1+\sqrt{2})&\frac{1}{\sqrt{2}}&0\\0&0&1\end{bmatrix}.\end{aligned}$$

So,

$$Q_1=\begin{bmatrix}1&0&0&0\\0&1-\left(1-\frac{1}{\sqrt{2}}\right)(1+\sqrt{2})^2&-(1-\frac{1}{\sqrt{2}})(1+\sqrt{2})&0\\0&-\left(1-\frac{1}{\sqrt{2}}\right)(1+\sqrt{2})&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0\\0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\0&0&0&1\end{bmatrix},$$

and

$$A_1\triangleq Q_1AQ_1=\begin{bmatrix}1&1&1&1\\-\sqrt{2}&-\sqrt{2}&\frac{1}{\sqrt{2}}&-\sqrt{2}\\0&-\sqrt{2}&-\frac{1}{\sqrt{2}}&0\\0&-1&2&1\end{bmatrix}Q_1=\begin{bmatrix}1&-\sqrt{2}&0&1\\-\sqrt{2}&\frac{1}{2}&\frac{3}{2}&-\sqrt{2}\\0&\frac{3}{2}&\frac{1}{2}&0\\0&-\frac{1}{\sqrt{2}}&\frac{3}{\sqrt{2}}&1\end{bmatrix}.$$

Next, write A_1 in the form

$$A_1 = \left[\begin{array}{cc|c} \star & \star & \star\,\star \\ \blacksquare & \star & \star\,\star \\ -- & - & -- \\ & & \\ \mathbf{0} & \mathbf{x}_1 & A^{(2)} \end{array}\right], \quad \text{where} \quad \mathbf{x}_1 = \begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

The next step is to do a similarity transformation of A_1 using a matrix of the form

$$Q_2 = \left[\begin{array}{c|c} I_2 & \mathbf{0}^T \\ -- & -- \\ & \\ O & {}^{(2)}P \end{array}\right] = \left[\begin{array}{c|c} I_2 & \mathbf{0}^T \\ - - - & - - - \\ & \\ O & I_2 - \beta_{\mathbf{u}_1}\mathbf{u}_1\mathbf{u}_1^T \end{array}\right] = I_4 - \beta_{\mathbf{u}_1}\mathbf{v}_1\mathbf{v}_1^T,$$

where $\beta_{\mathbf{u}_1} = \dfrac{2}{||\mathbf{u}_1||^2}$. This step should produce the desired matrix in upper Hessenberg form,

$$A_2 \triangleq Q_2 A_1 Q_2 = \begin{bmatrix} \star & \star & \star & \star \\ \blacksquare & \star & \star & \star \\ 0 & \blacksquare & \star & \star \\ 0 & 0 & \blacksquare & \star \end{bmatrix}.$$

In this problem,

$$\mathbf{u}_1 = \mathbf{x}_1 + (Sign(x_{1,1})||\,\mathbf{x}_1\,||)\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \\ -\frac{1}{\sqrt{2}} \end{bmatrix} + \frac{\sqrt{11}}{2}\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3+\sqrt{11}}{2} \\ \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \quad \beta_{\mathbf{u}_1} = ... = 2 - \frac{6}{\sqrt{11}}.$$

so

$${}^{(2)}P = I_2 - \frac{2}{||\mathbf{u}_1||^2}\mathbf{u}_1\mathbf{u}_1^T = I_2 - \left(2 - \frac{6}{\sqrt{11}}\right)\begin{bmatrix} \frac{3+\sqrt{11}}{2} \\ \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{3+\sqrt{11}}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix} = ... = \begin{bmatrix} -\frac{3}{\sqrt{11}} & \frac{\sqrt{2}}{\sqrt{11}} \\ \\ \frac{\sqrt{2}}{\sqrt{11}} & \frac{3}{\sqrt{11}} \end{bmatrix}.$$

So,

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \\ 0 & 1 & 0 & 0 \\ \\ 0 & 0 & -\frac{3}{\sqrt{11}} & \frac{\sqrt{2}}{\sqrt{11}} \\ \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{11}} & \frac{3}{\sqrt{11}} \end{bmatrix},$$

and

$$A_2 \triangleq Q_2 A_1 Q_2 = \begin{bmatrix} 1 & -\sqrt{2} & 0 & 1 \\ \\ -\sqrt{2} & \frac{1}{2} & \frac{3}{2} & -\sqrt{2} \\ \\ 0 & -\frac{\sqrt{11}}{2} & \frac{3}{2\sqrt{11}} & \frac{\sqrt{2}}{\sqrt{11}} \\ \\ 0 & 0 & \frac{5\sqrt{2}}{\sqrt{11}} & \frac{3}{\sqrt{11}} \end{bmatrix} Q_2 = \begin{bmatrix} 1 & -\sqrt{2} & \frac{\sqrt{2}}{\sqrt{11}} & \frac{3}{\sqrt{11}} \\ \\ -\sqrt{2} & \frac{1}{2} & -\frac{13}{2\sqrt{11}} & -\frac{3}{\sqrt{22}} \\ \\ 0 & -\frac{\sqrt{11}}{2} & -\frac{5}{22} & \frac{9}{11\sqrt{2}} \\ \\ 0 & 0 & -\frac{12\sqrt{2}}{\sqrt{11}} & \frac{19}{11} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.000000000 & -1.414213562 & 0.426401433 & 0.904534034 \\ \\ -1.414213562 & 0.500000000 & -1.959823740 & -0.639602149 \\ \\ 0.000000000 & -1.658312395 & -0.227272727 & 0.578541912 \\ \\ 0.000000000 & 0.000000000 & 1.542778432 & 1.727272727 \end{bmatrix}.$$

8.5.7.3. Write

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 1 \\ 0 & -1 & -2 & 1 \end{bmatrix} = \left[\begin{array}{c|c} a_{11} & \mathbf{y}^T \\ -- & -- \\ & \\ \mathbf{x} & A^{(3)} \end{array}\right], \quad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The first step is to do a similarity transformation of A using a matrix of the form

$$Q_1 = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ -- & -- \\ & \\ \mathbf{0} & {}^{(3)}P \end{array}\right] = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ --- & --- \\ & \\ \mathbf{0} & I_3 - \beta_{\mathbf{u}}\mathbf{u}\mathbf{u}^T \end{array}\right] = I_4 - \beta_{\mathbf{u}}\mathbf{v}\mathbf{v}^T,$$

where $\beta_{\mathbf{u}} = \dfrac{2}{||\mathbf{u}||^2}$. This step should produce

$$A_1 \triangleq Q_1 A Q_1 = \begin{bmatrix} \star & \star & \star & \star \\ \blacksquare & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{bmatrix}.$$

In this problem,

$$\mathbf{u} = \mathbf{x} + (Sign(x_1)||\,\mathbf{x}\,||) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \beta_{\mathbf{u}_1} = ... = 2 + \frac{6}{\sqrt{11}}.$$

We have

$$\begin{aligned} {}^{(3)}P &= I_3 - \frac{2}{||\mathbf{u}||^2}\mathbf{u}\mathbf{u}^T = I_3 - \left(1 - \frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{bmatrix} [\,1+\sqrt{2} \quad 1 \quad 0\,] \\ &= \begin{bmatrix} 1 - \left(1 - \frac{1}{\sqrt{2}}\right)(1+\sqrt{2})^2 & -(1-\frac{1}{\sqrt{2}})(1+\sqrt{2}) & 0 \\ -\left(1 - \frac{1}{\sqrt{2}}\right)(1+\sqrt{2}) & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

So,

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \left(1 - \frac{1}{\sqrt{2}}\right)(1+\sqrt{2})^2 & -(1-\frac{1}{\sqrt{2}})(1+\sqrt{2}) & 0 \\ 0 & -\left(1 - \frac{1}{\sqrt{2}}\right)(1+\sqrt{2}) & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$A_1 \triangleq Q_1 A Q_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\sqrt{2} & \frac{1}{\sqrt{2}} & 0 & -\sqrt{2} \\ 0 & -\frac{3}{\sqrt{2}} & 0 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} Q_1 = \begin{bmatrix} 1 & -\sqrt{2} & 0 & 1 \\ -\sqrt{2} & -\frac{1}{2} & -\frac{1}{2} & -\sqrt{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}.$$

Next, write A_1 in the form

$$A_1 = \left[\begin{array}{cc|c} \star & \star & \star\,\star \\ \blacksquare & \star & \star\,\star \\ -- & - & -- \\ & & \\ \mathbf{0} & \mathbf{x}_1 & A^{(2)} \end{array}\right], \quad \text{where} \quad \mathbf{x}_1 = \begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \\ \frac{3}{\sqrt{2}} \end{bmatrix}.$$

The next step is to do a similarity transformation of A_1 using a matrix of the form

$$Q_2 = \left[\begin{array}{c|c} I_2 & \mathbf{0}^T \\ -- & -- \\ & \\ O & {}^{(2)}P \end{array}\right] = \left[\begin{array}{c|c} I_2 & \mathbf{0}^T \\ --- & --- \\ & \\ O & I_2 - \beta_{\mathbf{u}_1}\mathbf{u}_1\mathbf{u}_1^T \end{array}\right] = I_4 - \beta_{\mathbf{u}_1}\mathbf{v}_1\mathbf{v}_1^T.$$

This step should produce the desired matrix in upper Hessenberg form,

$$A_2 \triangleq Q_2A_1Q_2 = \begin{bmatrix} \star & \star & \star & \star \\ \blacksquare & \star & \star & \star \\ 0 & \blacksquare & \star & \star \\ 0 & 0 & \blacksquare & \star \end{bmatrix}.$$

In this problem,

$$\mathbf{u}_1 = \mathbf{x}_1 + (Sign(x_1)||\,\mathbf{x}_1\,||)\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \\ \frac{3}{\sqrt{2}} \end{bmatrix} + \frac{3\sqrt{3}}{2}\begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3+3\sqrt{3}}{2} \\ \\ \frac{3}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \beta_{\mathbf{u}_1} = ... = \frac{2}{27}(3-\sqrt{3}).$$

so

$${}^{(2)}P = I_2 - \frac{2}{||\mathbf{u}_1||^2}\mathbf{u}_1\mathbf{u}_1^T = I_2 - \frac{2}{27}(3-\sqrt{3})\begin{bmatrix} \frac{3+3\sqrt{3}}{2} \\ \\ \frac{3}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{3+3\sqrt{3}}{2} & \frac{3}{\sqrt{2}} \end{bmatrix} = ... = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ \\ -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

So,

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

and

$$A_2 \triangleq Q_2A_1Q_2 = \begin{bmatrix} 1 & -\sqrt{2} & 0 & 1 \\ -\sqrt{2} & -\frac{1}{2} & -\frac{1}{2} & -\sqrt{2} \\ 0 & -\frac{3\sqrt{3}}{2} & -\frac{1}{2\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & -\frac{2\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} Q_2 = \begin{bmatrix} 1 & -\sqrt{2} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\sqrt{2} & -\frac{1}{2} & \frac{5}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{3\sqrt{3}}{2} & \frac{5}{6} & -\frac{1}{3\sqrt{2}} \\ 0 & 0 & \frac{\sqrt{2}}{3} & \frac{5}{3} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.000000000 & -1.414213562 & -0.816496581 & 0.577350269 \\ -1.414213562 & -0.500000000 & 1.443375673 & -0.408248290 \\ 0.000000000 & -2.598076211 & 0.833333333 & -0.235702260 \\ 0.000000000 & 0.000000000 & 0.471404521 & 1.666666667 \end{bmatrix}.$$

8.5.7.5. Denote $A_0 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. The Mathematica command QRDecomposition[A_0] yields QR decomposition

$$A_0 = Q_0R_0 = \left(\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & \sqrt{2} & 1 \\ -\sqrt{3} & \sqrt{2} & 1 \\ 0 & -\sqrt{2} & 2 \end{bmatrix}\right)\begin{bmatrix} \sqrt{2} & -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & -\frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

The QR algorithm defines the next iterate by

$$A_1 = R_0Q_0 = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & -\frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{bmatrix} \left(\frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 2 \end{bmatrix} \right)$$

$$= \frac{1}{3\sqrt{6}} \begin{bmatrix} 6\sqrt{6} & 3 & -3\sqrt{2} \\ -9 & 5\sqrt{6} & -\sqrt{3} \\ 0 & -\sqrt{3} & \sqrt{6} \end{bmatrix} \approx \begin{bmatrix} 2 & 0.408248290 & -0.577350269 \\ -1.224744871 & 1.666666667 & -0.235702260 \\ 0 & -0.235702260 & 0.333333333 \end{bmatrix}.$$

The Mathematica command QRDecomposition[A_1] yields QR decomposition

$$A_1 = Q_1R_1 = \left(\frac{1}{6\sqrt{165}} \begin{bmatrix} 12\sqrt{30} & 23\sqrt{3} & \sqrt{33} \\ -18\sqrt{5} & 46\sqrt{2} & 2\sqrt{22} \\ 0 & -11 & 23\sqrt{11} \end{bmatrix} \right) \begin{bmatrix} \frac{\sqrt{11}}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{11}} & -\frac{\sqrt{3}}{\sqrt{22}} \\ 0 & \frac{\sqrt{30}}{\sqrt{11}} & -\frac{7}{\sqrt{165}} \\ 0 & 0 & \frac{1}{\sqrt{15}} \end{bmatrix}.$$

The QR algorithm defines the next iterate by

$$A_2 = R_1Q_1 = \frac{1}{990\sqrt{11}} \begin{bmatrix} 2250\sqrt{11} & 258\sqrt{110} & -264\sqrt{10} \\ -270\sqrt{110} & -1457\sqrt{11} & -1111 \\ 0 & -121 & 253\sqrt{11} \end{bmatrix}$$

$$\approx \begin{bmatrix} 2.272727273 & 0.824108724 & -0.254256690 \\ -0.862439362 & 1.471717172 & -0.338362731 \\ 0 & -0.036851387 & 0.255555556 \end{bmatrix}.$$

8.5.7.7. With initial guess $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we calculate

$$\widehat{\mathbf{x}}_1 = A\mathbf{x}_0 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.000000000 \\ 2.000000000 \end{bmatrix}, \quad M_1 = \widehat{x}_{1,1} = 3.000000000, \ \mathbf{x}_1 = M_1^{-1}\widehat{\mathbf{x}}_1 = \begin{bmatrix} 1.0000000000 \\ 0.6666666667 \end{bmatrix},$$

$$\widehat{\mathbf{x}}_2 = A\mathbf{x}_1 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1.0000000000 \\ 0.6666666667 \end{bmatrix} = \begin{bmatrix} 3.6666666667 \\ 3.3333333333 \end{bmatrix}, \quad M_2 = \widehat{x}_{1,2} = 3.6666666667,$$

$$\mathbf{x}_2 = M_2^{-1}\widehat{\mathbf{x}}_2 = \begin{bmatrix} 1.0000000000 \\ 0.9090909091 \end{bmatrix},$$

$$\widehat{\mathbf{x}}_3 = A\mathbf{x}_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1.0000000000 \\ 0.9090909091 \end{bmatrix} = \begin{bmatrix} 3.9090909091 \\ 3.8181818181 \end{bmatrix}, \quad M_3 = \widehat{x}_{1,3} = 3.9090909091,$$

$$\mathbf{x}_3 = M_3^{-1}\widehat{\mathbf{x}}_3 = \begin{bmatrix} 1.0000000000 \\ 0.9767441860 \end{bmatrix},$$

$$\widehat{\mathbf{x}}_4 = A\mathbf{x}_3 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1.0000000000 \\ 0.9767441860 \end{bmatrix} = \begin{bmatrix} 3.9767441860 \\ 3.9534883721 \end{bmatrix}, \quad M_4 = \widehat{x}_{1,4} = 3.9767441860,$$

$$\mathbf{x}_4 = M_4^{-1}\widehat{\mathbf{x}}_4 = \begin{bmatrix} 1.0000000000 \\ 0.9941520468 \end{bmatrix},$$

$$\widehat{\mathbf{x}}_5 = A\mathbf{x}_4 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1.0000000000 \\ 0.9941520468 \end{bmatrix} = \begin{bmatrix} 3.9941520468 \\ 3.9883040936 \end{bmatrix}, \quad M_5 = \widehat{x}_{1,5} = 3.9941520468,$$

$$\mathbf{x}_5 = M_5^{-1}\widehat{\mathbf{x}}_1 = \begin{bmatrix} 1.0000000000 \\ 0.9985358712 \end{bmatrix},$$

It appears that slowly $M_k \to 4$, the dominant eigenvalue, and $\mathbf{x}_k \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, a corresponding eigenvector.

8.5.7.9. We want to explain why $\mathbf{u}$ being any non-zero vector and $\beta_{\mathbf{u}} \triangleq \frac{2}{||\mathbf{u}||^2}$ together imply that $Q_{\mathbf{u}} \triangleq I - \beta_{\mathbf{u}}\mathbf{u}\mathbf{u}^T$ is a Householder matrix, that is, is of the form $Q \triangleq I - 2\mathbf{q}\mathbf{q}^T$ where $\mathbf{q}$ is a real, unit vector. How? Define $\mathbf{q} = \frac{1}{||\mathbf{u}||}\,\mathbf{u}$. Then

$$Q_{\mathbf{u}} \triangleq I - \beta_{\mathbf{u}}\mathbf{u}\mathbf{u}^T = I - \frac{2}{||\mathbf{u}||^2}\mathbf{u}\mathbf{u}^T = I - 2\mathbf{q}\mathbf{q}^T$$

is a Householder matrix.

Section 8.6.1

8.6.1.1. $(\delta\Delta\delta)f_i = (\delta\Delta)\delta f_i = \delta\big(\Delta(f_{i+0.5} - f_{i-0.5})\big) = \delta\big((f_{i+1.5} - f_{i+0.5}) - (f_{i+0.5} - f_{i-0.5})\big)$

$$= \delta(f_{i+1.5} - 2f_{i+0.5} + f_{i-0.5}) = (f_{i+2} - f_{i+1}) + (-2f_{i-1}) + 2f_i) + (f_i - f_{i-1})$$

$$= f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}.$$

8.6.1.3. By Taylor's Theorem, there exists an ξ with $x_i \le \xi \le x_{i+1}$ such that

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 + \frac{f'''(\xi)}{3!}\,h^3, \quad \text{that is,} \quad f_{i+1} = f_i + hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 + \frac{f'''(\xi)}{6}\,h^3.$$

Likewise, there exists an η with $x_{i-1} \le \eta \le x_i$ such that

$$f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 - \frac{f'''(\eta)}{3!}\,h^3, \quad \text{that is,} \quad f_{i-1} = f_i - hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 - \frac{f'''(\eta)}{6}\,h^2,$$

and there exists an ζ with $x_{i-2} \le \zeta \le x_i$ such that

$$f(x_{i-2}) = f(x_i) - 2hf'(x_i) + \frac{f''(x_i)}{2}\,(2h)^2 - \frac{f''(\zeta)}{3!}\,(2h)^3, \text{ that is, } f_{i-2} = f_i - 2hf'(x_i) + f''(x_i)2h^2 - \frac{f''(\zeta)}{6}\,8h^3.$$

It follows that

$$f''(x_i) - \frac{1}{3h^2}\,(f_{i+1} - 3f_{i-1} + 2f_{i-2}) = f''(x_i) - \frac{1}{3h^2}\Big(f_i + hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 + \frac{f'''(\xi)}{6}\,h^3$$
$$-3\left(f_i - hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 - \frac{f'''(\eta)}{6}\,h^2\right) + 2\left(f_i - 2hf'(x_i) + f''(x_i)2h^2 - \frac{4}{3}\,f'''(\zeta)h^3\right)\Big)$$
$$= \frac{1}{3h^2}\Big(3h^2 f''(x_i) - f_i + 3f_i - 2f_i - hf'(x_i) - 3hf'(x_i) + 4hf'(x_i)$$
$$-\frac{f''(x_i)}{2}\,h^2 + 3\cdot\frac{f''(x_i)}{2}\,h^2 - 2f''(x_i)2h^2 - \frac{f'''(\xi)}{6}\,h^3 - 3\,\frac{f'''(\eta)}{6}\,h^2 + 2\cdot\frac{4}{3}\,f'''(\zeta)h^3\Big)$$
$$= \frac{1}{3h^2}\Big(-\frac{f'''(\xi)}{6}\,h^3 - 3\cdot\frac{f'''(\eta)}{6}\,h^2 + 2\cdot\frac{4}{3}\,f'''(\zeta)h^3\Big) = \frac{h}{18}\left(-f'''(\xi) - 3f'''(\eta) + 16f'''(\zeta)\right) = \mathcal{O}(h).$$

So, the error is of order one, that is, $|f''(x_i) - \frac{1}{3h^2}\,(f_{i+1} - 3f_{i-1} + 2f_{i-2})\,| = \mathcal{O}(h)$.

8.6.1.5. By Taylor's Theorem, there exists an ξ with $x_i \le \xi \le x_{i+1}$ such that

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{f''(\xi)}{2}\,h^2, \quad \text{that is,} \quad f_{i+1} = f_i + hf'(x_i) + \frac{f''\xi)}{2}\,h^2.$$

Likewise, there exists an η with $x_{i-1} \le \eta \le x_i$ such that

$$f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{f''(\eta)}{2}\,h^2, \quad \text{that is,} \quad f_{i-1} = f_i - hf'(x_i) + \frac{f''(\eta)}{2}\,h^2.$$

It follows that

$$f'(x_i) - (\alpha h)^{-1}\,(f_{i+1} - 4f_i + 3f_{i-1})$$
$$= f'(x_i) - \frac{1}{\alpha h}\left(f_i + hf'(x_i) + \frac{f''\xi)}{2}\,h^2 - 4f_i + 3\Big(f_i - hf'(x_i) + \frac{f''(\eta)}{2}\,h^2\Big)\right)$$
$$= \frac{1}{\alpha h}\left(\alpha hf'(x_i) - f_i + 4f_i - 3f_i - hf'(x_i) + 3hf'(x_i) - 3\cdot\frac{f''\xi)}{2}\,h^2 - 3\cdot\frac{f''(\eta)}{2}\,h^2\right)$$
$$= \frac{1}{\alpha h}\left((\alpha + 2)hf'(x_i) - \frac{f''\xi)}{2}\,h^2 - \frac{3}{2}\,f''(\eta)\,h^2\right).$$

Only if $\alpha = -2$ does the formula give a good approximation for $f'(x_i)$. If $\alpha = -2$, then we have

$$f'(x_i) - (-2h)^{-1}\,(f_{i+1} - 3f_i + 2f_{i-1}) = \frac{1}{-2h}\left(-\frac{f''\xi)}{2}\,h^2 - \frac{3}{2}\,f''(\eta)\,h^2\right) = h\left(\frac{1}{2}\,f''\xi) + \frac{3}{2}\,f''(\eta)\right) = \mathcal{O}(h).$$

So, $\alpha = -2$ gives error of order one, that is, $|f'(x_i) - \frac{1}{-2h}\,(f_{i+1} - 3f_i + 2f_{i-1})\,| = \mathcal{O}(h)$.

8.6.1.7. By Taylor's Theorem, there exists an ξ with $x_i \le \xi \le x_{i+1}$ such that

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 + \frac{f'''(\xi)}{3!}\,h^3, \quad \text{that is,} \quad f_{i+1} = f_i + hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 + \frac{f'''(\xi)}{6}\,h^3.$$

Likewise, there exists an η with $x_{i-2} \le \eta \le x_i$ such that

$$f(x_{i-2}) = f(x_i) - 2hf'(x_i) + \frac{f''(x_i)}{2}\,(2h)^2 - \frac{f''(\eta)}{3!}\,(2h)^3, \text{ that is, } f_{i-2} = f_i - 2hf'(x_i) + f''(x_i)2h^2 - \frac{f''(\eta)}{6}\,8h^3.$$

It follows that

$$f''(x_i) - \frac{1}{\alpha h^2}\,(2f_{i+1} - 3f_i + f_{i-2}) = f''(x_i) - \frac{1}{\alpha h^2}\Big(2\big(f_i + hf'(x_i) + \frac{f''(x_i)}{2}\,h^2 + \frac{f'''(\xi)}{6}\,h^3\big)$$

$$-3f_i + \big(f_i - 2hf'(x_i) + f''(x_i)2h^2 - \frac{4}{3}\,f'''(\eta)h^3\big)\Big)$$

$$= \frac{1}{\alpha h^2}\Big(\alpha h^2 f''(x_i) - 2f_i + 3f_i - f_i - 2hf'(x_i) + 2hf'(x_i) - f''(x_i)\,h^2 - f''(x_i)2h^2 - \frac{f'''(\xi)}{3}\,h^3 + \frac{4}{3}\,f'''(\zeta)h^3\Big)$$

$$= \frac{1}{\alpha h^2}\Big((\alpha - 3)h^2 f''(x_i) - \frac{f'''(\xi)}{3}\,h^3 + \frac{4}{3}\,f'''(\zeta)h^3\Big).$$

Only if $\alpha = 3$ does the formula give a good approximation for $f''(x_i)$. If $\alpha = 3$, then we have

$$f'(x_i) - (3h^2)^{-1}\,(2f_{i+1} - 3f_i + f_{i-2}) = -\frac{1}{3}\,h \cdot \big(-f'''(\xi) + 4f'''(\zeta)\big) = \mathcal{O}(h).$$

So, $\alpha = 3$ gives error of order one, that is, $|f''(x_i) - \frac{1}{3h^2}\,(2f_{i+1} - 3f_i + f_{i-2})\,| = \mathcal{O}(h)$.

8.6.1.9. Because we want to choose the constants a, b, c so as to make the approximation $f'(x_i) \approx af_{i-2} + bf_i + cf_{i+1}$ have the highest order, when we use Taylor's Theorem we should include more than the minimum numbers of terms implies by approximating the first derivative.

By Taylor's Theorem, there exists an ξ with $x_i \le \xi \le x_{i+1}$ such that

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + h^2\,\frac{f''(x_i)}{2} + \frac{f'''(\xi)}{3!}\,h^3, \quad \text{that is,} \quad f_{i+1} = f_i + hf'(x_i) + h^2\,\frac{f''(x_i)}{2} + \frac{f'''(\xi)}{3!}\,h^3.$$

Likewise, there exists an η with $x_{i-2} \le \eta \le x_i$ such that

$$f(x_{i-2}) = f(x_i) - 2hf'(x_i) + (2h)^2\frac{f''(x_i)}{2} - \frac{f'''(\eta)}{3!}(2h)^3, \quad \text{that is,}$$

$$f_{i-2} = f_i - 2hf'(x_i) + 2h^2 f''(x_i) - \frac{4}{3}\,f'''(\eta)h^3.$$

It follows that

$$f'(x_i) - (af_{i-2} + bf_i + cf_{i+1})$$

$$= f'(x_i) - a\Big(f_i - 2hf'(x_i) + 2h^2\,f''(x_i) - \frac{4}{3}\,f'''(\eta)\,h^3\Big) - bf_i - c\Big(f_i + hf'(x_i) + h^2\,\frac{f''(x_i)}{2} + \frac{f'''(\xi)}{3!}\,h^3\Big)$$

$$= -(a+b+c)f_i + (1 + 2ah - ch)f'(x_i) - (2a + \frac{1}{2}\,c)h^2 f''(x_i) + \frac{h^3}{6}\,\big(8af'''(\xi) - cf'''(\eta)\big)\,.$$

In order to get a good approximation, we need at least that

$$(1)\ (a+b+c) = 0 \quad \text{and} \quad (2)\ (1 + 2ah - ch) = 0;$$

additionally, we can get a better approximation if we also have

$$(3)\ (2a + \frac{1}{2}\,c) = 0.$$

This gives a system of three equations in three unknowns, a, b, and c:

$$\left\{ \begin{array}{rrrl} a & +b & +c & = 0 \\ 2ha & & -hc & = -1 \\ 2a & & +\frac{1}{2}\,c & = 0 \end{array} \right\},$$

which we solve using row reduction:

$$
\left[\begin{array}{rrrr} 1 & 1 & 1 \,| & 0 \\ 2h & 0 & -h \,| & -1 \\ 2 & 0 & \frac{1}{2} \,| & 0 \end{array}\right]
\underset{\substack{h^{-1}R_2 \to R_2 \\ -2R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3}}{\sim}
\left[\begin{array}{rrrr} 1 & 1 & 1 \,| & 0 \\ 0 & -2 & -3 \,| & -h^{-1} \\ 0 & -2 & -\frac{3}{2} \,| & 0 \end{array}\right]
\underset{\substack{-\frac{1}{2}R_2 \to R_2 \\ -R_2+R_1 \to R_1 \\ -2R_2+R_3 \to R_3}}{\sim}
\left[\begin{array}{rrrr} 1 & 0 & -1/2 \,| & -h^{-1}/2 \\ 0 & 1 & 3/2 \,| & h^{-1}/2 \\ 0 & 0 & 3/2 \,| & h^{-1} \end{array}\right]
$$

This implies $c = \dfrac{2}{3h}$; using back substitution, we get successively $b = -\dfrac{1}{2h}$ and $a = -\dfrac{1}{6h}$.

This gives

$$
|f'(x_i) - \frac{1}{6h}\left(-f_{i-2} - 3f_i + 4f_{i+1}\right)| = \frac{h^3}{6}\big(8af'''(\xi) - cf'''(\eta)\big) = \frac{h^3}{6}\Big(-\frac{1}{6h}\cdot 8f'''(\xi) - \frac{2}{3h}\,f'''(\eta)\Big) = \mathcal{O}(h^2),
$$

that is, $f'(x_i) \approx \dfrac{1}{6h}\left(-f_{i-2} - 3f_i + 4f_{i+1}\right)$ gives error of order two.

Section 8.7.7

8.7.7.1. (a) Using initial time $t_0 = 0$ and step size $h = 0.5$, hence $t_i = 0.5i$, and $f(t, y) = t^2 - y^2$, we calculate using Euler's Method:

$$\left\{\begin{array}{l} y(0) = y_0 = y(t_0) = \frac{1}{2}, \\ \\ y(0.5) \approx y_1 = y_0 + hf(t_0, y_0) = \frac{1}{2} + 0.5\left(0^2 - \left(\frac{1}{2}\right)^2\right) = 0.375, \\ \\ y(1.0) \approx y_2 = y_1 + hf(t_1, y_1) = 0.375 + 0.5\left(0.5^2 - (0.375)^2\right) = 0.4296875. \end{array}\right\}$$

(b) Using initial time $t_0 = 0$ and step size $h = -0.5$, hence $t_i = -0.5i$, and $f(t, y) = t^2 - y^2$, we calculate using Euler's Method:

$$\left\{\begin{array}{l} y(0) = y_0 = y(t_0) = \frac{1}{2}, \\ \\ y(-0.5) \approx y_1 = y_0 - hf(t_0, y_0) = \frac{1}{2} - 0.5\left(0^2 - \left(\frac{1}{2}\right)^2\right) = 0.625, \\ \\ y(-1.0) \approx y_2 = y_1 - hf(t_1, y_1) = 0.625 - 0.5\left((-0.5)^2 - (0.625)^2\right) = 0.6953125. \end{array}\right\}$$

(c) see figure

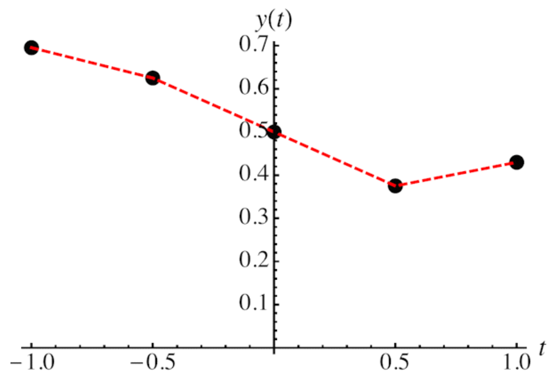


Figure 1: Answer for problem 8.7.7.1(c)

8.7.7.3. (a) Using initial time $t_0 = 1$ and step size $h_1 = 0.2$, hence $t_i = 1 + 0.2i$, and $f(t, y) = y - t$, we calculate using Euler's Method:

$$\left\{\begin{array}{l} y(1) = y_0 = y(t_0) = 3, \\ \\ y(1.2) \approx y_1 = y_0 + hf(t_0, y_0) = 3 + 0.2\,(3 - 1) = 3.4, \\ \\ y(1.4) \approx y_2 = y_1 + hf(t_1, y_1) = 3.4 + 0.2\,(3.4 - 1.2) = 3.88, \\ \\ y(1.6) \approx y_3 = y_2 + hf(t_2, y_2) = 3.88 + 0.2\,(3.88 - 1.4) = 4.376. \end{array}\right\}$$

(b) Using initial time $t_0 = 1$ and step size $h_4 = 0.6$, hence $t_i = 1 + 0.6i$, and $f(t, y) = y - t$, we calculate using the Runge-Kutta Method of order four:

$$\left\{\begin{array}{l} y(1) = y_0 = y(t_0) = 3, \\ \\ \quad k_1 = hf(t_0, y_0) = hf(1, 3) = 0.6\,(3 - 1) = 1.2, \\ \\ \quad k_2 = hf(t_{0.5}, y_0 + \frac{1}{2}\,k_1) = hf\big(1.3, 3 + \frac{1}{2}(1.2)\big) = 0.6\,(3.6 - 1.3) = 1.38, \\ \\ \quad k_3 = hf(t_{0.5}, y_0 + \frac{1}{2}\,k_2) = hf\big(1.3, 3 + \frac{1}{2}(1.38)\big) = 0.6\,(3.69 - 1.3) = 1.434, \\ \\ \quad k_4 = hf(t_1, y_0 + k_3) = hf\big(1.6, 3 + 1.434\big) = 0.6\,(4.434 - 1.6) = 1.7004, \\ \\ y(1.6) \approx y_1 = y_0 + \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4) = 3 + \frac{1}{6}\,(1.2 + 2(1.38) + 2(1.434) + 1.7004) = 4.4214. \end{array}\right\}$$

(c) To find the exact solution of $\dot{y} - y = -t,\ y(1) = 3$, use the integrating factor $\mu(t) = \exp\big(\int(-1)dt\big) = e^{-t}$ to get

$$\frac{d}{dt}\big[\, e^{-t}y\,\big] = e^{-t}\dot{y} - e^{-t}y = -te^{-t},$$

and then integrate to get $e^{-t}y = \displaystyle\int -te^{-t}\,dt = (t+1)e^{-t} + c_1$, that is, $y = t + 1 + c_1e^t$, where c_1 is an arbitrary constant.

The IC implies $3 = y(1) = 1 + 1 + c_1e^1$, so $c_1 = e^{-1}$. The exact solution is $y(t) = t + 1 + e^{-1+t}$.

(d) The exact value of interest is $y(1.6) = 1.6 + 1 + e^{-1+1.6} \approx 4.4221188$. Comparing the exact value $y(1.6)$ to the approximate values produced in parts (a) and (b), the Runge-Kutta method of order four with step size $h_4 = 0.6$ is inaccurate by less than 0.02% relative error while Euler's method with step size $h_1 = 0.2$ is inaccurate by about 1.04% relative error. So, even using a larger step size, that Runge-Kutta method was very much more accurate than Euler's method.

8.7.7.5. (a) Using initial time $t_0 = 1$ and step size $h_1 = 0.2$, hence $t_i = 1 + 0.2i$, and $f(t, y) = -2y + e^{-t}$, we calculate using Euler's Method:

$$\left\{\begin{array}{l} y(1) = y_0 = y(t_0) = 3, \\[6pt] y(1.2) \approx y_1 = y_0 + hf(t_0, y_0) = 3 + 0.2\left(-2\cdot 3 + e^{-1}\right) \approx 1.873575888, \\[6pt] y(1.4) \approx y_2 = y_1 + hf(t_1, y_1) = 1.873575888 + 0.2\left(-2\cdot 1.873575888 + e^{-1.2}\right) \approx 1.184384375, \\[6pt] y(1.6) \approx y_3 = y_2 + hf(t_2, y_2) = 1.184384375 + 0.2\left(-2\cdot 1.184384375 + e^{-1.4}\right) \approx 0.759950018. \end{array}\right\}$$

(b) Using initial time $t_0 = 1$ and step size $h_2 = 0.6$, hence $t_i = 1 + 0.6i$, and $f(t, y) = y - t$, we calculate using the Runge-Kutta Method of order two given in (8.68):

$$\left\{\begin{array}{l} y(1) = y_0 = y(t_0) = 3, \\[6pt] \quad k_1 = \frac{h}{2}\,f(t_0, y_0) = \frac{h}{2}\,f(1, 3) = 0.3\left(-2\cdot 3 + e^{-1}\right) \approx -1.689636168, \\[6pt] \quad k_2 = hf(t_{0.5}, y_0 + k_1) = hf\big(1.3, 3 - 1.689636168\big) = 0.6\left(-2\cdot(1.310363832) + e^{-1.3}\right) \\ \qquad\qquad \approx -1.408917523, \\[6pt] y(1.6) \approx y_1 = y_0 + k_2 \approx 3 - 1.408917523 = 1.591082477. \end{array}\right\}$$

(c) To find the exact solution of $\dot{y} + 2y = e^{-t},\ y(1) = 3$, use the integrating factor $\mu(t) = \exp\big(\int 2dt\big) = e^{2t}$ to get

$$\frac{d}{dt}\big[\, e^{2t}y\,\big] = e^{2t}\dot{y} + 2e^{2t}y = e^{2t}e^{-t} = e^t,$$

and then integrate to get $e^{2t}y = \displaystyle\int e^t\,dt = e^t + c_1$, that is, $y = e^{-t} + c_1e^{-2t}$, where c_1 is an arbitrary constant.

The IC implies $3 = y(1) = e^{-1} + c_1e^{-2}$, so $c_1 = 3e^2 - e$. The exact solution is $y(t) = e^{-t} + \big(3e - 1\big)e^{1-2t}$.

(d) The exact value of interest is $y(1.6) = e^{-1.6} + \big(3e - 1\big)e^{1-3.2} \approx 0.994675995$. Comparing the exact value $y(1.6)$ to the approximate values produced in parts (a) and (b), the the result of (8.68), a Runge-Kutta method of order two, with step size $h_2 = 0.6$ is inaccurate by about 60% relative error. Euler's method with step size $h_1 = 0.2$ is inaccurate by about 23.6% relative error. So, using a larger step size, a Runge-Kutta method of order two is much less accurate than Euler's method.

Note that Euler's method has a *global error*, over several time steps, that is $\mathcal{O}\big(h\big)$. According to Example 8.16 in Section 8.6, this Runge-Kutta method of order two has a *local error*, over one time step, of $\mathcal{O}\big(h^3\big)$. But $h_2 = 0.6$ has $h_2^3 = 0.216$. This explains why this Runge-Kutta method of order two, over one large time step, may do poorly compared to Euler's method over three smaller time steps. But, in truth, both a step size of $h = 0.2$ for Euler's method and $h_2 = 0.6$ for this Runge-Kutta method of order two are too large for this IVP!

8.7.7.7. $\dot{y} = y^2 - t \Rightarrow \ddot{y} = \dfrac{d}{dt}\big[\,y^2 - t\,\big] = 2y\dot{y} - 1 = 2y(y^2 - t) - 1 = 2y^3 - 2ty - 1$. To get local error order to be $\mathcal{O}(h^3)$, Taylor's Method of order two for this ODE is

$$y_{i+1} = y_i + h\dot{y}_i + \frac{h^2}{2}\,\ddot{y}_i = y_i + h\cdot(y_i^2 - t_i) + \frac{h^2}{2}\cdot\big(-1 - 2t_iy_i + 2y_i^3\big)\,.$$

8.7.7.9. $\dot{y} = ye^{\cos(t-y)} \Rightarrow$

$$\begin{aligned}\ddot{y} &= \frac{d}{dt}\big[\, ye^{\cos(t-y)}\,\big] = \dot{y}\cdot e^{\cos(t-y)} + ye^{\cos(t-y)}\cdot\big(-\sin(t-y)\cdot(1-\dot{y})\big)\\ &= ye^{\cos(t-y)}e^{\cos(t-y)} + ye^{\cos(t-y)}\cdot\big(-\sin(t-y)\big)\cdot\big(1-ye^{\cos(t-y)}\big)\\ &= ye^{\cos(t-y)}\Big(e^{\cos(t-y)} - \sin(t-y) + y\sin(t-y)e^{\cos(t-y)}\Big)\\ &= ye^{\cos(t-y)}\Big(\big(1+y\sin(t-y)\big)e^{\cos(t-y)} - \sin(t-y)\Big).\end{aligned}$$

To get local error order to be $\mathcal{O}(h^3)$, Taylor's Method of order two for this ODE is

$$y_{i+1} = y_i + h\dot{y}_i + \frac{h^2}{2}\,\ddot{y}_i$$

that is,

$$y_{i+1} = y_i + h\cdot y_ie^{\cos(t_i-y_i)} + \frac{h^2}{2}\cdot y_ie^{\cos(t_i-y_i)}\cdot\Big(\big(1+y_i\sin(t_i-y_i)\big)e^{\cos(t_i-y_i)} - \sin(t_i-y_i)\Big).$$

8.7.7.11. (a) $\dot{y} + 2y = \cos 3t \Rightarrow$

$$\ddot{y} = \frac{d}{dt}\big[-2y+\cos 3t\,\big] = -2\dot{y} - 3\sin 3t = -2(-2y+\cos 3t) - 3\sin 3t = 4y - 2\cos 3t - 3\sin 3t.$$

To get the local error order to be $\mathcal{O}(h^3)$, Taylor's Method of order two for this ODE is

$$y_{i+1} = y_i + h\dot{y}_i + \frac{h^2}{2}\,\ddot{y}_i$$

that is,

$$y_{i+1} = y_i + h\cdot(-2y_i + \cos 3t_i) + \frac{h^2}{2}\cdot\Big(4y_i - 2\cos 3t_i - 3\sin 3t_i\Big).$$

(b) Use IC $y_0 = y(0) = 2$, $h = 0.25$, and the result of part (a) in an EXCEL spreadsheet to get the results in the table:

Table 34: Approximate solution of Problem 8.7.7.11(b): Taylor's Method with step size $h = 0.25$

		Taylor's method	Exact solution
i	t_i	y_i	$y(t_i)$
0	0.00	1.00000000	1.00000000
1	0.25	0.81250000	0.78308702
2	0.50	0.58110053	0.55235640
3	0.75	0.28293590	0.27171570
4	1.00	−0.01389198	−0.00522591
5	1.25	−0.20753608	−0.18868213
6	1.50	−0.22998106	−0.21588648
7	1.75	−0.09161892	−0.09388164
8	2.00	0.11927931	0.09873585

(c) To find the exact solution of $\dot{y} + 2y = \cos 3t,\ y(0) = 1$, use the integrating factor $\mu(t) = \exp\left(\int 2dt\right) = e^{2t}$ to get

$$\frac{d}{dt}\big[\,e^{2t}y\,\big] = e^{2t}\dot{y} + 2e^{2t}y = e^{2t}\cos 3t,$$

and then integrate to get

$$e^{2t}y = \int e^{2t}\cos 3t,\, dt = \frac{e^{2t}}{2^2+3^2}\,(2\cos 3t + 3\sin 3t) + c_1,$$

where c_1 is an arbitrary constant. The general solution is

$$y = c_1 e^{-2t} + \frac{1}{13}\,(2\cos 3t + 3\sin 3t),$$

where c_1 is an arbitrary constant.

The IC implies $1 = y(0) = c_1 + \frac{2}{13}$, so $c_1 = \frac{11}{13}$. The exact solution is

$$y = \frac{11}{13}\,e^{-2t} + \frac{1}{13}\,(2\cos 3t + 3\sin 3t).$$

The desired exact solution function value:

$$y(2.0) = \frac{11}{13}\,e^{-4} + \frac{1}{13}\,(2\cos 6 + 3\sin 6) \approx 0.0987358543.$$

(d) The exact solution's value $y(2.0) \approx 0.0987358543$ is pretty well approximated in (b), where $h = 0.25$ and $y(2.0) \approx y_8 \approx 0.11927931$.

8.7.7.13. The difference equation has characteristic equation $r^2 + 2h\,r - 1 = 0$, which has roots $r = -h \pm \sqrt{h^2+1}$, so the solutions are of the form $y_i = c_1(-h + \sqrt{h^2+1})^i + c_2(-h - \sqrt{h^2+1})^i$. With $c_1 = 0, c_2 = 1$, the solution is $y_i = (-\alpha)^i$. For $h > 0$, we have that $\alpha \triangleq h + \sqrt{h^2+1} > 0 + \sqrt{h^2+1} > 1$, so this solution of the difference equation has $|y_i| \to \infty$ as $i \to \infty$. This is different from all of the solutions of the original, ODE, which have $y(t) \to 0$ as $t \to \infty$.

8.7.7.15. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. For this system, Euler's method with time step size h, which is real and non-zero, gives approximate solutions using $\mathbf{x}_0 = \mathbf{x}(t_0)$ and

$$\mathbf{x}_{k+1} = \mathbf{x}_k + hA\mathbf{x}_k = (I + hA)\mathbf{x}_k,$$

hence

$$\mathbf{x}_k = (I + hA)^k\mathbf{x}_0,\ k = 0, 1, 2, ...$$

As in Section 5.7, we can use the eigenvalues and eigenvectors of the matrix

$$(I + hA) = \begin{bmatrix} 1 & h \\ -h & 1 \end{bmatrix}.$$

To find the eigenvalues, use

$$0 = |\,I + hA - \lambda I\,| = \begin{vmatrix} 1-\lambda & h \\ -h & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + h^2,$$

so the eigenvalues are $\lambda_{1,2} = 1 \pm ih$, and the eigenvectors corresponding to λ_1 are found using

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -ih & h & 0 \\ -h & -ih & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ after } \tfrac{i}{h}\,R_1 \to R_1,\ hR_1 + R_2 \to R_2$$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = 1 + ih$.

Using λ_1 and $\mathbf{v}_1$ we can find two linearly independent solutions: As in Example 5.33, we calculate that $\lambda_1 = 1 + ih = \alpha + i\nu = \rho(\cos\omega + i\sin\omega)$, where

$$\rho = \sqrt{1^2 + h^2} = \sqrt{1 + h^2} \quad \text{and} \quad \tan\omega = \frac{h}{1}.$$

Because $(\alpha, \nu) = (1, h)$ is, for $h > 0$ or $h < 0$, in the first or fourth quadrant, respectively, $\omega = \arctan(h)$. We have

$$\rho^k(\cos\omega k + i\sin\omega k)\mathbf{v}_1 = (1 + h^2)^{k/2}\left(\cos\big(k\arctan(h)\big) + i\,\sin\cos\big(k\arctan(h)\big)\right)\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= (1 + h^2)^{k/2}\left(\begin{bmatrix} \sin\big(k\arctan(h)\big) \\ \cos\big(k\arctan(h)\big) \end{bmatrix} + i\begin{bmatrix} -\cos\big(k\arctan(h)\big) \\ \sin\big(k\arctan(h)\big) \end{bmatrix}\right).$$

The system of difference equations that are Euler's Method for the system has general solution which is a linear combination of the real and imaginary parts of $\rho^k(\cos\omega k + i\sin\omega k)\mathbf{v}_1$, that is,

$$\mathbf{x}_k = (1 + h^2)^{k/2}\left(c_1\begin{bmatrix} \sin\big(k\arctan(h)\big) \\ \cos\big(k\arctan(h)\big) \end{bmatrix} + c_2\begin{bmatrix} -\cos\big(k\arctan(h)\big) \\ \sin\big(k\arctan(h)\big) \end{bmatrix}\right),\ k = 0, 1, 2, ...,$$

where c_1, c_2=arbitrary constants.

We can rewrite the general solution using the amplitude-phase form. Equate the first component of $\mathbf{x}_k$:

$$x_{1,k} = (1+h^2)^{k/2}\left(-c_2\cos\big(k\arctan(h)\big) + c_1\sin\big(k\arctan(h)\big)\right)$$
$$= (1+h^2)^{k/2}A_1\cos\big(k\arctan(h)-\delta_1\big),$$

where $A_1 = \sqrt{(-c_2)^2+c_1^2}$ and $\tan\delta_1 = \dfrac{c_1}{-c_2}$. We can also equate

$$x_{2,k} = (1+h^2)^{k/2}\left(c_1\cos\big(k\arctan(h)\big) + c_2\sin\big(k\arctan(h)\big)\right)$$
$$= (1+h^2)^{k/2}A_2\sin\big(k\arctan(h)-\delta_2\big)$$
$$= (1+h^2)^{k/2}\left(A_2\sin\big(k\arctan(h)\big)\cos\delta_2 - A_2\cos\big(k\arctan(h)\big)\sin\delta_2\right)$$
$$= (1+h^2)^{k/2}\left(-A_2\sin\delta_2\cos\big(k\arctan(h)\big) + A_2\cos\delta_2\sin\big(k\arctan(h)\big)\right),$$

where $A_2 = \sqrt{c_1^2+c_2^2} = A_1$ and $\tan\delta_2 = \dfrac{A_2\sin\delta_2}{A_2\cos\delta_2} = \dfrac{-c_1}{c_2} = \tan\delta_1$.

So, either $\delta_2=\delta_1$ or $\delta_2=\delta_1+\pi$ or $\delta_2=\delta_1-\pi$. That, along with the fact that $A_2=A_1$ and $\sin(\theta\pm\pi) = -\sin(\theta)$, imply that

$$x_{2,k} = (1+h^2)^{k/2}A_2\sin\big(k\arctan(h)-\delta_2\big) = \pm(1+h^2)^{k/2}A_1\sin\big(k\arctan(h)-\delta_1\big).$$

So,

$$\left(\frac{x_{1,k}}{A_1}\right)^2 + \left(\frac{x_{2,k}}{A_1}\right)^2 = (1+h^2)^k\cos^2\!\big(k\arctan(h)-\delta_1\big) + (1+h^2)^k\sin^2\!\big(k\arctan(h)-\delta_1\big) \equiv (1+h^2)^k.$$

So, in $\mathbb{R}^2$, the solutions of the system of difference equations, $\mathbf{x}_k$ spiral away from $(0,0)$ as $k\to\infty$.

As to the original system, which is in companion form, it's not too hard to derive that the general solution is

$$\mathbf{x}(t) = \begin{bmatrix} A\cos\big(t-\delta\big) \\ -A\sin\big(t-\delta\big)\end{bmatrix}.$$

This implies that $||\mathbf{x}(t)|| \equiv A^2$, so the solutions of the ODE system are circles in $\mathbb{R}^2$.

8.7.7.17. Denote $f_i = f(t_i)$, as in Section 8.3. Continuing,

$$y_2 = y_0 + \frac{h}{2}f_0 + \frac{h}{2}f_1 + \frac{h}{2}f_1 + \frac{h}{2}f_2 = y_0 + \frac{h}{2}\left(f_0+2f_1+f_2\right), ...,$$
$$y_n = y_0 + \frac{h}{2}\left(f_0+2f_1+2f_2+...+2f_{n-1}+f_n\right) \approx y_0 + \int_0^1 f(t)\,dt,$$

using the Trapezoidal Rule approximation of the definite integral. So, in a sense, the Modified Euler's Method is a generalization of the Trapezoidal Rule.

8.7.7.19. Note that we are asked to show that the local error is $\mathcal{O}\big(h^4\big)$, so we must include enough terms in approximations to get such an error.

The **Runge-Kutta Method of order three** is given in (8.74):

$$(\star)\quad \left\{\begin{array}{l} y_0 = y(t_0) \\ y_{i+1} \triangleq y_i + \frac{1}{6}(k_1+4k_2+k_3),\ \text{where} \\ k_1 \triangleq hf(t_i,y_i), \\ k_2 \triangleq hf(t_{i+0.5}, y_i+\frac{1}{2}k_1),\ \text{and} \\ k_3 \triangleq hf(t_{i+1}, y_i - k_1 + 2k_2) \end{array}\right\}.$$

Recall that when evaluating *local* error we assume $y_i = y(t_i)$. Equation $(\star)$ gives

$$(\star\star)\quad y_{i+1} = y_i + \frac{h}{6}\left(f(t_i,y_i) + 4f\Big(t_i+\frac{h}{2}, y_i+\frac{1}{2}k_1\Big) + f(t_i+h, y_i-k_1+2k_2)\right).$$

Taylor Series for a function of two variables, found below in Theorem 13.7 in Section 13.2, implies that

$$k_2 = hf\Big(t_i+\frac{h}{2}, y_i+\frac{1}{2}k_1\Big) = h\Big(f(t_i,y_i)+\frac{h}{2}\,\frac{\partial f}{\partial t}(t_i,y_i)+\frac{1}{2}\,k_1\,\frac{\partial f}{\partial y}(t_i,y_i)+\frac{1}{2}\Big(\frac{h}{2}\Big)^2\frac{\partial^2 f}{\partial t^2}(t_i,y_i)$$

$$+\frac{h}{2}\cdot\frac{1}{2}\,k_1\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+\frac{1}{2}\Big(\frac{1}{2}\,k_1\Big)^2\frac{\partial^2 f}{\partial y^2}(t_i,y_i)+\mathcal{O}\left(h^4+|h|^3|k_1|+h^2k_1^2\right).$$

But, $k_1 = hf(t_i,y_i) = hf_i$, so

$$(\star\star\star)\qquad k_2{=}hf_i+\frac{h^2}{2}\,\frac{\partial f}{\partial t}(t_i,y_i)+\frac{h^2}{2}\,f_i\,\frac{\partial f}{\partial y}(t_i,y_i)+\frac{h^3}{8}\,\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+\frac{h^3}{4}\,f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+\frac{h^3}{8}\,f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i)+\mathcal{O}\left(h^4\right).$$

Also

$$f(t_i+h, y_i-k_1+2k_2) = f(t_i,y_i)+h\frac{\partial f}{\partial t}(t_i,y_i)+(-k_1+2k_2)\frac{\partial f}{\partial y}(t_i,y_i)+\frac{1}{2}\,h^2\frac{\partial^2 f}{\partial t^2}(t_i,y_i)$$

$$+h\cdot(-k_1+2k_2)\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+\frac{1}{2}(-k_1+2k_2)^2\frac{\partial^2 f}{\partial y^2}(t_i,y_i)$$

$$+\mathcal{O}\left(|h|^3+h^2(|k_1|+|k_2|)+|h|(k_1^2+k_2^2)\right).$$

Substitute into this $k_1 = hf(t_i,y_i) = hf_i$ and the formula for k_2 given in $(\star\star\star)$ to get

$$k_3 = hf(t_i+h, y_i-k_1+2k_2) = hf_i+h^2\frac{\partial f}{\partial t}(t_i,y_i)+$$

$$+h^2\Big(-f_i+2\Big(f_i+\frac{h}{2}\,\frac{\partial f}{\partial t}(t_i,y_i)+\frac{h}{2}\,f_i\,\frac{\partial f}{\partial y}(t_i,y_i)+\frac{h^2}{8}\,\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+\frac{h^2}{4}\,f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+\frac{h^2}{8}\,f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i)\Big)\Big)\cdot\frac{\partial f}{\partial y}(t_i,y_i)$$

$$+\frac{1}{2}\,h^3\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+h^3\Big(-f_i+2\Big(f_i+\frac{h}{2}\,\frac{\partial f}{\partial t}(t_i,y_i)+\frac{h}{2}\,f_i\,\frac{\partial f}{\partial y}(t_i,y_i)+\frac{h^2}{8}\,\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+\frac{h^2}{4}\,f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+$$

$$+\frac{h^2}{8}\,f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i)\Big)\Big)\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)$$

$$+\frac{1}{2}\,h^3\Big(-f_i+2\Big(f_i+\frac{h}{2}\,\frac{\partial f}{\partial t}(t_i,y_i)+\frac{h}{2}\,f_i\,\frac{\partial f}{\partial y}(t_i,y_i)+\frac{h^2}{8}\,\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+\frac{h^2}{4}\,f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+\frac{h^2}{8}\,f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i)\Big)\Big)^2\cdot\frac{\partial^2 f}{\partial y^2}(t_i,y_i)$$

$$+\mathcal{O}\Big(h^4\Big)$$

so

$$k_3 = hf_i+h^2\frac{\partial f}{\partial t}(t_i,y_i)+h^2f_i\,\frac{\partial f}{\partial y}(t_i,y_i)+h^3\,\frac{\partial f}{\partial t}(t_i,y_i)\cdot\frac{\partial f}{\partial y}(t_i,y_i)+h^3\,f_i\cdot\Big(\frac{\partial f}{\partial y}(t_i,y_i)\Big)^2+$$

$$+\frac{1}{2}\,h^3\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+h^3f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+\frac{1}{2}\,h^3f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i)+\mathcal{O}\big(h^4\big).$$

Combining this with $(\star\star)$ gives

$$y_{i+1} = y_i+\frac{1}{6}(k_1+4k_2+k_3)$$

$$= y_i+\frac{1}{6}\Big(hf_i+4\Big(hf_i+\frac{h^2}{2}\,\frac{\partial f}{\partial t}(t_i,y_i)+\frac{h^2}{2}\,f_i\,\frac{\partial f}{\partial y}(t_i,y_i)+\frac{h^3}{8}\,\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+\frac{h^3}{4}\,f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+$$

$$+\frac{h^3}{8}\,f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i)\Big)$$

$$+hf_i+h^2\frac{\partial f}{\partial t}(t_i,y_i)+h^2f_i\,\frac{\partial f}{\partial y}(t_i,y_i)+h^3\,\frac{\partial f}{\partial t}(t_i,y_i)\cdot\frac{\partial f}{\partial y}(t_i,y_i)+h^3\,f_i\cdot\Big(\frac{\partial f}{\partial y}(t_i,y_i)\Big)^2+$$

$$+\frac{1}{2}\,h^3\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+h^3f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+\frac{1}{2}\,h^3f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i)+\mathcal{O}\big(h^4\big).$$

By combining terms, this can be rewritten as

$$y_{i+1} = y_i+hf_i+\frac{h^2}{2}\,\frac{\partial f}{\partial t}(t_i,y_i)+\frac{h^2}{2}\,f_i\,\frac{\partial f}{\partial y}(t_i,y_i)+\frac{h^3}{6}\,\frac{\partial^2 f}{\partial t^2}(t_i,y_i)+\frac{h^3}{3}\,f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)+\frac{h^3}{6}\,f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i)+$$

$$+\frac{h^3}{6}\,\frac{\partial f}{\partial t}(t_i,y_i)\cdot\frac{\partial f}{\partial y}(t_i,y_i)+\frac{h^3}{6}\,f_i\cdot\Big(\frac{\partial f}{\partial y}(t_i,y_i)\Big)^2+\mathcal{O}\big(h^4\big)$$

On the other hand, Taylor's Theorem in the form of (8.65), along with (8.67) and (8.70), imply

$$y(t_{i+1}) = y_i + hf_i + \frac{h^2}{2}\dot{f}_i + \frac{h^3}{6}\ddot{f}_i + \mathcal{O}(h^4) = y_i + hf_i + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_i,y_i) + \frac{\partial f}{\partial y}(t_i,y_i)\cdot f_i\right) +$$

$$+\frac{h^3}{6}\Big(\frac{\partial^2 f}{\partial t^2}(t_i,y_i) + 2\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i)\cdot f_i + \frac{\partial f}{\partial y}(t_i,y_i)\cdot\left(\frac{\partial f}{\partial t}(t_i,y_i) + \frac{\partial f}{\partial y}(t_i,y_i)\cdot f_i\right) +$$

$$+\frac{\partial^2 f}{\partial y^2}(t_i,y_i)\cdot(f_i)^2\Big) + \mathcal{O}(h^4).$$

This can be rewritten as

$$y(t_{i+1}) = y_i + hf_i + \frac{h^2}{2}\,\frac{\partial f}{\partial t}(t_i,y_i) + \frac{h^2}{2}f_i\,\frac{\partial f}{\partial y}(t_i,y_i) + \frac{h^3}{6}\,\frac{\partial^2 f}{\partial t^2}(t_i,y_i) + \frac{h^3}{3}\,f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i) +$$

$$+\frac{h^3}{6}\,f_i^2\frac{\partial^2 f}{\partial y^2}(t_i,y_i) + \frac{h^3}{6}\,\frac{\partial f}{\partial t}(t_i,y_i)\cdot\frac{\partial f}{\partial y}(t_i,y_i) + \frac{h^3}{6}\,f_i\left(\frac{\partial f}{\partial y}(t_i,y_i)\right)^2 + \mathcal{O}(h^4).$$

So, the local error is

$$|y(t_{i+1}) - y_{i+1}| = \Big|\cancel{y_i} + \cancel{hf_i} + \cancel{\frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_i,y_i) + \frac{\partial f}{\partial y}(t_i,y_i)\cdot f_i\right)}$$

$$\cancel{+\frac{h^3}{6}\left(\frac{\partial^2 f}{\partial t^2}(t_i,y_i) + 2f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i) + f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i) + \frac{\partial f}{\partial t}(t_i,y_i)\cdot\frac{\partial f}{\partial y}(t_i,y_i) + f_i\left(\frac{\partial f}{\partial y}(t_i,y_i)\right)^2\right)} + \mathcal{O}(h^4)$$

$$-\cancel{y_i} - \cancel{hf_i} + \cancel{\frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_i,y_i) + \frac{\partial f}{\partial y}(t_i,y_i)f_i\right)}$$

$$\cancel{-\frac{h^3}{6}\left(\frac{\partial^2 f}{\partial t^2}(t_i,y_i) + 2f_i\,\frac{\partial^2 f}{\partial t\partial y}(t_i,y_i) + f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i,y_i) + \frac{\partial f}{\partial t}(t_i,y_i)\cdot\frac{\partial f}{\partial y}(t_i,y_i) + f_i\left(\frac{\partial f}{\partial y}(t_i,y_i)\right)^2\right)} + \mathcal{O}(h^4)\Big|,$$

that is, the local error is $\mathcal{O}(h^4)$.

Section 8.8.4

8.8.4.1. (a) Define $t_i = ih$, $y_i = y(t_i)$, $i = 0, ..., 4$, where $h = 0.25$. Using the central difference approximation for the first derivative term, the replacement equations are, for $i = 1, 2, 3$,

$$h^{-2}\left(y_{i+1} - 2y_i + y_{i-1}\right) + 2\sqrt{2}\,\cos\left(\frac{2\pi ih}{3}\right)y_i = 0$$

that is,

$$y_{i-1} + \left(-2 + h^2 \cdot 2\sqrt{2}\,\cos\left(\frac{2\pi ih}{3}\right)\right)y_i + y_{i+1} = 0.$$

The BCs are $0 = y(0) = y_0$ and $-1 = y(1) = y_4$. The system is

$$\begin{bmatrix} -2 + \frac{\sqrt{2}}{8} \cdot \frac{\sqrt{3}}{2} & 1 & 0 \\ 1 & -2 + \frac{\sqrt{2}}{8} \cdot \frac{1}{2} & 1 \\ 0 & 1 & -2 + \frac{\sqrt{2}}{8} \cdot 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -y_0 \\ 0 \\ -y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) The approximate solution is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 + \frac{\sqrt{6}}{16} & 1 & 0 \\ 1 & -2 + \frac{\sqrt{2}}{16} & 1 \\ 0 & 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -.3111164480 \\ -.5746031117 \\ -.7873015559 \end{bmatrix}.$$

The piecewise linear approximate solution is the red, dashed graph in the figure.

(c) There is no exact solution of this non-constant coefficients linear second order ODE. But we did use the Mathematica command *NDSolve* for the ODE-BVP to find a more accurate approximate solution and it agrees well with the coarse approximation we found.

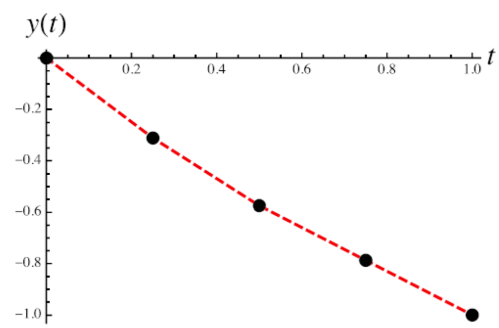


Figure 2: Answer for problem 8.8.4.1

8.8.4.3. (a) Define $x_i = ih$, $y_i = y(x_i)$, $i = 0, ..., 4$, where $h = 0.5$. The replacement equations are, for $i = 1, 2, 3$,

$$h^{-2}\left(y_{i+1} - 2y_i + y_{i-1}\right) + y_i = x_i$$

that is,

$$y_{i-1} + (-2 + h^2)y_i + y_{i+1} = ih^3.$$

The BCs are $-1 = y(0) = y_0$ and $5 = y(2) = y_4$. The system is

$$\begin{bmatrix} -2 + \frac{1}{4} & 1 & 0 \\ 1 & -2 + \frac{1}{4} & 1 \\ 0 & 1 & -2 + \frac{1}{4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} - y_0 \\ \frac{2}{8} \\ \frac{3}{8} - y_4 \end{bmatrix}$$

(b) The approximate solution is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} & 1 & 0 \\ 1 & -\frac{7}{4} & 1 \\ 0 & 1 & -\frac{7}{4} \end{bmatrix}^{-1} \begin{bmatrix} \frac{9}{8} \\ \frac{2}{8} \\ -\frac{37}{8} \end{bmatrix} = \frac{1}{238} \begin{bmatrix} 239 \\ 686 \\ 1021 \end{bmatrix} \approx \begin{bmatrix} 1.00420168 \\ 2.88235294 \\ 4.28991597 \end{bmatrix}.$$

The piecewise linear approximate solution is the red, dashed graph in the figure.

(c) The method of undetermined coefficients for the ODE gives $y_h = c_1 \cos x + c_2 \sin x$ and, after some calculations, $y_P = x$. Plug $y = y_h + y_P = x + c_1 \cos x + c_2 \sin x$ into the BCs to get

$$\left\{ \begin{array}{r} -1 = y(0) = c_1 \\ 5 = y(2) = c_1 \cos 2 + c_2 \sin 2 + 2 \end{array} \right\}.$$

The exact solution of $(\star)$ is

$$y(t) = x - \cos x + \frac{3 + \cos 2}{\sin 2} \cdot \sin x,$$

whose graph is given in the figure. The results from part (a) look very good, for example, $y_1 = 1.00420168$ vs. the exact value $y(.5) \approx 0.984749672$, $y_2 = 2.88235294$ vs. the exact value $y(1) \approx 2.85081572$, and $y_3 = 4.28991597$ vs. the exact value $y(1.5) \approx 4.26373753$.

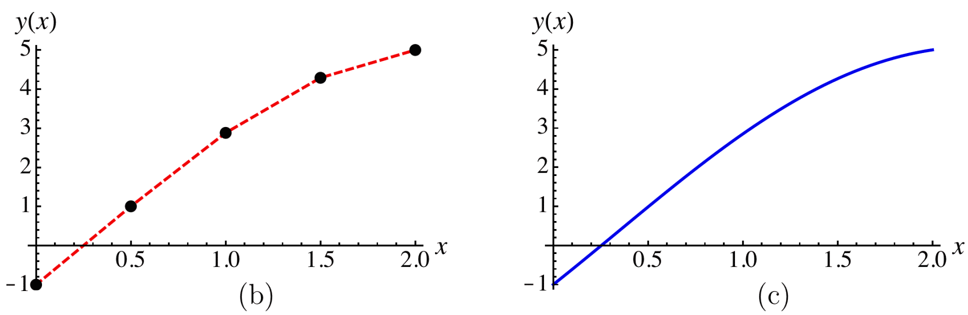


Figure 3: Answer for problem 8.8.4.3(b) and (c)

8.8.4.5. (a) Define $x_i = ih$, $y_i = y(x_i)$, $i = 0, ..., 4$, where $h = 0.25$. One BC is $5 = y(1) = y_4$. The other BC is $3 = y'(0) \approx (2h)^{-1}(y_1 - y_{-1})$, that is, $3 \cdot 2h = y_1 - y_{-1}$, using the central difference approximation of the derivative.

To solve for the fictitious value $y(-h) \approx y_{-1}$ we include it in the difference approximation of the ODE at $x = 0$. The replacement equations are, for $i = 0, 1, 2, 3$,

$$h^{-2}\left(y_{i+1} - 2y_i + y_{i-1}\right) + 2y_i = 0$$

that is,

$$y_{i-1} + (-2 + 2h^2)y_i + y_{i+1} = 0,$$

as well as the BC

$$-y_{-1} - y_1 = 3 \cdot 2h.$$

The system is

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -2+\frac{1}{8} & 1 & 0 & 0 \\ 0 & 1 & -2+\frac{1}{8} & 1 & 0 \\ 0 & 0 & 1 & -2+\frac{1}{8} & 1 \\ 0 & 0 & 0 & 1 & -2+\frac{1}{8} \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6h \\ 0 \\ 0 \\ 0 \\ -y_4 \end{bmatrix}$$

(b) The approximate solution is

$$\begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -2+\frac{1}{8} & 1 & 0 & 0 \\ 0 & 1 & -2+\frac{1}{8} & 1 & 0 \\ 0 & 0 & 1 & -2+\frac{1}{8} & 1 \\ 0 & 0 & 0 & 1 & -2+\frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} \frac{6}{4} \\ 0 \\ 0 \\ 0 \\ -5 \end{bmatrix} \approx \begin{bmatrix} 17.3529170 \\ 19.3097781 \\ 18.8529170 \\ 16.0394412 \\ 11.2210353 \end{bmatrix}.$$

The fictitious value y_{-1} does not appear in the final conclusion: The piecewise linear approximate solution is the red, dashed graph in the figure.

(c) The linear homogeneous ODE has general solution $y(x) = c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x)$. Plug this into the BCs to get

$$\left\{ \begin{array}{l} 3 = y'(0) = \sqrt{2}\,c_2 \\ 5 = y(1) = c_1 \cos\sqrt{2} + c_2 \sin\sqrt{2} \end{array} \right\}.$$

The exact solution of $(\star)$ is

$$y(x) = \left(5 - \frac{3}{\sqrt{2}} \sin\sqrt{2}\right) \cdot \frac{\cos(\sqrt{2}\,x)}{\cos\sqrt{2}} + \frac{3}{\sqrt{2}} \cdot \sin(\sqrt{2}\,x),$$

whose graph is given in the figure. The results from part (a) look good, for example, $y_0 = 19.3097781$ vs. the exact value $y(0) \approx 18.6261587$, $y_1 = 18.8529170$ vs. the exact value $y(.25) \approx 18.2085721$, $y_2 = 16.0394413$ vs. the exact value $y(.5) \approx 15.5385246$, and $y_3 = 11.2210353$ vs. the exact value $y(.75) \approx 10.94630976$.

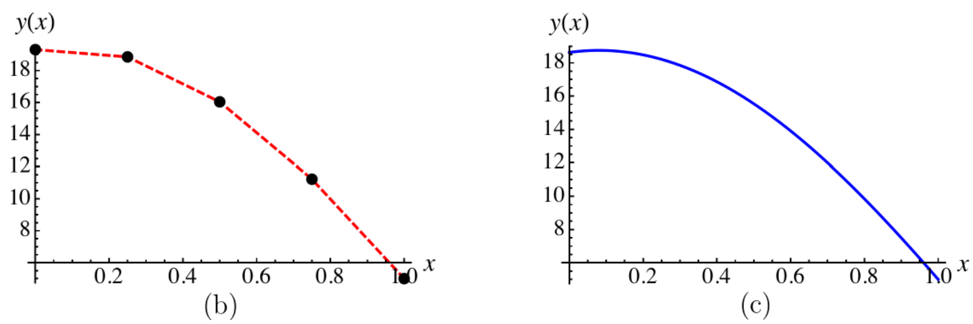


Figure 4: Answer for problem 8.8.4.5(b) and (c)

8.8.4.7. Define $x_i = ih$, $y_i = y(x_i)$, $i = 0, ..., 4$, where $h = 0.25$. One BC is $0 = y(1) = y_4$. The other BC is $1 = y(0) - y'(0) \approx y_0 - (h)^{-1}(y_0 - y_{-1})$, that is, $h = y_{-1} + (h-1)y_0$, using the backwards difference approximation of the derivative.

To solve for the fictitious value $y(-h) \approx y_{-1}$ we include it in the difference approximation of the ODE at $x = 0$. The replacement equations are, for $i = 0, 1, 2, 3$,

$$h^{-2}\,(y_{i+1} - 2y_i + y_{i-1}) = -x_i$$

that is,

$$y_{i-1} - 2y_i + y_{i+1} = -h^2 x_i = -ih^3,$$

as well as the BC

$$y_{-1} + (h-1)y_0 = h.$$

$$\begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} h \\ 0 \\ -h^3 \\ -2h^3 \\ -3h^3 - y_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{1}{64} \\ -\frac{2}{64} \\ -\frac{3}{64} \end{bmatrix}$$

(b) The approximate solution is

$$\begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{1}{64} \\ -\frac{2}{64} \\ -\frac{3}{64} \end{bmatrix} \approx \begin{bmatrix} .68359375 \\ .57812500 \\ .47265625 \\ .35156250 \\ .19921875 \end{bmatrix}.$$

The piecewise linear approximate solution is the red, dashed graph in the figure.

(c) Integrate the ODE twice to get general solution $y(x) = c_1 x + c_2 - \frac{1}{6}x^3$. Plug this and $y'(x) = c_1 - \frac{1}{2}x^2$ into the BCs to get

$$\left\{ \begin{array}{l} 1 = y(0) - y'(0) = c_2 - c_1 \\ 0 = y(1) = c_1 + c_2 - \frac{1}{6} \end{array} \right\}.$$

The exact solution of $(\star)$ is

$$y(x) = -\frac{5}{12}x + \frac{7}{12} - \frac{1}{6}x^3.$$

The solid curve in the figure is the exact solution. The thick dashed curve is the solution using finite differences and agrees well with the exact solution.

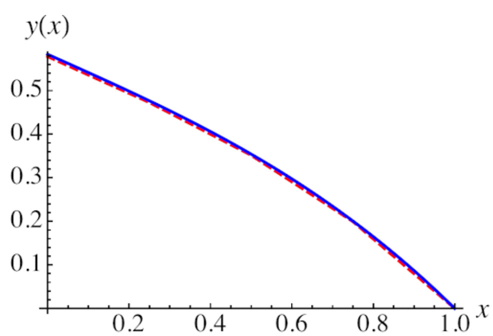


Figure 5: Answer for problem 8.8.4.7

8.8.4.9. Define $r_i = ih$, $R_i = R(r_i)$, $i = 0, ..., 4$, where $h = 0.25$. Using the central difference approximation for the first derivative term, the replacement equations are, for $i = 1, 2, 3$,

$$h^{-2}\left(R_{i+1} - 2R_i + R_{i-1}\right) + (2h)^{-1}\frac{2}{r_i + 1}\left(R_{i+1} - R_{i-1}\right) + \lambda\frac{1}{(r+1)^4}R_i = 0,$$

that is,

$$\Big(1 - \frac{h}{r_i + 1}\Big)R_{i-1} - 2R_i + \Big(1 + \frac{h}{r_i + 1}\Big)R_{i+1} = \frac{\lambda h^2}{(r_i + 1)^4}R_i.$$

The BCs are $0 = y(0) = R_0$ and $0 = y(1) = R_4$. The system is a *generalized* eigenvalue problem, specifically $A\mathbf{R} = \lambda B\mathbf{R}$, specifically

$$\begin{bmatrix} -2 & \frac{6}{5} & 0 \\ \frac{5}{6} & -2 & \frac{7}{6} \\ 0 & \frac{6}{7} & -2 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = -\frac{\lambda}{16} \begin{bmatrix} \frac{256}{625} & 0 & 0 \\ 0 & \frac{16}{81} & 0 \\ 0 & 0 & \frac{256}{2401} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}.$$

Using the Mathematica command Eigensystems[{A, B}], we found the approximate eigenvalues and eigenvectors $\mathbf{R} \triangleq [\, R_0 \quad R_1 \quad R_2 \quad R_3 \quad R_4\,]^T$:

$$\begin{aligned} \lambda_1 &\approx 37.607284106899115, \quad \mathbf{R} \approx [0 \quad 0.690872837 \quad 0.643898300 \quad 0.328769984 \quad 0]^T \\ \lambda_2 &\approx 138.80108530222495, \quad \mathbf{R} \approx [0 \quad -0.621341452 \quad 0.560288352 \quad 0.547733296 \quad 0]^T \\ \lambda_3 &\approx 363.84163059087587, \quad \mathbf{R} \approx [0 \quad 0.128025550 \quad -0.531340229 \quad 0.837428815 \quad 0]^T \end{aligned}$$

and the corresponding numerical eigenfunctions (dashed) curves) in the figure.

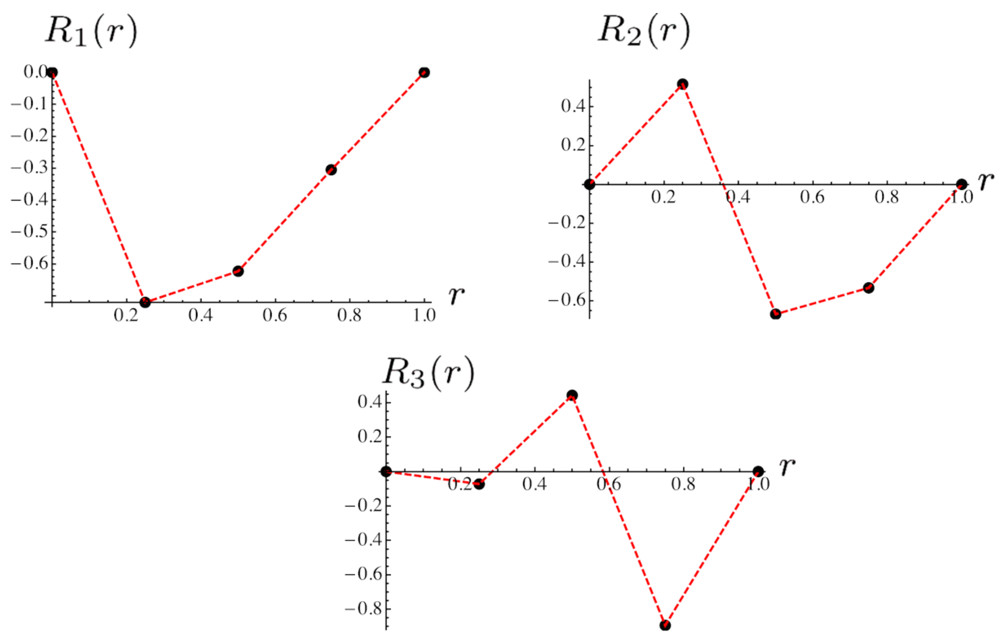


Figure 6: Answer for problem 8.8.4.9

Section 8.9.6

8.9.6.1. (a) For $N = 2$, $h = \frac{1}{2}$, and our approximate solution is

$$y_2(x) = \sum_{j=-1}^{3} z_j C_j(x),$$

where we will solve for the constants $z_{-1}, z_0, ..., z_4$. Using (8.94), the boundary conditions require two equations

$$(1) \quad 0 = y(0) = y_2(x_0) = \frac{1}{6}\left(z_{-1} + 4z_0 + z_1\right)$$

and

$$(2) \quad -1 = y(1) = y_2(x_2) = \frac{1}{6}\left(z_1 + 4z_2 + z_3\right).$$

Note that $x_j = j \cdot \frac{1}{2}$, for $j = 0, 1, 2$. The replacement equations $(3) - (5)$ for the ODE are, respectively, for $j = 0, 1, 2$,

$$(3) - (5) \quad 4\left(z_{j-1} - 2z_j + z_{j+1}\right) + 2\sqrt{2}\cos\left(\frac{2\pi j \cdot \frac{1}{2}}{3}\right) \cdot \frac{1}{6}\left(z_{j-1} + 4z_j + z_{j+1}\right) = 0.$$

We include replacement equations for the ODE at the endpoints $x = x_0$ and $x = x_2$ because the spline functions are twice continuously differentiable at the endpoints.

The system is

$$\begin{bmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ 4 + \frac{\sqrt{2}}{3} & -8 + \frac{4\sqrt{2}}{3} & 4 + \frac{\sqrt{2}}{3} & 0 & 0 \\ 0 & 4 + \frac{\sqrt{2}}{6} & -8 + \frac{4\sqrt{2}}{6} & 4 + \frac{\sqrt{2}}{6} & 0 \\ 0 & 0 & 4 - \frac{\sqrt{2}}{6} & -8 - \frac{4\sqrt{2}}{6} & 4 - \frac{\sqrt{2}}{6} \end{bmatrix} \begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) The approximate solution of the system is

$$\begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \approx \begin{bmatrix} 0.564829708 \\ 0 \\ -0.564829708 \\ -0.941074435 \\ -1.670872552 \end{bmatrix}$$

The approximate solution of the ODE-BVP is

$$y_2(x) \approx 0.564829708C_{-1}(x) - 0.564829708C_1(x) - 0.941074435C_2(x) - 1.670872552C_3(x).$$

The solid curve in the figure is an approximate solution produced by Mathematica. The thick dashed curve is the solution using cubic B-splines. The fact that the latter solution is twice continuously differentiable is apparent and agrees pretty well with Mathematica's approximate solution.

(c) There is no exact solution of this non-constant coefficients linear second order ODE. But we did use the Mathematica command *NDSolve* for the ODE-BVP to find a more accurate approximate solution and it agrees well with the coarse approximation we found.

8.9.6.3. (a) For $N = 4$, $h = \frac{1}{2}$, and our approximate solution is

$$y_4(x) = \sum_{j=-1}^{3} z_j C_j(x),$$

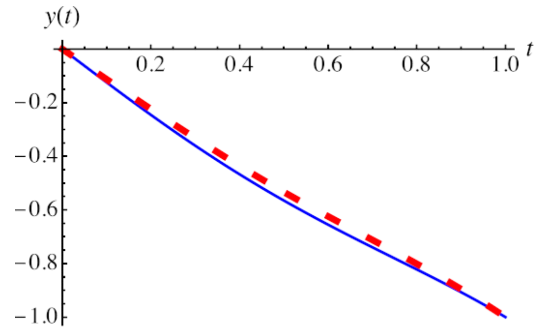


Figure 7: Answer for problem 8.9.6.1

where we will solve for the constants $z_{-1}, z_0, ..., z_4$. Using (8.94), the boundary conditions require two equations

$$(1) \quad -1 = y(0) = y_4(x_0) = \frac{1}{6}\left(z_{-1} + 4z_0 + z_1\right)$$

and

$$(2) \quad 3 = y(1) = y_2(x_4) = \frac{1}{6}\left(z_3 + 4z_4 + z_5\right).$$

The replacement equations (3) − (7) for the ODE are, respectively, for $j = 0, 1, ..., 4$,

$$(3) - (7) \quad 4\left(z_{j-1} - 2z_j + z_{j+1}\right) + \frac{1}{6}\left(z_{j-1} + 4z_j + z_{j+1}\right) = x_j.$$

We include replacement equations for the ODE at the endpoints $x = x_0$ and $x = x_4$ because the spline functions are twice continuously differentiable at the endpoints.

Note that $x_j = \dfrac{j}{2}$, for $j = 0, 1, ..., 4$. The system is

$$\begin{bmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ 4+\frac{1}{6} & -8+\frac{4}{6} & 4+\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 4+\frac{1}{6} & -8+\frac{4}{6} & 4+\frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 4+\frac{1}{6} & -8+\frac{4}{6} & 4+\frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 4+\frac{1}{6} & -8+\frac{4}{6} & 4+\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 4+\frac{1}{6} & -8+\frac{4}{6} & 4+\frac{1}{6} \end{bmatrix} \begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ \frac{1}{2} \\ 1 \\ \frac{3}{2} \\ 2 \end{bmatrix}.$$

(b) The approximate solution of the system is

$$\begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \approx \begin{bmatrix} -1.719908174 \\ -1.041666667 \\ 0.113425159 \\ 0.962038387 \\ 2.046612720 \\ 3 \\ 3.953387280 \end{bmatrix}$$

The approximate solution of the ODE-BVP is

$$y_4(x) \approx -1.719908174C_{-1}(x) - 1.041666667C_0(x) + 0.113425159C_1(x) + 0.962038387C_2(x)$$
$$+2.046612720C_3(x) + 3C_4(x) + 3.953387280C_5(x).$$

(c) The method of undetermined coefficients for the ODE gives $y_h = c_1 \cos x + c_2 \sin x$ and, after some calculations, $y_P = x$. Plug $y = y_h + y_P = x + c_1 \cos x + c_2 \sin x$ into the BCs to get

$$\left\{ \begin{array}{l} -1 = y(0) = c_1 \\ \quad 5 = y(2) = c_1 \cos 2 + c_2 \sin 2 + 2, \end{array} \right\}$$

The exact solution of ($\star$) is

$$y(t) = x - \cos x + \frac{3 + \cos 2}{\sin 2} \cdot \sin x.$$

The solid curve in the figure is the exact solution, as plotted by Mathematica. The thick dashed curve is the solution using cubic B-splines. The fact that the latter solution is twice continuously differentiable is apparent and agrees with well with the exact solution.

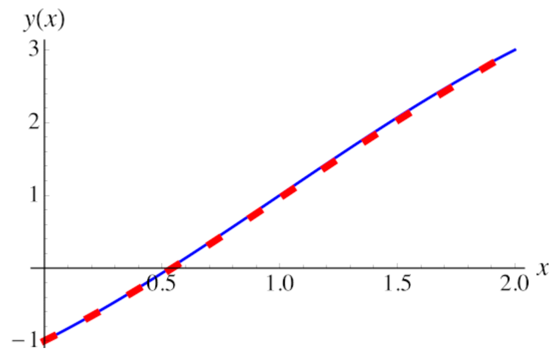


Figure 8: Answer for problem 8.9.6.3

8.9.6.5. For $N = 4$, $h = \frac{1}{4}$, and our approximate solution is

$$y_4(x) = \sum_{j=-1}^{3} z_j C_j(x),$$

where we will solve for the constants $z_{-1}, z_0, ..., z_4$. Using (8.94), the boundary conditions require two equations

$$(1) \quad 1 = y(0) - y'(0) = \frac{1}{6}\left(z_{-1} + 4z_0 + z_1\right) - (2h)^{-1}\left(-z_{-1} + z_1\right)$$

and

$$(2) \quad 0 = y(1) = y_4(x_4) = \frac{1}{6}\left(z_3 + 4z_4 + z_5\right).$$

The replacement equations (3) − (7) for the ODE are, respectively, for $j = 0, 1, ..., 4$,

$$(3) - (7) \quad 16\left(z_{j-1} - 2z_j + z_{j+1}\right) = -x_j.$$

We include replacement equations for the ODE at the endpoints $x = x_0$ and $x = x_4$ because the spline functions are twice continuously differentiable at the endpoints.

Note that $x_j = \dfrac{j}{4}$, for $j = 0, 1, ..., 4$. The system is

$$\begin{bmatrix} \frac{1}{6}+2 & \frac{4}{6} & \frac{1}{6}-2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ 16 & -32 & 16 & 0 & 0 & 0 & 0 \\ 0 & 16 & -32 & 16 & 0 & 0 & 0 \\ 0 & 0 & 16 & -32 & 16 & 0 & 0 \\ 0 & 0 & 0 & 16 & -32 & 16 & 0 \\ 0 & 0 & 0 & 0 & 16 & -32 & 16 \end{bmatrix} \begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{4} \\ -\frac{2}{4} \\ -\frac{3}{4} \\ -\frac{4}{4} \end{bmatrix}.$$

(b) The approximate solution is

$$\begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \approx \begin{bmatrix} 0.6875 \\ 0.583333333 \\ 0.479166667 \\ 0.359375 \\ 0.208333333 \\ 0.0104166667 \\ -0.25 \end{bmatrix}$$

The approximate solution of the ODE-BVP is

$$y_4(x) \approx 0.6875C_{-1}(x) + 0.583333333C_0(x) + 0.479166667C_1(x) + 0.359375C_2(x)$$
$$+0.208333333C_3(x) + 0.0104166667C_4(x) - 0.25C_5(x).$$

(c) Integrate the ODE twice to get general solution $y(x) = c_1 x + c_2 - \frac{1}{6}\,x^3$. Plug this and $y'(x) = c_2 - \frac{1}{2}\,x^2$ into the BCs to get

$$\left\{ \begin{array}{l} 1 = y(0) - y'(0) = c_2 - c_1 \\ 0 = y(1) = c_1 + c_2 - \frac{1}{6} \end{array} \right\}$$

The exact solution of ($\star$) is

$$y(x) = -\frac{5}{12}\,x + \frac{7}{12} - \frac{1}{6}\,x^3.$$

The solid curve in the figure is the exact solution, as plotted by Mathematica. The thick dashed curve is the solution using cubic B-splines. The fact that the latter solution is twice continuously differentiable is apparent and agrees pretty well with the exact solution.

8.9.6.7. For $k = 1$, we defined $N_{j,1}(x) \triangleq \left\{ \begin{array}{ll} 1, & x_j \le x \le x_{j+1} \\ 0, & \text{all other } x \end{array} \right\}$.

The recursive definition (8.103) is, for $k = 2$,

$$N_{j,2}(x) = \omega_{j,2}(x)N_{j,1}(x) + \big(1 - \omega_{j+1,2}(x)\big)N_{j+1,1}(x)$$

where $\omega_{j,2}(x) = \frac{x - x_j}{x_{j+1} - x_j}$. For $j = 0, ..., N-2$, using the definition of $N_{j,1}(x)$, $N_{j,2}(x)$ can be rewritten as

$$N_{j,2}(x) = \omega_{j,2}(x) \cdot \left\{ \begin{array}{ll} 1, & x_j \le x \le x_{j+1} \\ 0, & \text{all other } x \end{array} \right\} + \big(1 - \omega_{j+1,2}(x)\big) \cdot \left\{ \begin{array}{ll} 1, & x_{j+1} \le x \le x_{j+2} \\ 0, & \text{all other } x \end{array} \right\},$$

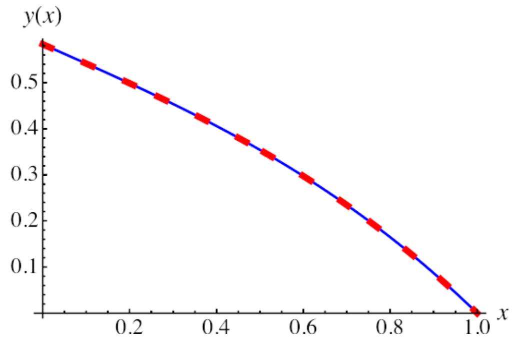


Figure 9: Answer for problem 8.9.6.5

Using the definition of $\omega_{j,2}(x)$, we have

$$N_{j,2}(x) = \frac{x-x_j}{x_{j+1}-x_j}\cdot\left\{\begin{array}{ll}1, & x_j \le x \le x_{j+1}\\ \\ 0, & \text{all other } x\end{array}\right\} + \left(1-\frac{x-x_{j+1}}{x_{j+1+1}-x_{j+1}}\right)\cdot\left\{\begin{array}{ll}1, & x_{j+1} \le x \le x_{j+2}\\ \\ 0, & \text{all other } x\end{array}\right\}$$

$$= \left\{\begin{array}{ll}\dfrac{x-x_j}{x_{j+1}-x_j}, & x_j \le x \le x_{j+1}\\ \\ 0, & \text{all other } x\end{array}\right\} + \left\{\begin{array}{ll}\left(1-\dfrac{x-x_{j+1}}{x_{j+2}-x_{j+1}}\right), & x_{j+1} \le x \le x_{j+2}\\ \\ 0, & \text{all other } x\end{array}\right\}$$

$$= \left\{\begin{array}{ll}\dfrac{x-x_j}{x_{j+1}-x_j}, & x_j \le x \le x_{j+1}\\ \\ \left(1-\dfrac{x-x_{j+1}}{x_{j+2}-x_{j+1}}\right), & x_{j+1} \le x \le x_{j+2}\\ \\ 0, & \text{all other } x\end{array}\right\} = \left\{\begin{array}{ll}\dfrac{x-x_j}{x_{j+1}-x_j}, & x_j \le x \le x_{j+1}\\ \\ \dfrac{x_{j+2}-x}{x_{j+2}-x_{j+1}}, & x_{j+1} \le x \le x_{j+2}\\ \\ 0, & \text{all other } x\end{array}\right\} = N_{j,2}(x),$$

as we wanted to show.

8.9.6.9. Choose any x in the interval $[x_0, x_N]$. Then there is an integer j with $x_{j-1} \le x \le x_j$ and $1 \le j \le N$. Note that for this value of j, (8.84) gives

$$L_N(x) = y_{j-1} + \frac{y_j - y_{j-1}}{x_j - x_{j-1}}\cdot(x - x_{j-1}).$$

Because $T_i(x) = 0$ for $i < j-1$ and for $i > j$, (8.101) gives

$$\begin{aligned} f_N(x) = y_{j-1}T_{j-1}(x) + y_jT_j(x) &= y_{j-1}\cdot\left(1-\frac{x-x_{j-1}}{x_j-x_{j-1}}\right) + y_j\cdot\frac{x-x_{j-1}}{x_j-x_{j-1}}\\ &= y_{j-1} - y_{j-1}\cdot\frac{x-x_{j-1}}{x_j-x_{j-1}} + y_j\cdot\frac{x-x_{j-1}}{x_j-x_{j-1}}\\ &= (y_j - y_{j-1})\cdot\frac{x-x_{j-1}}{x_j-x_{j-1}} = L_N(x).\end{aligned}$$

So, yes, (8.102) and (8.84) give the same piecewise, linear approximation of $f(x)$ for $x_0 \le x \le x_N$.

8.9.6.11. In the figure, (a) shows the curve and the control points; (b) shows that and the control polygon, too.

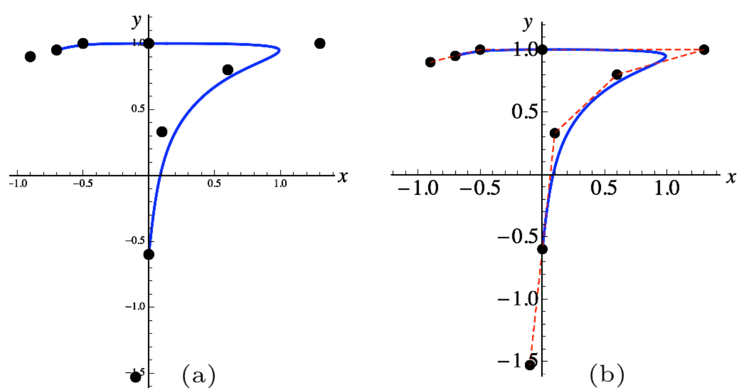


Figure 10: Answer for problem 8.9.6.11

Chapter Nine

Section 9.1

9.1.8.1. $L = 3$, $f(x)$ is an odd function, and

$$f(x) = \left\{ \begin{array}{cc} x, & 0 < x < 1 \\ \frac{3-x}{2}, & 1 < x < 3 \end{array} \right\},$$

so $a_n = 0$ for $n = 0, 1, 2, ...$ and

$$b_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\left(\int_0^1 x\,\sin\left(\frac{n\pi x}{3}\right)dx + \int_1^3 \frac{3-x}{2}\,\sin\left(\frac{n\pi x}{3}\right)dx\right)$$

$$= \frac{2}{3}\Big[\,\frac{x\,\cos\left(\frac{n\pi x}{3}\right)}{-n\pi/3} + \frac{\sin\left(\frac{n\pi x}{3}\right)}{(n\pi/3)^2}\,\Big]_0^1 + \frac{1}{3}\Big[\,\frac{(3-x)\cos\left(\frac{n\pi x}{3}\right)}{-n\pi/3} - \frac{\sin\left(\frac{n\pi x}{3}\right)}{(n\pi/3)^2}\,\Big]_1^3$$

$$= \frac{2}{3}\Big(\frac{\cancel{\cos\left(\frac{n\pi}{3}\right)} - 0}{-n\pi/3} + \frac{\sin\left(\frac{n\pi}{3}\right) - 0}{(n\pi/3)^2}\Big) + \frac{1}{3}\Big(\frac{0 - \cancel{2\cos\left(\frac{n\pi}{3}\right)}}{-n\pi/3} - \frac{0 - \sin\left(\frac{n\pi}{3}\right)}{(n\pi/3)^2}\Big) = \frac{\sin\left(\frac{n\pi}{3}\right)}{(n\pi/3)^2}.$$

So,

$$f(x) \doteq f_s(x) = \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\sin\left(\frac{n\pi x}{3}\right).$$

9.1.8.3. $f(x) = x$ is an odd function, so $a_n = 0$ for $n = 0, 1, 2, ...$ and

$$b_n = \frac{2}{L}\int_0^L x\,\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\Big[\,\frac{x\,\cos\left(\frac{n\pi x}{L}\right)}{-n\pi/L} + \frac{\sin\left(\frac{n\pi x}{L}\right)}{(n\pi/L)^2}\,\Big]_0^L = \frac{2}{L}\Big(\frac{L\,\cos n\pi - 0}{-n\pi/L} + \frac{0-0}{(n\pi/L)^2}\Big) = -\frac{2L}{n\pi}\,(-1)^n$$

$$= \frac{2(-1)^{n+1}L}{n\pi}.$$

So,

$$x \doteq f_s(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}L}{n\pi}\sin\left(\frac{n\pi x}{L}\right).$$

9.1.8.5. see figure

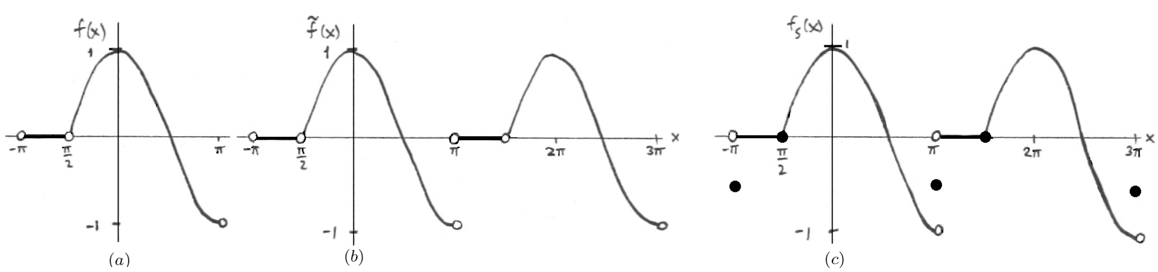


Figure 1: Problem 9.1.8.5

9.1.8.7. see figure

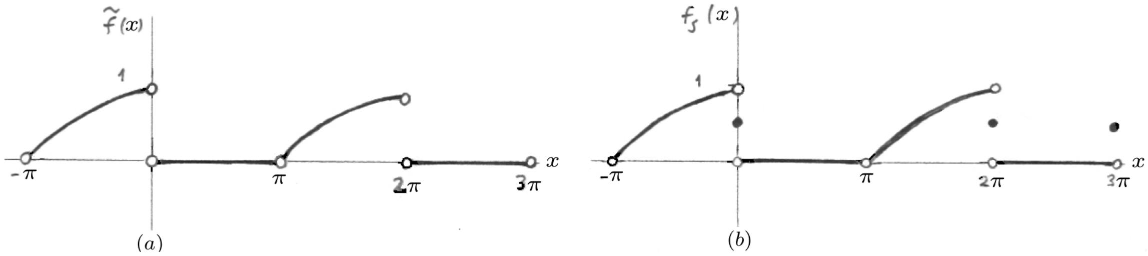


Figure 2: Problem 9.1.8.7

9.1.8.9. $f(x) = |x|$ is an even function, so for $n = 1, 2, 3, ...,$ $b_n = 0$, and

$$a_0 = \frac{2}{L}\int_0^L |x|\, dx = \frac{2}{L}\int_0^L x\, dx = \frac{2}{L}\Big[\,\frac{1}{2}x^2\,\Big]_0^L = L,$$

and

$$a_n = \frac{2}{L}\int_0^L x\,\cos\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\Big[\,\frac{x\,\sin\big(\frac{n\pi x}{L}\big)}{n\pi/L} + \frac{\cos\big(\frac{n\pi x}{L}\big)}{(n\pi/L)^2}\,\Big]_0^L = \frac{2}{L}\Big(\frac{0-0}{-n\pi/L} + \frac{\cos n\pi - 1}{(n\pi/L)^2}\Big) = \frac{-2L}{(n\pi)^2}\big(1-(-1)^n\big).$$

So,

$$|x| \doteq f_s(x) = \frac{L}{2} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\big(1-(-1)^n\big)\cos\Big(\frac{n\pi x}{L}\Big).$$

But $1-(-1)^n = 0$ if n is even. If n is odd then $n = 2k-1$ for some positive integer k and then $1-(-1)^n = 1-(-1)^{2k-1} = 2$. So,

$$|x| \doteq f_s(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\Big(\frac{(2k-1)\pi x}{L}\Big).$$

9.1.8.11. $L = \pi$, so $f(x) \doteq f_s(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(n\pi x) + b_n\sin(n\pi x)\right)$. We can rewrite

$$f(x) = 1 + \cos^2 x\cdot\sin(x - \frac{\pi}{3}),\ -\frac{\pi}{2} < x < \frac{\pi}{2} = 1 + \frac{1}{2}(1+\cos 2x)\cdot(\sin x\,\cos\frac{\pi}{3} - \cos x\,\sin\frac{\pi}{3})$$

$$= 1 + \frac{1}{2}(1+\cos 2x)\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right) = 1 + \frac{1}{4}\sin x - \frac{\sqrt{3}}{4}\cos x + \frac{1}{4}\sin x\cos 2x - \frac{\sqrt{3}}{4}\cos x\,\cos 2x$$

$$= 1 + \frac{1}{4}\sin x - \frac{\sqrt{3}}{4}\cos x + \frac{1}{4}\cdot\frac{1}{2}\big(\sin(x-2x) + \sin(x+2x)\big) - \frac{\sqrt{3}}{4}\cdot\frac{1}{2}\big(\cos(x-2x) + \cos(x+2x)\big)$$

$$= 1 + \frac{1}{4}\sin x - \frac{\sqrt{3}}{4}\cos x + \frac{1}{8}\big(\sin(-x) + \sin 3x\big) - \frac{\sqrt{3}}{8}\big(\cos(-x) + \cos 3x\big)$$

$$= 1 + \frac{1}{4}\sin x - \frac{\sqrt{3}}{4}\cos x - \frac{1}{8}\sin x + \frac{1}{8}\sin 3x - \frac{\sqrt{3}}{8}\cos x - \frac{\sqrt{3}}{8}\cos 3x$$

$$= 1 + \frac{1}{8}\sin x - \frac{3\sqrt{3}}{8}\cos x + \frac{1}{8}\sin 3x - \frac{\sqrt{3}}{8}\cos 3x.$$

9.1.8.13. $f(x) = x\,\cos x$ is an odd function, so $a_n = 0$ for $n = 0, 1, 2, ...$ and

$$b_n = \frac{2}{L}\int_0^\pi x\,\cos x\,\sin nx\,dx = \frac{2}{\pi}\int_0^\pi x\cdot\frac{1}{2}\big(\sin\big((n-1)x\big) + \sin(n+1)x\big)dx$$

$$= \frac{1}{\pi}\int_0^\pi \Big(x\,\sin\big((n-1)x\big) + x\,\sin\big((n+1)x\big)\Big)dx.$$

For $n = 1$, we calculate

$$b_1 = \frac{1}{\pi}\int_0^\pi x\,\sin 2x\,dx = \frac{1}{\pi}\Big[-\frac{1}{2}\,x\cos 2x + \frac{1}{4}\sin 2x\Big]_0^\pi. = \frac{1}{\pi}\Big(-\frac{1}{2}(\pi-0) + \frac{1}{4}(0-0)\Big) = -\frac{1}{2}.$$

For $n \neq 1$, we calculate

$$b_n = \frac{1}{\pi}\int_0^\pi \Big(x\,\sin\big((n-1)x\big) + x\,\sin\big((n+1)x\big)\Big)dx$$

$$= \frac{1}{\pi}\Big[\frac{x\cos\big((n-1)x\big)}{-(n-1)} + \frac{\sin\big((n-1)x\big)}{(n-1)^2}\Big]_0^\pi + \frac{1}{\pi}\Big[\frac{x\cos\big((n+1)x\big)}{-(n+1)} + \frac{\sin\big((n+1)x\big)}{(n+1)^2}\Big]_0^\pi$$

$$= \frac{1}{\pi}\Big(\frac{\pi(-1)^{n-1} - 0}{-(n-1)} + \frac{0-0}{(n-1)^2} + \frac{\pi(-1)^{n+1} - 0}{-(n+1)} + \frac{0-0}{(n+1)^2}\Big).$$

Because $1 = (-1)^n\cdot(-1)^n$, it follows that $(-1)^{-n} = (-1)^n$. So, if $n \neq 1$,

$$b_n = \frac{1}{\pi}\Big(\frac{-\pi(-1)^n}{-(n-1)} + \frac{-\pi(-1)^n}{-(n+1)}\Big) = (-1)^n\Big(\frac{1}{n-1} + \frac{1}{n+1}\Big) = \frac{2n}{n^2-1}\,(-1)^n.$$

So,

$$x\,\cos x \doteq f_s(x) = -\frac{1}{2}\,\sin x + \sum_{n=2}^\infty \frac{2n}{n^2-1}(-1)^n\sin nx.$$

9.1.8.15. (a) $f(x)$ is an odd function and $L = \pi$, so o $a_n = 0$ for $n = 0, 1, 2, ...$ and

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\,\sin nx\,dx.$$

Using

$$f(x) = \left\{\begin{array}{ll} x, & 0 \leq x \leq \frac{\pi}{2} \\ \\ \frac{\pi}{2}, & \frac{\pi}{2} \leq x \leq \pi \end{array}\right\},$$

we calculate

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x\,\sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^\pi \frac{\pi}{2}\,\sin nx\,dx = \frac{2}{\pi}\,\Big[\frac{x\cos nx}{-n} + \frac{\sin nx}{n^2}\Big]_0^{\pi/2} + \Big[\frac{\cos nx}{-n}\Big]_{\pi/2}^\pi$$

$$= \frac{2}{\pi}\Big(-\frac{\pi}{2n}\cos\frac{n\pi}{2} + 0 + \frac{1}{n^2}\sin\frac{n\pi}{2} - 0\Big) - \frac{1}{n}\Big((-1)^n - \cos\frac{n\pi}{2}\Big) = -\cancel{\frac{1}{n}\cos\frac{n\pi}{2}} + \frac{2}{n^2\pi}\,\sin\frac{n\pi}{2} - \frac{1}{n}\,(-1)^n + \cancel{\frac{1}{n}\cos\frac{n\pi}{2}}$$

$$= \frac{2}{n^2\pi}\,\sin\frac{n\pi}{2} + \frac{1}{n}\,(-1)^{n+1}.$$

So,

$$f(x) \doteq f_s(x) = \sum_{n=1}^\infty\Big(\frac{(-1)^{n+1}}{n} + \frac{2}{n^2\pi}\sin\Big(\frac{n\pi}{2}\Big)\Big)\sin nx$$

(b) see figure

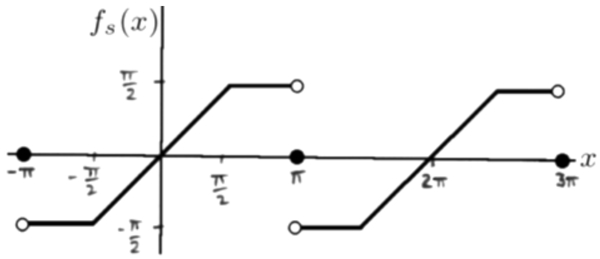


Figure 3: Problem 9.1.8.15(b)

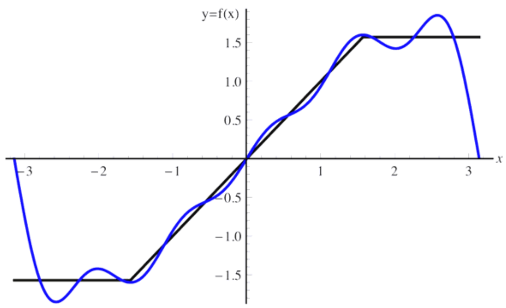


Figure 4: Problem 9.1.8.15(d)

(c) $g(x) = \left(1 + \dfrac{2}{\pi}\right)\sin x - \dfrac{1}{2}\sin 2x + \left(\dfrac{1}{3} - \dfrac{2}{9\pi}\right)\sin 3x\ - \dfrac{1}{4}\sin 4x + \left(\dfrac{1}{5} + \dfrac{2}{25\pi}\right)\sin 5x.$

(d) see figure: $g(x)$ is the oscillating function; $f(x)$ is the original, sigmoidal function.

9.1.8.17. (a) $f(x)$ is neither odd nor even, and $L = \pi$. We calculate

$$a_0 = \frac{1}{L}\int_{-L}^{L} e^{-x}\,dx = \frac{1}{L}\Big[-e^{-x}\Big]_{-L}^{L} = \frac{e^L - e^{-L}}{L}, a_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\cdot\frac{1}{1+\left(\frac{n\pi}{L}\right)^2}\Big[e^{-x}\cdot\left(\frac{n\pi}{L}\sin\frac{n\pi x}{L} - \cos\frac{n\pi x}{L}\right)\Big]_{-L}^{L} = \frac{1}{L}\cdot\frac{1}{1+\left(\frac{n\pi}{L}\right)^2}\left(0 - e^{-L}(-1)^n - 0 + e^{L}(-1)^n\right)$$

$$= \frac{e^L - e^{-L}}{L}\cdot\frac{1}{1+\left(\frac{n\pi}{L}\right)^2}(-1)^n$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\cdot\frac{1}{1+\left(\frac{n\pi}{L}\right)^2}\Big[e^{-x}\cdot\left(-\frac{n\pi}{L}\cos\frac{n\pi x}{L} - \sin\frac{n\pi x}{L}\right)\Big]_{-L}^{L}$$

$$= \frac{1}{L}\cdot\frac{1}{1+\left(\frac{n\pi}{L}\right)^2}\left(-e^{-L}\cdot\frac{n\pi}{L}(-1)^n - 0 + e^{L}\cdot\frac{n\pi}{L}(-1)^n + 0\right) = \frac{e^L - e^{-L}}{L}\cdot\frac{1}{1+\left(\frac{n\pi}{L}\right)^2}\cdot\frac{n\pi}{L}\cdot(-1)^n.$$

So,

$$f(x) \doteq f_s(x) = \frac{e^L - e^{-L}}{L}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+\left(\frac{n\pi}{L}\right)^2}\left(\cos\left(\frac{n\pi x}{L}\right) + \frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\right)\right)$$

(b) see figure

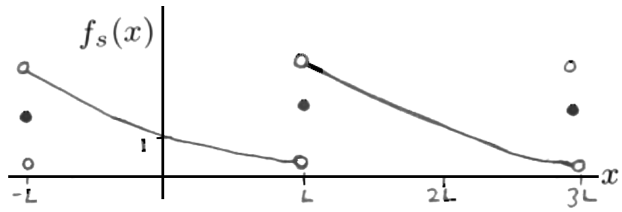


Figure 5: Problem 9.1.8.17(b)

9.1.8.19. (a) $f(x)$ is neither even nor odd, and $L = \pi$. We calculate

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} \frac{x}{2}\,dx\right) = \frac{1}{\pi}\left(\left[\frac{1}{2}x^2\right]_{-\pi}^{0} + \left[\frac{1}{4}x^2\right]_{0}^{\pi}\right) = ... = -\frac{\pi}{4},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\,\cos nx\,dx + \int_{0}^{\pi} \frac{x}{2}\,\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{x\,\cos nx}{-n\pi} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{2}\left[\frac{x\,\cos nx}{-n\pi} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}\right) = \frac{1}{\pi}\Big(0 - \frac{(-1)^n}{n} + 0 - 0 - \frac{(-1)^n}{2n} - 0 + 0 - 0\Big)$$

$$= -\frac{3}{2n\pi}\,(-1)^n,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\,\sin nx\,dx + \int_{0}^{\pi} \frac{x}{2}\,\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{x\,\sin nx}{n\pi} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{2}\left[\frac{x\,\sin nx}{n\pi} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right) = \frac{1}{\pi}\Big(0 - 0 + \frac{1-(-1)^n}{n^2} + 0 - 0 + \frac{(-1)^n - 1}{2n^2}\Big)$$

$$= \frac{1}{2n^2\pi}\Big(1 - (-1)^n\Big).$$

But $1 - (-1)^n = 0$ if n is even. If n is odd then $n = 2k - 1$ for some positive integer k and then $1 - (-1)^n = 1 - (-1)^{2k-1} = 2$. So,

$$f(x) \doteq f_s(x) = -\frac{\pi}{8} - \frac{3}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos nx + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2k-1)^2}\sin\big((2k-1)x\big)$$

(b) see the figure.

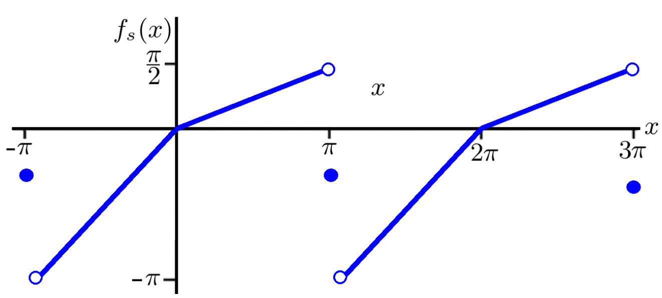


Figure 6: Problem 9.1.8.19(a)

9.1.8.21. (a) $f(x)$ is neither even nor odd, and $L = 2$. Using

$$f(x) = \begin{Bmatrix} x, & -2 < x < 1 \\ \\ 2-x, & 1 < x < 2 \end{Bmatrix},$$

we calculate

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_{-2}^{1} x\,dx + \int_{1}^{2}(2-x)dx\right) = \frac{1}{2}\left(\Big[\frac{1}{2}x^2\Big]_{-2}^{1} + \Big[2x - \frac{1}{2}x^2\Big]_{1}^{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - 2 + (4-2) - \frac{1}{2}(4-1)\right) = -\frac{1}{2},$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\,\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\left(\int_{-2}^{1} x\;\cos\frac{n\pi x}{2}\,dx + \int_{1}^{2}(2-x)\cos\frac{n\pi x}{2}\,dx\right)$$

$$= \frac{1}{2}\Big[\frac{x\,\sin\frac{n\pi x}{2}}{n\pi/2} + \frac{\cos\frac{n\pi x}{2}}{(n\pi/2)^2}\Big]_{-2}^{1} + \frac{1}{2}\Big[\frac{(2-x)\,\sin\frac{n\pi x}{2}}{n\pi/2} - \frac{\cos\frac{n\pi x}{2}}{(n\pi/2)^2}\Big]_{1}^{2}$$

$$= \frac{1}{2}\Big(\cancel{\frac{\sin\frac{n\pi}{2} - 0}{n\pi/2}} + \frac{\cos\frac{n\pi}{2} - (-1)^n}{(n\pi/2)^2} + \cancel{\frac{0 - \sin\frac{n\pi}{2}}{n\pi/2}} - \frac{(-1)^n - \cos\frac{n\pi}{2}}{(n\pi/2)^2}\Big) = \frac{4}{\pi^2}\cdot\frac{1}{n^2}\left((-1)^{n+1} + \cos\left(\frac{n\pi}{2}\right)\right),$$

and

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\,\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left(\int_{-2}^{1} x\;\sin\frac{n\pi x}{2}\,dx + \int_{1}^{2}(2-x)\sin\frac{n\pi x}{2}\,dx\right)$$

$$= \frac{1}{2}\Big[\frac{x\,\cos\frac{n\pi x}{2}}{-n\pi/2} + \frac{\sin\frac{n\pi x}{2}}{(n\pi/2)^2}\Big]_{-2}^{1} + \frac{1}{2}\Big[\frac{(2-x)\,\cos\frac{n\pi x}{2}}{-n\pi/2} - \frac{\sin\frac{n\pi x}{2}}{(n\pi/2)^2}\Big]_{1}^{2}$$

$$= \frac{1}{2}\Big(\frac{\cancel{\cos\frac{n\pi}{2}} + 2(-1)^n}{-n\pi/2} + \frac{\sin\frac{n\pi}{2} - 0}{(n\pi/2)^2} + \frac{0 - \cancel{\cos\frac{n\pi}{2}}}{-n\pi/2} - \frac{0 - \sin\frac{n\pi}{2}}{(n\pi/2)^2}\Big) = \left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

So,

$$f(x) \sim f_s(x) = -\frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left((-1)^{n+1} + \cos\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{2}\right)$$

$$+\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi x}{2}\right).$$

(b) see the figure.

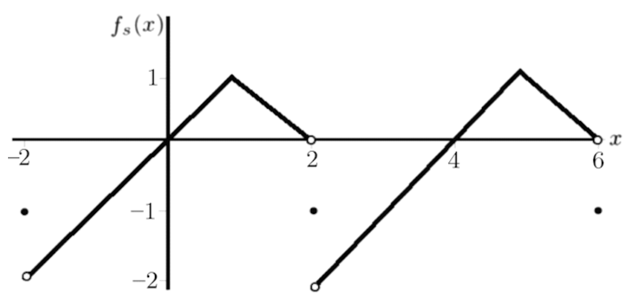


Figure 7: Problem 9.1.8.21(b)

9.1.8.23. First, rewrite

$$b_n = \frac{1}{L}\left(\int_{-L}^{0} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

Next, In the first integral make the substitution $x = -y$ to get

$$\int_{-L}^{0} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{L}^{0} f(-y)\sin\left(\frac{n\pi(-y)}{L}\right)(-dy) = \int_{0}^{L} f(-y)\sin\left(-\frac{n\pi y}{L}\right)dy.$$

After that, use the fact that $\sin\theta$ is odd in θ and the assumption that $f(-y) = f(y)$ to conclude that

$$\int_{-L}^{0} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{0}^{L} f(-y)\sin\left(-\frac{n\pi y}{L}\right)dy = \int_{0}^{L} f(y)\Big(-\sin\left(\frac{n\pi y}{L}\right)\Big)dy = -\int_{0}^{L} f(y)\sin\left(\frac{n\pi y}{L}\right)dy.$$

To summarize,

$$\begin{aligned} b_n &= \frac{1}{L}\left(\int_{-L}^{0} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right) \\ &= \frac{1}{L}\left(-\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right) = 0, \end{aligned}$$

as we were asked to explain.

Section 9.2

9.2.4.1. (a) $L = \pi$ so the Fourier sine series has the form $f(x) \doteq f_{sin}(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{2} - \cos 2x\right)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\Big(\frac{1}{2}\sin nx - \cos 2x\,\sin nx\Big)dx \\ &= \frac{1}{\pi}\int_0^{\pi}\Big(\sin nx - \sin\big((n-2)x\big) - \sin\big((n+2)x\big)\Big)dx. \end{aligned}$$

If $n = 1$, we calculate $b_1 = \frac{1}{\pi}\int_0^{\pi}\Big(\sin 2x - \sin 4x\Big)dx = \frac{1}{\pi}\Big[\frac{\cos 2x}{-2} - \frac{\cos 4x}{-4}\Big]_0^{\pi} = 0.$

If $n \neq 1$, we calculate

$$b_n = \frac{1}{\pi}\Big[\frac{\cos nx}{-n} - \frac{\cos\big((n-2)x\big)}{-(n-2)} - \frac{\cos\big((n+2)x\big)}{-(n+2)}\Big]_0^{\pi} = \frac{1}{\pi}\Big(\frac{(-1)^n - 1}{-n} - \frac{(-1)^{(n-2)} - 1}{-(n-2)} - \frac{(-1)^{(n+2)} - 1}{-(n+2)}\Big)$$

$$= \frac{1}{\pi}\Big(\frac{(-1)^n - 1}{-n} - \frac{(-1)^n - 1}{-(n-2)} - \frac{(-1)^n - 1}{-(n+2)}\Big) = \frac{1 - (-1)^n}{\pi}\cdot\Big(\frac{1}{n} - \frac{1}{n-2} - \frac{1}{n+2}\Big)$$

$$= \frac{1 - (-1)^n}{\pi}\cdot\frac{(n-2)(n+2) - n(n+2) - n(n-2)}{n(n-2)(n+2)} = \frac{1 - (-1)^n}{\pi}\cdot\frac{-n^2 - 4}{n(n-2)(n+2)} = -\frac{1 - (-1)^n}{\pi}\cdot\frac{4 + n^2}{n(-4 + n^2)}.$$

But $1 - (-1)^n = 0$ if n is even. If n is odd then $n = 2k - 1$ for some positive integer k, and then $1 - (-1)^n = 1 - (-1)^{2k-1} = 2$. So, the Fourier sine series is

$$f_{sin}(x) = -\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{4 + (2k-1)^2}{(2k-1)(-4 + (2k-1)^2)}\sin\big((2k-1)x\big)$$

(b) Because $L = \pi$, $f_{cos}(x) = \frac{1}{2} - \cos 2x$ is its own Fourier cosine series,

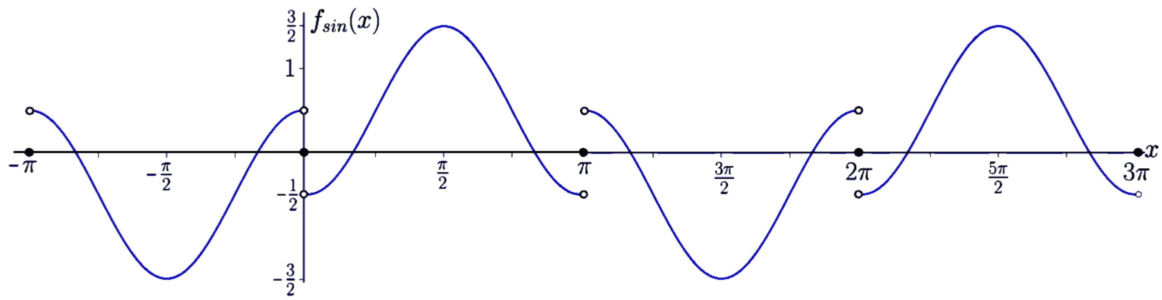


Figure 8: Problem 9.2.4.1

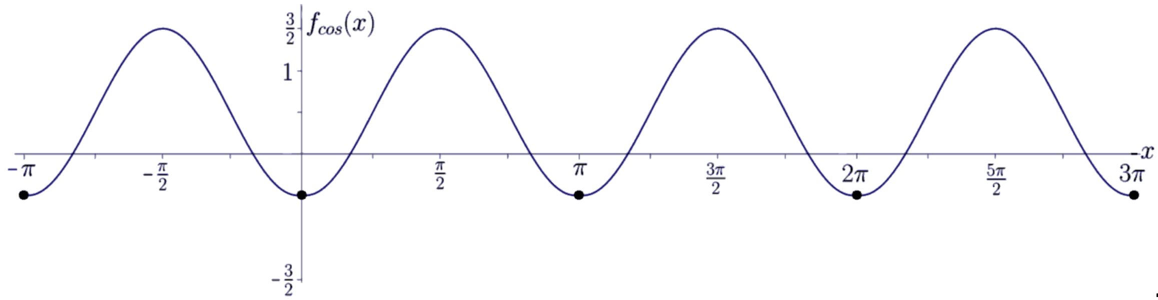


Figure 9: Problem 9.2.4.1

(c) see the figures

9.2.4.3. (a) $L = 3$. Using

$$f(x) = \left\{ \begin{array}{rl} 2, & 0 < x < 1 \\ \\ -1, & 1 < x < 3 \end{array} \right\},$$

we calculate

$$b_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\left(\int_0^1 f(x)\sin\left(\frac{n\pi x}{3}\right)dx + \int_1^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx\right)$$

$$= \frac{2}{3}\left(\int_0^1 2\sin\left(\frac{n\pi x}{3}\right)dx + \int_1^3 (-1)\sin\left(\frac{n\pi x}{3}\right)dx\right) = \frac{2}{3}\left(\left[\frac{2\cos\left(\frac{n\pi x}{3}\right)}{-n\pi/3}\right]_0^1 - \left[\frac{\cos\left(\frac{n\pi x}{3}\right)}{-n\pi/3}\right]_1^3\right)$$

$$= \frac{2}{3}\cdot\frac{3}{n\pi}\Big(-2\cos\left(\frac{n\pi}{3}\right)+2+(-1)^n-\cos\left(\frac{n\pi}{3}\right)\Big) = \frac{2}{\pi}\cdot\frac{1}{n}\Big(2+(-1)^n-3\cos\frac{n\pi}{3}\Big).$$

So, the Fourier sine series representation is

$$f(x) \doteq f_{sin}(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Big(2+(-1)^n-3\cos\frac{n\pi}{3}\Big)\sin\frac{n\pi x}{3}$$

(b) We calculate

$$a_0 = \frac{2}{3}\int_0^3 f(x)dx = \frac{2}{3}\left(\int_0^1 f(x)dx + \int_1^3 f(x)dx\right) = \frac{2}{3}\left(\int_0^1 2dx + \int_1^3(-1)dx\right) = \frac{2}{3}\left(2+(-2)\right) = 0,$$

and

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\left(\int_0^1 f(x)\cos\left(\frac{n\pi x}{3}\right)dx + \int_1^3 f(x)\cos\left(\frac{n\pi x}{3}\right)dx\right)$$

$$= \frac{2}{3}\left(\int_0^1 2\cos\left(\frac{n\pi x}{3}\right)dx + \int_1^3 (-1)\cos\left(\frac{n\pi x}{3}\right)dx\right) = \frac{2}{3}\left(\Big[\frac{2\sin\left(\frac{n\pi x}{3}\right)}{n\pi/3}\Big]_0^1 - \Big[\frac{\sin\left(\frac{n\pi x}{3}\right)}{n\pi/3}\Big]_1^3\right)$$

$$= \frac{2}{3}\cdot\frac{3}{n\pi}\Big(2\sin\left(\frac{n\pi}{3}\right) - 0 - 0 + \sin\left(\frac{n\pi}{3}\right)\Big) = \frac{6}{n\pi}\sin\left(\frac{n\pi}{3}\right).$$

So, the Fourier cosine series representation is

$$f(x) \doteq f_s(x) = \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi x}{3}\right).$$

(c) See figures

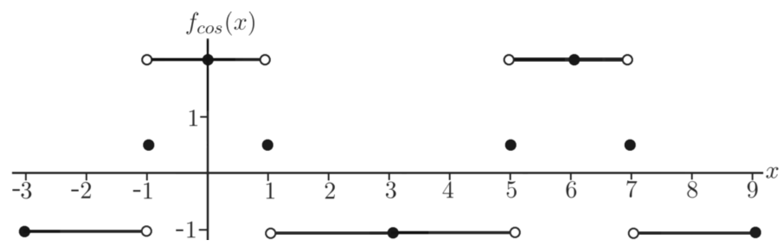


Figure 10: Problem 9.2.4.3(c)

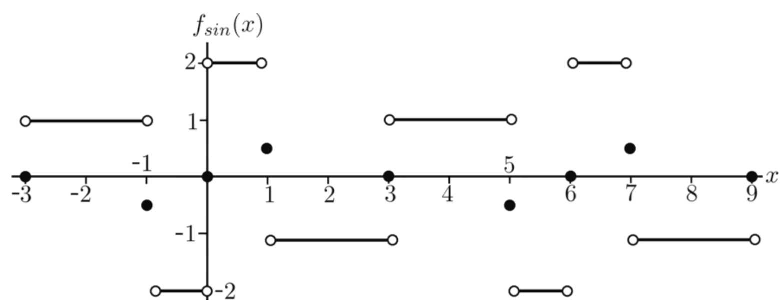


Figure 11: Problem 9.2.4.3(c)

9.2.4.5. We calculate

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\int_0^{L/2} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{2}{L}\left(\int_0^{L/2} 2x\,\sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{2}{L}\Big(\Big[\frac{2x\,\cos\frac{n\pi x}{L}}{-n\pi/L} + \frac{2\,\sin\frac{n\pi x}{L}}{(n\pi/L)^2}\Big]_0^{L/2} + \Big[\frac{(L-x)\cos\frac{n\pi x}{L}}{-n\pi/L} - \frac{\sin\frac{n\pi x}{L}}{(n\pi/L)^2}\Big]_{L/2}^{L}\Big)$$

$$= \frac{2}{L}\Big(\frac{L\cos\frac{n\pi}{2} - 0}{-n\pi/L} + \frac{2\sin\frac{n\pi}{2} - 0}{(n\pi/L)^2} + \frac{0 - \frac{L}{2}\,\cos\frac{n\pi}{2}}{-n\pi/L} - \frac{0 - \sin\frac{n\pi}{2}}{(n\pi/L)^2}\Big)$$

$$= -\frac{L}{n\pi}\cos\frac{n\pi}{2} + \frac{6L}{n^2\pi^2}\sin\frac{n\pi}{2}. = \frac{L}{n^2\pi^2}\left(-n\pi\cos\frac{n\pi}{2} + 6\sin\frac{n\pi}{2}\right).$$

So, the Fourier sine series is

$$f_{sin}(x) = \frac{L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(-n\pi\cos\frac{n\pi}{2} + 6\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{L},$$

and see the figure.

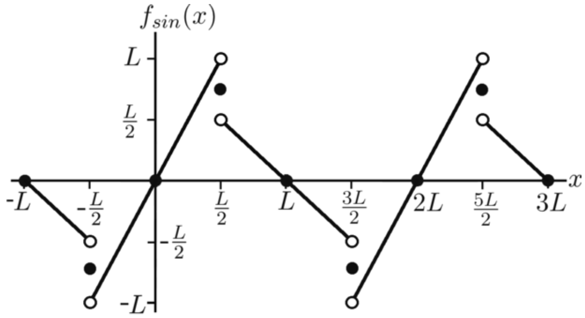


Figure 12: Problem 9.2.4.5

9.2.4.7. $L = \pi$, so $a_0 = \dfrac{2}{L}\displaystyle\int_0^{\pi}\sin x\,dx = \dfrac{2}{\pi}\Big[-\cos x\Big]_0^{\pi} = \dfrac{2}{\pi}\big(1-(-1)\big) = \dfrac{4}{\pi}$, and

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\,\cos nx\,dx = \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}\big(\sin(x-nx)+\sin(x+nx)\big)dx.$$

If $n = 1$, we calculate that

$$a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\,\cos x\,dx = \frac{1}{\pi}\Big[\sin^2 x\Big]_0^{\pi} = 0.$$

If $n \neq 1$, then

$$a_n = \frac{1}{\pi}\cdot\Big[\frac{\cos\big((1-n)x\big)}{-(1-n)} + \frac{\cos\big((1+n)x\big)}{-(1+n)}\Big]_0^{\pi} = \frac{1}{\pi}\cdot\Big(\frac{\cos\big((1-n)\pi\big)-1}{-(1-n)} + \frac{\cos\big((1+n)\pi\big)-1}{-(1+n)}\Big)$$

$$= \frac{1}{\pi}\cdot\Big(\frac{(-1)^{1-n}-1}{-(1-n)} + \frac{(-1)^{1+n}-1}{-(1+n)}\Big).$$

Because $1 = (-1)^n\cdot(-1)^n$, it follows that $(-1)^{-n} = (-1)^n$. So, if $n \neq 1$,

$$a_n = \frac{1}{\pi}\cdot\Big(\frac{-(-1)^{-n}-1}{-(1-n)} + \frac{-(-1)^{n}-1}{-(1+n)}\Big) = \frac{1}{\pi}\Big(\frac{1}{1-n}+\frac{1}{1+n}\Big)\big(1+(-1)^n\big) = \frac{1}{\pi}\,\frac{2}{1-n^2}\big(1+(-1)^n\big).$$

So, the Fourier cosine series representation is

$$f(x) \doteq f_{cos}(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{1-n^2}\big(1+(-1)^n\big)\cos nx.$$

But $1+(-1)^n = 0$ if n is odd. If n is even then $n = 2k$ for some positive integer k and then $1+(-1)^n = 1+(-1)^{2k} = 2$. So, the Fourier cosine series is given by

$$f(x) \doteq f_{cos}(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos 2kx,$$

and see the figure.

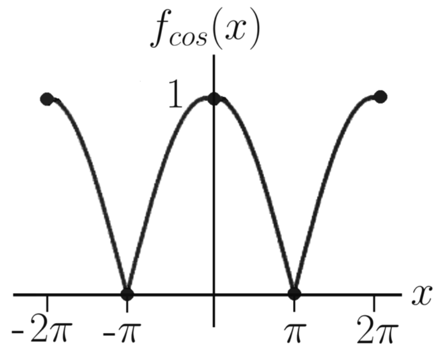


Figure 13: Problem 9.2.4.7

9.2.4.9. $L = \frac{1}{2}$. We calculate

$$a_0 = \frac{2}{1/2}\int_0^\pi f(x)dx = 4\int_0^\pi (1-2x)dx = 4\Big[x - x^2\Big]_0^\pi = 4\Big(\pi - \pi^2 - 0 + 0\Big) = 4\pi(1-\pi),$$

and

$$a_n = \frac{2}{1/2}\int_0^{1/2} f(x)\cos 2n\pi x\,dx = 4\int_0^{1/2}(1-2x)\cos 2n\pi x\,dx = 4\Big[\frac{(1-2x)\sin 2n\pi x}{2n\pi} + \frac{-2\cos 2n\pi x}{4n^2\pi^2}\Big]_0^{1/2}$$

$$= 4\Big(0 - 0 - \frac{2\big((-1)^n - 1\big)}{4n^2\pi^2}\Big) = \frac{2}{n^2\pi^2}\Big(1-(-1)^n\Big).$$

But $1-(-1)^n = 0$ if n is even. If n is odd then $n = 2k-1$ for some positive integer k and then $1-(-1)^n = 1-(-1)^{2k-1} = 2$. So, the Fourier cosine series is

$$f(x) \doteq f_{cos}(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\big(2(2k-1)\pi x\big),$$

and see the figure.

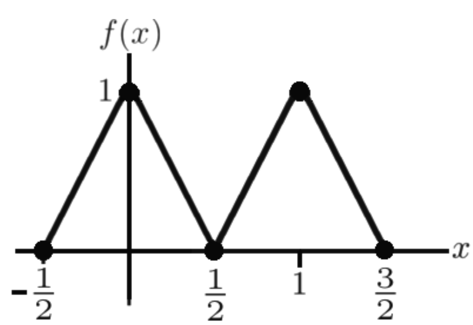


Figure 14: Problem 9.2.4.9

9.2.4.11. Recall from Example 9.5 in Section 9.1 that, with $L = \pi$,

$$f(t) \doteq f_s(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\big((2k-1)t\big).$$

A convergence theorem, specifically Theorem 9.1, says that $f(t) = f_s(t)$ except at t being an integer multiple of π.

Our ODE can be rewritten as

$$\dot{y} + 3y = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\big((2k-1)t\big)$$

Look for a particular solution in the form

$$y_p(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\,y_{k,p}(t)$$

where $y_{k,p}(t)$ satisfies

$$\dot{y}_{k,p} + 3y_{k,p} = \sin\big((2k-1)t\big),\ \ k = 1, 2, ... \tag{1}$$

The method of undetermined coefficients for (1) gives solution form

$$y_{k,p}(t) = A\cos\big((2k-1)t\big) + B\sin\big((2k-1)t\big),$$

where A, B are constants to be determined. Plug this into the ODE and sort terms to get

$$\begin{aligned}\underline{0}\cos\big((2k-1)t\big) + \underline{1}\sin\big((2k-1)t\big) &= \dot{y}_{k,p} + 3y_{k,p}\\ &= (3A + (2k-1)B)\cos\big((2k-1)t\big) + (-(2k-1)A + 3B)\sin\big((2k-1)t\big).\end{aligned}$$

So, we need

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 3 & 2k-1 \\ -(2k-1) & 3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{9+(2k-1)^2}\begin{bmatrix} 3 & -(2k-1) \\ 2k-1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{9+(2k-1)^2}\begin{bmatrix} -(2k-1) \\ 3 \end{bmatrix}.$$

The solutions, for $k = 1, 2, ...$, are

$$y_{k,p}(t) = \frac{1}{9+(2k-1)^2}\left(-(2k-1)\cos\big((2k-1)t\big) + 3\sin\big((2k-1)t\big)\right).$$

Thus

$$y_p(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\,y_{k,p}(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)(9+(2k-1)^2)}\cdot\left(-(2k-1)\cos\big((2k-1)t\big) + 3\sin\big((2k-1)t\big)\right).$$

The general solution of the ODE is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-3t} + y_p(t).$$

The initial condition yields

$$0 = y(0) = c_1 + y_p(0) = c_1 + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)(9+(2k-1)^2)}\left(-(2k-1)\right),$$

hence

$$c_1 = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)(9+(2k-1)^2)}\cdot(2k-1).$$

By adding two infinite series together, the solution of the ODE is

$$y(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)(9+(2k-1)^2)}\left((2k-1)e^{-3t} - (2k-1)\cos\big((2k-1)t\big) + 3\sin\big((2k-1)t\big)\right).$$

The figures show graphs of the approximation of the solution $y(t)$ by a finite sum

$$y_N(t) \triangleq \frac{4}{\pi}\sum_{k=1}^{N} \frac{1}{(2k-1)(4+(2k-1)^2)}\left((2k-1)e^{-3t}-(2k-1)\cos\left((2k-1)t\right)+3\sin\left((2k-1)t\right)\right)$$

for $N = 2$ and $N = 8$. In each picture the approximation is drawn as a dashed curve and the exact solution (see Example 4.34 in Section 4.5) is drawn as a solid curve. In (a) the approximation uses $N = 2$ and in (b) the approximation uses $N = 8$.

The exact solution was graphed using the Mathematica commands

$$\mathsf{f[t\_] := Evaluate[}\ \frac{\mathsf{t}}{\mathsf{Abs[t]}}\,(-1)^{\left(\mathsf{IntegerPart[}\frac{\mathsf{t}}{\pi}\mathsf{]}\right)}\ \mathsf{]},$$

and

$$\mathsf{a{=}\ Plot}\Big[\int_0^{\mathsf{t}} \mathsf{e}^{-3(\mathsf{t-u})}\mathsf{f[t-u]du}, \{\mathsf{t}, 0, 6\pi\}, \mathsf{AxesLabel} \rightarrow \{\mathsf{t}, \text{"}\mathsf{y(t)}\text{"}, \mathsf{Text[FontSize} \rightarrow \mathsf{Large]}\},$$

$$\mathsf{AxesStyle} \rightarrow \mathsf{Thickness[0.00315]}, \mathsf{PlotPoints} \rightarrow 100, \mathsf{PlotStyle} \rightarrow \{\mathsf{Black}, \mathsf{Thickness[0.005]}\}, \mathsf{Exclusions} \rightarrow \{0, 6\pi\}\Big]$$

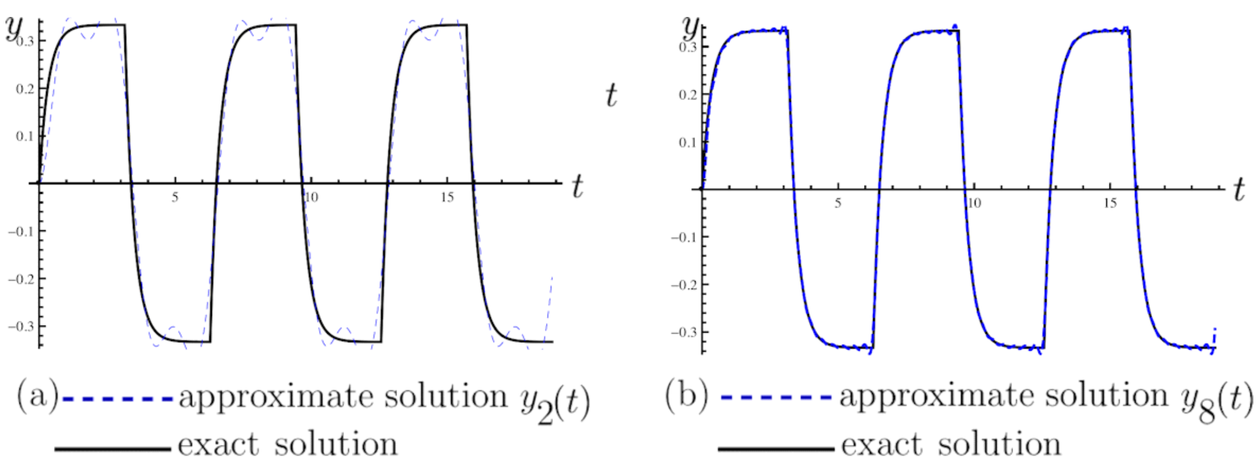


Figure 15: Problem 9.2.4.11

9.2.4.13. Recall from Example 9.5 that, with $L = \pi$,

$$f(t) \doteq f_s(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\left((2k-1)t\right).$$

A convergence theorem, specifically Theorem 9.1, says that $f(t) = f_s(t)$ except at t being an integer multiple of π.

Our ODE can be rewritten as

$$\dot{y} + 3y = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\left((2k-1)t\right)$$

Look for a particular solution in the form

$$y_p(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\,y_{k,p}(t)$$

where $y_{k,p}(t)$ satisfies

$$\ddot{y}_{k,p} + 2y_{k,p} = \sin\big((2k-1)t\big),\ k = 1, 2, ... \tag{1}$$

The method of undetermined coefficients for (1) gives solution form

$$y_{k,p}(t) = A\cos\big((2k-1)t\big) + B\sin\big((2k-1)t\big),$$

where A, B are constants to be determined. Plug this into the ODE and sort terms to get

$$\begin{aligned}\underline{0}\cos\big((2k-1)t\big) + \underline{1}\sin\big((2k-1)t\big) &= \ddot{y}_{k,p} + 2y_{k,p}\\ &= \big(-(2k-1)^2+2\big)\big)\, A\cos\big((2k-1)t\big) + \big(-(2k-1)^2+2\big)\big)\, B\sin\big((2k-1)t\big).\end{aligned}$$

So, we need $A = 0$ and $B = \dfrac{1}{-(2k-1)^2+2}$.

The solutions, for $k = 1, 2, ...$, are

$$y_{k,p}(t) = \frac{1}{-(2k-1)^2+2}\sin\big((2k-1)t\big)$$

Thus

$$y_p(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\,y_{k,p}(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)\big(-(2k-1)^2+2\big)}\cdot\sin\big((2k-1)t\big).$$

The general solution of the ODE is

$$y(t) = y_h(t) + y_p(t) = c_1\cos(\sqrt{2}\,t) + c_2\sin(\sqrt{2}\,t) + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)\big(-(2k-1)^2+2\big)}\cdot\sin\big((2k-1)t\big),$$

hence

$$\dot{y}(t) = -\sqrt{2}\,c_1\sin(\sqrt{2}\,t) + \sqrt{2}\,c_2\sin(\sqrt{2}\,t) + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{\big(-(2k-1)^2+2\big)}\cdot\cos\big((2k-1)t\big),$$

The first initial condition yields

$$0 = y(0) = c_1 + y_p(0) = c_1 + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)\big(-(2k-1)^2+2\big)}\cdot 0 = c_1,$$

hence $c_1 = 0$.

The second initial condition yields

$$1 = \dot{y}(0) = \sqrt{2}\,c_2 + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{\big(-(2k-1)^2+2\big)}$$

hence

$$c_2 = \frac{1}{\sqrt{2}} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}\big(-(2k-1)^2+2\big)}.$$

By combining two infinite series, we have that the solution of the ODE is

$$y(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t) + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}(2k-1)\big(-(2k-1)^2+2\big)}\left(-(2k-1)\sin(\sqrt{2}\,t) + \sqrt{2}\,\sin\big((2k-1)t\big)\right).$$

The figure show graphs of the approximation of the solution $y(t)$ by a finite sum

$$y_N(t) \triangleq \frac{4}{\pi}\sum_{k=1}^{N}\frac{1}{(2k-1)(4+(2k-1)^2)}\big((2k-1)e^{-3t} - (2k-1)\cos\big((2k-1)t\big) + 3\sin\big((2k-1)t\big)\big)$$

for $N = 2$ and $N = 8$. In each picture the approximation is drawn as a dashed curve and the exact solution (see Example 4.33 in Section 4.5 and problem 5.4.1.12) is drawn as a solid curve. In (a) the approximation uses $N = 2$ and in (b) the approximation uses $N = 8$.

The exact solution was graphed using the Mathematica commands

$$\mathsf{f[t\_] := Evaluate[\ \frac{t}{Abs[\,t\,]}\,(-1)^{(IntegerPart[\,\frac{t}{\pi}\,])}\ ],}$$

$$\mathsf{g[t\_] := \frac{Sin[\,t\sqrt{2}\,]}{\sqrt{2}} + \int_0^t \frac{Sin[\,(t-u)\sqrt{2}\,]}{\sqrt{2}}\,f[t-u]du}$$

and

$$\mathsf{a= Plot\Big[g[t], \{t, 0, 6\pi\}, AxesLabel \to\{t, "y(t)", Text[FontSize \to Large]\},}$$

$$\mathsf{AxesStyle \to Thickness[0.00315], PlotPoints \to 100, PlotStyle \to \{Black, Thickness[0.005]\}, Exclusions \to \{0, 6\pi\}\Big]}$$

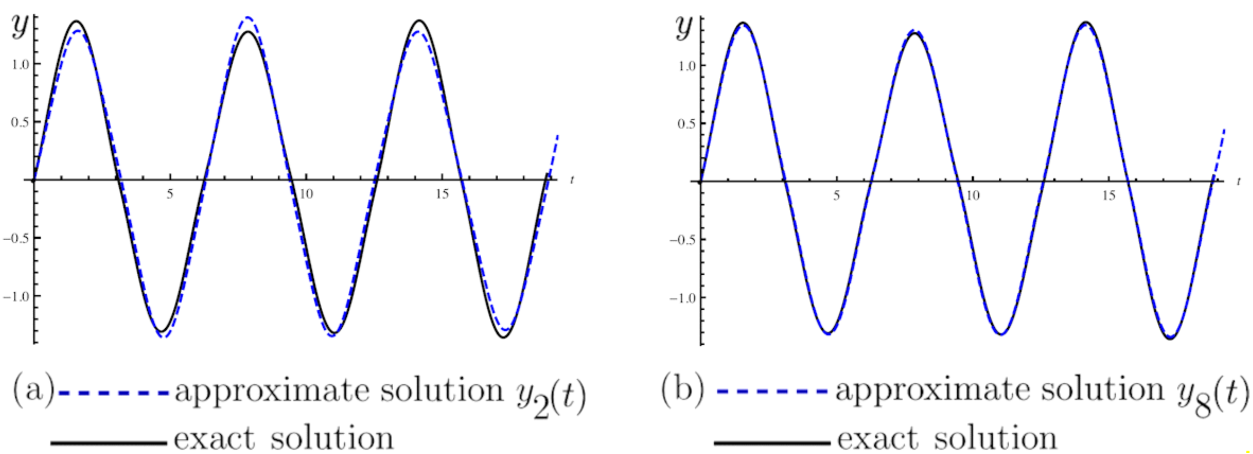


Figure 16: Problem 9.2.4.13

Section 9.3

9.3.3.1. *Case 1:* If $\lambda = 0$, then the differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) = 0$, whose solutions are $X = c_1 + c_2x$, for arbitrary constants c_1, c_2. In this case, $X'(x) = c_2$.

Applying the first BC gives $0 = X'(0) = c_2$. Applying the second BC gives $0 = X'(L) = c_2$. So, both BCs are satisfied if, and only if, $c_2 = 0$. When $\lambda = 0$, the ODE-BVP has non-trivial solution $X(x) \equiv c_1 = c_1 \cdot 1$. So, $\lambda_0 = 0$ is an eigenvalue for this problem, with corresponding eigenfunction $X_0(x) \equiv 1$.

Case 2: If $\lambda > 0$, rewrite $\lambda = \omega^2$, where $\omega \triangleq \sqrt{\lambda} > 0$. The differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) + \omega^2 X(x) = 0$, whose solutions are $X = c_1 \cos \omega x + c_2 \sin \omega x$, for arbitrary constants c_1, c_2. In this case, $X'(x) = -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x$.

Applying the first BC gives $0 = X'(0) = -\omega c_1 \cdot 0 + \omega c_2 \cdot 1 = \omega c_2$. Because $\omega > 0$, this implies $c_2 = 0$. So, $X = c_1 \cos \omega x$. Applying the second BC gives

$$0 = X'(L) = -\omega c_1 \sin \omega L.$$

Because $\omega > 0$ and we need $c_1 \neq 0$ in order to have an eigenfunction, $\lambda > 0$ is an eigenvalue if, and only if, $\sin(\omega L) = 0$. So, we have the trivial solution for the function $X(x)$ unless ω satisfies the "characteristic equation"

$$\sin(\omega L) = 0.$$

Trigonometry implies that there are infinitely many values of ω that make this true: $\omega = \dfrac{n\pi}{L}$, any integer n.

Now, $n = 0$ is an integer but gives $\lambda = \omega^2 = 0^2$, which is not allowed in Case 2. Additionally, while any integer $n < 0$, say $n = -m$, does give $\omega = \frac{-m\pi}{L}$ that satisfies $\sin(\omega L) = 0$, it turns out that $n < 0$ gives no eigenfunction for $X(x)$ beyond the ones we get for $n > 0$. Why? Because, if $n = -m$ then $X(x) = \cos \omega x = \cos\left(\frac{-m\pi x}{L}\right) = \cos\left(\frac{m\pi x}{L}\right)$, which duplicates the eigenfunction $X = \cos\left(\frac{m\pi x}{L}\right)$.

The case $\lambda > 0$ gives *eigenvalues* $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$, $n = 1, 2, ...$, and corresponding *eigenfunctions* $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, ...$.

Case 3: If $\lambda < 0$, rewrite $\lambda = -\omega^2$, where $\omega \triangleq \sqrt{-\lambda}$. The differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) - \omega^2 X(x) = 0$, whose solutions are $X = c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$, for arbitrary constants c_1, c_2. In this case, $X'(x) = \omega c_1 \sinh \omega x + \omega c_2 \cosh \omega x$.

Applying the first BC gives $0 = X'(0) = \omega c_1 \cdot 0 + \omega c_2 \cdot 1 = \omega c_2$. Because $\omega > 0$, this implies $c_2 = 0$. So, $X = c_1 \cosh \omega x$. Applying the second BC gives

$$0 = X'(L) = \omega c_1 \sinh \omega L.$$

Note that $\omega > 0$ implies $\sinh \omega L > 0$. This implies $c_1 = 0$, so there is no eigenfunction if $\lambda < 0$.

9.3.3.3. *Case 1:* If $\lambda = 0$, then the differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) = 0$, whose solutions are $X = c_1 + c_2 x$, for arbitrary constants c_1, c_2. In this case, $X'(x) = c_2$.

Applying the first BC gives $0 = X'(0) = c_2$, hence $c_2 = 0$ and $X(x) \equiv c_1$. Applying the second BC gives $0 = X(L) = c_1$. So, both BCs are satisfied if, and only if, $c_1 = c_2 = 0$. When $\lambda = 0$, the ODE-BVP has only the trivial solution. So, $\lambda = 0$ is not an eigenvalue for this problem.

Case 2: If $\lambda > 0$, rewrite $\lambda = \omega^2$, where $\omega \triangleq \sqrt{\lambda} > 0$. The differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) + \omega^2 X(x) = 0$, whose solutions are $X = c_1 \cos \omega x + c_2 \sin \omega x$, for arbitrary constants c_1, c_2. In this case, $X'(x) = -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x$.

Applying the first BC gives $0 = X'(0) = -\omega c_1 \cdot 0 + \omega c_2 \cdot 1 = \omega c_2$. Because $\omega > 0$, this implies $c_2 = 0$. So, $X = c_1 \cos \omega x$. Applying the second BC gives

$$0 = X(L) = c_1 \cos \omega L.$$

Because $\omega > 0$ and we need $c_1 \neq 0$ in order to have an eigenfunction, $\lambda > 0$ is an eigenvalue if, and only if, $\cos(\omega L) = 0$. So, we have the trivial solution for the function $X(x)$ unless ω satisfies the "characteristic equation"

$$\cos(\omega L) = 0.$$

Trigonometry implies that there are infinitely many values of ω that make this true: $\omega = \dfrac{(n - \frac{1}{2})\pi}{L}$, any integer n.

While any integer $n \leq 0$, say $n = -m$, does give $\omega = \dfrac{-(m + \frac{1}{2})\pi}{L}$ that satisfies $\cos(\omega L) = 0$, it turns out that $n \leq 0$ gives no eigenfunction for $X(x)$ beyond the ones we get for $n > 0$. Why? Because, if $n = -m$ then $X(x) = \cos \omega x = \cos\left(\frac{-(m+\frac{1}{2})\pi x}{L}\right) = \cos\left(\frac{(m+\frac{1}{2})\pi x}{L}\right)$, which duplicates the eigenfunction $X = \cos\left(\frac{(m+\frac{1}{2})\pi x}{L}\right)$.

The case $\lambda > 0$ gives *eigenvalues* $\lambda_n = \left(\dfrac{(n - \frac{1}{2})\pi}{L}\right)^2$, $n = 1, 2, ...$, and corresponding *eigenfunctions* $X_n(x) = \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$, $n = 1, 2, ...$

Case 3: If $\lambda < 0$, rewrite $\lambda = -\omega^2$, where $\omega \triangleq \sqrt{-\lambda}$. The differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) - \omega^2 X(x) = 0$, whose solutions are $X = c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$, for arbitrary constants c_1, c_2. In this case, $X'(x) = \omega c_1 \sinh \omega x + \omega c_2 \cosh \omega x$.

Applying the first BC gives $0 = X'(0) = \omega c_1 \cdot 0 + \omega c_2 \cdot 1 = \omega c_2$. Because $\omega > 0$, this implies $c_2 = 0$. So, $X = c_1 \cosh \omega x$. Applying the second BC gives

$$0 = X(L) = c_1 \cosh \omega L.$$

Note that $\omega > 0$ implies $\cosh \omega L > 0$. This implies $c_1 = 0$, so there is no eigenfunction if $\lambda < 0$.

9.3.3.5. Suppose $\lambda > 0$ is an eigenvalue for (9.20-9.21). Let $\omega = \sqrt{\lambda} > 0$, so ODE (9.20) becomes $X''+\omega^2 X = 0$, whose solutions are

$$X = c_1 \cos \omega x \, + c_2 \sin \omega x,$$

where c_1, c_2 are arbitrary constants. In this case, $X = -\omega c_1 \sin \omega x \, + \omega c_2 \cos \omega x$.

Plug the solutions into the BCs (9.21) to get

$$\left\{ \begin{array}{l} 0 = \epsilon_0 X(0) - \epsilon_1 X'(0) = \epsilon_0 c_1 - \epsilon_1 \omega \, c_2, \\ \\ 0 = \gamma_0 X(L) + \gamma_1 X'(L) = \gamma_0 (c_1 \cos \omega L + c_2 \sin \omega L) + \gamma_1 \omega (-c_1 \sin \omega L + c_2 \cos \omega L) \end{array} \right\},$$

that is,

$$\left\{ \begin{array}{c} 0 = \epsilon_0 c_1 - \epsilon_1 \omega \, c_2, \\ \\ 0 = (\gamma_0 \cos \omega L - \gamma_1 \omega \sin \omega L) c_1 + (\gamma_0 \sin \omega L + \gamma_1 \omega \cos \omega L) c_2 \end{array} \right\}.$$

This can be written using a matrix and vector:

$$\left[\begin{array}{cc} \epsilon_0 & -\omega \, \epsilon_1 \\ \\ \gamma_0 \cos \omega L - \gamma_1 \omega \sin \omega L & \gamma_0 \sin \omega L + \gamma_1 \omega \cos \omega L \end{array} \right] \left[\begin{array}{c} c_1 \\ \\ c_2 \end{array} \right].$$

There is an eigenvalue if, and only if, the system has a non-trivial solution for $\left[\begin{array}{c} c_1 \\ c_2 \end{array} \right]$, which is true if and only if

$$0 = \det \left[\begin{array}{cc} \epsilon_0 & -\omega \, \epsilon_1 \\ \\ \gamma_0 \cos \omega L - \gamma_1 \omega \sin \omega L & \gamma_0 \sin \omega L + \gamma_1 \omega \cos \omega L \end{array} \right]$$

$$= \epsilon_0 (\gamma_0 \sin \omega L + \gamma_1 \omega \cos \omega L) + \omega \, \epsilon_1 (\gamma_0 \cos \omega L - \gamma_1 \omega \sin \omega L) = (\epsilon_0 \gamma_0 - \lambda \, \epsilon_1 \gamma_1) \sin(\sqrt{\lambda} \, L) + (\epsilon_0 \gamma_1 + \epsilon_1 \gamma_0) \sqrt{\lambda} \, \cos(\sqrt{\lambda} \, L).$$

that is if and only if λ satisfies characteristic equation (9.24).

9.3.3.7. The ODE is not (9.20) in Section 9.3, so we have to work from scratch, as in Example 9.14 in Section 9.3.

Plug $X = e^{sx}$ into the ODE to get

$$0 = s^2 + 2s + (\lambda + 1) = (s + 1)^2 + \lambda.$$

Case 1: If $\lambda = 0$, then the roots are $s = -1, -1$, so the solutions of the differential equation are $X = c_1 e^{-x} + c_2 x e^{-x}$, for arbitrary constants c_1, c_2. In this case, $X' = -c_1 e^{-x} + c_2 (1 - x) e^{-x}$.

Applying the first BC gives $0 = X'(0) = -c_1 + c_2$, which implies $c_2 = c_1$, hence $X = c_1 \big(e^{-x} + x e^{-x} \big)$ and $X' = c_1 \big(-e^{-x} + (1 - x) e^{-x} \big) = -c_1 x e^{-x}$. Applying the second BC gives $0 = X'(L) = -c_1 L e^{-L}$, which implies $c_1 = 0$. So, both BCs are satisfied if, and only if, $c_1 = c_2 = 0$. When $\lambda = 0$, the ODE-BVP has only the trivial solution. So, $\lambda = 0$ is not an eigenvalue for this problem.

Case 2: If $\lambda > 0$, rewrite $\lambda = \omega^2$, where $\omega \triangleq \sqrt{\lambda} > 0$. The roots are $s = -1 \pm i\omega$. The differential equation has solutions $X = e^{-x} (c_1 \cos \omega x + c_2 \sin \omega x)$, for arbitrary constants c_1, c_2. In this case,

$$X' = e^{-x} \big(-c_1 (\cos \omega x + \omega \, \sin \omega x) + c_2 (\omega \, \cos \omega x - \sin \omega x) \big)$$

Applying the first BC gives $0 = X'(0) = -c_1 \cdot 1 + \omega \, c_2 \cdot 1 = -c_1 + \omega \, c_2$, which implies $c_1 = \omega \, c_2$. So, $X = c_2 e^{-x} (\omega \, \cos \omega x + \sin \omega x)$ and $X' = -c_2 (\omega^2 + 1) e^{-x} \sin \omega x$. Applying the second BC gives

$$0 = X'(L) = -c_2 (\omega^2 + 1) e^{-L} \sin \omega L.$$

Because $\omega > 0$ and we need $c_2 \neq 0$ in order to have an eigenfunction, $\lambda > 0$ is an eigenvalue if, and only if, $\sin(\omega L) = 0$.

Trigonometry implies that there are infinitely many values of ω that make this true: $\omega = \dfrac{n\pi}{L}$, any integer n.

While any integer $n \leq 0$, say $n = -m$, does give $\omega = \dfrac{-m\pi}{L}$ that satisfies $\sin(\omega L) = 0$, it turns out that $n \leq 0$ gives no eigenfunction for $X(x)$ beyond the ones we get for $n > 0$. Why? Because, if $n = -m$ then

$$X = e^{-x}(\omega \cos\omega x + \sin\omega x) = e^{-x}\Big(\frac{-m\pi}{L}\cos\left(\frac{-m\pi x}{L}\right) + \sin\left(\frac{-m\pi x}{L}\right)\Big)$$

$$= e^{-x}\Big(-\frac{m\pi}{L}\cos\left(\frac{m\pi x}{L}\right) - \sin\left(\frac{m\pi x}{L}\right)\Big) = -e^{-x}\Big(\frac{m\pi}{L}\cos\frac{m\pi x}{L} + \sin\frac{m\pi x}{L}\Big)$$

which essentially duplicates the eigenfunction $X = e^{-x}\Big(\frac{m\pi}{L}\cos\left(\frac{m\pi x}{L}\right) + \sin\left(\frac{m\pi x}{L}\right)\Big)$.

The case $\lambda > 0$ gives *eigenvalues* $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$, $n = 1, 2, ...$, and corresponding *eigenfunctions*

$$X_n(x) = e^{-x}\Big(\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\Big),$$

$n = 1, 2, ...$

Case 3: If $\lambda < 0$, rewrite $\lambda = -\omega^2$, where $\omega \triangleq \sqrt{-\lambda} > 0$. The roots are $s = -1 \pm \omega$. The differential equation has solutions $X = e^{-x}\big(c_1\cosh(\omega x) + c_2\sinh(\omega x)\big)$, for arbitrary constants c_1, c_2. In this case,

$$X' = e^{-x}\big(c_1(-\cosh\omega x + \omega\sinh\omega x) + c_2(\omega\cosh\omega x - \sinh\omega x)\big)$$

Applying the first BC gives $0 = X'(0) = -c_1 \cdot 1 + \omega c_2 \cdot 1 = -c_1 + \omega c_2$, which implies $c_1 = \omega c_2$. So, $X = c_2 e^{-x}(\omega\cosh\omega x + \sinh\omega x)$ and $X' = c_2(\omega^2 + 1)e^{-x}\sinh\omega x$. Applying the second BC gives

$$0 = X'(L) = c_2(\omega^2 + 1)e^{-L}\sinh\omega L.$$

Because $\omega > 0$ implies $\sinh\omega L > 0$, and we need $c_2 \neq 0$ in order to have an eigenfunction, $\lambda > 0$ gives an eigenvalue if, and only if, $\omega = 1$, hence $\lambda_0 = -\omega^2 = -1$, for which the corresponding eigenfunction is

$$X_0(x) = e^{-x}\cdot(\cosh x + \sinh x) = e^{-x}\cdot\left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right) = e^{-x}\cdot(e^x) \equiv 1.$$

In retrospect, it is at least "familiar" that we should have the constant eigenfunction $X_0(x) \equiv 1$ for the homogeneous Neumann BCs.

9.3.3.9. The functions $\cos\big((n - \frac{1}{2})x\big)$, $n = 1, 2, ...$ are orthogonal on the interval $0 < x < \pi$. As on page 687, we have a generalized Fourier expansion

$$f(x) \doteq \sum_{n=1}^{\infty} c_n \cos\Big((n - \frac{1}{2})x\Big),\ 0 < x < \frac{\pi}{2},$$

where

$$c_n = \frac{\int_0^\pi f(x)\cos\Big((n - \frac{1}{2})x\Big)dx}{\int_0^\pi |\cos\Big((n - \frac{1}{2})x\Big)|^2 dx} = \frac{2}{\pi}\int_0^\pi f(x)\cos\Big((n - \frac{1}{2})x\Big)dx.$$

Here, $f(x) = \sin x$. We calculate

$$c_n = \frac{2}{\pi}\int_0^\pi \sin x\,\cos\Big((n - \frac{1}{2})x\Big)dx = \frac{2}{\pi}\int_0^\pi \frac{1}{2}\Big(\sin\Big((1 - (n - \frac{1}{2}))x\Big) + \sin\Big((1 + (n - \frac{1}{2}))x\Big)\Big)dx$$

$$= \frac{1}{\pi}\int_0^\pi \left(\sin\left(\left(\frac{3}{2}-n\right)x\right) + \sin\left(\left(\frac{1}{2}+n\right)x\right)\right) dx = \frac{1}{\pi}\left[\frac{\cos\left(\left(\frac{3}{2}-n\right)x\right)}{-(\frac{3}{2}-n)} + \frac{\cos\left(\left(\frac{1}{2}+n\right)x\right)}{-(\frac{1}{2}+n)}\right]_0^\pi$$

$$= \frac{1}{\pi}\left(\frac{\cos\left(\left(\frac{3}{2}-n\right)\pi\right)-1}{-(\frac{3}{2}-n)} + \frac{\cos\left(\left(\frac{1}{2}+n\right)\pi\right)-1}{-(\frac{1}{2}+n)}\right) = \frac{1}{\pi}\left(\frac{0-1}{-(\frac{3}{2}-n)} + \frac{0-1}{-(\frac{1}{2}+n)}\right) = \frac{1}{\pi}\left(-\frac{1}{n-\frac{3}{2}} + \frac{1}{n+\frac{1}{2}}\right)$$

$$= \frac{1}{\pi}\cdot\frac{-2}{(n-\frac{3}{2})(n+\frac{1}{2})} = \frac{-2}{\pi}\cdot\frac{1}{n^2-n-\frac{3}{4}}\cdot\frac{4}{4} = -\frac{8}{\pi}\cdot\frac{1}{4n^2-4n-3}.$$

So, the generalized Fourier expansion is

$$f(x) \doteq -\frac{8}{\pi}\sum_{n=1}^\infty \frac{1}{4n^2-4n-3}\cos\left((n-\frac{1}{2})x\right).$$

9.3.3.11. The ODE is not (9.20), so we have to work from scratch, as in Example 9.14 in Section 9.3. The problem assumes that $\mu > 0$.

Plug $X = e^{sx}$ into the ODE to get

$$0 = s^2 + 2\mu s + 2\mu^2,$$

so

$$s = \frac{-2\mu \pm \sqrt{(2\mu)^2 - 8\mu^2}}{2} = -\mu \pm i\mu.$$

The differential equation has solutions $X = e^{-\mu x}(c_1\cos\mu x + c_2\sin\mu x)$, for arbitrary constants c_1, c_2. In this case,

$$X' = e^{-\mu x}\big(-c_1(\mu\cos\omega x + \mu\sin\omega x) + c_2(\mu\cos\omega x - \mu\sin\omega x)\big),$$

that is,

$$X' = \mu e^{-\mu x}\big(-c_1(\cos\omega x + \sin\omega x) + c_2(\cos\omega x - \sin\omega x)\big),$$

Applying the first BC gives $0 = X(0) = c_1\cdot 1 + c_2\cdot 0 = c_1$, so $X = c_2 e^{-\mu x}\sin\mu x$ and $X' = c_2\mu\, e^{-\mu x}(\cos\mu x - \sin\mu x)$. Applying the second BC gives

$$0 = X'(L) = c_2\mu\, e^{-\mu L}(\cos\mu L - \sin\mu L).$$

Because $\mu > 0$ and we need $c_2 \neq 0$ in order to have an eigenfunction, $\lambda > 0$ is an eigenvalue if, and only if, $\cos\mu L - \sin\mu L = 0$, which is the characteristic equation.

Because $\cos\mu L = \sin\mu L = 0$ is impossible, the characteristic equation can be rewritten as

$$1 = \tan\mu L.$$

The eigenvalues are $\mu_n = \frac{1}{L}\left(\frac{\pi}{4} + n\pi\right)$, $n = 0, 1, 2, \ldots$.

Section 9.4

9.4.3.1. $L = \pi$, so $\omega_n = n$ and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i\omega_n x}\, dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0}\cos 3x\, e^{-inx}\, dx + \int_0^\pi 0\cdot e^{-inx}\, dx\right) = \frac{1}{2\pi}\int_{-\pi}^{0}\frac{e^{i3x}+e^{-i3x}}{2}\, e^{-inx}\, dx$$

$$= \frac{1}{4\pi}\int_{-\pi}^{0}\left(e^{i(3-n)x} + e^{-i(3+n)x}\right) dx. \quad (\star)$$

If $n = 3$,

$$c_3 = \frac{1}{4\pi}\int_{-\pi}^{0}\left(1+e^{-i6x}\right)dx = \frac{1}{4\pi}\Big[\,x+\frac{e^{-i6x}}{-i6}\,\Big]_{-\pi}^{0} = \frac{1}{4\pi}\Big(\,(0-(-\pi)+\frac{1-1}{-i6}\Big) = \frac{1}{4}.$$

If $n = -3$,

$$c_{-3} = \frac{1}{4\pi}\int_{-\pi}^{0}\left(e^{i6x}+1\right)dx = \frac{1}{4\pi}\Big[\,\frac{e^{i6x}}{i6}+x\,\Big]_{-\pi}^{0} = \frac{1}{4\pi}\Big(\,\frac{1-1}{i6}+(0-(-\pi)\Big) = \frac{1}{4}.$$

Using ($\star$) above, if $|n| \neq 3$

$$c_n = \frac{1}{4\pi}\Big[\,\frac{e^{i(3-n)x}}{i(3-n)}+\frac{e^{-i(3+n)x}}{-i(3+n)}\,\Big]_{-\pi}^{0} = \frac{1}{4\pi}\Big(\,\frac{1-e^{-i(3-n)\pi}}{i(3-n)}+\frac{1-e^{i(3+n)\pi}}{-i(3+n)}\Big) = \frac{1}{4\pi}\Big(\,\frac{1-(-1)^{3-n}}{i(3-n)}+\frac{1-(-1)^{3+n}}{-i(3+n)}\Big)$$

$$= \frac{1}{4\pi}\Big(\,\frac{1+(-1)^n}{i(3-n)}+\frac{1+(-1)^n}{-i(3+n)}\Big) = \frac{1+(-1)^n}{4\pi}\cdot\Big(\,\frac{1}{i(3-n)}-\frac{1}{i(3+n)}\Big) = \frac{1+(-1)^n}{4\pi i}\cdot\Big(\,\frac{1}{3-n}-\frac{1}{3+n}\Big)$$

$$= \frac{1+(-1)^n}{4\pi i}\cdot\frac{2n}{9-n^2}.$$

Using $\dfrac{1}{i} = -i$ and the fact that

$$1+(-1)^n = \left\{\begin{array}{ll} 2, & n = 2k \\ \\ 0, & n = 2k-1 \end{array}\right\},$$

if $|n| \neq 3$ then $c_n = 0$ for odd n and

$$c_{2k} = \frac{2\cdot(-i)}{4\pi}\cdot\frac{2\cdot 2k}{9-(2k)^2} = \frac{i2}{\pi}\cdot\frac{k}{(2k)^2-9}.$$

The complex Fourier series representation is

$$f(x) \doteq \frac{1}{4}\Big(e^{-i3x}+e^{-i3x}\Big)+\frac{i2}{\pi}\sum_{k=-\infty}^{\infty}\frac{k}{4k^2-9}\,e^{i2kx}.$$

9.4.3.3. $L = \pi$, so $\omega_n = n$ and the complex Fourier series is the function itself, because

$$f(x) = 1-i3\sin 2x+\frac{1}{2}\,\cos 5x = 1-i3\cdot\frac{e^{i2x}-e^{-i2x}}{i2}+\frac{1}{2}\cdot\frac{e^{i5x}+e^{-i5x}}{2}$$

$$= \frac{1}{4}e^{-i5x}+\frac{3}{2}\,e^{-i2x}+1-\frac{3}{2}\,e^{i2x}+\frac{1}{4}e^{-i5x}.$$

9.4.3.5. $f(t) = Step(t-1) - Step(t-3) = \left\{\begin{array}{ll} 0, & t<1 \\ 1, & t>1 \end{array}\right\} - \left\{\begin{array}{ll} 0, & t<3 \\ 1, & t>3 \end{array}\right\} = \left\{\begin{array}{ll} 0, & t\leq 1 \\ 1, & 1<t<3 \\ 0, & t\geq 1 \end{array}\right\}$, so the Fourier transform of $f(t)$ is

$$\mathcal{F}[\,f(t)\,] \triangleq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}}\int_{1}^{3} e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}}\Big[\,\frac{e^{-i\omega t}}{-i\omega}\,\Big]_{1}^{3} = \frac{i}{\omega\sqrt{2\pi}}\left(e^{-i3\omega}-e^{-i\omega}\right).$$

9.4.3.7. $\mathcal{F}[\,f(t)\,] \triangleq \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt = \tfrac{1}{\sqrt{2\pi}}\Big(\int_{-\infty}^{0} 0\cdot e^{-i\omega t}\,dt + \int_0^\infty e^{-\alpha t}\sin(\omega_0 t)\,e^{-i\omega t}\,dt\Big)$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\alpha t}\cdot \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{i2}\cdot e^{-i\omega t}\,dt,$$

which we can break up into two integrals.

The first improper integral is

$$\int_0^\infty e^{-\alpha t}\cdot e^{i\omega_0 t}e^{-i\omega t}\,dt = \int_0^\infty e^{-(\alpha+i(\omega-\omega_0))t}\,dt = \lim_{b\to\infty}\int_0^b e^{-(\alpha+i(\omega-\omega_0))t}\,dt$$

$$= \lim_{b\to\infty}\left[\frac{1}{-\big(\alpha+i(\omega-\omega_0)\big)}\,e^{-(\alpha+i(\omega-\omega_0))t}\right]_0^b = \lim_{b\to\infty}\frac{1}{-\big(\alpha+i(\omega-\omega_0)\big)}\Big(e^{-(\alpha+i(\omega-\omega_0))b} - 1\Big)$$

$$= \frac{1}{-\big(\alpha+i(\omega-\omega_0)\big)}(0-1) = \frac{1}{\big(\alpha+i(\omega-\omega_0)\big)},$$

because the constant $\alpha > 0$ was assumed to be positive.

Similarly,

$$\int_0^\infty e^{-\alpha t}\cdot e^{-i\omega_0 t}e^{-i\omega t}\,dt = \int_0^\infty e^{-(\alpha+i(\omega+\omega_0))t}\,dt = \ldots = \frac{1}{\big(\alpha+i(\omega+\omega_0)\big)},$$

because the constant $\alpha > 0$ was assumed to be positive.

Putting the two integrals back together gives

$$\mathcal{F}[\,f(t)\,] = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{i2}\bigg(\frac{1}{\big(\alpha+i(\omega-\omega_0)\big)} - \frac{1}{\big(\alpha+i(\omega+\omega_0)\big)}\bigg) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{i2}\,\frac{\alpha+i(\omega+\omega_0) - \big(\alpha+i(\omega-\omega_0)\big)}{\big(\alpha+i(\omega-\omega_0)\big)\big(\alpha+i(\omega+\omega_0)\big)}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{i2}\,\frac{i2\omega_0}{\big((\alpha+i\omega)-i\omega_0\big)\big((\alpha+i\omega)+i\omega_0\big)} = \frac{\omega_0}{\sqrt{2\pi}}\cdot\frac{1}{(\alpha+i\omega)^2+\omega_0^2} = \frac{\omega_0}{\sqrt{2\pi}}\cdot\frac{1}{\alpha^2+i2\alpha\omega-\omega^2+\omega_0^2}$$

$$= \frac{\omega_0}{\sqrt{2\pi}\,\big(\alpha^2+\omega_0^2-\omega^2+i2\alpha\omega\big)}.$$

9.4.3.9. $\mathcal{F}[\,e^{-at}\,Step(t)] \triangleq \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-at}\,Step(t)\,e^{-i\omega t}dt = \tfrac{1}{\sqrt{2\pi}}\int_0^\infty\, e^{-at}\,e^{-i\omega t}dt \triangleq \lim_{R\to\infty}\,\tfrac{1}{\sqrt{2\pi}}\int_0^R e^{-(a+i\omega)t}dt$

$$= \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_0^R = \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left(\frac{e^{-(a+i\omega)R}}{-(a+i\omega)} - \frac{1}{-(a+i\omega)}\right) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{a+i\omega}.$$

9.4.3.11. Example 9.21's conclusion in Section 9.4 of

$$\mathcal{F}[\,f(x)\,] = \sqrt{\frac{2}{\pi}}\cdot\frac{1-e^{-i3\omega\pi}}{4-\omega^2},$$

for $|\omega| \neq 2$, looks like the $F(\omega)$ we're given, except for two aspects of problem 9.4.3.11: (a) there is no factor of $\sqrt{\dfrac{2}{\pi}}$, and (b) the numerator of $F(\omega)$ is $1-e^{-i5\pi\omega}$ instead of $1-e^{-i3\pi\omega}$.

Method 1: Table 9.2's entry F.15 [with the correction in the errata page] is a generalization of the result of Example 9.21.

$$\mathcal{F}\left[\left\{\begin{array}{ll}\sin kx, & 0 \le x \le \tau \\ 0, & \text{all other } x\end{array}\right\}\right] = \sqrt{\frac{1}{2\pi}}\cdot\frac{1}{k^2-\omega^2}\Big(e^{-i\omega\tau}(-i\omega\sin k\tau - k\cos k\tau) + k\Big)$$

We should choose $k = 2$ so that the denominator of $4 - \omega^2$ fits the form $k^2 - \omega^2$. We should choose $\tau = 5\pi$ so that the $e^{-i\omega\tau}$ would fit the form $e^{-i5\pi\omega}$. So, Table 9.2's entry F.15 [with the correction in the errata page] gives, as a particular example,

$$\mathcal{F}\left[\left\{\begin{array}{ll}\sin 2x, & 0 \le x \le 5\pi \\ \\ 0, & \text{all other } x\end{array}\right\}\right] = \sqrt{\frac{1}{2\pi}} \cdot \frac{1}{4-\omega^2}\Big(e^{-i\omega 5\pi}(-i\omega \sin 2\cdot 5\pi - 2\cos 2\cdot 5\pi) + 2\Big).$$

Using the facts that $\sin 10\pi = 0$ and $\cos 10\pi = 1$, this implies that

$$\mathcal{F}\left[\left\{\begin{array}{ll}\sin 2x, & 0 \le x \le 5\pi \\ \\ 0, & \text{all other } x\end{array}\right\}\right] = \sqrt{\frac{1}{2\pi}} \cdot \frac{1}{4-\omega^2}\big(-2e^{-i\omega 5\pi} + 2\big) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{4-\omega^2}\big(1 - e^{-i5\pi\omega}\big).$$

This implies that

$$\mathcal{F}^{-1}\left[\frac{1 - e^{-i5\pi\omega}}{4-\omega^2}\right] = \sqrt{\frac{\pi}{2}} \cdot \left\{\begin{array}{ll}\sin 2x, & 0 \le x \le 5\pi \\ \\ 0, & \text{all other } x\end{array}\right\}.$$

<u>Method 2</u>: The 3π that appears in the formula for the function $f(x)$ in Example 9.21 in Section 9.4 appears in the Fourier transform in the $e^{-i3\pi\omega}$. This suggests modifying Example 9.21 to find the Fourier transform of

$$g(x) \triangleq \left\{\begin{array}{ll}\sin 2x, & 0 \le x \le 5\pi \\ \\ 0, & \text{all other } x\end{array}\right\}:$$

We can calculate, from the definition of Fourier transform, that

$$\mathcal{F}\,[\,g(x)\,] = ... = \sqrt{\frac{2}{\pi}} \cdot \frac{1 - e^{-i5\pi\omega}}{4-\omega^2}.$$

[Please check for yourself the work needed in the = ... =.]

From the Fourier transform of $g(x)$ we can find the inverse transform

$$\mathcal{F}^{-1}\left[\frac{1 - e^{-i5\pi\omega}}{4-\omega^2}\right] = \sqrt{\frac{\pi}{2}} \cdot \left\{\begin{array}{ll}\sin 2x, & 0 \le x \le 5\pi \\ \\ 0, & \text{all other } x\end{array}\right\}.$$

9.4.3.13. Example 9.19's conclusion in Section 9.4 of

$$\mathcal{F}[\,f(t)\,] = a \cdot \sqrt{\frac{2}{\pi}} \cdot \left\{\begin{array}{ll}\dfrac{\sin(a\omega)}{a\omega}, & \omega \ne 0 \\ \\ 1, & \omega = 0\end{array}\right\},$$

looks a lot like the $F(\omega)$ whose inverse transform we're given, except for two aspects of problem 9.4.3.13: (a) there is no factor of $\sqrt{\dfrac{2}{\pi}}$, and (b) the numerator of $F(\omega)$ is $\sin b\omega$ instead of $\sin a\omega$.

We can take care of (b) easily by changing a to b, and we can take care of (a) by multiplying by an appropriate factor: Example 9.19 in Section 9.4 implies

$$\mathcal{F}\left[\sqrt{\frac{\pi}{2}}\left\{\begin{array}{ll}1, & -b < t < b \\ \\ 0, & |t| > b\end{array}\right\}\right] = \sqrt{\frac{\pi}{2}} \cdot b \cdot \sqrt{\frac{2}{\pi}} \cdot \left\{\begin{array}{ll}\dfrac{\sin(b\omega)}{b\omega}, & \omega \ne 0 \\ \\ 1, & \omega = 0\end{array}\right\},$$

so

$$\mathcal{F}^{-1}\left[\,\frac{\sin b\omega}{\omega}\,\right] = \sqrt{\frac{\pi}{2}} \cdot \left\{ \begin{array}{ll} 1, & -b < t < b \\ & \\ 0, & |t| > b \end{array} \right\}.$$

9.4.3.15. Hints: Use the time delay Theorem 9.11 to find $\mathcal{F}[\,p(t-2n)\,]$, use the fact that $\displaystyle\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ if $|r| < 1$, and use the fact that $e^{-n} = (e^{-1})^n$.

$$\mathcal{F}[\,g(t)\,] = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\omega}{\omega} \cdot \frac{1}{1 - e^{-1+i2\omega}}$$

9.4.3.17. (a) Using Table 9.2's entry F.2, we calculate

$$\mathcal{F}[\,f(x)\cos kx\,](\omega) = \mathcal{F}\left[\,f(x)\cdot\frac{1}{2}(e^{ikx} + e^{-ikx})\right](\omega) = \frac{1}{2}\left(\mathcal{F}[\,f(x)\,e^{ikx}\,](\omega) + \mathcal{F}[\,f(x)\,e^{-ikx}\,](\omega)\right)$$

$$= \frac{1}{2}\left(F(\omega-k) + F(\omega+k)\right).$$

(b) Using Table 9.2's entry F.2, we calculate

$$\mathcal{F}[\,f(x)\sin kx\,](\omega) = \mathcal{F}\left[\,f(x)\cdot\frac{1}{i2}(e^{ikx} - e^{-ikx})\right](\omega) = \frac{1}{12}\left(\mathcal{F}[\,f(x)\,e^{ikx}\,](\omega) - \mathcal{F}[\,f(x)\,e^{-ikx}\,](\omega)\right)$$

$$= \frac{1}{i2}\left(F(\omega-k) - F(\omega+k)\right).$$

9.4.3.19. The result of problem 9.4.3.18 implies that

$$\mathcal{F}^{-1}\left[\left\{ \begin{array}{ll} \dfrac{\cos(\omega b)-1}{\omega b}, & \omega\neq 0 \\ & \\ 0, & \omega = 0 \end{array} \right\}\right] = i\sqrt{\frac{\pi}{2}} \cdot \left\{ \begin{array}{rr} 1, & -b < t < 0 \\ -1, & 0 < t < b \\ 0, & |t| > b \end{array} \right\}.$$

Using Theorem 9.9(a),

$$\mathcal{F}\left[\frac{\cos(bt)-1}{bt}\right](\omega) = \mathcal{F}^{-1}\left[\frac{\cos(bt)-1}{bt}\right](-\omega) = i\sqrt{\frac{\pi}{2}} \cdot \left\{ \begin{array}{rl} 1, & -b < -\omega < 0 \\ -1, & 0 < -\omega < b \\ 0, & |-\omega| > b \end{array} \right\}$$

$$= i\sqrt{\frac{\pi}{2}} \cdot \left\{ \begin{array}{rl} 1, & 0 < \omega < b \\ -1, & -b < \omega < 0 \\ 0, & |\omega| > b \end{array} \right\}$$

$$= i\sqrt{\frac{\pi}{2}} \cdot \left\{ \begin{array}{rl} -1, & -b < \omega < 0 \\ 1, & 0 < \omega < b \\ 0, & |\omega| > b \end{array} \right\}.$$

Section 9.5

9.5.3.1. Let $\omega = e^{i2\pi/N}$. Then

$$f(t) = \sin\left(\frac{4\pi t}{N}\right) = \frac{1}{2i}\left(e^{i4\pi k/N} - e^{-i4\pi k/N}\right) = -\frac{i}{2}e^{i4\pi k/N} + \frac{i}{2}e^{-i4\pi k/N} = -\frac{i}{2}\omega^{2k} + \frac{i}{2}\omega^{-2k} = -\frac{i}{2}\omega^{2k} + \frac{i}{2}\omega^{(N-2)k}.$$

It follows that

$$\mathbf{f} = -\frac{i}{2}\begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \vdots \\ \omega^{2(N-1)} \end{bmatrix} + \frac{i}{2}\begin{bmatrix} 1 \\ \omega^{N-2} \\ \omega^{2(N-2)} \\ \vdots \\ \omega^{(N-1)(N-2)} \end{bmatrix},$$

so

$$\mathbf{F} = DFT[\,\mathbf{f}\,] = \frac{i\sqrt{N}}{2}\left(-\mathbf{d}^{(2)} + \mathbf{d}^{(N-2)}\right) = \frac{i\sqrt{N}}{2}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

that is, $F_2 = -\frac{i\sqrt{N}}{2}$, $F_{N-2} = \frac{i\sqrt{N}}{2}$ and $F_\ell = 0$ for all other indices ℓ.

9.5.3.3. Let $\omega = e^{i2\pi/N}$. Then $f(t) = a^t = e^{-i2\pi k_0 t/N} = \omega^{-k_0 t}$, so

$$\mathbf{f} = \begin{bmatrix} 1 \\ \omega^{-k_0} \\ \omega^{-2k_0} \\ \vdots \\ \omega^{-(N-1)k_0} \end{bmatrix} = \begin{bmatrix} 1 \\ \omega^{N-k_0} \\ \omega^{2N-2k_0} \\ \vdots \\ \omega^{(N-1)N-(N-1)k_0} \end{bmatrix} = \begin{bmatrix} 1 \\ \omega^{N-k_0} \\ \omega^{2(N-k_0)} \\ \vdots \\ \omega^{(N-1)(N-k_0)} \end{bmatrix}.$$

We calculate that

$$\mathbf{F} = DFT[\,\mathbf{f}\,] = \sqrt{N}\mathbf{d}^{(N-k_0)} = \sqrt{N}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow (N-k_0)\,place \\ \\ \\ \\ \end{matrix},$$

that is, $F_{N-k_0} = \sqrt{N}$ and $F_\ell = 0$ for all other indices ℓ.

9.5.3.5. (a) For $N = 4$, $\omega^{-1} = e^{-i2\pi/4} = e^{-i\pi/2} = -i$. On the graph, it appears that $f_0 \approx -0.5$, $f_1 = f(1\,ms) \approx 1.0$, $f_2 = f(2\,ms) \approx 0.8$, and $f_3 = f(3\,ms) \approx 1.0$, so $\mathbf{f} \approx \begin{bmatrix} -0.5 \\ 1.0 \\ 0.8 \\ 1.0 \end{bmatrix}$. Calculating by hand gives

$$DFT[\,\mathbf{f}\,] = \frac{1}{\sqrt{4}}\begin{bmatrix} -0.5 & +1.0 & +0.8 & +1.0 \\ -0.5 & +1.0(-i) & +0.8(-1) & +1.0(i) \\ -0.5 & +1.0(-1) & +0.8(1) & +1.0(-1) \\ -0.5 & +1.0(i) & +0.8(-1) & +1.0(-i) \end{bmatrix} = \begin{bmatrix} 1.2 & +i0 \\ -0.65 & +i0 \\ -0.85 & +i0 \\ -0.65 & -i0 \end{bmatrix}$$

(b) For $N = 8$, $\omega^{-1} = e^{-i2\pi/8} = e^{-i\pi/4} = \frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}$. On the graph, it appears that $f_0 \approx -0.5$, $f_1 = f(0.5\,ms) \approx 0.3$, $f_2 = f(1\,ms) \approx 1.0$, $f_3 = f(1.5\,ms) \approx 1.1$, $f_0 000000000000000000000000004 = f(2\,ms) \approx 0.8$, $f_5 = f(2.5\,ms) \approx 0.8$, $f_6 = f(3\,ms) \approx 1.0$, and $f_7 = f(3.5\,ms) \approx 1.4$, so

$$\mathbf{f} \approx \begin{bmatrix} -0.5 \\ 0.3 \\ 1.0 \\ 1.1 \\ 0.8 \\ 0.8 \\ 1.0 \\ 1.4 \end{bmatrix}.$$

Calculating by hand (!) gives

$DFT[\,\mathbf{f}\,]$

$$\approx \frac{1}{\sqrt{8}} \begin{bmatrix}
-0.5 +0.3 & +1.0 & +1.1 & +0.8 & +0.8 & +1.0 & +1.4 \\
-0.5 +0.3(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}) & +1.0(-i) & +1.1(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}) & +0.8(-1) & +0.8(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) & +1.0(i) & +1.4(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) \\
-0.5 +0.3(-i) & +1.0(-1) & +1.1(i) & +0.8(1) & +0.8(-i) & +1.0(-1) & +1.4(i) \\
-0.5 +0.3(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}) & +1.0(i) & +1.1(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}) & +0.8(-1) & +0.8(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) & +1.0(-i) & +1.4(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) \\
-0.5 +0.3(-1) & +1.0(1) & +1.1(-1) & +0.8(1) & +0.8(-1) & +1.0(1) & +1.4(-1) \\
-0.5 +0.3(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) & +1.0(-i) & +1.1(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) & +0.8(-1) & +0.8(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}) & +1.0(i) & +1.4(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}) \\
-0.5 +0.3(i) & +1.0(-1) & +1.1(-i) & +0.8(1) & +0.8(i) & +1.0(-1) & +1.4(-i) \\
-0.5 +0.3(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) & +1.0(i) & +1.1(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}) & +0.8(-1) & +0.8(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}) & +1.0(-i) & +1.4(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}})
\end{bmatrix}$$

$$\approx \begin{bmatrix} 2.1 & +i0 \\ -0.51 & +i0.20 \\ -0.60 & +i0.49 \\ -0.41 & +i0.20 \\ -0.46 & +i0 \\ -0.41 & -i0.20 \\ -0.60 & -i0.49 \\ -0.51 & -i0.20 \end{bmatrix},$$

after rounding off to two significant digits.

9.5.3.7. (a) For $N = 4$, the table gives gives us $\mathbf{f} = \begin{bmatrix} f(0\,ms) \\ f(1\,ms) \\ f(2\,ms) \\ f(3\,ms) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ -2 \end{bmatrix}$. Using Mathematica$^{\text{TM}}$we get

$$DFT[\,\mathbf{f}\,] = \begin{bmatrix} 1+i0 \\ 1-i2 \\ 1+i0 \\ 1+i2 \end{bmatrix}.$$

(b) For $N=8$, the table gives gives us $\mathbf{f}=\begin{bmatrix} f(0\,ms)\\ f(0.5\,ms)\\ f(1\,ms)\\ f(1.5\,ms)\\ f(2\,ms)\\ f(2.5\,ms)\\ f(3\,ms)\\ f(3.5\,ms)\end{bmatrix}=\begin{bmatrix}2\\3\\2\\2\\0\\-1\\-2\\-1\end{bmatrix}$. Mathematica$^{\text{TM}}$gives $DFT[\,\mathbf{f}\,]=$

$\begin{bmatrix} 1.8 & +i0\\ 0.96 & -i3.2\\ 0.71 & -i0.35\\ 0.46 & -i0.34\\ -0.35 & +i0\\ 0.46 & +i0.34\\ 0.71 & +i0.35\\ 0.96 & +i3.2\end{bmatrix}$, after rounding off to two significant digits.

9.5.3.9. For $N=4$, $\omega^{-1}=e^{-i2\pi/4}=e^{-i\pi/2}=-i$. We were given $\mathbf{a}=\begin{bmatrix}a_0\\ \vdots\\ a_3\end{bmatrix}=\begin{bmatrix}1\\ -0.6-0.8i\\ -0.28+0.96i\\ 0.936-0.352i\end{bmatrix}$.

Calculating by hand gives

$$DFT[\,\mathbf{f}\,]=\frac{1}{\sqrt{4}}\begin{bmatrix} 1 & +(-0.6-0.8i) & +(-0.28+0.96i) & +(0.936-0.352i)\\ 1 & +(-0.6-0.8i)(-i) & +(-0.28+0.96i)(-1) & +(0.936-0.352i)(i)\\ 1 & +(-0.6-0.8i)(-1) & +(-0.28+0.96i)(1) & +(0.936-0.352i)(-1)\\ 1 & +(-0.6-0.8i)(i) & +(-0.28+0.96i)(-1) & +(0.936-0.352i)(-i)\end{bmatrix}=\begin{bmatrix}0.528 & -i0.0960\\ 0.416 & +i0.288\\ 0.192 & +i1.056\\ 0.864 & -i1.248\end{bmatrix},$$

after rounding off to two significant digits.

9.5.3.11. Suppose $F_{N-\ell}=\overline{F_\ell}$, for $\ell=0,...,N$, where $DFT_N[\mathbf{f}]=\mathbf{F}=\{F_\ell\}_{\ell=0}^{N-1}$. We will explain why $\mathbf{f}$ is a sequence of real numbers. Denote $\omega=e^{i2\pi/N}$, as usual.

The synthesis equation is

$$f_k=\frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}F_\ell\,\omega^{k\ell},$$

for $k=0,1,...,N-1$. It follows from $\overline{\omega}=\omega^{-1}$ that

$$\overline{f_k}=\overline{\frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}F_\ell\,\omega^{k\ell}}=\frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}\overline{F_\ell}\,\overline{\omega}^{k\ell}=\frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}F_{N-\ell}\,\omega^{-k\ell}=\frac{1}{\sqrt{N}}\left(\sum_{\ell=1}^{N-1}F_{N-\ell}\,\omega^{-k\ell}+F_0\right).$$

[We used the "wrap around" convention that $F_{-\ell}\triangleq F_{N-\ell}$ if $\ell=0,...,N$.]

The change of index of summation $m=N-\ell$, that is, $\ell=N-m$, gives

$$\overline{f_k}=\frac{1}{\sqrt{N}}\left(\sum_{m=1}^{N-1}F_m\,\omega^{-k(N-m)}+F_0\right)=\frac{1}{\sqrt{N}}\left(\sum_{m=1}^{N-1}F_m\,\omega^{-kN}\omega^{km}+F_0\right)=\frac{1}{\sqrt{N}}\left(\sum_{m=1}^{N-1}F_m\,(1)^{-k}\cdot\omega^{km}+F_0\right)$$

$$=\frac{1}{\sqrt{N}}\left(\sum_{m=1}^{N-1}F_m\,\omega^{km}\right)=f_k.$$

So, $\overline{f_k}=f_k$ for all k, that is $\mathbf{f}$ is a sequence of real numbers.

Section 9.6

9.6.4.1. Similar to Example 9.30 in Section 9.6, multiply the ODE $0 = (p(x)X'(x))' + (\lambda s(x) + q(x))X(x) = 0$ by the eigenfunction $X(x)$ and then integrate over the interval $a < x < b$ to get

$$\int_a^b \lambda s(x)\big(X(x)\big)^2\,dx = -\int_a^b X(x)\,(p(x)X'(x))'\,dx - \int_a^b q(x)\big(X(x)\big)^2\,dx,$$

after rearranging terms. Next, use integration by parts to get

$$(\star)\quad \lambda\int_a^b s(x)\big(X(x)\big)^2 dx = -\big[\,p(x)X(x)X'(x)\,\big]_a^b + \int_a^b \Big(p(x)X'(x)\Big)X'(x)\,dx - \int_a^b q(x)\big(X(x)\big)^2 dx$$

$$= -p(b)X(b)X'(b) + p(a)X(a)X'(a) + \int_a^b p(x)\big(X'(x)\big)^2 dx - \int_a^b q(x)\big(X(x)\big)^2 dx.$$

In the last result of Theorem 9.15 in Section 9.6, we are assuming that $\epsilon_0, \epsilon_1 > 0$, so the first BC satisfied by the eigenfunction $X(x)$ implies that

$$X'(a) = \frac{\epsilon_0}{\epsilon_1}\,X(a).$$

In addition, we are assuming that this is a regular Sturm-Liouville problem, as defined in Definition 9.5, so either $\gamma_0 \neq 0$ or $\gamma_1 \neq 0$, as well as assuming that $p(x) > 0$ on $[a, b]$. Also, for the last result of Theorem 9.15 in Section 9.6, we are assuming also that $\gamma_0, \gamma_1 \geq 0$. It follows that either γ_1 or $\gamma_0 > 0$. For the moment, let us assume that $\gamma_1 > 0$. [At the end of the work we will mention how the rest of the work would proceed if, instead, we assume that $\gamma_0 > 0$.]

The assumption that $\gamma_1 > 0$, along with the second BC satisfied by the eigenfunction $X(x)$, implies that

$$X'(b) = -\frac{\gamma_0}{\gamma_1}\,X(b).$$

This and $(\star)$ imply that

$$(\star\star)\quad \lambda\int_a^b s(x)\big(X(x)\big)^2 dx = \frac{\gamma_0 p(b)}{\gamma_1}\,\big(X(b)\big)^2 + \frac{\epsilon_0 p(a)}{\epsilon_1}\,\big(X(a)\big)^2 + \int_a^b p(x)\big(X'(x)\big)^2 dx - \int_a^b q(x)\big(X(x)\big)^2 dx.$$

In the last result of Theorem 9.15 in Section 9.6, we are assuming that $\epsilon_0, \epsilon_1 > 0,\ \gamma_0, \gamma_1 \geq 0$ and $q(x) \leq 0$ on the interval $[\,a, b\,]$. It follows from $(\star\star)$, along with the assumption that the Sturm-Liouville problem is regular, that

$$(\star\star\star)\quad \lambda\int_a^b s(x)\big(X(x)\big)^2 dx \geq 0.$$

Recall that in Definition 9.5 we are assuming that $s(x) \geq 0$ and is not identically zero on $[\,a, b\,]$. It follows that $\int_a^b s(x)\big(X(x)\big)^2 dx \geq 0$. That, and $(\star\star\star)$, together imply that $\lambda \geq 0$. So, there can be no negative eigenvalue. The next to last thing we need to explain is why $\lambda = 0$ cannot be an eigenvalue.

If $\lambda = 0$ were an eigenvalue, then $(\star\star)$ would imply that

$$(\star\star\star\star)\quad 0 = \frac{\gamma_0 p(b)}{\gamma_1}\,\Big(X(b)\Big)^2 + \frac{\epsilon_0 p(a)}{\epsilon_1}\,\Big(X(a)\Big)^2 + \int_a^b p(x)\big(X'(x)\big)^2 dx - \int_a^b q(x)\big(X(x)\big)^2 dx.$$

Now, the previous assumptions mentioned, that $\epsilon_0, \epsilon_1 > 0,\ \gamma_0, \gamma_1 \geq 0$ and $q(x) \leq 0$ on the interval $[\,a, b\,]$, along with another part of Definition 9.5, namely that $p(x) > 0$ on $[\,a, b\,]$, imply that every one of the terms $\frac{\gamma_0 p(b)}{\gamma_1}\,\Big(X(b)\Big)^2$, $\frac{\epsilon_0 p(a)}{\epsilon_1}\,\Big(X(a)\Big)^2$, $\int_a^b p(x)\big(X'(x)\big)^2 dx$, and $-\int_a^b q(x)\big(X(x)\big)^2 dx$ are non-negative.

It follows from that and $(\star\star\star\star)$ that $0 = \frac{\gamma_0 p(b)}{\gamma_1}\,\Big(X(b)\Big)^2$, $0 = \frac{\epsilon_0 p(a)}{\epsilon_1}\,\Big(X(a)\Big)^2$, $0 = \int_a^b p(x)\big(X'(x)\big)^2 dx$, and $0 = -\int_a^b q(x)\big(X(x)\big)^2 dx$. The third of these equalities, along with the facts that both $\big(X'(x)\big)^2 \geq 0$ and $p(x) > 0$ on $[\,a, b\,]$ imply that $X'(x) \equiv 0$ on $[\,a, b\,]$. This implies that $X(x)$ is constant on $[\,a, b\,]$.

But $0 = \dfrac{\epsilon_0 p(a)}{\epsilon_1}\Big(X(a)\Big)^2$, along with the assumptions that $\epsilon_0, \epsilon_1 > 0$, implies that $X(a) = 0$. So, the constant $X(x)$ is identically zero on $[\,a, b\,]$. So, $\lambda = 0$ cannot be an eigenvalue.

The last thing to mention is how we would proceed if, instead of assuming that $\gamma_1 > 0$, we assume that $\gamma_0 > 0$. In that case, we would replace the term $-p(b)X(b)X'(b)$ by $\dfrac{\gamma_1 p(b)}{\gamma_0}\Big(X'(b)\Big)^2$, which is also non-negative because we are assuming that $\gamma_0, \gamma_1 \ge 0$, $p(x) > 0$ on $[\,a, b\,]$, and that $\gamma_1 > 0$. The term $\dfrac{\gamma_1 p(b)}{\gamma_0}\Big(X'(b)\Big)^2$ has the same sign properties and implications as the term $\dfrac{\gamma_0 p(b)}{\gamma_1}\Big(X(b)\Big)^2$ would have had in the subsequent reasoning.

9.6.4.3. (a) The ODE can be rewritten to be the Cauchy-Euler ODE $r^2R''+rR'+\lambda R = 0$, whose characteristic equation is $0 = n(n-1) + n + \lambda = n^2 + \lambda$.
Case 1: $\lambda = -\omega^2$ and $\omega > 0$, then $R = c_1 r^{\omega} + c_2 r^{-\omega}$. The BCs require

$$\left\{\begin{array}{c} 0 = R'(a) = \omega(c_1 a^{\omega-1} - c_2 a^{-\omega-1}) \\ \\ 0 = R'(b) = \omega(c_1 b^{\omega-1} - c_2 b^{-\omega-1}) \end{array}\right\},$$

hence

$$\begin{bmatrix} a^{\omega} & -a^{-\omega} \\ b^{\omega} & -b^{-\omega} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Because $\begin{vmatrix} a^{\omega} & -a^{-\omega} \\ b^{\omega} & -b^{-\omega} \end{vmatrix} = \left(\dfrac{b}{a}\right)^{\omega} - \left(\dfrac{a}{b}\right)^{\omega}$ and $a < b$, the determinant is never zero and so there is no eigenvalue $\lambda < 0$.

Case 2: $\lambda = 0$, then $R = c_1 + c_2 \ln r$. The BCs require $0 = R'(a) = c_2\,\frac{1}{a}$ and $0 = R'(b) = c_2\,\frac{1}{b}$, so all that is required is $c_2 = 0$. So, $\lambda = 0$ is an eigenvalue.

[We were not asked for the eigenfunctions, but we mention that for $\lambda = 0$ the eigenfunctions are given by $R(r) = c_1 R_0(r)$. where $R_0(r) \equiv 1$ and c_1 is an arbitrary nonzero constant.]

Case 3: $\lambda = \omega^2$ and $\omega > 0$, then $R(r) = c_1 \cos(\omega \ln r) + c_2 \sin(\omega \ln r)$, hence
$R'(r) = \dfrac{\omega}{r}\left(-c_1 \sin(\omega \ln r) + c_2 \cos(\omega \ln r)\right)$. The BCs require

$$\left\{\begin{array}{c} 0 = R'(a) = \dfrac{\omega}{a}(-c_1 \sin(\omega \ln a) + c_2 \cos(\omega \ln a)) \\ \\ 0 = R'(b) = \dfrac{\omega}{b}(-c_1 \sin(\omega \ln b) + c_2 \cos(\omega \ln b)) \end{array}\right\},$$

hence

$$\begin{bmatrix} -\sin(\omega \ln a) & \cos(\omega \ln a) \\ -\sin(\omega \ln b) & \cos(\omega \ln b) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Because

$$\begin{vmatrix} -\sin(\omega \ln a) & \cos(\omega \ln a) \\ -\sin(\omega \ln b) & \cos(\omega \ln b) \end{vmatrix} = \sin(\omega \ln b)\cos(\omega \ln a) - \cos(\omega \ln b)\sin(\omega \ln a) = \sin\big(\omega(\ln b - \ln a)\big) = \sin\left(\omega \ln \frac{b}{a}\right)$$

and $a < b$, the eigenvalues are $\lambda_n = \omega_n^2 = \left(\dfrac{n\pi}{\ln(b/a)}\right)^2$.

[We were not asked for the eigenfunctions, but we mention that for $\lambda = \lambda_n$, we can use the adjugate matrix method to find that the corresponding eigenfunctions are

$$R_n(r) = \cos(\omega_n\ \ln b)\cos(\omega_n\ \ln r) + \sin(\omega_n\ \ln b)\sin(\omega_n\ \ln r) = \cos\left(\frac{n\pi}{\ln(b/a)}(\ln b - \ln r)\right) = \cos\left(\frac{n\pi\ \ln(b/r)}{\ln(b/a)}\right).]$$

(b) According to Theorem 9.15, the orthogonality relation for the eigenfunctions $\{R_0(r); R_n(r)\}$ for the regular Sturm-Liouville problem

$$\left\{\begin{array}{c} \dfrac{d}{dr}\left[r\,\dfrac{dR}{dr}\right] + \lambda\,\dfrac{1}{r}\,R(r) = 0, \\ \\ R'(a) = R'(b) = 0 \end{array}\right\}:$$

is that, for $n \neq m$,

$$\int_a^b R_n(r)R_m(r)\,\frac{1}{r}\,dr = 0.$$

9.6.4.5. (a) To find the eigenvalues, we consider three separate cases: $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$.
Case 1: If $\lambda = 0$, then the differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) = 0$, whose solutions are $X = c_1 + c_2 x$, for arbitrary constants c_1, c_2. It follows that $X'(x) = c_2$. The BCs require

$$\left\{\begin{array}{c} 0 = X\left(\frac{\pi}{4}\right) - 3X'\left(\frac{\pi}{4}\right) = c_1 + c_2\,\frac{\pi}{4} - 3c_1 = -2c_1 + \frac{\pi}{4}c_2 \\ \\ 0 = X\left(\frac{3\pi}{4}\right) + X'\left(\frac{3\pi}{4}\right) = c_1 + c_2\,\frac{3\pi}{4} + c_1 = 2c_1 + \frac{3\pi}{4}\,c_2 \end{array}\right\},$$

hence

$$\begin{bmatrix} -2 & \frac{\pi}{4} \\ \\ 2 & \frac{3\pi}{4} \end{bmatrix}\begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}.$$

Because $\begin{vmatrix} -2 & \frac{\pi}{4} \\ 2 & \frac{3\pi}{4} \end{vmatrix} = -\dfrac{3\pi}{2} - \dfrac{\pi}{2} = -2\pi \neq 0$, so $\lambda = 0$ is not an eigenvalue.
Case 2: If $\lambda > 0$, it will turn out to be convenient to rewrite $\lambda = \omega^2$, where $\omega \triangleq \sqrt{\lambda}$. Then the differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) + \omega^2 X(x) = 0$, the undamped harmonic oscillator differential equation of Section 3.3 whose solutions are $X = c_1\cos\omega x + c_2\sin\omega x$, for constants c_1, c_2. The BCs require

$$\left\{\begin{array}{c} 0 = X\left(\frac{\pi}{4}\right) - 3X'\left(\frac{\pi}{4}\right) = c_1\cos\left(\frac{\omega\pi}{4}\right) + c_2\sin\left(\frac{\omega\pi}{4}\right) - 3\omega\left(-c_1\sin\left(\frac{\omega\pi}{4}\right) + c_2\cos\left(\frac{\omega\pi}{4}\right)\right) \\ \\ 0 = X\left(\frac{3\pi}{4}\right) + X'\left(\frac{3\pi}{4}\right) = c_1\cos\left(\frac{3\omega\pi}{4}\right) + c_2\sin\left(\frac{3\omega\pi}{4}\right) + \omega\left(-c_1\sin\left(\frac{3\omega\pi}{4}\right) + c_2\cos\left(\frac{3\omega\pi}{4}\right)\right) \end{array}\right\},$$

hence

$$\begin{bmatrix} \cos\left(\frac{\omega\pi}{4}\right) + 3\omega\sin\left(\frac{\omega\pi}{4}\right) & \sin\left(\frac{\omega\pi}{4}\right) - 3\omega\cos\left(\frac{\omega\pi}{4}\right) \\ \\ \cos\left(\frac{3\omega\pi}{4}\right) - \omega\sin\left(\frac{3\omega\pi}{4}\right) & \sin\left(\frac{3\omega\pi}{4}\right) + \omega\cos\left(\frac{3\omega\pi}{4}\right) \end{bmatrix}\begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Because $\left[\begin{array}{cc} \cos\left(\frac{\omega\pi}{4}\right) + 3\omega\sin\left(\frac{\omega\pi}{4}\right) & \sin\left(\frac{\omega\pi}{4}\right) - 3\omega\cos\left(\frac{\omega\pi}{4}\right) \\ \\ \cos\left(\frac{3\omega\pi}{4}\right) - \omega\sin\left(\frac{3\omega\pi}{4}\right) & \sin\left(\frac{3\omega\pi}{4}\right) + \omega\cos\left(\frac{3\omega\pi}{4}\right) \end{array}\right|$

$$= \left(\cos\left(\frac{\omega\pi}{4}\right) + 3\omega\sin\left(\frac{\omega\pi}{4}\right)\right)\left(\sin\left(\frac{3\omega\pi}{4}\right) + \omega\cos\left(\frac{3\omega\pi}{4}\right)\right)$$

$$-\left(\sin\left(\frac{\omega\pi}{4}\right) - 3\omega\cos\left(\frac{\omega\pi}{4}\right)\right)\left(\cos\left(\frac{3\omega\pi}{4}\right) - \omega\sin\left(\frac{3\omega\pi}{4}\right)\right)$$

$$= ... = 4\omega\left(\cos\left(\frac{3\omega\pi}{4}\right)\cos\left(\frac{\omega\pi}{4}\right) + \sin\left(\frac{3\omega\pi}{4}\right)\sin\left(\frac{\omega\pi}{4}\right)\right)$$

$$+(1-3\omega^2)\left(\sin\left(\frac{3\omega\pi}{4}\right)\cos\left(\frac{\omega\pi}{4}\right) - \cos\left(\frac{3\omega\pi}{4}\right)\sin\left(\frac{\omega\pi}{4}\right)\right)$$

$$= 4\omega\cos\left(\frac{2\omega\pi}{4}\right) + (1-3\omega^2)\sin\left(\frac{2\omega\pi}{4}\right).$$

So, the characteristic equation is

$$0 = (1-3\lambda)\sin\left(\frac{\pi\sqrt{\lambda}}{2}\right) + \sqrt{\lambda}\,\cos\left(\frac{\pi\sqrt{\lambda}}{2}\right).$$

It is not possible to find the exact eigenvalues, but we could approximate them using a root finding method from Section 8.1.

Case 3: If $\lambda < 0$, it will turn out to be convenient to rewrite $\lambda = -\omega^2$, where $\omega \triangleq \sqrt{-\lambda}$. Then the differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) - \omega^2 X(x) = 0$, whose solutions are $X = c_1\cosh(\omega x) + c_2\sinh(\omega x)$, for arbitrary constants c_1, c_2. The BCs require

$$\left\{\begin{array}{l} 0 = X\left(\frac{\pi}{4}\right) - 3X'\left(\frac{\pi}{4}\right) = c_1\cosh\left(\frac{\omega\pi}{4}\right) + c_2\sinh\left(\frac{\omega\pi}{4}\right) - 3\omega\left(c_1\sinh\left(\frac{\omega\pi}{4}\right) + c_2\cosh\left(\frac{\omega\pi}{4}\right)\right) \\ \\ 0 = X\left(\frac{3\pi}{4}\right) + X'\left(\frac{3\pi}{4}\right) = c_1\cosh\left(\frac{3\omega\pi}{4}\right) + c_2\sinh\left(\frac{3\omega\pi}{4}\right) + \omega\left(c_1\sinh\left(\frac{3\omega\pi}{4}\right) + c_2\cosh\left(\frac{3\omega\pi}{4}\right)\right) \end{array}\right\},$$

hence

$$\begin{bmatrix} \cosh\left(\frac{\omega\pi}{4}\right) - 3\omega\sinh\left(\frac{\omega\pi}{4}\right) & \sinh\left(\frac{\omega\pi}{4}\right) - 3\omega\cosh\left(\frac{\omega\pi}{4}\right) \\ \\ \cosh\left(\frac{3\omega\pi}{4}\right) + \omega\sinh\left(\frac{3\omega\pi}{4}\right) & \sinh\left(\frac{3\omega\pi}{4}\right) + \omega\cosh\left(\frac{3\omega\pi}{4}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Because $\left|\begin{matrix} \cosh\left(\frac{\omega\pi}{4}\right) - 3\omega\sinh\left(\frac{\omega\pi}{4}\right) & \sinh\left(\frac{\omega\pi}{4}\right) - 3\omega\cosh\left(\frac{\omega\pi}{4}\right) \\ \\ \cosh\left(\frac{3\omega\pi}{4}\right) + \omega\sinh\left(\frac{3\omega\pi}{4}\right) & \sinh\left(\frac{3\omega\pi}{4}\right) + \omega\cosh\left(\frac{3\omega\pi}{4}\right) \end{matrix}\right|$

$$\begin{aligned}
&= \left(\cosh\left(\frac{\omega\pi}{4}\right) - 3\omega\sinh\left(\frac{\omega\pi}{4}\right)\right)\left(\sinh\left(\frac{3\omega\pi}{4}\right) + \omega\cosh\left(\frac{3\omega\pi}{4}\right)\right) \\
&\quad - \left(\sinh\left(\frac{\omega\pi}{4}\right) - 3\omega\cosh\left(\frac{\omega\pi}{4}\right)\right)\left(\cosh\left(\frac{3\omega\pi}{4}\right) + \omega\sinh\left(\frac{3\omega\pi}{4}\right)\right) \\
&= \ldots = 4\omega\left(\cosh\left(\frac{3\omega\pi}{4}\right)\cosh\left(\frac{\omega\pi}{4}\right) - \sinh\left(\frac{3\omega\pi}{4}\right)\sinh\left(\frac{\omega\pi}{4}\right)\right) \\
&\quad + (1+3\omega^2)\left(\sinh\left(\frac{3\omega\pi}{4}\right)\cosh\left(\frac{\omega\pi}{4}\right) - \cosh\left(\frac{3\omega\pi}{4}\right)\sinh\left(\frac{\omega\pi}{4}\right)\right) \\
&= 4\omega\cosh\left(\frac{2\omega\pi}{4}\right) + (1+3\omega^2)\sinh\left(\frac{2\omega\pi}{4}\right).
\end{aligned}$$

Because $\sinh\left(\frac{\omega\pi}{2}\right) > 0$ and $\cosh\left(\frac{\omega\pi}{2}\right) > 0$ for all $\omega > 0$, there is no eigenvalue for $\lambda < 0$.

(b) In the context of Theorem 9.15 in Section 9.6, the ODE $X'' + \lambda X = 0$ has $s(x) \equiv 1$. So, the orthogonality relation is that $0 = \int_{\pi/4}^{3\pi/4} X_n(x) X_m(x)\,dx$ for $n \neq m$.

9.6.4.7. (a) (i) For $\lambda = 0$ the ODE becomes $X''''(x) = 0$, so $X(x) = c_0 + c_1 x + \frac{1}{2}c_2 x^2 + \frac{1}{6}c_3 x^3$, where the c_i's are arbitrary constants. Substitute $X(x)$ and $X'(x) = c_1 + c_2 x + \frac{1}{2}c_3 x$ into the first two BCs to get

$$\left\{\begin{array}{ll} 0 = & X(0) = c_0 \\ 0 = & X'(0) = c_1 \end{array}\right\}.$$

Substitute these into $X(x)$ to reduce it to $X(x) = \frac{1}{2}c_2 x^2 + \frac{1}{6}c_3 x^3$. Substitute $X(x)$ and $X'(x) = c_2 x + \frac{1}{2}c_3 x^2$ into the last two BCs to get the system of equations

$$\left\{\begin{array}{ll} 0 = X(L) = \frac{1}{2}c_2 L^2 & +\frac{1}{6}c_3 L^3 \\ \\ 0 = X'(L) = c_2 L & +\frac{1}{2}c_3 L^2 \end{array}\right\},$$

that is,

$$\begin{bmatrix} \frac{1}{2}\,L^2 & \frac{1}{6}\,L^3 \\ \\ L & \frac{1}{2}\,L^2 \end{bmatrix} \begin{bmatrix} c_2 \\ \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}.$$

Because

$$\begin{vmatrix} \frac{1}{2}\,L^2 & \frac{1}{6}\,L^3 \\ \\ L & \frac{1}{2}\,L^2 \end{vmatrix} = \frac{1}{12}\,L^3 \neq 0,$$

it follows that $c_2 = c_3 = 0$. This yields $X(x) \equiv 0$, so $\lambda = 0$ is not an eigenvalue of this fourth order ODE-BVP.

(ii) Suppose $\lambda < 0$. Define $\omega = (-\lambda)^{1/4}$ for convenience. The ODE $X'''' - \omega^4 X = 0$ has characteristic polynomial $s^4 - \omega^4 = (s^2 - \omega^2)(s^2 + \omega^2)$, which has roots $s = \pm\omega, \pm i\omega$. The solutions of the ODE are

$$X(x) = c_1 \cosh(\omega x) + c_2 \sinh(\omega x) + c_3 \cos(\omega x) + c_4 \sin(\omega x).$$

Substitute that and $X'(x) = \omega\big(c_1 \sinh(\omega x) + c_2 \cosh(\omega x) - c_3 \sin(\omega x) + c_4 \cos(\omega x)\big)$ into the first two BCs to get

$$\left\{ \begin{array}{rl} 0 = & X(0) = c_1 + c_3 \\ 0 = & X'(0) = c_2 + c_4 \end{array} \right\},$$

which implies $c_1 = -c_3$ and $c_2 = -c_4$. Substitute these into $X(x)$ to reduce it to

$$X(x) = c_3\big(\cos(\omega x) - \cosh(\omega x)\big) + c_4\big(\sin(\omega x) - \sinh(\omega x)\big).$$

Substitute $X(x)$ and $X'(x) = \omega\Big(c_3\big(-\sin(\omega x) - \sinh(\omega x)\big) + c_4\big(\cos(\omega x) - \cosh(\omega x)\big)\Big)$ into the last two BCs to get the system of equations

$$\left\{ \begin{array}{rll} 0 = & X(L) & = c_3\big(\cos(\omega L) - \cosh(\omega L)\big) + c_4\big(\sin(\omega L) - \sinh(\omega L)\big) \\ \\ 0 = & X'(L) & = \omega\Big(c_3\big(-\sin(\omega L) - \sinh(\omega L)\big) + c_4\big(\cos(\omega L) - \cosh(\omega L)\big)\Big) \end{array} \right\},$$

or, equivalently,

$$(\star\star) \begin{bmatrix} \cos(\omega L) - \cosh(\omega L) & \sin(\omega L) - \sinh(\omega L) \\ -\sin(\omega L) - \sinh(\omega L) & \cos(\omega L) - \cosh(\omega L) \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

There is a non-trivial solution for $X(x)$ if, and only if,

$$\begin{aligned} 0 &= \begin{vmatrix} \cos(\omega L) - \cosh(\omega L) & \sin(\omega L) - \sinh(\omega L) \\ -\sin(\omega L) - \sinh(\omega L) & \cos(\omega L) - \cosh(\omega L) \end{vmatrix} \\ &= \big(\cos(\omega L) - \cosh(\omega L)\big)\big(\cos(\omega L) - \cosh(\omega L)\big) - \big(\sin(\omega L) - \sinh(\omega L)\big)\big(-\sin(\omega L) - \sinh(\omega L)\big) \\ &= \Big(\cos^2(\omega L) + \sin^2(\omega L)\Big) + \Big(\cosh^2(\omega L) - \sinh^2(\omega L)\Big) - 2\cosh(\omega L)\cos(\omega L) \end{aligned}$$

that is, if and only if,

$$0 = 1 - \cosh(\omega L)\cos(\omega L).$$

This could also be rewritten as

$$\cos(\theta) = \frac{1}{\cosh(\theta)},$$

where $\theta = \omega L$. Graphed below are the functions $\cos(\theta)$, in a dashed curve, and $\frac{1}{\cosh(\theta)}$, in a solid curve. Their points of intersection give eigenvalues.

There are infinitely many eigenvalues $\lambda_n = -\omega_n^4$, where $\omega_n = \frac{\theta_n}{L} \to \infty$, as $n \to \infty$. Because $\cosh(\theta) \to \infty$, as $\theta \to \infty$, the roots $\theta_n \sim \big(n - \frac{1}{2}\big)\pi$ as $n \to \infty$.

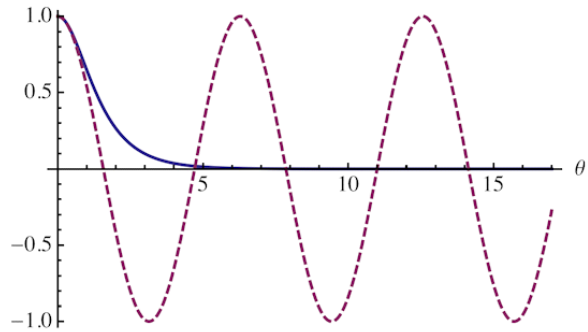


Figure 17: Problem 9.6.4.7

(iii) Suppose $\lambda > 0$. Define $\omega = (\lambda)^{1/4}$ for convenience. The ODE $X'''' + \omega^4 X = 0$ has characteristic polynomial $s^4 + \omega^4$, which implies $s^2 = \pm i\omega = \pm e^{i\pi/2}$ and thus the roots are $s = \pm e^{i\pi/4}, \pm e^{i5\pi/4}$. These make for quite a complicated general solution of the ODE. It might be easier to just use the method of Example 9.30 to explain why $\lambda > 0$ cannot be an eigenvalue:

Suppose that λ is an eigenvalue. Multiply through the ODE by $X(x)$ and integrate on the interval $0 < x < L$ to get

$$0 = \int_0^L X(x)X''''(x)dx + \lambda \int_0^L \big(X(x)\big)^2 dx.$$

Integrate by parts to get

$$0 = \Big[X(x)X'''(x)\Big]_0^L - \int_0^L X'(x)X'''(x)dx + \lambda \int_0^L \big(X(x)\big)^2 dx.$$

The BCs $X(0) = X(L) = 0$ explains why this implies that

$$\lambda \int_0^L \big(X(x)\big)^2 dx = \int_0^L X'(x)X'''(x)dx,$$

and another use of integration by parts gives

$$\lambda \int_0^L \big(X(x)\big)^2 dx = \Big[X'(x)X''(x)\Big]_0^L - \int_0^L X''(x)X''(x)dx.$$

The BCs $X'(0) = X'(L) = 0$ explains why this implies that

$$(\star\star\star) \quad \lambda \int_0^L \big(X(x)\big)^2 dx = -\int_0^L X''(x)\big)^2 dx.$$

Because λ is an eigenvalue, we cannot have the continuous function $X(x) \equiv 0$ on the interval $0 < x < L$. It follows that $\int_0^L \big(X(x)\big)^2 dx > 0$. This and $(\star\star\star)$ imply that $\lambda \le 0$. This contradictions our assumption that $\lambda > 0$, so there cannot be an eigenvalue $\lambda > 0$.

(b) Regarding orthogonality, we can use the method of Example 9.29: Suppose λ_n and λ_m are two distinct eigenvalues, with corresponding eigenfunctions $X_n(x)$ and $X_m(x)$. From the ODE, with λ replaced by λ_n and $X(x)$ replaced by $X_n(x)$, we get

$$(1) \quad X_n''''(x) + \lambda_n X_n(x) = 0, \; a < x < b,$$

and similarly we get

$$(2) \quad X_m''''(x) + \lambda_m X_m(x) = 0, \; a < x < b.$$

Multiply ODE (1) by $X_m(x)$, and subtract from that $X_n(x)$ times ODE (2) to get

$$0 = X_m(x)X_n''''(x) + \lambda_n X_m(x)X_n(x) - X_n(x)X_m''''(x) - \lambda_m X_n(x)X_m(x).$$

Integrate from 0 to L to get

$$(\star)\quad 0=\int_0^L 0\,dx=\int_0^L \big(X_m(x)X_n''''(x)-X_n(x)X_m''''(x)\big)dx+(\lambda_n-\lambda_m)\int_0^L X_n(x)X_m(x)dx.$$

For the first term, integration by parts gives

$$\int_0^L \big(X_m(x)X_n''''(x)-X_n(x)X_m''''(x)\big)dx$$

$$=\Big[\,X_m(x)X_n'''(x)-X_n(x)X_m'''(x)\,\Big]_0^L-\int_0^L\big(X_m'(x)X_n'''(x)-X_n'(x)X_m'''(x)\big)dx$$

$$=X_m(L)X_n'''(L)-X_n(L)X_m'''(L)-X_m(0)X_n'''(0)+X_n(0)X_m'''(0)-\int_0^L\big(X_m'(x)X_n'''(x)-X_n'(x)X_m'''(x)\big)dx.$$

Using the BCs $X_n(L)=X_m(L)=X_n(0)=X_m(L)=0$, followed by another use of integration by parts, we have

$$\int_0^L \big(X_m(x)X_n''''(x)-X_n(x)X_m''''(x)\big)dx=0-\int_0^L\big(X_m'(x)X_n'''(x)-X_n'(x)X_m'''(x)\big)dx$$

$$=\Big[\,X_m'(x)X_n''(x)-X_n'(x)X_m''(x)\,\Big]_0^L-\int_0^L\big(X_m''(x)X_n''(x)-X_n''(x)X_m''(x)\big)dx=0\,,$$

using the BCs $X_n'(L)=X_m'(L)=X_n'(0)=X_m'(L)=0$, and canceling $\big(X_m'(x)X_n'''(x)-X_n'(x)X_m'''(x)\big)\equiv 0$.

So, $(\star)$ reduces to

$$0=(\lambda_n-\lambda_m)\int_0^L X_n(x)X_m(x)dx.$$

Because $\lambda_n\neq\lambda_m$, we can divide through by $(\lambda_n-\lambda_m)$ to get the orthogonality relation

$$0=\int_0^L X_n(x)X_m(x)\,dx,\ \text{ for } n\neq m.$$

9.6.4.9. The graph of $g(\omega)\triangleq\sqrt{\dfrac{2}{3}}+\dfrac{\tan(\sqrt{2}\omega)}{\tan(\sqrt{3}\omega)}$, where $\omega\triangleq\sqrt{\lambda}$, shown below, *appears* to give infinitely many eigenvalues $\lambda_n=\omega_n^2$, where $\omega_n\sim n$ as $n\to\infty$.

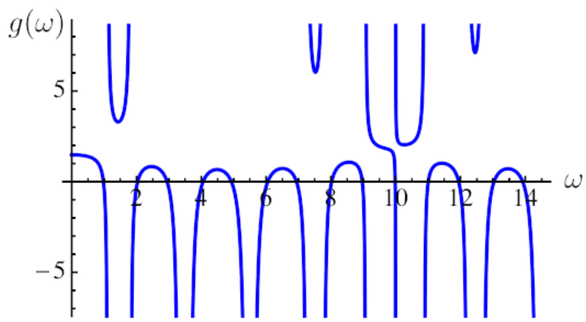


Figure 18: Problem 9.6.4.9

9.6.4.11. We are assuming that $0<\lambda_n=\omega_n^2$. The general solution of the ODE is $X(x)=c_1\cos(\omega_n x)+c_2\sin(\omega_n x)$. Substitute this and $X(x)=-c_1\omega_n\sin(\omega_n x)+c_2\omega_n\cos(\omega_n x)$ into the two BCs to get

$$\left\{\begin{array}{l} 0=\epsilon_0 X(a)-\epsilon_1 X'(a)=\epsilon_0\big(c_1\cos(\omega_n a)+c_2\sin(\omega_n a)\big)-\epsilon_1\big(-c_1\omega_n\sin(\omega_n a)+c_2\omega_n\cos(\omega_n a)\big)\\ \\ 0=\gamma_0 X(b)+\gamma_1 X'(b)=\gamma_0\big(c_1\cos(\omega_n b)+c_2\sin(\omega_n b)\big)+\gamma_1\big(-c_1\omega_n\sin(\omega_n b)+c_2\omega_n\cos(\omega_n b)\big)\end{array}\right\},$$

or, equivalently,

$$(\star)\qquad \begin{bmatrix} \epsilon_0\cos(\omega_n a)+\epsilon_1\omega_n\sin(\omega_n a) & \epsilon_0\sin(\omega_n a)-\epsilon_1\omega_n\cos(\omega_n a) \\ \\ \gamma_0\cos(\omega_n b)-\gamma_1\omega_n\sin(\omega_n b) & \gamma_0\sin(\omega_n b)+\gamma_1\omega_n\cos(\omega_n b) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

There is a non-trivial solution for $X(x)$ if, and only if,

$$0 = \begin{vmatrix} \epsilon_0\cos(\omega_n a)+\epsilon_1\omega_n\sin(\omega_n a) & \epsilon_0\sin(\omega_n a)-\epsilon_1\omega_n\cos(\omega_n a) \\ \\ \gamma_0\cos(\omega_n b)-\gamma_1\omega_n\sin(\omega_n b) & \gamma_0\sin(\omega_n b)+\gamma_1\omega_n\cos(\omega_n b) \end{vmatrix}$$

The adjugate of the 2×2 matrix in $(\star)$ is

$$\begin{bmatrix} \gamma_0\sin(\omega_n b)+\gamma_1\omega_n\cos(\omega_n b) & -\epsilon_0\sin(\omega_n a)+\epsilon_1\omega_n\cos(\omega_n a) \\ \\ -\gamma_0\cos(\omega_n b)+\gamma_1\omega_n\sin(\omega_n b) & \epsilon_0\cos(\omega_n a)+\epsilon_1\omega_n\sin(\omega_n a) \end{bmatrix}.$$

If the first column of the adjugate matrix is nonzero it gives eigenfunctions
$X_n(x) = \big(\gamma_0\,\sin(\omega_n b)+\gamma_1\,\omega_n\cos(\omega_n b)\big)\cos(\omega_n x)\ +\big(-\gamma_0\,\cos(\omega_n b)+\gamma_1\,\omega_n\sin(\omega_n b)\big)\sin(\omega_n x).$

If the second column of the adjugate is nonzero it gives eigenfunctions
$X_n(x) = \big(-\epsilon_0\,\sin(\omega_n a)+\epsilon_1\,\omega_n\cos(\omega_n a)\big)\cos(\omega_n x)\ +\big(\epsilon_0\,\cos(\omega_n a)+\epsilon_1\,\omega_n\sin(\omega_n a)\big)\sin(\omega_n x).$

Note that at least one of the columns of the adjugate matrix must be a nonzero vector in $\mathbb{R}^2$, because of Theorem 1.33 in Section 1.7 and the fact that Theorem 9.15 says that "...the only eigenfunctions are the nonzero multiples of a single eigenfunction...," hence the 2×2 matrix in $(\star)$ cannot have rank equal to zero.

Section 9.7

9.7.1.1. Begin by calculating the left hand side, that is, by substituting $\beta X(x)$ into the Rayleigh quotient, as given in (9.83) in Section 9.7:

$$\mathcal{R}_{SL}[\,\beta X(x)\,] \triangleq \frac{-\big[\,p(x)\beta X(x)\beta X'(x)\,\big]_a^b + \displaystyle\int_a^b \Big(p(x)\big(\beta X'(x)\big)^2 - q(x)\big(\beta X(x)\big)^2\Big)dx}{\displaystyle\int_a^b s(x)\big(\beta X(x)\big)^2 dx}$$

$$= \frac{-\cancel{\beta^2}\big[\,p(x)X(x)X'(x)\,\big]_a^b + \cancel{\beta^2}\displaystyle\int_a^b \Big(p(x)\big(X'(x)\big)^2 - q(x)\big(X(x)\big)^2\Big)dx}{\cancel{\beta^2}\displaystyle\int_a^b s(x)\big(X(x)\big)^2 dx}$$

$$= \frac{-\big[\,p(x)X(x)X'(x)\,\big]_a^b + \displaystyle\int_a^b \Big(p(x)\big(X'(x)\big)^2 - q(x)\big(X(x)\big)^2\Big)dx}{\displaystyle\int_a^b s(x)\big(X(x)\big)^2 dx} \triangleq \mathcal{R}_{SL}[\,X(x)\,]\,.$$

9.7.1.3. Using (9.83) in Section 9.7, we have that the Rayleigh quotient is

$$\mathcal{R}_{SL}[X(x)] = \frac{-\big(X(L)X'(L)-X(0)X'(0)\big) + \displaystyle\int_0^L \big(X'(x)\big)^2 dx}{\displaystyle\int_0^L \big(X(x)\big)^2 dx} = \frac{h\big(X(L)\big)^2 + \displaystyle\int_0^L \big(X'(x)\big)^2 dx}{\displaystyle\int_0^L \big(X(x)\big)^2 dx}\,.$$

As in Example 9.35 in Section 9.7, the family of trial functions$X(x) = 1 + \mu x$ gives

$$\mathcal{R}_{SL}[\,1+\mu x\,] = \frac{h(1+\mu L)^2 + \int_0^L \mu^2 dx}{\displaystyle\int_0^L (1+\mu x)^2\,dx} = \frac{h(1+2\mu L+\mu^2L^2)+\mu^2 L}{L+\mu L^2+\frac{1}{3}\mu^2L^3} = \frac{h(1+2\mu L+\mu^2L^2)+\mu^2 L}{L(1+\mu L+\frac{1}{3}\mu^2L^2)} \triangleq F(\mu).$$

An estimate for the minimum eigenvalue λ_1 would be the minimum of $F(\mu)$ over $-\infty < \mu < \infty$, for a given h and L.

9.7.1.5. Using Table 9.1 in Section 9.3, we see that $X(x) \triangleq \cos \dfrac{\pi x}{L}$ is the simplest function of the form $\cos \omega x$ that satisfies the BCs $X'(0) = X'(L) = 0$.

Using (9.83) in Section 9.7, we have that the Rayleigh quotient is

$$\mathcal{R}_{SL}(X(x)) = \frac{\displaystyle\int_0^L \big(X'(x)\big)^2 dx}{\displaystyle\int_0^L x\big(X(x)\big)^2 dx}\,.$$

This gives as an estimate for minimum eigenvalue

$$\lambda_1 \approx \mathcal{R}_{SL}\Big[\,\cos\frac{\pi x}{L}\,\Big] = \frac{\displaystyle\int_0^L \Big(-\frac{\pi}{L}\,\sin\frac{\pi x}{L}\Big)^2 dx}{\displaystyle\int_0^L x\Big(\cos\frac{\pi x}{L}\Big)^2 dx} = \ldots = \frac{\pi^2/(2L)}{L^2/4} = \frac{2\pi^2}{L^3}\,.$$

9.7.1.7. (a) Using (9.83) in Section 9.7, we have that the Rayleigh quotient is

$$\mathcal{R}_{SL}(X) = \frac{\displaystyle\int_0^L AC(X'(x))^2 dx}{\displaystyle\int_0^L x^2(X(x))^2 dx}$$

(b) Because $X(x)$ satisfies the ODE, two things are true: (1) $X''(x) = -\dfrac{P^2}{AC}\big(x^2X(x)\big)$, for $0 < x < L$, and (2) $X(x)$ is continuous on the interval $0 \le x \le L$. The latter implies that there exists a finite number α such that

$$\alpha = X(0^+) \triangleq \lim_{x\to 0^+} X(x).$$

Combining this with (1) implies that there exists

$$X''(0^+) = \lim_{x\to 0^+} -\frac{P^2}{AC}\big(x^2X(x)\big) = -\frac{P^2}{AC}\big(0^2\cdot X(0^+)\big) = 0.$$

(c) Because $X(x)$ satisfies the ODE, two things are true: (1) $X''(x) = -\dfrac{P^2}{AC}\big(x^2X(x)\big)$, for $0 < x < L$, and (2) $X(x)$ is continuous on the interval $0 \le x \le L$. The latter implies that there exists a finite number α such that

$$\alpha = X(L^-) \triangleq \lim_{x\to L^-} X(x).$$

Combining this with (1) and the BC $X(L) = 0$ to see that

$$X''(L^-) = \lim_{x\to L^-} -\frac{P^2}{AC}\big(x^2X(x)\big) = -\frac{P^2}{AC}\big(L^2\cdot 0\big) = 0.$$

(d) Take the derivative with respect to x of both sides of the ODE to conclude that $ACX''' + P^2(2xX + x^2X') = 0$. This, along with the BC $X'(0) = 0$ and the fact that that there exists a finite number α such that $\alpha = X(0^+) \triangleq \lim_{x\to 0^+} X(x)$, implies that

$$X'''(0^+) = \lim_{x\to 0^+} -\frac{P^2}{AC}\big(2xX + x^2X'\big) = -\frac{P^2}{AC}\big(2\cdot 0\cdot X(0^+) + 0^2\cdot 0)\big) = 0.$$

(e) Look for a single trial function of the form

$$q(x) = \alpha + \beta x + \frac{1}{2}\,\xi x^2 + \frac{1}{6}\,\eta x^3 + \frac{1}{24}\,\gamma x^4 + \frac{1}{120}\,\mu x^5$$

that satisfies the five BCs: Use the facts that

$$q'(x) = \beta + \xi x + \frac{1}{2}\,\eta x^2 + \frac{1}{6}\,\gamma x^3 + \frac{1}{24}\,\mu x^4 \quad \text{and} \quad q''(x) = \xi + \eta x + \frac{1}{2}\,\gamma x^2 + \frac{1}{6}\,\mu x^3$$

to get

$$\left\{\begin{array}{l} 0 = q'(0) = \beta \\ 0 = q''(0) = \xi \\ 0 = q'''(0) = \eta \end{array}\right\}.$$

and thus

$$q(x) = \alpha + \frac{1}{24}\,\gamma x^4 + \frac{1}{120}\,\mu x^5. \quad \text{and} \quad q''(x) = \frac{1}{2}\,\gamma x^2 + \frac{1}{6}\,\mu x^3.$$

The remaining two BCs are

$$\left\{\begin{array}{l} 0 = q(L) = \alpha + \frac{1}{24}\,\gamma L^4 + \frac{1}{120}\,\mu L^5 \\ \\ 0 = q''(L) = \frac{1}{2}\,\gamma L^2 + \frac{1}{6}\,\mu L^3 \end{array}\right\}.$$

It follows from $q''(L) = 0$ that $\mu = -\frac{3}{L}\,\gamma$, hence $q(L) = 0$ implies that

$$\alpha = -\frac{1}{24}\,\gamma L^4 - \frac{1}{120}\,\mu L^5 = -\frac{1}{24}\,\gamma L^4 - \frac{1}{120}\cdot\left(-\frac{3}{L}\,\gamma\right)L^5 = \left(-\frac{1}{24} + \frac{1}{40}\right)\gamma L^4 = -\frac{1}{60}\,\gamma L^4,$$

so

$$q(x) = -\frac{1}{60}\,\gamma L^4 + \frac{1}{24}\,\gamma x^4 - \frac{1}{120}\cdot\frac{3}{L}\,\gamma x^5 = -\frac{1}{120}\,L^4\,\gamma\left(2 - 5\left(\frac{x}{L}\right)^4 + 3\left(\frac{x}{L}\right)^5\right).$$

For convenience, choose $\gamma = -120L^{-4}$. So,

$$q(x) = 2 - 5\left(\frac{x}{L}\right)^4 + 3\left(\frac{x}{L}\right)^5$$

is an example of a fifth degree polynomial that satisfies the five BCs.

Using (9.83) in Section 9.7, we have that the Rayleigh quotient is

$$\mathcal{R}_{SL}(X(x)) = \frac{AC\displaystyle\int_0^L \big(X'(x)\big)^2 dx}{\displaystyle\int_0^L x^2\big(X(x)\big)^2 dx}.$$

This gives as an estimate for minimum eigenvalue

$$P_1^2 = \lambda_1 \approx \mathcal{R}_{SL}\Big[\,2 - 5\left(\frac{x}{L}\right)^4 + 3\left(\frac{x}{L}\right)^5\,\Big] = \frac{AC\displaystyle\int_0^L \left(\frac{1}{L}\Big(-20\left(\frac{x}{L}\right)^3 + 15\left(\frac{x}{L}\right)^4\Big)\right)^2 dx}{\displaystyle\int_0^L x^2\left(2 - 5\left(\frac{x}{L}\right)^4 + 3\left(\frac{x}{L}\right)^5\right)^2 dx}$$

$$= \frac{50AC/(7L)}{1325L^3/3003} = \frac{50\cdot 3003AC}{1325\cdot 7L^4},$$

so
$$P_1 \approx \sqrt{\frac{50\cdot 3003AC}{1325\cdot 7L^4}} = \sqrt{\frac{50\cdot 3003}{1325\cdot 7}}\cdot\frac{\sqrt{AC}}{L^2} = ... = \sqrt{\frac{858}{53}}\cdot\frac{\sqrt{AC}}{L^2}\,.$$

Section 9.8

9.8.4.1. Assume that the Fourier sine series of f on the interval $[\,0, L\,]$ is given by
$$f(x) \doteq f_{sin}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$
where $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$. Define $\langle f(x),\ f(x)\rangle \triangleq \displaystyle\int_0^L |f(x)|^2 dx$.

Method I: Using work similar to that which established Theorem 9.17, using orthogonality of $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ for positive integers $m \neq n$, we have
$$\int_0^L |f(x)|^2 dx = \langle f(x),\ f(x)\rangle = \left\langle \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right), \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right\rangle$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_m b_n \left\langle\ \sin\left(\frac{m\pi x}{L}\right),\ \sin\left(\frac{n\pi x}{L}\right)\right\rangle = \sum_{n=1}^{\infty} b_n b_n\,\frac{L}{2} = L\cdot\frac{1}{2}\sum_{n=1}^{\infty} b_n^2,$$
hence
$$\frac{1}{L}\int_0^L |f(x)|^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.$$

Method II: Using the result of Theorem 9.17 specifically for a function $f(x)$ that is odd on the interval $[-L, L]$, hence $a_n = 0$ for $n = 0, 1, 2, ...$, oddness of $f(x)$ implies that
$$\frac{1}{L}\int_0^L |f(x)|^2 dx = \frac{1}{2L}\int_{-L}^L |f(x)|^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = 0 + \frac{1}{2}\sum_{n=1}^{\infty}(0 + b_n^2),$$
hence
$$\frac{1}{L}\int_0^L |f(x)|^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.$$

9.8.4.3. Here, $a_0 = 1$, $a_n = e^{-n}$, and $b_n = e^{-2n}$, for $n = 1, 2, 3, ...$. Using sums of geometric series, Theorem 9.17 implies that
$$Power = \int_0^{2L} R\big(I(t)\big)^2 = \frac{L}{2} + L\sum_{n=1}^{N}\left(e^{-2n} + e^{-4n}\right) = \frac{L}{2} + L\sum_{n=1}^{N}\left((e^{-2})^n + (e^{-4})^n\right)$$
$$= L\left(\frac{1}{2} + \frac{1}{1 - e^{-2}} + \frac{1}{1 - e^{-4}}\right).$$

9.8.4.5. Ex.1: For example using Section 9.1's Example 9.5's result for a square wave function with, specifically, $L = \pi$,
$$f(x) = \left\{\begin{array}{rc} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{array}\right\},$$

whose Fourier series is

$$f(x) \doteq f_s(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin\left(\frac{(2k-1)\pi x}{L}\right).$$

and Parseval's Theorem 9.18, we get

$$1 = \frac{1}{2\pi}\int_0^{\pi} |f(x)|^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{\pi(2k-1)}\right)^2 = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right).$$

It follows that

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+... \,.$$

Ex.2: For example, using Section 9.1's Example 9.3's result result for $f(x)=(x-1)^2$ defined only on the interval $-2<x<2$, with $L=2$, whose Fourier series is

$$(x-1)^2 \doteq f_s(x) = \frac{7}{3} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{16}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{8}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right)$$

and Parseval's Theorem 9.17, we get

$$\frac{61}{5} = \frac{1}{2\cdot 2}\int_{-2}^{2}|f(x)|^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right) = \frac{49}{9} + \frac{32}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{4}{n^4\pi^2}+\frac{1}{n^2}\right).$$

It follows that

$$\sum_{n=1}^{\infty}\left(\frac{4}{\pi^2 n^4}+\frac{1}{n^2}\right) = \frac{\pi^2}{32}\left(\frac{61}{5}-\frac{49}{9}\right) = \frac{19\pi^2}{90}\,.$$

One could stop there, or go on to use it, along with the result of problem 9.8.4.4, to get another identity:

$$\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^4} = \frac{19\pi^2}{90} - \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{19\pi^2}{90} - \frac{\pi^2}{6} = \frac{4\pi^2}{90},$$

hence

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}\,.$$

9.8.4.7. For any square integrable function $f(x)$, express it in its Fourier series on the interval $-\pi<x<\pi$. Parseval's Theorem 9.17 gives

$$\infty > ||\, f\, ||_2^2 = \frac{\pi a_0^2}{2} + \pi\sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right).$$

By the "divergence test" for infinite series of real numbers, the fact that the sum converges implies that $a_n \to 0$ as $n\to\infty$. But,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\,\langle f, \sin nx\rangle.$$

So, for all square integrable functions f,

$$\langle f, \frac{1}{\sqrt{\pi}}\sin(nx)\rangle \to 0 = \langle f, 0\rangle \text{ as } n\to\infty,$$

which says that the sequence $\left\{\frac{1}{\sqrt{\pi}}\sin(nx)\right\}_{n=1}^{\infty}$ converges weakly to the zero function in $L^2(-\pi,\pi)$.

Chapter Ten

Section 10.1.3

10.1.3.1. For an arbitrary control volume $\mathcal{V}$ we have (10.3), that is,

$$\iiint_{\mathcal{V}} \frac{\partial e}{\partial t}\, dV = -\oiint_{\mathcal{S}} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS + \iiint_{\mathcal{V}} Q(\mathbf{r},t)\, dV.$$

For an arbitrary rectangular slice $\mathcal{V} = \{(x,y,z) : 0<x<a, 0<y<b, \alpha<z<\beta\}$, where $\beta>\alpha$, the surface $\mathcal{S}$ that bounds $\mathcal{V}$ consists of six parts:

(1) $\mathcal{S}_1$, the rectangle $z=0, 0<x<a, 0<y<b$, on which $\widehat{\mathbf{n}} = -\hat{\boldsymbol{k}}$,

(2) $\mathcal{S}_2$, the rectangle $z=H, 0<x<a, 0<y<b$, on which $\widehat{\mathbf{n}} = \hat{\boldsymbol{k}}$,

(3) $\mathcal{S}_3$, the rectangle $x=0, 0<y<b, \alpha<z<\beta$, on which $\widehat{\mathbf{n}} = -\hat{\boldsymbol{\imath}}$,

(4) $\mathcal{S}_4$, the rectangle $x=a, 0<y<b, \alpha<z<\beta$, on which $\widehat{\mathbf{n}} = \hat{\boldsymbol{\imath}}$,

(5) $\mathcal{S}_3$, the rectangle $y=0, 0<x<a, 0<z<H$, on which $\widehat{\mathbf{n}} = -\hat{\boldsymbol{\jmath}}$, and

(6) $\mathcal{S}_3$, the rectangle $y=b, 0<x<a, 0<z<H$, on which $\widehat{\mathbf{n}} = \hat{\boldsymbol{\jmath}}$.

So,

$$\begin{aligned}
-\oiint_{\mathcal{S}} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS = &-\iint_{\mathcal{S}_1} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS - \iint_{\mathcal{S}_2} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS - \iint_{\mathcal{S}_3} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS - \iint_{\mathcal{S}_4} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS \\
&- \iint_{\mathcal{S}_5} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS - \iint_{\mathcal{S}_6} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS \\
= &-\int_0^b \int_0^a \mathbf{q}(x,y,\alpha,t) \bullet (-\hat{\boldsymbol{k}}) dx\, dy - \int_0^b \int_0^a \mathbf{q}(x,y,\beta,t) \bullet (\hat{\boldsymbol{k}}) dx\, dy \\
&- \int_\alpha^\beta \int_0^b \mathbf{q}(0,y,z,t) \bullet (-\hat{\boldsymbol{\imath}}) dy\, dz - \int_\alpha^\beta \int_0^b \mathbf{q}(a,y,z,t) \bullet (-\hat{\boldsymbol{\imath}}) dy\, dz \\
&- \int_\alpha^\beta \int_0^a \mathbf{q}(x,0,z,t) \bullet (-\hat{\boldsymbol{\jmath}}) dx\, dz - \int_\alpha^\beta \int_0^a \mathbf{q}(x,b,z,t) \bullet (\hat{\boldsymbol{\jmath}}) dx\, dz
\end{aligned}$$

So (10.3) is

$$\int_\alpha^\beta \int_0^b \int_0^a \frac{\partial e}{\partial t}(x,y,z,t) dxdydz = \int_0^b \int_0^a \mathbf{q}_z(x,y,\alpha,t) dxdy - \int_0^b \int_0^a \mathbf{q}_z(x,y,\beta,t) dxdy + \int_\alpha^\beta \int_0^b \mathbf{q}_x(0,y,z,t) dydz$$

$$-\int_\alpha^\beta \int_0^b \mathbf{q}_x(a,y,z,t) dydz + \int_\alpha^\beta \int_0^a \mathbf{q}_y(x,0,z,t) dxdz - \int_\alpha^\beta \int_0^a \mathbf{q}_y(x,b,z,t) dxdz + \int_\alpha^\beta \int_0^b \int_0^a Q(x,y,z,t) dxdydz\,.$$

If we assume that $e, \mathbf{q}$, and Q do not depend on z and we assume that $q_z \equiv 0$, that is, there is no heat flux out of the slab at any point on the top surface, $\mathcal{S}_2$, or the bottom surface, $\mathcal{S}_1$, then we can integrate to get

$$(\star) \quad (\beta-\alpha) \int_0^b \int_0^a \frac{\partial e}{\partial t}(x,y,t) dxdy = (\beta-\alpha) \int_0^b \mathbf{q}_x(0,y,t) dy - (\beta-\alpha) \int_0^b \mathbf{q}_x(a,y,t) dy$$

$$+(\beta-\alpha) \int_0^a \mathbf{q}_y(x,0,t) dx - (\beta-\alpha) \int_0^a \mathbf{q}_y(x,b,t) dx + (\beta-\alpha) \int_0^b \int_0^a Q(x,y,t) dxdy\,.$$

The first two terms on the right hand side can be rewritten using

$$q_x(0,y,t) - q_x(a,y,t) = \int_0^a \frac{\partial q_x}{\partial x}(x,y,t)\,dx,$$

and the next two terms on the right hand side can be rewritten using

$$q_y(x,0,t) - q_y(x,b,t) = \int_0^b \frac{\partial q_y}{\partial y}(x,y,t)\,dy,$$

After dividing through by the positive constant $(\beta-\alpha)$ and moving terms to the left hand side, $(\star)$ implies that

$$\int_0^b\int_0^a \frac{\partial e}{\partial t}(x,y,t)dxdy - \int_0^b\int_0^a \frac{\partial q_x}{\partial x}(x,y,t)\,dxdy - \int_0^b\int_0^a \frac{\partial q_y}{\partial y}(x,y,t)\,dxdy - \int_0^b\int_0^a Q(x,y,t)dxdy = 0.$$

This being true for all α,β satisfying $0<\alpha<\beta<H$ implies the PDE

$$\frac{\partial e}{\partial t}(x,y,t) = -\frac{\partial q_x}{\partial x}(x,y,t) - \frac{\partial q_y}{\partial y}(x,y,t) + Q(x,y,t),$$

the two space dimensional version of the heat balance equation.

10.1.3.3. For an arbitrary control volume $\mathcal{V}$ we derived (10.3), that is,

$$\iiint_{\mathcal{V}} \frac{\partial e}{\partial t}\,dV = -\oiint_{\mathcal{S}} \mathbf{q}(\mathbf{r},t)\bullet\widehat{\mathbf{n}}\,dS + \iiint_{\mathcal{V}} Q(\mathbf{r},t)\,dV.$$

It is natural to rewrite an arbitrary cylindrical slice $\mathcal{V} = \{(x,y,z) : a<x<b, 0\le y^2+z^2\le R^2\}$ in an alternative cylindrical coordinates as $\mathcal{V} = \{(x,\mathrm{r},\vartheta) : a<x<b,\ 0\le \mathrm{r}<R, -\pi<\vartheta\le\pi\}$. The surface $\mathcal{S}$ that bounds $\mathcal{V}$ consists of three parts, similarly to Example 7.36 in Section 7.5: (1) $\mathcal{S}_-$, the disk $x=a$, $0\le y^2+z^2\le R^2$, on which $\widehat{\mathbf{n}} = -\hat{\boldsymbol{\imath}}$, (2) $\mathcal{S}_+$, the disk $x=b$, $0\le y^2+z^2\le R^2$, on which $\widehat{\mathbf{n}} = \hat{\boldsymbol{\imath}}$, and (3) $\tilde{\mathcal{S}}$, the lateral surface $a<x<b$, $\mathrm{r}=R$, $-\pi<\vartheta\le\pi$, on which $\widehat{\mathbf{n}} = \hat{\mathbf{e}}_\mathrm{r} = \cos\vartheta\,\hat{\boldsymbol{\imath}} + \sin\vartheta\,\hat{\boldsymbol{\jmath}}$. In these cylindrical coordinates, $\mathbf{q} = q_x\,\hat{\boldsymbol{\imath}} + q_\mathrm{r}\,\hat{\mathbf{e}}_\mathrm{r} + q_\vartheta\,\hat{\mathbf{e}}_\vartheta$.

So,

$$-\oiint_{\mathcal{S}} \mathbf{q}(\mathbf{r},t)\bullet\widehat{\mathbf{n}}\,dS = -\iint_{\mathcal{S}_-}\mathbf{q}(\mathbf{r},t)\bullet\widehat{\mathbf{n}}\,dS - \iint_{\mathcal{S}_+}\mathbf{q}(\mathbf{r},t)\bullet\widehat{\mathbf{n}}\,dS - \iint_{\tilde{\mathcal{S}}}\mathbf{q}(\mathbf{r},t)\bullet\widehat{\mathbf{n}}\,dS$$

$$= -\int_{-\pi}^{\pi}\int_0^R \mathbf{q}(a,\mathrm{r},\vartheta,t)\bullet(-\hat{\boldsymbol{\imath}})\mathrm{r}d\mathrm{r}d\vartheta - \int_{-\pi}^{\pi}\int_0^R \mathbf{q}(b,\mathrm{r},\vartheta,t)\bullet(\hat{\boldsymbol{\imath}})\mathrm{r}d\mathrm{r}d\vartheta - \int_a^b\int_{-\pi}^{\pi}\mathbf{q}(x,R,\vartheta,t)\bullet\hat{\mathbf{e}}_\mathrm{r}\,d\vartheta dx$$

So (10.3) is

$$\int_a^b\int_{-\pi}^{\pi}\int_0^R \frac{\partial e}{\partial t}(x,\mathrm{r},\vartheta,t)\mathrm{r}\,d\mathrm{r}d\vartheta dx = \int_{-\pi}^{\pi}\int_0^R \mathbf{q}(a,\mathrm{r},\vartheta,t)\bullet\hat{\boldsymbol{\imath}}\,\mathrm{r}d\mathrm{r}d\vartheta + \int_{-\pi}^{\pi}\int_0^R \mathbf{q}(b,\mathrm{r},\vartheta,t)\bullet(-\hat{\boldsymbol{\imath}})\mathrm{r}d\mathrm{r}d\vartheta$$

$$-\int_a^b\int_{-\pi}^{\pi}\mathbf{q}(x,R,\vartheta,t)\bullet\hat{\mathbf{e}}_\mathrm{r}\,d\vartheta dx + \int_a^b\int_{-\pi}^{\pi}\int_0^R Q(x,\mathrm{r},\vartheta,t)\mathrm{r}\,d\mathrm{r}d\vartheta dx$$

$$= \int_{-\pi}^{\pi}\int_0^R q_x(a,\mathrm{r},\vartheta,t)\mathrm{r}\,d\mathrm{r}\,d\vartheta - \int_{-\pi}^{\pi}\int_0^R q_x(b,\mathrm{r},\vartheta,t)\mathrm{r}\,d\mathrm{r}\,d\vartheta - \int_a^b\int_{-\pi}^{\pi} q_\mathrm{r}(x,R,\vartheta,t)d\vartheta\,dx$$

$$+\int_a^b\int_{-\pi}^{\pi}\int_0^R Q(x,\mathrm{r},\vartheta,t)\mathrm{r}\,d\mathrm{r}\,d\vartheta\,dx.$$

But, unlike Example 10.2 in Section 10.1, here we are assuming that the *lateral* side of the rod is not insulated but instead is losing heat according to Newton's Law of Cooling. This implies that on the lateral surface of the rod,

$$q_\mathrm{r}(x,R,\vartheta,t) = -k(x)\big(u(x,R,\vartheta,t) - u_0(x,\vartheta,t)\big),$$

where $u(x, R, \theta, t)$ is the temperature on the surface of the rod and $u_0(x, \theta, t)\big)$ is the temperature in the medium surrounding the rod. So, (10.3) becomes

$$\int_a^b \int_{-\pi}^{\pi} \int_0^R \frac{\partial e}{\partial t}(x, \mathrm{r}, \vartheta, t)\mathrm{r}\, d\mathrm{r} d\vartheta dx = \int_{-\pi}^{\pi} \int_0^R q_x(a, \mathrm{r}, \vartheta, t)\mathrm{r}\, d\mathrm{r}\, d\vartheta - \int_{-\pi}^{\pi} \int_0^R q_x(b, \mathrm{r}, \vartheta, t)\mathrm{r}\, d\mathrm{r}\, d\vartheta$$

$$+ \int_a^b \int_{-\pi}^{\pi} k(x)\big(u(x, R, \vartheta, t) - u_0(x, \vartheta, t)\big)\, R\, d\vartheta\, dx + \int_a^b \int_{-\pi}^{\pi} \int_0^R Q(x, \mathrm{r}, \vartheta, t)\mathrm{r}\, d\mathrm{r}\, d\vartheta\, dx.$$

Assume, as in Example 10.2 in Section 10.1, that $e, \mathbf{q}$, and Q do not depend on r or ϑ. In addition, assume that medium's temperature, $u_0(x, \vartheta, t)\big)$, does not depend on r or ϑ. Then we can integrate to get

$$\pi R^2 \int_a^b \frac{\partial e}{\partial t}(x, t)dx = -\pi R^2 \left(q_x(b, t) - q_x(a, t)\right) + 2\pi R \int_a^b k(x)\big(u(x, R, t) - u_0(x, t)\big)\, dx + \pi R^2 \int_a^b Q(x, t)dx.$$

The first two terms on the right hand side can be rewritten using

$$q_x(b, t) - q_x(a, t) = \int_a^b \frac{\partial q_x}{\partial x}(x, t)\, dx.$$

After dividing through by the constant πR^2 and moving terms to the left hand side, we get

$$\int_a^b \left(\frac{\partial e}{\partial t}(x, t) + \frac{\partial q_x}{\partial x}(x, t) - \frac{2}{R} k(x)\big(u(x, t) - u_0(x, t)\big) - Q(x, t) \right) dx = 0.$$

This being true for all a, b satisfying $0 < a < b < L$ implies the PDE

$$\frac{\partial e}{\partial t}(x, t) = -\frac{\partial q_x}{\partial x}(x, t) + \frac{2}{R} k(x)\big(u(x, t) - u_0(x, t)\big) + Q(x, t),$$

the one space dimensional version of a heat balance equation modified to include heat loss through the lateral side of the rod by Newton's Law of Cooling.

Section 10.2.4

10.2.4.1. Multiplication of the ODE through by r gives

$$\frac{d}{dr}\left[r\,\frac{du}{dr}\right]=-r$$

and then indefinite integration gives

$$r\,\frac{du}{dr}=-\frac{1}{2}\,r^2+c,$$

where c is an arbitrary constant. Division through by r gives

$$\frac{du}{dr}=-\frac{1}{2}\,r+\frac{c}{r}.$$

Indefinite integration gives

$$u=-\frac{1}{4}\,r^2+c\,\ln r+c_2,$$

and then use the boundary conditions to get

$$\left\{\begin{array}{c} T_0=u(a)=-\tfrac{1}{4}\,a^2+c\ln a+c_2 \\ \\ 0=u'(b)=-\dfrac{1}{2}\,b+\dfrac{c}{b} \end{array}\right\},$$

that is,

$$\left\{\begin{array}{c} c\,\ln a+c_2=T_0+\tfrac{1}{4}\,a^2 \\ \\ \dfrac{c}{b}=\dfrac{1}{2}\,b \end{array}\right\}.$$

The second equation implies $c=\frac{1}{2}\,b^2$. Substitute that into the first equation to get

$$\frac{1}{2}\,b^2\ln a+c_2=T_0+\frac{1}{4}\,a^2,$$

so

$$c_2=T_0+\frac{1}{4}\,a^2-\frac{1}{2}\,b^2\ln a.$$

The solution of the ODE-BVP is

$$u(r)=-\frac{1}{4}\,r^2+\frac{1}{2}\,b^2\ln r+T_0+\frac{1}{4}\,a^2-\frac{1}{2}\,b^2\ln a,$$

that is,

$$u(r)=-\frac{1}{4}\left(r^2-a^2\right)+\frac{b^2}{2}\ln\left(\frac{r}{a}\right)+T_0,\ \ a<r<b,$$

after noting that $r>0$.

10.2.4.3. $\dfrac{\partial u}{\partial t}\equiv 0$ implies that the equilibrium temperature $u=u(x)$ satisfies the ODE-BVP

$$\left\{\begin{array}{c} 0=u''(x)-\eta(u-\bar{T}), 0<x<L, \\ \\ u(0)=T_0, u(L)=T_1 \end{array}\right\}.$$

Note that η was assumed to be a positive constant.

The ODE can be rewritten as

$$(\star)\ \ u''(x)-\eta u=-\eta\,\bar{T}.$$

The method of undetermined coefficients easily finds a particular solution $u_p = \bar{T}$. The corresponding homogeneous ODE is $u''(x) - \eta u = 0$, whose solutions are

$$u_h = c_1 \cosh(\sqrt{\eta}\, x) + c_2 \sinh(\sqrt{\eta}\, x),$$

so the general solution of ODE $(\star)$ is

$$u = u_p + u_h = \bar{T} + c_1 \cosh(\sqrt{\eta}\, x) + c_2 \sinh(\sqrt{\eta}\, x),$$

where c_1 and c_2 are arbitrary constants. Plug the general solution into the two BCs to get

$$\left\{ \begin{array}{l} T_0 = u(0) = \bar{T} + c_1 \\ \\ T_1 = u(L) = \bar{T} + c_1 \cosh(\sqrt{\eta}\, L) + c_2 \sinh(\sqrt{\eta}\, L) \end{array} \right\}.$$

The first equation implies $c_1 = T_0 - \bar{T}$. Substitute that into the second equation to get

$$T_1 = u(L) = \bar{T} + (T_0 - \bar{T}) \cosh(\sqrt{\eta}\, L) + c_2 \sinh(\sqrt{\eta}\, L).$$

Because $\eta > 0$ implies $\sinh(\sqrt{\eta}\, L)$, we can solve for c_2:

$$c_2 = \frac{1}{\sinh(\sqrt{\eta}\, L)} \big(T_1 - \bar{T} - (T_0 - \bar{T}) \cosh(\sqrt{\eta}\, L) \big).$$

The equilibrium temperature distribution, that is the solution of the ODE-BVP, is

$$u(x) = \bar{T} + (T_0 - \bar{T}) \cosh(\sqrt{\eta}\, x) + \frac{T_1 - \bar{T} - (T_0 - T) \cosh(\sqrt{\eta}\, L)}{\sinh(\sqrt{\eta}\, L)} \sinh(\sqrt{\eta}\, x),$$

for $0 < x < L$.

10.2.4.5. $\dfrac{\partial u}{\partial t} \equiv 0$ implies that the equilibrium temperature $u = u(x)$ satisfies the ODE-BVP

$$\left\{ \begin{array}{c} u''(x) + 4u = 0,\ 0 < x < L, \\ \\ u(0) = T_0 \neq 0, u'(L) = 0 \end{array} \right\}.$$

(Insulated at the right end implies $u'(L) = 0$.) The general solution of the ODE is

$$u = c_1 \cos 2x + c_2 \sin 2x,$$

where c_1 and c_2 are arbitrary constants. Plug the general solution into the two BCs to get

$$\left\{ \begin{array}{l} T_0 = u(0) = c_1 \\ \\ 0 = u'(L) = 2c_2 \cos 2L \end{array} \right\}.$$

So, $c_1 = T_0$, but there are two cases concerning c_2: (a) If $\cos 2L = 0$ then c_2 is arbitrary; if $\cos 2L \neq 0$ then $c_2 = 0$.

(a) There is exactly one equilibrium solution when $L \neq \frac{1}{2} \cdot \left(n - \frac{1}{2}\right) \pi$ for *all* integers n.

(b) There are infinitely many equilibrium solutions when $L = \frac{1}{2} \cdot \left(n - \frac{1}{2}\right) \pi$ for *any* integer n.

(c) There is no no value of L for which there is no equilibrium solution.

10.2.4.7. The source is $Q(x) = \cos\left(\dfrac{\pi x}{L}\right)$ and $\dfrac{\partial u}{\partial t} \equiv 0$, implying that the equilibrium temperature $u = u(x)$ satisfies the ODE-BVP

$$\left\{ \begin{array}{c} \kappa\, u''(x) + \cos\left(\frac{\pi x}{L}\right) = 0,\ 0 < x < L, \\ \\ u'(0) = 0, u(L) = T_1 \end{array} \right\}.$$

(Insulated at the left end implies $u'(0) = 0$.) Note that the thermal conductivity, κ, is assumed to be constant.

Integrate the ODE
$$u'' = -\frac{1}{\kappa}\cos\left(\frac{\pi x}{L}\right)$$
once to get
$$u' = -\frac{L}{\pi\,\kappa}\sin\left(\frac{\pi x}{L}\right) + c_1,$$
where c_1 is an arbitrary constant. Plug this into the first BC to get
$$0 = u'(0) = -\frac{L}{\pi\,\kappa}\cdot 0 + c_1,$$
so $c_1 = 0$ and
$$u' = -\frac{L}{\pi\,\kappa}\sin\left(\frac{\pi x}{L}\right).$$
Integrate a second time to get
$$u = \frac{L^2}{\pi^2\,\kappa}\cos\left(\frac{\pi x}{L}\right) + c_2$$
where c_2 is an arbitrary constant. Plug this into the second BC to get
$$T_1 = u(L) = \frac{L^2}{\pi^2\,\kappa}\cos\pi + c_2 = -\frac{L^2}{\pi^2\,\kappa} + c_2,$$
which we can solve for c_2.

The equilibrium temperature distribution is
$$u(x) = T_1 + \frac{L^2}{\pi^2\kappa}\left(1 + \cos\left(\frac{\pi x}{L}\right)\right).$$

10.2.4.9. Because $\nabla^2 u$ is on the right hand side of the PDE, the material is homogeneous. Because the PDE has the form
$$\frac{\partial u}{\partial t} = \nabla^2 u - Q_0$$
(in spherical coordinates), the homogeneous material has thermal conductivity, specific heat, and mass density all equal to one, in appropriate units.

In spherical coordinates,
$$\mathcal{V} = \{(\rho, \phi, \theta) :\ 0 \le \theta \le 2\pi, \frac{\pi}{2} \le \phi \le \pi, 0 \le \rho < a\}$$
is the lower half of the ball of radius a whose center is at the origin. The absence of θ from the PDE says that the problem is circularly symmetric around the $z-$axis.

The term $-Q_0$ models that the material is absorbing heat at a rate Q_0. The boundary condition $\frac{\partial u}{\partial \rho}(a, \phi, t) = 0$ implies that the material is insulated on the spherical surface $\rho = a$ for $z \le 0$. The BC $\frac{1}{\rho}\frac{\partial u}{\partial \phi}\Big(\rho, \frac{\pi}{2}, t\Big) = -\Big(u\Big(\rho, \frac{\pi}{2}, t\Big) - 20\Big)$ models that on the $z = 0$ disk surface the solid is losing heat at a rate equal to the difference between its temperature and the medium's constant temperature of $20°$, all in appropriate units.

The IC $u(\rho, \phi, 0) = 40$ implies that the initial temperature of the material is uniformly $40°$.

10.2.4.11. (a) At $x = 0$, $\widehat{\mathbf{n}} = -\hat{\boldsymbol{\imath}}$. So, the heat flux out of the left end at $x = 0$ is
$$\widehat{\mathbf{n}} \bullet \mathbf{q}(0, t) = -\hat{\boldsymbol{\imath}} \bullet (-\kappa(0)\boldsymbol{\nabla} u(0, t) = \kappa(0)\frac{\partial u}{\partial x}(0, t).$$

Physically, the heat conductance $\kappa(0) > 0$, so the heat flux out of the left end is positive if, and only if, $\dfrac{\partial u}{\partial x}(0,t) > 0$.

(b) At $x = L$, $\widehat{\mathbf{n}} = \hat{\boldsymbol{\imath}}$. So, the heat flux out of the right end at $x = L$ is

$$\widehat{\mathbf{n}} \bullet \mathbf{q}(L,t) = \hat{\boldsymbol{\imath}} \bullet (-\kappa(L)\boldsymbol{\nabla} u(L,t) = -\kappa(L)\frac{\partial u}{\partial x}(L,t).$$

Physically, the heat conductance $\kappa(L) > 0$, so the heat flux out of the right end is positive if, and only if, $\dfrac{\partial u}{\partial x}(0,t) < 0$.

10.2.4.13. $\dfrac{\partial u}{\partial t} \equiv 0$ because we are solving for the equilibrium temperature distribution. There is no source term and the problem is circularly symmetric, so $u = u(r)$ satisfies the ODE-BVP

$$\left\{ \begin{array}{c} \kappa\,\dfrac{1}{r}\,\dfrac{d}{dr}\left[r\,\dfrac{du}{dr} \right] = 0,\ a < r < b, \\ \\ u(a) = 0, u'(b) = u'_b \end{array} \right\},$$

for some constant u'_b.

We are given that the total rate of heat flow out of the annulus through the circle $r = b$ is

$$\beta = \oint_{\mathcal{C}} \mathbf{q}(L) \bullet \widehat{\mathbf{n}} \cdot (ds) = \int_0^{2\pi} -\kappa \frac{\partial u}{\partial n}\Big|_{r=b} \cdot (b\,d\theta).$$

On $r = b$, the outward unit normal vector is $\hat{\mathbf{e}}_r$, so

$$\frac{\partial u}{\partial n}\Big|_{r=b} = \hat{\mathbf{e}}_r \bullet \boldsymbol{\nabla} u\Big|_{r=b} = \hat{\mathbf{e}}_r \bullet \left(\frac{\partial u}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\,\frac{\partial u}{\partial \theta}\,\hat{\mathbf{e}}_\theta \right)\Big|_{r=b} = \frac{\partial u}{\partial r}(b),$$

so

$$\beta = \int_0^{2\pi} -\kappa \frac{\partial u}{\partial n}\Big|_{r=b} b\,d\theta = \int_0^{2\pi} -\kappa \frac{\partial u}{\partial r}(b) \cdot b\,d\theta = -2\pi\, b\, \kappa\, u'_b,$$

because $\dfrac{\partial u}{\partial r}\Big|_{r=b}$ does not depend on θ. Solving for u'_b gives

$$u'_b = -\frac{\beta}{2\pi\, b\, \kappa}.$$

The ODE can be rewritten as

$$\frac{d}{dr}\left[r\,\frac{du}{dr} \right] = 0,$$

whose solutions are

$$r\,\frac{du}{dr} = c_1, \quad \text{that is,} \quad u'(r) = \frac{c_1}{r},$$

where c_1 is an arbitrary constant. Integrate a second time to get

$$u = \int \frac{c_1}{r}\,dr = c_1 \ln r + c_2,$$

where c_2 is an arbitrary constant.

Plug $u'(r)$ into the condition on u'_b to get

$$-\frac{\beta}{2\pi\, b\, \kappa} = u'_b = \frac{c_1}{b},$$

which implies

$$c_1 = -\frac{\beta}{2\pi\,\kappa},$$

so

$$u = -\frac{\beta}{2\pi\,\kappa}\,\ln r + c_2.$$

Plug this into the first BC to get

$$0 = u(a) = -\frac{\beta}{2\pi\,\kappa}\,\ln a + c_2.$$

So, the solution of the ODE-BVP is

$$u = -\frac{\beta}{2\pi\,\kappa}\,\ln r + \frac{\beta}{2\pi\,\kappa}\,\ln a = \frac{\beta}{2\pi\,\kappa}\,\ln\frac{a}{r}.$$

So,

$$T = u(b) = \frac{\beta}{2\pi\kappa}\,\ln\left(\frac{a}{b}\right).$$

10.2.4.15. Perfect thermal contact requires continuity of temperature, hence

$$\lim_{x\to 0^-} u(x,t) = \lim_{y\to 0^+} v(y,t) = \lim_{x\to 0^+} w(x,t),$$

and zero net heat flux at the origin. The latter is, using Fourier's Law of heat conduction,

$$\lim_{x\to 0^-}\left(-\kappa_1\,\frac{\partial u}{\partial x}(x,t)\right) + \lim_{y\to 0^+}\left(\kappa_2\,\frac{\partial v}{\partial y}(y,t)\right) + \lim_{x\to 0^+}\left(\kappa_3\,\frac{\partial w}{\partial x}(x,t)\right) = 0.$$

So, the BCs are

$$u(0^-,t) = v(0^+,t) = w(0^+,t),$$

and

$$-\kappa_1\,\frac{\partial u}{\partial x}(0^-,t) + \kappa_2\,\frac{\partial v}{\partial y}(0^+,t) + \kappa_3\,\frac{\partial w}{\partial x}(0^+,t) = 0.$$

The latter can be written as

$$\kappa_1\,\frac{\partial u}{\partial x}(0^-,t) = \kappa_2\,\frac{\partial v}{\partial y}(0^+,t) + \kappa_3\,\frac{\partial w}{\partial x}(0^+,t),$$

which is analogous to Kirchoff's law on continuity of electrical current at a node in a circuit.

10.2.4.17. The interface is at $\theta = \frac{\pi}{4}$. Continuity of temperature at the interface implies

$$u_1\Big(r, \frac{\pi}{4}^{\,+}, t\Big) = u_2\Big(r, \frac{\pi}{4}^{\,-}, t\Big).$$

Along a radial line segment, the normal derivative is

$$\widehat{\mathbf{n}} \bullet \boldsymbol{\nabla} u = \hat{\mathbf{e}}_\theta \bullet \left(\frac{\partial u}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\,\frac{\partial u}{\partial \theta}\,\hat{\mathbf{e}}_\theta\right) = \frac{1}{r}\,\frac{\partial u}{\partial \theta}.$$

Continuity of the total flux out of sector $\mathcal{D}_1$ and into sector $\mathcal{D}_2$ implies that, by integrating on the radial segment $\theta = \frac{\pi}{4}$ from $r = 0$ to $r = a$,

$$\int_0^a \frac{\kappa_1}{r}\cdot\frac{\partial u_1}{\partial \theta}\Big(r, \frac{\pi}{4}^{\,+}\Big)dr = \int_0^a \frac{\kappa_2}{r}\cdot\frac{\partial u_2}{\partial \theta}\Big(r, \frac{\pi}{4}^{\,-}\Big)dr.$$

10.2.4.19. $E = c\varrho\, A\int_0^L u(x,t)\,dx$ implies

$$\dot{E}(t) = \frac{d}{dt}\left[c\varrho\, A\int_0^L u(x,t)\,dx\right],$$

where c, ϱ, A are constants because the material is assumed to be homogeneous and, as usual, we assume L is a constant. So,

$$\dot{E}(t) = c\varrho\, A\int_0^L \frac{\partial u}{\partial t}(x,t)\,dx = c\varrho\, A\int_0^L \Big(\alpha\,\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{1}{c\varrho}\,Q(x)\Big)\,dx = \alpha c\varrho\, A\int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)\ dx + A\int_0^L Q_0\,dx$$

$$= \kappa\, A\left[\,\frac{\partial u}{\partial x}(x,t)\,\right]_0^L + A\Big[\,Q_0\, x\,\Big]_0^L = \kappa\, A\left(\,\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\,\right) + ALQ_0 = \kappa\, A\,(0-0) + ALQ_0.$$

Indefinite integration of the ODE, $\dot{E}(t) = ALQ_0$, gives

$$E(t) = ALQ_0 t + c,$$

where c is an arbitrary constant. Using the IC $E(0) = E_0$ we see that

$$E(t) = ALQ_0 t + E_0,$$

which fits the form $E(t) = E(0) + \beta t$ if and only if

$$Q_0 = \frac{\beta}{AL}.$$

10.2.4.21. Because U satisfies $\dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2} - U$ and $T(x,t) = e^{-x}U(x,t)$,

$$\frac{\partial T}{\partial t} = e^{-x}\cdot\frac{\partial U}{\partial t} = -e^{-x}U + e^{-x}\,\frac{\partial^2 U}{\partial x^2}.$$

Also, the product rule gives

$$\frac{\partial T}{\partial x} = -e^{-x}U(x,t) + e^{-x}\,\frac{\partial U}{\partial x}$$

and

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left[-e^{-x}U(x,t) + e^{-x}\,\frac{\partial U}{\partial x}\right] = e^{-x}U(x,t) - e^{-x}\,\frac{\partial U}{\partial x} - e^{-x}\,\frac{\partial U}{\partial x} + e^{-x}\,\frac{\partial^2 U}{\partial x^2}$$

$$= -e^{-x}U(x,t) + 2e^{-x}U(x,t) - 2e^{-x}\,\frac{\partial U}{\partial x} + e^{-x}\,\frac{\partial^2 U}{\partial x^2} = -e^{-x}U(x,t) + e^{-x}\,\frac{\partial^2 U}{\partial x^2} - 2\left(-e^{-x}U + e^{-x}\,\frac{\partial U}{\partial x}\right)$$

$$= \frac{\partial T}{\partial t} - 2\frac{\partial T}{\partial x}.$$

Also,

$$T(0,t) = e^{-0}U(0,t) = 1\cdot 0 = 0,$$
$$T(L,t) = e^{-L}U(L,t) = 1\cdot e^{-L}\cdot 0 = 0,$$

and

$$T(0,t) = e^{-x}\,U(x,0) = e^{-x}\left(\,e^{x}f(x)\,\right) = f(x).$$

(b) We will not be able to do this part until Section 11.1.

10.2.4.23. $\dfrac{\partial u}{\partial t} \equiv 0$ because we are solving for the equilibrium temperature distribution. There is no source term and the problem is spherically symmetric, so $u = u(\rho)$ satisfies the ODE-BVP

$$\left\{ \begin{array}{c} \kappa \dfrac{1}{\rho^2} \dfrac{d}{d\rho} \left[\rho^2 \dfrac{du}{d\rho} \right] = 0,\ a < \rho < b, \\ \\ u(a) = 0, u'(b) = u'_b \end{array} \right\},$$

for some constant u'_b.

We are given that the total rate of heat flow out of the annulus through the sphere $\rho = b$ is

$$\beta = \oiint_{\mathcal{C}} \mathbf{q}(L) \bullet \widehat{\mathbf{n}}\,(dS) = \int_0^{2\pi} \int_0^{\pi} -\kappa \frac{\partial u}{\partial n}\Big|_{\rho=b} \cdot (b^2 \sin\phi\, d\phi\, d\theta).$$

The outward unit normal vector is $\hat{\mathbf{e}}_\rho$, so

$$\frac{\partial u}{\partial n}\Big|_{\rho=b} = \hat{\mathbf{e}}_\rho \bullet \boldsymbol{\nabla} u\Big|_{\rho=b} = \hat{\mathbf{e}}_\rho \bullet \left(\frac{\partial u}{\partial \rho}\, \hat{\mathbf{e}}_\rho + \frac{1}{\rho}\, \frac{\partial f}{\partial \phi}\, \hat{\mathbf{e}}_\phi + \frac{1}{\rho \sin\phi}\, \frac{\partial f}{\partial \theta} \right)\Big|_{\rho=b} = \frac{\partial u}{\partial \rho}(b),$$

so

$$\beta = \int_0^{2\pi} \int_0^{\pi} -\kappa \frac{\partial u}{\partial n}\Big|_{\rho=b} \cdot (b^2 \sin\phi\, d\phi\, d\theta) = \int_0^{2\pi} \int_0^{\pi} -\kappa \frac{du}{d\rho}(b) \cdot b^2 \sin\phi\, d\phi\, d\theta = ... = -4\pi\, b^2 \kappa\, u'_b,$$

because $\dfrac{\partial u}{\partial \rho}\Big|_{\rho=b}$ does not depend on ϕ or θ. Solving for u'_b gives

$$u'_b = -\frac{\beta}{4\pi\, b^2 \kappa}.$$

The ODE can be rewritten as

$$\frac{d}{d\rho} \left[\rho^2 \frac{du}{d\rho} \right] = 0,$$

whose solutions are

$$\rho^2 \frac{du}{d\rho} = c_1, \quad \text{that is,} \quad u'(\rho) = \frac{c_1}{\rho^2},$$

where c_1 is an arbitrary constant. Integrate a second time to get

$$u = \int \frac{c_1}{\rho^2}\, d\rho = -c_1 \cdot \frac{1}{\rho} + c_2,$$

where c_2 is an arbitrary constant.

Plug $u'(\rho)$ into the condition on u'_b to get

$$-\frac{\beta}{4\pi\, b^2 \kappa} = u'_b = \frac{c_1}{b^2},$$

which implies

$$c_1 = -\frac{\beta}{4\pi\, \kappa},$$

so

$$u = \frac{\beta}{4\pi\, \kappa} \cdot \frac{1}{\rho} + c_2.$$

Plug this into the first BC to get

$$20 = u(a) = \frac{\beta}{4\pi\, \kappa} \cdot \frac{1}{a} + c_2.$$

So, the solution of the ODE-BVP is

$$u = 20 + \frac{\beta}{4\pi\, \kappa} \cdot \frac{1}{\rho} - \frac{\beta}{4\pi\, \kappa} \cdot \frac{1}{a} = 20 - \frac{\beta}{4\pi\, \kappa} \left(\frac{1}{a} - \frac{1}{\rho} \right).$$

So,

$$T = u(b) = 20 - \frac{\beta}{4\pi\kappa} \left(\frac{1}{a} - \frac{1}{b} \right).$$

Section 10.3.4

10.3.4.1. Integrate Laplace's equation over the interval $[0,L]$ to get

$$0=\int_0^L 0\cdot dx=\int_0^L \frac{d^2u}{dx^2}(x)=\left[\frac{du}{dx}\right]_0^L=u'(L)-u'(0).$$

10.3.4.3. The circularly symmetric electric potential $V=V(r)$ satisfies the ODE-BVP

$$\left\{\begin{array}{c}\kappa\,\frac{1}{r}\,\frac{d}{dr}\left[r\,\frac{dV}{dr}\right]=0,\ 1.3<r<2.5,\\ \\ V(1.3)=V_0, V(2.5)=V_1\end{array}\right\},$$

assuming distance are measured in cm. The ODE can be rewritten as

$$\frac{d}{dr}\left[r\,\frac{dV}{dr}\right]=0,$$

whose solutions are

$$r\,\frac{dV}{dr}=c_1,\quad \text{that is,}\quad V'(r)=\frac{c_1}{r},$$

where c_1 is an arbitrary constant. Integrate a second time to get

$$V=\int\frac{c_1}{r}\,dr=c_1\ln r+c_2,$$

where c_2 is an arbitrary constant.

Plug V into the first BCs to get

$$\left\{\begin{array}{c}V_0=V(1.3)=c_1\ln 1.3+c_2,\\ \\ V_1=V(2.5)=c_1\ln 2.5+c_2\end{array}\right\},$$

which implies

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}\ln 1.3 & 1\\ \ln 2.5 & 1\end{bmatrix}^{-1}\begin{bmatrix}V_0\\ V_1\end{bmatrix}=\frac{1}{\ln 1.3-\ln 2.5}\begin{bmatrix}1 & -1\\ -\ln 2.5 & \ln 1.3\end{bmatrix}\begin{bmatrix}V_0\\ V_1\end{bmatrix}=\frac{-1}{\ln 2.5-\ln 1.3}\begin{bmatrix}V_0-V_1\\ V_1\ln 1.3-V_0\ln 2.5\end{bmatrix}.$$

The solution is

$$V(r)=\frac{1}{\ln 2.5-\ln 1.3}\left((-V_0+V_1)\ln r-V_1\ln 1.3+V_0\ln 2.5\right)$$

$$=\frac{1}{\ln 2.5-\ln 1.3}\left(V_1(\ln r-\ln 1.3)+V_0(\ln 2.5-\ln r)\right)=\frac{1}{\ln(2.5/1.3)}\left(V_1\ln\left(\frac{r}{1.3}\right)+V_0\ln\left(\frac{2.5}{r}\right)\right).$$

10.3.4.5. The solvability condition for Laplace's equation in the domain enclosed by curve $\mathcal{C}$ is $0=\oint_{\mathcal{C}}\frac{\partial u}{\partial n}\,ds$. Here, the positively oriented curve $\mathcal{C}$ consists of four line segments: $y=0, 0\le x\le L$, $x=L, 0\le y\le H$, $y=H, 0\le x\le L$, and $x=0, 0\le y\le H$, as shown in the figure. Only on the last two is $\frac{\partial u}{\partial n}$ not identically zero.

We parametrize those two line segments and give the corresponding outward unit normal vectors as

$$\mathcal{C}_3:\ \mathbf{r}=(L-t)\hat{\boldsymbol{\imath}}+H\,\hat{\boldsymbol{\jmath}}, 0\le t\le L:\quad \widehat{\mathbf{n}}=\hat{\boldsymbol{\jmath}}$$

and

$$\mathcal{C}_4: \mathbf{r} = 0\cdot\hat{\boldsymbol{\imath}} + (H-t)\hat{\boldsymbol{\jmath}}, 0 \le t \le H: \quad \widehat{\mathbf{n}} = -\hat{\boldsymbol{\imath}}.$$

The solvability condition is

$$0 = \oint_{\mathcal{C}} \frac{\partial u}{\partial n}\, ds = \int_{\mathcal{C}_3} \boldsymbol{\nabla} u \bullet \hat{\boldsymbol{\jmath}}\, ds + \int_{\mathcal{C}_4} \boldsymbol{\nabla} u \bullet (-\hat{\boldsymbol{\imath}})\, ds = \int_{\mathcal{C}_3} \frac{\partial u}{\partial y}\, ds + \int_{\mathcal{C}_4} \Big(-\frac{\partial u}{\partial x}\Big) ds$$

$$= \int_0^L \frac{\partial u}{\partial y}(L-t, H)\, dt - \int_0^H \frac{\partial u}{\partial x}(0, H-t)\, dt = \int_0^L k\, dt - \int_0^H (H-t)\, dt = -kL + \Big[Ht - \frac{1}{2}t^2\Big]_0^H$$

$$= -kL + \frac{1}{2}H^2.$$

So, the only value of k for which there is a solution is $k = \dfrac{H^2}{2L}$.

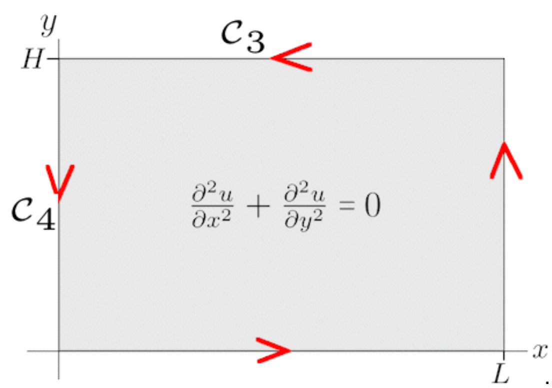


Figure 1: Problem 10.3.4.5

10.3.4.7. The boundary of the domain consists of two parts: $\mathcal{C}_+$, the positively oriented circle $r = a$, and $\mathcal{C}_-$, the negatively oriented rectangle inside the circle. The solvability condition is that

$$0 = \oint_{\mathcal{C}_+} \frac{\partial u}{\partial n}\, ds + \oint_{\mathcal{C}_-} \frac{\partial u}{\partial n}\, ds$$

Parametrize

$$\mathcal{C}_+: \mathbf{r} = a\,\hat{\mathbf{e}}_\theta, -\pi \le \theta \le \pi,$$

on which $\widehat{\mathbf{n}} = \hat{\mathbf{e}}_r$, and

$$\mathcal{C}_0: \mathbf{r} = t\,\hat{\boldsymbol{\imath}} + H\,\hat{\boldsymbol{\jmath}}, -L \le t \le L,$$

on which $\widehat{\mathbf{n}} = -\hat{\boldsymbol{\jmath}}$.

Only on $\mathcal{C}_+$ and $\mathcal{C}_0$ are the normal derivatives of the temperature not identically zero.

The solvability condition is

$$0 = \oint_{\mathcal{C}_+} \frac{\partial u}{\partial n}\, ds + \int_{\mathcal{C}_0} \frac{\partial u}{\partial n}\, ds = \int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(a,\theta)\cdot(a\, d\theta) + \int_{-L}^{L} -\frac{\partial u}{\partial y}(t, H)\cdot(dt),$$

that is,

$$0 = a\int_{-\pi}^{\pi} f(\theta)\, d\theta - \int_{-L}^{L} g(t)\, dt,$$

or, equivalently,

$$0 = a\int_{-\pi}^{\pi} f(\theta)\, d\theta - \int_{-L}^{L} g(x)\, dx.$$

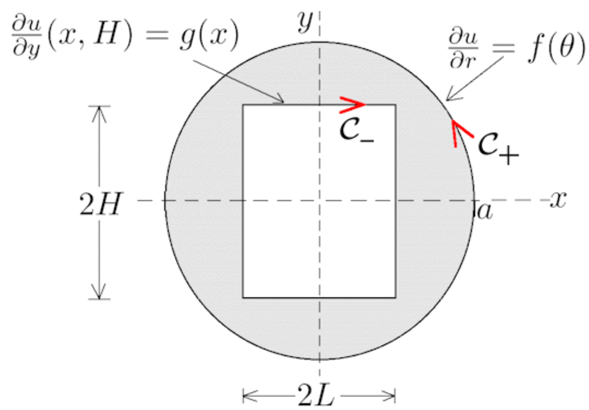


Figure 2: Problem 10.3.4.7

10.3.4.9. $\dfrac{\partial u}{\partial t}\equiv 0$ because we are solving for the equilibrium temperature distribution, so $u=u(r)$ should solve the ODE

$$0=\frac{1}{r}\frac{d}{dr}\left[r\,\frac{du}{dr}\right]-4,$$

that is,

$$\frac{d}{dr}\left[r\,\frac{du}{dr}\right]=4r.$$

Indefinite integration implies

$$r\,\frac{du}{dr}=2r^2+c_1,$$

that is,

$$\frac{du}{dr}=2r+\frac{c_1}{r},$$

where c_1 is an arbitrary constant. Indefinite integration a second time implies

$$u=r^2+c_1\ln r+c_2,$$

where c_2 is an arbitrary constant.

The finiteness BC, $|\,u(0^+,t)\,|<\infty$, implies $c_1=0$, so

$$u=r^2+c_2,\quad \text{hence}\quad u'(r)=2r.$$

Plug this into the second BC to get

$$5=u'(a)=2a.$$

(a) Only for $a=\frac{5}{2}$ is there an equilibrium temperature distribution.
(b) If $a=\frac{5}{2}$, the equilibrium temperature distribution is $u(r)=r^2+c_1$, where c_1 is an arbitrary constant.

10.3.4.11. The ODE-BVP for the equilibrium temperature distribution in problem 10.2.4.7 is

$$\left\{\begin{matrix}\kappa\,u''(x)+\cos\left(\frac{\pi x}{L}\right)=0,\ 0<x<L,\\ \\ u'(0)=0,u(L)=T_1\end{matrix}\right\}.$$

Note that the thermal conductivity, κ, is assumed to be constant. Integrate the ODE once to get

$$0=\int_0^L\Big(\kappa\,u''(x)+\cos\left(\frac{\pi x}{L}\right)\Big)dx=\Big[\kappa u'(x)+\frac{L}{\pi}\,\sin\left(\frac{\pi x}{L}\right)\Big]_0^L$$

$$= \kappa\big(u'(L) - u'(0)\big) + 0 - 0 = \kappa\big(u'(L) - 0\big) = \kappa u'(L),$$

after using the boundary condition $u'(0) = 0$.

(a) The solvability condition is $u'(L) = 0$.

(b) rate of heat flow out of the right end is

$$\mathbf{q}(L) \bullet \widehat{\mathbf{n}} = (-\kappa \boldsymbol{\nabla} u) \bullet (+\hat{\boldsymbol{\imath}}) = -\kappa \frac{du}{dx}(L) = 0,$$

that is, is zero when the temperature is at equilibrium.

10.3.4.13. Integrate both sides of the anisotropic Laplace's equation over $\mathcal{V}$ and then use the divergence theorem to get

$$0 = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \bullet (\kappa(\mathbf{r})\boldsymbol{\nabla} u) = \oiint_{\mathcal{S}} (\kappa(\mathbf{r})\boldsymbol{\nabla} u) \bullet \widehat{\mathbf{n}}\, dS = \oiint_{\mathcal{S}} g(\mathbf{r})\, dS.$$

So, the solvability condition is that $0 = \oiint_{\mathcal{S}} g(\mathbf{r})\, dS$.

10.3.4.15. Begin by using the Calculus I chain rule for the substitution $r = \dfrac{1}{p}$ to get

$$\frac{\partial U}{\partial p}(p,\theta) = \frac{\partial}{\partial p}\Big[\, u\Big(\frac{1}{p},\theta\Big)\Big] = \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) \cdot \frac{\partial r}{\partial p} = \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) \cdot \frac{\partial}{\partial p}\Big[\frac{1}{p}\Big] = -\frac{1}{p^2} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big).$$

Using the product rule,

$$\frac{\partial^2 U}{\partial p^2}(p,\theta) = \frac{\partial}{\partial p}\Big[\frac{\partial U}{\partial p}\Big] = \frac{\partial}{\partial p}\Big[-\frac{1}{p^2} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big)\Big] = \frac{2}{p^3} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) - \frac{1}{p^2} \cdot \frac{\partial}{\partial p}\Big[\frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big)\Big].$$

Using the Calculus I chain rule for the substitution $r = \dfrac{1}{p}$ gives

$$\frac{\partial^2 U}{\partial p^2}(p,\theta) = \frac{2}{p^3} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) - \frac{1}{p^2} \cdot \frac{\partial}{\partial r}\Big[\frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big)\Big] \cdot \frac{\partial r}{\partial p} = \frac{2}{p^3} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) - \frac{1}{p^2} \cdot \frac{\partial^2 u}{\partial r^2}\Big(\frac{1}{p},\theta\Big) \cdot \Big(-\frac{1}{p^2}\Big)$$

$$= \frac{1}{p^4} \cdot \frac{\partial^2 u}{\partial r^2}\Big(\frac{1}{p},\theta\Big) + \frac{2}{p^3} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big).$$

Because $u = u(r,\theta)$ satisfies Laplace's equation, that is,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\,\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\left(r\,\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

we have

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r} \cdot \frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

So,

$$\frac{\partial^2 U}{\partial p^2}(p,\theta) = \frac{1}{p^4} \cdot \frac{\partial^2 u}{\partial r^2}\Big(\frac{1}{p},\theta\Big) + \frac{2}{p^3} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) = \frac{1}{p^4}\left(-\frac{1}{r} \cdot \frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + \frac{2}{p^3}\frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big)$$

$$= \frac{1}{p^4}\left(-p \cdot \frac{\partial u}{\partial r} - p^2 \cdot \frac{\partial^2 u}{\partial \theta^2}\right) + \frac{2}{p^3}\frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) = -\frac{1}{p^3} \cdot \frac{\partial u}{\partial r} - \frac{1}{p^2} \cdot \frac{\partial^2 u}{\partial \theta^2} + \frac{2}{p^3} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big)$$

that is,

$$(\star) \quad \frac{\partial^2 U}{\partial p^2}(p,\theta) = \frac{1}{p^3} \cdot \frac{\partial u}{\partial r} - \frac{1}{p^2} \cdot \frac{\partial^2 u}{\partial \theta^2}.$$

But, our first result was that

$$\frac{\partial U}{\partial p}(p,\theta) = -\frac{1}{p^2}\cdot\frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big),$$

so

$$\frac{1}{p^3}\cdot\frac{\partial u}{\partial r} = -\frac{1}{p}\cdot\frac{\partial U}{\partial p}(p,\theta).$$

Substitute this into $(\star)$ to get

$$\frac{\partial^2 U}{\partial p^2}(p,\theta) = -\frac{1}{p}\cdot\frac{\partial U}{\partial p}(p,\theta) - \frac{1}{p^2}\cdot\frac{\partial^2 u}{\partial \theta^2},$$

that is,

$$\frac{\partial^2 U}{\partial p^2}(p,\theta) + \frac{1}{p}\cdot\frac{\partial U}{\partial p}(p,\theta) + \frac{1}{p^2}\cdot\frac{\partial^2 U}{\partial \theta^2}(p,\theta) = 0,$$

that is $U = U(p,\theta)$ satisfies Laplace's equation, after noting that

$$\frac{\partial^2 U}{\partial \theta^2}(p,\theta) = \frac{\partial^2 u}{\partial \theta^2}\Big(\frac{1}{p},\theta\Big).$$

Section 10.4.6

10.4.6.1. Starting from Section 10.4's equation (10.58),

$$\varrho(X)\,\frac{\partial^2\mathbf{r}}{\partial t^2}=\frac{\partial}{\partial X}\left[\beta\left(||\frac{\partial\mathbf{r}}{\partial X}(X,t)||\right)\frac{\partial\mathbf{r}}{\partial X}\right]+\mathbf{f}$$

we assume that the mass density is $\varrho=\varrho(X)$, the tension is $T=T(X)=\beta\left(||\frac{\partial\mathbf{r}}{\partial X}(X,t)||\right)$, the body force is $\mathbf{f}=\mathbf{0}$, and we approximate $\mathbf{r}(X,t)\approx X\hat{\boldsymbol{\imath}}+y(X,t)\hat{\boldsymbol{\jmath}}\approx x\hat{\boldsymbol{\imath}}+y(x,t)\hat{\boldsymbol{\jmath}}$. Then the wave equation becomes

$$\varrho(x)\,\frac{\partial^2 y(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left[T(x)\,\frac{\partial y(x,t)}{\partial x}\right].$$

10.4.6.3. Tension $T=21.3\,lbs=(21.3\,lbs)\left(\dfrac{9.80665\,N}{2.205\,lb}\right)\approx 94.73090476\,N$ and string length is $L\approx 0.352\,m$.

wave speed $c=\sqrt{T/\varrho}$, and the fundamental mode has vibration frequency

$$440\cdot 2\pi=440Hz=\omega=\frac{\pi c}{L}=\frac{\pi\sqrt{T/\varrho}}{L}\quad\Rightarrow\quad\sqrt{\varrho}=\frac{\sqrt{T}}{880L}.$$

$$\Rightarrow\quad \varrho\approx\frac{94.94620227}{(880\cdot 0.352)^2}\approx 9.872815337\times 10^{-4}\quad\Rightarrow\quad \varrho\approx 9.87\times 10^{-4}\,kg/m,\text{ to three significant digits}$$

10.4.6.5. Following the hint, we first note that the matrix multiplication that has the effect of swapping the variables y and z is given by the elementary matrix

$$E\triangleq\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},$$

related to work in Sections 1.2 and 6.6.

We calculate that

$$E\left[\begin{array}{cc|c}a&b&0\\c&d&0\\\hline 0&0&0\end{array}\right]E^T=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\left[\begin{array}{cc|c}a&b&0\\c&d&0\\\hline 0&0&0\end{array}\right]\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}a&0&b\\c&0&d\\0&0&0\end{bmatrix}=\begin{bmatrix}a&0&b\\0&0&0\\c&0&d\end{bmatrix},$$

as was desired. So, let $\mathcal{O}=E=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$.

10.4.6.7. With the strain tensor given in (10.75) in Section 10.4, that is,

$$(\star)\quad[\varepsilon]=[\varepsilon_{ij}]=\begin{bmatrix}\alpha&0&0\\0&-\alpha\nu&0\\0&0&-\alpha\nu\end{bmatrix},$$

Hooke's law says that the stress tensor is

$$[\tau] = C[\varepsilon],$$

and (10.74) in Section 10.4 says that

$$[\tau] = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})I + 2\mu[\varepsilon].$$

Substituting in the $[\varepsilon]$ given in $(\star)$, we have

$$[\tau] = \lambda(\alpha - \alpha\nu - \alpha\nu)I + 2\mu[\varepsilon] = \lambda\alpha(1-2\nu)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} \alpha & 0 & 0 \\ 0 & -\alpha\nu & 0 \\ 0 & 0 & -\alpha\nu \end{bmatrix}$$

$$= \alpha \begin{bmatrix} \lambda(1-2\nu)+2\mu & 0 & 0 \\ 0 & \lambda(1-2\nu)-2\mu\nu & 0 \\ 0 & 0 & \lambda(1-2\nu)-2\mu\nu \end{bmatrix},$$

which is (10.76), as we were asked to explain.

10.4.6.9. Regarding (10.78), we have that

$$\frac{E}{2(1+\nu)} = \frac{1}{2(1+\nu)} \cdot E = \frac{1}{2(1+\nu)} \cdot \frac{\mu \cdot (3\lambda + 2\mu)}{\lambda + \mu} = \frac{1}{2(1 + \frac{\lambda}{2(\lambda+\mu)})} \cdot \frac{\mu \cdot (3\lambda + 2\mu)}{(\lambda+\mu)}$$

$$= \frac{\cancel{2}(\lambda+\mu)}{\cancel{2}(2(\lambda+\mu)+\lambda)} \cdot \frac{\mu \cdot (3\lambda+2\mu)}{(\lambda+\mu)} = \frac{(\lambda+\mu)}{(3\lambda+2\mu)} \cdot \frac{\mu \cdot (3\lambda+2\mu)}{(\lambda+\mu)} = \mu,$$

as were asked to explain, and hence

$$\frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{E}{2(1+\nu)} \cdot \frac{2\nu}{(1-2\nu)} = \mu \cdot \frac{2\nu}{(1-2\nu)} = \mu \cdot \frac{\cancel{2}\frac{\lambda}{\cancel{2}(\lambda+\mu)}}{(1-\cancel{2}\frac{\lambda}{\cancel{2}(\lambda+\mu)})} = \mu \cdot \frac{\frac{\lambda}{(\lambda+\mu)}}{(1-\frac{\lambda}{(\lambda+\mu)})}$$

$$= \mu \cdot \frac{\frac{\lambda}{(\lambda+\mu)}}{(1-\frac{\lambda}{(\lambda+\mu)})} \cdot \frac{(\lambda+\mu)}{(\lambda+\mu)} = \mu \cdot \frac{\lambda}{(\cancel{\lambda}+\mu)-\cancel{\lambda}} = \mu \cdot \frac{\lambda}{\mu} = \lambda,$$

as we were asked to explain.

10.4.6.11. We start with (10.57) in Section 10.4, $\varrho(X)\,\frac{\partial^2 \mathbf{r}}{\partial t^2}(X,t) = \frac{\partial \mathbf{T}}{\partial X}(X,t) + \mathbf{f}(X,t)$, and follow the instructions in the problem: First, take the dot product of both sides of that PDE with an unspecified "test function" $\boldsymbol{\eta} = \boldsymbol{\eta}(X,t)$, and then integrate both sides with respect to X over the interval $[\,0, L\,]$ to get

$$(\star) \quad \int_0^L \varrho(X)\,\frac{\partial^2 \mathbf{r}}{\partial t^2}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX = \int_0^L \frac{\partial \mathbf{T}}{\partial X}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX + \int_0^L \mathbf{f}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX\,.$$

Next, use integration by parts on the second integral to get

$$\int_0^L \frac{\partial \mathbf{T}}{\partial X}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX = \Big[\,\mathbf{T}(X,t) \bullet \boldsymbol{\eta}(X,t)\Big]_0^L - \int_0^L \mathbf{T}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial X}(X,t)\,dX\,.$$

Let us assume that the test function $\boldsymbol{\eta}$ satisfies the BCs

$$\boldsymbol{\eta}(0,t) = \boldsymbol{\eta}(L,t) = \mathbf{0},$$

hence

$$\int_0^L \frac{\partial \mathbf{T}}{\partial X}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX = -\int_0^L \mathbf{T}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial X}(X,t)\,dX\,.$$

So, $(\star)$ can be rewritten as

$$\int_0^L \varrho(X)\,\frac{\partial^2 \mathbf{r}}{\partial t^2}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX = -\int_0^L \mathbf{T}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial X}(X,t)\,dX + \int_0^L \mathbf{f}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX\,,$$

or, equivalently,

$$\int_0^L \left(\varrho(X)\,\frac{\partial^2 \mathbf{r}}{\partial t^2}(X,t) \bullet \boldsymbol{\eta}(X,t) + \mathbf{T}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial X}(X,t) - \mathbf{f}(X,t) \bullet \boldsymbol{\eta}(X,t)\right) dX = 0.$$

Next, integrate both sides of the equation with respect to t over the interval $[0,\infty)$ to get

$$(\star\star) \quad \int_0^\infty \int_0^L \left(\varrho(X)\,\frac{\partial^2 \mathbf{r}}{\partial t^2}(X,t) \bullet \boldsymbol{\eta}(X,t) + \mathbf{T}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial X}(X,t) - \mathbf{f}(X,t) \bullet \boldsymbol{\eta}(X,t)\right) dX\,dt = 0.$$

The penultimate step is to use integration by parts (with respect to t) on the first term in $(\star\star)$ to get

$$(\star\star\star) \quad \int_0^\infty \int_0^L \varrho(X)\,\frac{\partial^2 \mathbf{r}}{\partial t^2}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX\,dt = \lim_{b\to\infty}\left(\int_0^b \int_0^L \varrho(X)\,\frac{\partial^2 \mathbf{r}}{\partial t^2}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX\right)$$

$$= \lim_{b\to\infty}\left(\Big[\int_0^L \varrho(X)\,\frac{\partial \mathbf{r}}{\partial t}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX\,\Big]_0^b - \int_0^b \int_0^L \varrho(X)\,\frac{\partial \mathbf{r}}{\partial t}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial t}(X,t)\,dX\,dt\right).$$

Let us assume that all improper integrals appearing in the work exist. [This assumption can be replaced by assuming that the test function $\boldsymbol{\eta}(X,t)$ and the solution $\mathbf{r}(X,t)$ are in spaces of functions with suitable integrability, but this can be thought of as mathematical technicalities.] Also, since $\mathbf{r}(X,t)$ satisfies the boundary conditions (10.54) in Section 10.4,

$$\mathbf{r}(0,t) = \mathbf{0} \quad \text{and} \quad \mathbf{r}(L\,\hat{\boldsymbol{\imath}},t) = L\,\hat{\boldsymbol{\imath}}.$$

It follows that, interchanging[1] the operation of taking a partial derivative with evaluation at $X=0$ or at $X=L$

$$\frac{\partial \mathbf{r}}{\partial t}(0,t) \equiv \mathbf{0} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial t}(L,t) \equiv \mathbf{0}.$$

Then $(\star\star\star)$ becomes

$$(\star\star\star\star) \quad \int_0^\infty \int_0^L \varrho(X)\,\frac{\partial^2 \mathbf{r}}{\partial t^2}(X,t) \bullet \boldsymbol{\eta}(X,t)\,dX\,dt = 0 - \int_0^\infty \int_0^L \varrho(X)\,\frac{\partial \mathbf{r}}{\partial t}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial t}(X,t)\,dX\,dt.$$

Substitute this into $(\star\star)$ to get

$$(\star\star\star\,\star\star) \quad \int_0^\infty \int_0^L \left(-\varrho(X)\,\frac{\partial \mathbf{r}}{\partial t}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial t}(X,t) + \mathbf{T}(X,t) \bullet \frac{\partial \boldsymbol{\eta}}{\partial X}(X,t) - \mathbf{f}(X,t) \bullet \boldsymbol{\eta}(X,t)\right) dX\,dt = 0,$$

which is true for all suitably “nice enough " functions $\boldsymbol{\eta}(X,t)$. Equation $(\star\star\star\,\star\star)$ is the principle of virtual work, as we wished to explain.

[1] this is another mathematical technicality that is true as long as the function $\mathbf{r}(X,t)$ is continuously differentiable in (X,t).

Section 10.5.5

10.5.5.1. In Figure 10.23 in Section 10.5, we were given $\Phi(x) = u(x,0) = \left\{ \begin{array}{cc} 0, & x < -2 \\ x+2, & -2 \le x < 0 \\ 0, & 0 \le x \end{array} \right\}$. Because $\dfrac{\partial u}{\partial t}(x,0) \equiv 0$, the solution of the PDE is given by $u(x,t) = \frac{1}{2}\,\Phi(x-ct) + \frac{1}{2}\,\Phi(x+ct)$, where the wave speed is $c = \sqrt{T_0/\varrho_0} = \sqrt{4/0.01} = 20$.

(a) So, the solution of the problem is

$$u(x,t) = \frac{1}{2}\left\{ \begin{array}{cc} 0, & x-20t < -2 \\ x-20t+2, & -2 \le x-20t < 0 \\ 0, & 0 \le x-20t \end{array} \right\} + \frac{1}{2}\left\{ \begin{array}{cc} 0, & x+20t < -2 \\ x+20t+2, & -2 \le x+20t < 0 \\ 0, & 0 \le x+20t \end{array} \right\}$$

that is,

$$u(x,t) = \frac{1}{2}\left\{ \begin{array}{cc} 0, & x < -2+20t \\ x-20t+2, & -2+20t \le x < 20t \\ 0, & 20t \le x \end{array} \right\} + \frac{1}{2}\left\{ \begin{array}{cc} 0, & x < -2-20t \\ x+20t+2, & -2-20t \le x < -20t \\ 0, & -20t \le x \end{array} \right\}.$$

(b) At $t = 0.025$, the solution of the PDE is

$$u(x,0.025) = \frac{1}{2}\left\{ \begin{array}{cc} 0, & x < -1.5 \\ x+1.5, & -1.5 \le x < 0.5 \\ 0, & 0.5 \le x \end{array} \right\} + \frac{1}{2}\left\{ \begin{array}{cc} 0, & x < -2.5 \\ x+2.5, & -2.5 \le x < -0.5 \\ 0, & -0.5 \le x \end{array} \right\}$$

$$= \left\{ \begin{array}{cc} 0, & x < -2.5 \\ \frac{1}{2}(x+2.5), & -2.5 \le x < -1.5 \\ \frac{1}{2}(x+2.5) + \frac{1}{2}(x+1.5), & -1.5 \le x < -0.5 \\ \frac{1}{2}(x+1.5), & -0.5 \le x < 0.5 \\ 0, & 0.5 \le x \end{array} \right\},$$

whose graph is shown in the figure.

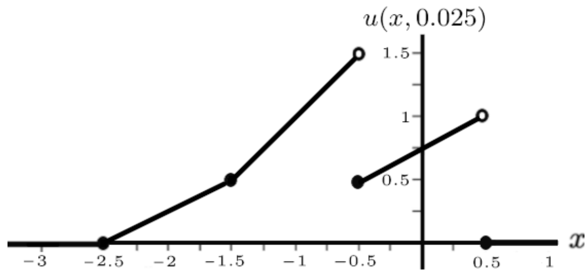


Figure 3: Problem 10.5.1: $u(x, 0.025)$

(c) At $t = 0.05$, the solution of the PDE is

$$u(x, 0.05) = \frac{1}{2}\left\{\begin{array}{cc} 0, & x < -1 \\ x+1, & -1 \le x < 1 \\ 0, & 1 \le x \end{array}\right\} + \frac{1}{2}\left\{\begin{array}{cc} 0, & x < -3 \\ x+3, & -3 \le x < -1 \\ 0, & -1 \le x \end{array}\right\} = \left\{\begin{array}{cc} 0, & x < -3 \\ \frac{1}{2}(x+3), & -3 \le x < -1 \\ \frac{1}{2}(x+1), & -1 \le x < 1 \\ 0, & 1 \le x \end{array}\right\},$$

10.5.5.3. We were given

$$\Phi(x) = u(x, 0) = \left\{\begin{array}{cc} 0, & x < -1 \\ x+1, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x \end{array}\right\}.$$

Because $\dfrac{\partial u}{\partial t}(x, 0) \equiv 0$, the solution of the PDE is given by

$$u(x, t) = \frac{1}{2}\,\Phi(x - ct) + \frac{1}{2}\,\Phi(x + ct),$$

where the wave speed is $c = 10$. So, the solution of the problem is

$$u(x, 0.07) = \frac{1}{2}\,\Phi(x - 0.7) + \frac{1}{2}\,\Phi(x + 0.7),$$

whose graph is shown in the figure.

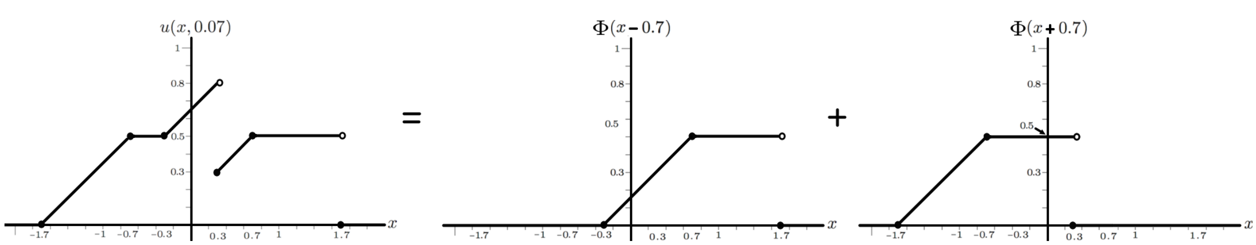


Figure 4: Problem 10.5.3: $u(x, 0.07)$

10.5.5.5. By Theorem 10.4 in Section 10.5, the solution is

$$u(x, t) = \frac{1}{40}\int_{x-20t}^{x+20t} \Psi(\xi)\, d\xi,$$

where $\Psi(x) = \dfrac{\partial y}{\partial t}(x, 0)$. Unfortunately, it is not true that $u(x,t) = \dfrac{1}{40}\displaystyle\int_{x-20t}^{x+20t} 20\, d\xi$.

Instead, we can write

$$u(x,t) = \int_{x-20t\le\xi\le x+20t} \Psi(\xi)\, d\xi = \frac{1}{2}\int_{x-20t\le\xi\le x+20t \text{ and } 0\le\xi<2} 1\, d\xi + \frac{1}{2}\int_{x-20t\le\xi\le x+20t \text{ and } (\xi<0 \text{ or } \xi\ge 2)} 0\, d\xi,$$

that is,

$$(\star) \quad u(x,t) = \frac{1}{2}\int_{[x-20t,x+20t]\cap[0,2)} 1\, d\xi.$$

The **intersection** symbol, $\cap$, says that the integral in $(\star)$ is performed over only those ξ that are in the *overlap* of the interval $[x-20t, x+20t]$ and the interval $[0,2)$. So, we need to study that overlap. For any fixed $t>0$, define the interval

$$J_x \triangleq [x-20t, x+20t] = \{\xi :\ x-20t \le \xi \le x+20t\}.$$

For any fixed $t>0$, the interval J_x "slides" along the ξ real line as x varies. In studying the intersection $[x-20t, x+20t]\cap[0,2)$, ξ must also be located in the interval $[0,2)$, which does not vary as x varies.

We will study that intersection by letting x increase and thus sliding J_x along until it overlaps the interval $[0,2)$ and then slides past the interval $[0,2)$. It's as if J_x is a pulse of material that moves in relation to the fixed interval $[0,2)$. The latter interval is the support of the initial velocity datum $\Psi(x)$.

Illustrated in the figure are the six and only possible relationships the interval J_x can have with the interval $[0,2)$:

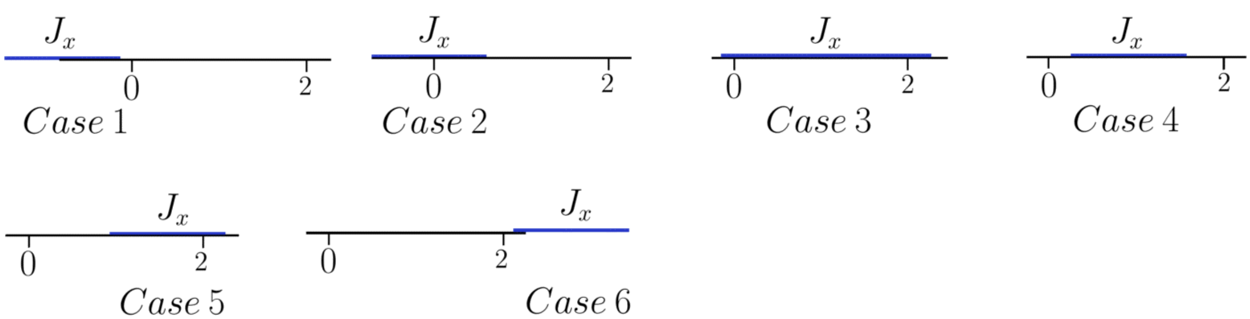


Figure 5: Interval cases

(1) J_x is to the left of $[0,2)$

(2) part of J_x is to the left of $[0,2)$ and part of J_x is contained in, but unequal to, $[0,2)$

(3) part of J_x is to the left of $[0,2)$, all of $[0,2)$ is contained in J_x, and part of J_x is to the right of $[0,2)$

(4) all of J_x is contained in, but unequal to, $[0,2)$

(5) part of J_x is contained in $[0,2)$ and part of J_x is to the right of $[0,2)$

(6) J_x is to the right of $[0,2)$.

Let us analyze if and when each of those six cases occurs, *for* $t>0$: Cases (3) and (4) have a "transitional" nature and will turn out to be useful to study first.

Case (3) can occur *only* if the length of the interval J_x is greater than or equal to 2, and, conversely, case (4) can occur *only* if the length of J_x is less than 2. The length of $J_x = [x-20t, x+20t]$ is $40t$. So, case (3) can occur only if $t \ge \frac{1}{20}$ and case (4) can occur only if $0 < t < \frac{1}{20}$. So, it makes sense to consider separately the cases (I) $0 < t < \frac{1}{20}$ and (II) $t \geq \frac{1}{20}$.

First, no matter whether $t \geq \frac{1}{20}$ or $0 < t < \frac{1}{20}$, cases (1) and (6) are the same in there being no overlap of J_x and $[0, 2)$.

Case (1) occurs when the right endpoint of the interval J_x is to the left of the interval $[0, 2)$, that is, when $x + 20t < 0$, that is, when $x < -20t$. In this case, $J_x \cap [0, 2) = \varnothing$, the **empty set**, so $u(x, t) = 0$. [The integral over nothing is nothing.]

Similarly, Case (6) occurs when the left endpoint of J_x is to the right of the interval $[0, 2)$, that is, when $x - 20t \geq 2$, that is, when $x \geq 2 + 20t$. In this case, $J_x \cap [0, 2) = \varnothing$, the empty set, so $u(x, t) = 0$.

(I) Next, suppose $0 < t < \frac{1}{20}$.

Case (2) occurs when J_x "straddles" $[0, 2)$ "on the left," specifically the right endpoint of J_x is at or to the right of the left endpoint of the interval $[0, 2)$ *and* the left endpoint of J_x is to the left of $[0, 2)$. This happens when both $x + 20t \geq 0$ and $x - 20t < 0$, that is, when $x \geq -20t$ and $x < 20t$, that is, when $-20t \leq x < 20t$. For $-20t \leq x < 20t$, the part of the interval J_x that overlaps $[0, 2)$ is $[0, x + 20t]$ or $[0, x + 20t)$, so

$$u(x, t) = \int_{J_x \cap [0,2)} \frac{1}{2}\, d\xi = \int_0^{x+20t} \frac{1}{2}\, d\xi = \frac{1}{2}(x + 20t).$$

Case (3) cannot occur because we supposed that $0 < t < \frac{1}{20}$.

Case (4) occurs when the left endpoint of J_x is at, or to the right of, 0 *and* the right endpoint of J_x is to the left of 2, that is, when $x - 20t \geq 0$ and $x + 20t < 2$, that is, when $x \geq 20t$ and $x < 2 - 20t$, that is, when $20t \leq x < 2 - 20t$. For $20t \leq x < 2 - 20t$, all of J_x lies inside $[0, 2)$. So, $J_x \cap [0, 2) = J_x$ and

$$u(x, t) = \int_{J_x \cap [0,2)} \frac{1}{2}\, d\xi = \int_{x-20t}^{x+20t} \frac{1}{2}\, d\xi = 20t.$$

Case (5) occurs when J_x straddles $[0, 2)$ on the right, specifically when (i) the right endpoint of J_x is to the right of the interval $[0, 2)$ *and* (ii) the left endpoint of J_x is in the interval $[0, 2)$. This happens when both (i) $x + 20t \geq 2$ and (ii) $0 \leq x - 20t < 2$, that is, when (i) $x \geq 2 - 20t$ and (ii)(A) $x \geq 20t$ and (ii)(B) $x < 2 + 20t$. Of the inequalities (i) $x \geq 2 - 20t$ and (ii)(A) $x \geq 20t$, (i) is more demanding because $2 - 20t > 20t$ follows from the assumption in part (a) that $0 < t < \frac{1}{20}$. So, case (5) occurs when (i) $x \geq 2 - 20t$ and (ii)(B) $x < 2 + 20t$, that is, when $2 - 20t \leq x < 2 + 20t$.

For $2 - 20t \leq x < 2 + 20t$, the part of the interval J_x that overlaps $[0, 2)$ is $[x - 20t, 2)$, so

$$u(x, t) = \int_{J_x \cap [0,2)} \frac{1}{2}\, d\xi = \int_{x-20t}^{2} \frac{1}{2}\, d\xi = \frac{1}{2}(2 - x + 20t).$$

Putting the six cases together, if $0 < t < \frac{1}{20}$ then

$$(*) \quad u(x, t) = \left\{ \begin{array}{lc} 0, & x < -20t \\ \frac{1}{2}(x + 20t), & -20t \leq x < 20t \\ 20t, & 20t \leq x < 2 - 20t \\ \frac{1}{2}(2 - x + 20t), & 2 - 20t \leq x < 2 + 20t \\ 0, & x \geq 2 + 20t \end{array} \right\}$$

(II) Second, suppose $t \geq \frac{1}{20}$. It makes sense to begin by studying the transitional case that can occur that couldn't occur in part (a).

Case (3) occurs when part of J_x is to the left of $[0, 2)$, all of $[0, 2)$ is contained in J_x, *and* part of J_x is to the right of $[0, 2)$. This occurs when the left endpoint of J_x, that is, $x - 20t$, is to the left of 0 and the right endpoint of J_x, that is, $x+20t$, is to the right of 2. This case occurs when both $x-20t<0$ and $x+20t\geq 2$, that is, when $x<20t$ and $x\geq 2-20t$.

So, case (3) occurs when $2-20t\leq x<20t$, in which case all of $[0,2)$ lies inside J_x, so $J_x\cap[0,2)=[0,2)$ and thus

$$u(x,t)=\int_{J_x\cap[0,2)}\frac{1}{2}\,d\xi=\int_0^2\frac{1}{2}\,d\xi=1.$$

Case (2) occurs when J_x straddles $[0,2)$ on the left, specifically when (i) the left endpoint of J_x is to the left of $[0,2)$ *and* (ii) the right endpoint of J_x is in the interval $[0,2)$. This happens when (i) $x-20t<0$ and (ii) $0\leq x+20t<2$, that is, (i) $x<20t$ and (ii)(A) $-20t\leq x$ and (ii)(B) $x<2-20t$. Of these three inequalities, (i) requires $x<20t$ and (ii)(B) requires $x<2-20t$; but, because we assumed that $t\geq\frac{1}{20}$, $(1-20t)<0$, so (ii)(B) implies $x<0$, which is more demanding than the requirement of (i) that $x<20t$. So, case (2) occurs when both $-20t\leq x$ and $x<2-20t$, that is, when $-20t\leq x<2-20t$.

In this case, the part of the interval J_x that overlaps $[0,2)$ is $[0,x+20t]$, so

$$u(x,t)=\int_{J_x\cap[0,2)}\frac{1}{2}\,d\xi=\int_0^{x+20t}\frac{1}{2}\,d\xi=\frac{1}{2}(x+20t).$$

Case (4) cannot occur because we supposed that $t\geq\frac{1}{20}$.

Case (5) occurs when the left endpoint of J_x is in $[0,2)$ and the right endpoint of J_x is to the right of 2. So, this requires (i) $0\leq x-20t<2$ and (ii) $x+20t\geq 2$, that is, when (i)(A) $x\geq 20t$ and (i)(B) $x<2+20t$, and (ii) $x\geq 2-20t$. We are assuming in part (b) that $t\geq\frac{1}{20}$, so, of the two inequalities (i)(A) $x\geq 20t$ and (ii) $x\geq 2-20t$, the more demanding requirement is that $x\geq 20t$. So, case (5) occurs when $20t\leq x<2+20t$.

In this case, the part of the interval J_x that overlaps $[0,2)$ is $[x-20t,2)$, so

$$u(x,t)=\int_{J_x\cap[0,2)}\frac{1}{2}\,d\xi=\int_{x-20t}^{2}\frac{1}{2}\,d\xi=\frac{1}{2}(2-x+20t).$$

Putting all of the cases that can occur together, for $t\geq\frac{1}{20}$,

$$u(x,t)=\left\{\begin{array}{lc} 0, & x<-20t \\ \frac{1}{2}(x+20t), & -20t\leq x<2-20t \\ 1, & 2-20t\leq x<20t \\ \frac{1}{2}(2-x+20t), & 20t\leq x<2+20t \\ 0, & x\geq 2+20t \end{array}\right\}.$$

This is the second half of the final conclusion for part (b).

(a) The graph of the solution $u(x,0.025)$ versus x follows from the formula in the case $0<t<\frac{1}{20}$, as given in $(*)$, specifically

$$u(x,0.025)=\left\{\begin{array}{lc} 0, & x<-0.5 \\ 0.5x+0.25, & -0.5\leq x<0.5 \\ 0.5, & 0.5\leq x<1.5 \\ 1.25-0.5x, & 1.5\leq x<2.5 \\ 0, & x\geq 2.5 \end{array}\right\}.$$

10.5.5.7. By Theorem 10.4 in Section 10.5, the solution is

$$u(x,t) = \frac{1}{40}\int_{x-20t}^{x+20t} \Psi(\xi)\, d\xi,$$

where $\Psi(x) = \dfrac{\partial y}{\partial t}(x,0)$. Unfortunately, it is not true that $u(x,t) = \dfrac{1}{40}\displaystyle\int_{x-20t}^{x+20t} 20\, d\xi$.

Instead, we can write

$$u(x,t) = \int_{x-20t\le\xi\le x+20t} \Psi(\xi)\, d\xi = \frac{1}{2}\int_{x-20t\le\xi\le x+20t \text{ and } 2\le\xi<3} 1\, d\xi + \frac{1}{2}\int_{x-20t\le\xi\le x+20t \text{ and } (\xi<2 \text{ or } \xi\ge 3)} 0\, d\xi,$$

that is,

$$(\star) \quad u(x,t) = \frac{1}{2}\int_{[x-20t,x+20t]\cap[2,3)} 1\, d\xi.$$

The **intersection** symbol, $\cap$, says that the integral in $(\star)$ is performed over only those ξ that are in the *overlap* of the interval $[x-20t, x+20t]$ and the interval $[2,3)$. So, we need to study that overlap. For any fixed $t>0$, define the interval

$$J_x \triangleq [x-20t, x+20t] = \{\xi : \, x-20t \le \xi \le x+20t\}.$$

For any fixed $t>0$, the interval J_x "slides" along the ξ real line as x varies. In studying the intersection $[x-20t, x+20t]\cap[2,3)$, ξ must also be located in the interval $[2,3)$, which does not vary as x varies.

We will study that intersection by letting x increase and thus sliding J_x along until it overlaps the interval $[2,3)$ and then slides past the interval $[2,3)$. It's as if J_x is a pulse of material that moves in relation to the fixed interval $[2,3)$. The latter interval is the support of the initial velocity datum $\Psi(x)$.

Illustrated in the figure are the six and only possible relationships the interval J_x can have with the interval $[2,3)$:

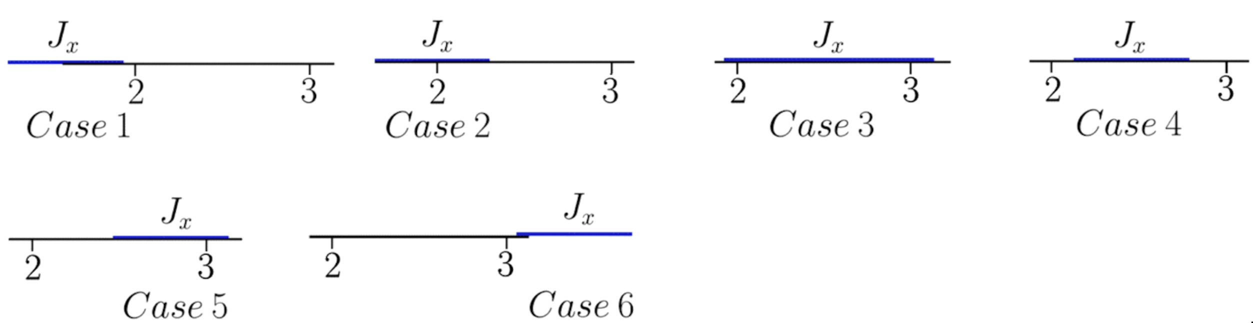


Figure 6: Interval cases

(1) J_x is to the left of $[2,3)$

(2) part of J_x is to the left of $[2,3)$ and part of J_x is contained in, but unequal to, $[2,3)$

(3) part of J_x is to the left of $[2,3)$, all of $[2,3)$ is contained in J_x, and part of J_x is to the right of $[2,3)$

(4) all of J_x is contained in, but unequal to, $[2,3)$

(5) part of J_x is contained in $[2,3)$ and part of J_x is to the right of $[2,3)$

(6) J_x is to the right of $[2,3)$.

Let us analyze if and when each of those six cases occurs, *for* $t > 0$: Cases (3) and (4) have a "transitional" nature and will turn out to be useful to study first.

Case (3) can occur *only* if the length of the interval J_x is greater than or equal to 1, and, conversely, case (4) can occur *only* if the length of J_x is less than 1. The length of $J_x = [x-20t, x+20t]$ is $40t$. So, case (3) can occur only if $t \geq \frac{1}{40}$ and case (4) can occur only if $0 < t < \frac{1}{40}$. So, it makes sense to consider separately the cases (I) $0 < t < \frac{1}{40}$ and (II) $t \geq \frac{1}{40}$.

First, no matter whether $t \geq \frac{1}{40}$ or $0 < t < \frac{1}{40}$, cases (1) and (6) are the same in there being no overlap of J_x and $[2,3)$.

Case (1) occurs when the right endpoint of the interval J_x is to the left of the interval $[2,3)$, that is, when $x + 20t < 2$, that is, when $x < 2 - 20t$. In this case, $J_x \cap [2,3) = \emptyset$, the **empty set**, so $u(x,t) = 0$. [The integral over nothing is nothing.]

Similarly, Case (6) occurs when the left endpoint of J_x is to the right of the interval $[2,3)$, that is, when $x - 20t \geq 3$, that is, when $x \geq 3 + 20t$. In this case, $J_x \cap [2,3) = \emptyset$, the empty set, so $u(x,t) = 0$.

(I) Next, suppose $0 < t < \frac{1}{40}$.

Case (2) occurs when J_x "straddles" $[2,3)$ "on the left," specifically the right endpoint of J_x is at or to the right of the left endpoint of the interval $[2,3)$ *and* the left endpoint of J_x is to the left of $[2,3)$. This happens when both $x + 20t \geq 2$ and $x - 20t < 2$, that is, when $x \geq 2 - 20t$ and $x < 2 + 20t$, that is, when $2 - 20t \leq x < 2 + 20t$. For $2 - 20t \leq x < 2 + 20t$, the part of the interval J_x that overlaps $[2,3)$ is $[2, x+20t]$ or $[2, x+20t)$, so

$$u(x,t) = \int_{J_x \cap [2,3)} \frac{1}{2}\,d\xi = \int_2^{x+20t} \frac{1}{2}\,d\xi = \frac{1}{2}(x + 20t - 2).$$

Case (3) cannot occur because we supposed that $0 < t < \frac{1}{40}$.

Case (4) occurs when the left endpoint of J_x is at, or to the right of, 2 *and* the right endpoint of J_x is to the left of 3, that is, when $x - 20t \geq 2$ and $x + 20t < 3$, that is, when $x \geq 2 + 20t$ and $x < 3 - 20t$, that is, when $2 + 20t \leq x < 3 - 20t$. For $2 + 20t \leq x < 3 - 20t$, all of J_x lies inside $[2,3)$. So, $J_x \cap [2,3) = J_x$ and

$$u(x,t) = \int_{J_x \cap [2,3)} \frac{1}{2}\,d\xi = \int_{x-20t}^{x+20t} \frac{1}{2}\,d\xi = 20t.$$

Case (5) occurs when J_x straddles $[2,3)$ on the right, specifically when (i) the right endpoint of J_x is to the right of the interval $[2,3)$ *and* (ii) the left endpoint of J_x is in the interval $[2,3)$. This happens when both (i) $x + 20t \geq 3$ and (ii) $2 \leq x - 20t < 3$, that is, when (i) $x \geq 3 - 20t$ and (ii)(A) $x \geq 2 + 20t$ and (ii)(B) $x < 3 + 20t$. Of the inequalities (i) $x \geq 3 - 20t$ and (ii)(A) $x \geq 2 + 20t$, (i) is more demanding because $3 - 20t > 2 + 20t$ follows from the assumption in part (a) that $0 < t < \frac{1}{40}$. So, case (5) occurs when (i) $x \geq 3 - 20t$ and (ii)(B) $x < 3 + 20t$, that is, when $3 - 20t \leq x < 3 + 20t$.

For $3 - 20t \leq x < 3 + 20t$, the part of the interval J_x that overlaps $[2,3)$ is $[x - 20t, 3)$, so

$$u(x,t) = \int_{J_x \cap [2,3)} \frac{1}{2}\,d\xi = \int_{x-20t}^{3} \frac{1}{2}\,d\xi = \frac{1}{2}(3 - x + 20t).$$

Putting the six cases together, if $0 < t < \frac{1}{40}$ then

$$(*) \quad u(x,t) = \left\{ \begin{array}{ll} 0, & x < 2 - 20t \\ \frac{1}{2}(x + 20t - 2), & 2 - 20t \leq x < 2 + 20t \\ 20t, & 2 + 20t \leq x < 3 - 20t \\ \frac{1}{2}(3 - x + 20t), & 3 - 20t \leq x < 3 + 20t \\ 0, & x \geq 3 + 20t \end{array} \right\}$$

(II) Second, suppose $t \ge \frac{1}{40}$. It makes sense to begin by studying the transitional case that can occur that couldn't occur in part (a).

Case (3) occurs when part of J_x is to the left of $[2, 3)$, all of $[2, 3)$ is contained in J_x, *and* part of J_x is to the right of $[2, 3)$. This occurs when the left endpoint of J_x, that is, $x - 20t$, is to the left of 0 and the right endpoint of J_x, that is, $x + 20t$, is to the right of 2. This case occurs when both $x - 20t < 2$ and $x + 20t \ge 3$, that is, when $x < 20t$ and $x \ge 2 - 20t$.

So, case (3) occurs when $3 - 20t \le x < 2 + 20t$, in which case all of $[2, 3)$ lies inside J_x, so $J_x \cap [2, 3) = [2, 3)$ and thus

$$u(x,t) = \int_{J_x \cap [2,3)} \frac{1}{2}\, d\xi = \int_0^2 \frac{1}{2}\, d\xi = 1.$$

Case (2) occurs when J_x straddles $[2, 3)$ on the left, specifically when (i) the left endpoint of J_x is to the left of $[2, 3)$ *and* (ii) the right endpoint of J_x is in the interval $[2, 3)$. This happens when (i) $x - 20t < 2$ and (ii) $2 \le x + 20t < 3$, that is, (i) $x < 2 + 20t$ and (ii)(A) $2 - 20t \le x$ and (ii)(B) $x < 3 - 20t$. Of these three inequalities, (i) requires $x < 2 + 20t$ and (ii)(B) requires $x < 3 - 20t$; but, because we assumed that $t \ge \frac{1}{40}$, $(3 - 20t) < 2 + 20t$, so (ii)(B) implies $x < 3 - 20t$, which is more demanding than the requirement of (i) that $x < 2 + 20t$. So, case (2) occurs when both $2 - 20t \le x$ and $x < 3 - 20t$, that is, when $2 - 20t \le x < 3 - 20t$.

In this case, the part of the interval J_x that overlaps $[2, 3)$ is $[2, x + 20t]$, so

$$u(x,t) = \int_{J_x \cap [2,3)} \frac{1}{2}\, d\xi = \int_2^{x+20t} \frac{1}{2}\, d\xi = \frac{1}{2}(x + 20t - 2).$$

Case (4) cannot occur because we supposed that $t \ge \frac{1}{40}$.

Case (5) occurs when the left endpoint of J_x is in $[2, 3)$ and the right endpoint of J_x is to the right of 3. So, this requires (i) $2 \le x - 20t < 3$ and (ii) $x + 20t \ge 3$, that is, when (i)(A) $x \ge 2 + 20t$ and (i)(B) $x < 3 + 20t$, and (ii) $x \ge 3 - 20t$. We are assuming in part (b) that $t \ge \frac{1}{40}$, so, of the two inequalities (i)(A) $x \ge 2 + 20t$ and (ii) $x \ge 3 - 20t$, the more demanding requirement is that $x \ge 2 + 20t$. So, case (5) occurs when $2 + 20t \le x < 3 + 20t$.

In this case, the part of the interval J_x that overlaps $[2, 3)$ is $[x - 20t, 3)$, so

$$u(x,t) = \int_{J_x \cap [2,3)} \frac{1}{2}\, d\xi = \int_{x-20t}^{3} \frac{1}{2}\, d\xi = \frac{1}{2}(3 - x + 20t).$$

Putting all of the cases that can occur together, for $t \ge \frac{1}{40}$,

$$u(x,t) = \left\{ \begin{array}{lc} 0, & x < 2 - 20t \\ \\ \frac{1}{2}(x + 20t - 2), & 2 - 20t \le x < 3 - 20t \\ \\ 1, & 3 - 20t \le x < 2 + 20t \\ \\ \frac{1}{2}(3 - x + 20t), & 2 + 20t \le x < 3 + 20t \\ \\ 0, & x \ge 3 + 20t \end{array} \right\}.$$

This is the second half of the final conclusion for part (b).

(a) The graph of the solution $u(x, 0.025)$ versus x follows from the formula in the case $t \ge \frac{1}{40}$. [Note that

$1 - 20t = 20t$ when $t = 0.025$]:

$$u(x, 0.025) = \left\{ \begin{array}{ll} 0, & x < 1.5 \\ & \\ 0.5x - 0.75, & 1.5 \le x < 2.5 \\ & \\ 1.75 - 0.5x, & 2.5 \le x < 3.5 \\ & \\ 0, & x \ge 2.5 \end{array} \right\}$$

Section 10.6.1

10.6.1.1. By Parseval's Theorem 9.21 in Section 9.8,

$$PE(t) \triangleq \int_0^L \frac{1}{2}\,T_0 \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{T_0 L}{4}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\left(a_n\cos\left(\frac{n\pi ct}{L}\right)+b_n\sin\left(\frac{n\pi ct}{L}\right)\right)^2.$$

The period is $\mathsf{T} = \dfrac{2\pi}{\pi c/L} = \dfrac{2L}{c}$. It follows that the time average value is given by

$$\overline{PE} = \frac{1}{2L/c}\int_0^{2L/c} PE(t)dt = \frac{c}{2L}\int_0^{2L/c}\left(\frac{T_0 L}{4}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\left(a_n\cos\left(\frac{n\pi ct}{L}\right)+b_n\sin\left(\frac{n\pi ct}{L}\right)\right)^2\right)dt$$

$$= \frac{cT_0}{8}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\int_0^{2L/c}\left(a_n\cos\left(\frac{n\pi ct}{L}\right)+b_n\sin\left(\frac{n\pi ct}{L}\right)\right)^2 dt$$

$$= \frac{cT_0}{8}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\int_0^{2L/c}\left(a_n^2\cos^2\left(\frac{n\pi ct}{L}\right)+2a_nb_n\sin\left(\frac{n\pi ct}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)+b_n^2\sin^2\left(\frac{n\pi ct}{L}\right)\right)dt$$

$$= \frac{cT_0}{8}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\int_0^{2L/c}\left(a_n^2\,\frac{1+\cos\left(\frac{2n\pi ct}{L}\right)}{2}+a_nb_n\sin\left(\frac{2n\pi ct}{L}\right)+b_n^2\,\frac{1-\cos\left(\frac{2n\pi ct}{L}\right)}{2}\right)dt$$

$$= \frac{cT_0}{8}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\left[a_n^2\left(\frac{t}{2}+\frac{\sin\left(\frac{2n\pi ct}{L}\right)}{4n\pi c/L}\right)-a_nb_n\,\frac{\cos\left(\frac{2n\pi ct}{L}\right)}{2n\pi c/L}+b_n^2\left(\frac{t}{2}-\frac{\sin\left(\frac{2n\pi ct}{L}\right)}{4n\pi c/L}\right)\right]_0^{2L/c}$$

$$= \frac{cT_0}{8}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\left(a_n^2\left(\frac{2L/c}{2}+\frac{0-0}{4n\pi c/L}\right)-a_nb_n\,\frac{1-1}{2n\pi c/L}+b_n^2\left(\frac{2L/c}{2}-\frac{0-0}{4n\pi c/L}\right)\right)$$

$$= \frac{T_0\pi^2 L}{8}\sum_{n=1}^{\infty} n^2\left(a_n^2+b_n^2\right).$$

On the other hand, by Parseval's Theorem 9.21 in Section 9.8,

$$KE(t) \triangleq \int_0^L \frac{1}{2}\varrho_0\left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{\varrho_0 L}{4}\sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)^2\left(-a_n\sin\left(\frac{n\pi ct}{L}\right)+b_n\cos\left(\frac{n\pi ct}{L}\right)\right)^2.$$

The period is $\mathsf{T} = \dfrac{2\pi}{\pi c/L} = \dfrac{2L}{c}$. It follows that the time average value is given by

$$\overline{KE} = \frac{1}{2L/c}\int_0^{2L/c} KE(t)dt = \frac{c}{2L}\int_0^{2L/c}\frac{\varrho_0 L}{4}\sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)^2\left(-a_n\cos\left(\frac{n\pi ct}{L}\right)+b_n\sin\left(\frac{n\pi ct}{L}\right)\right)^2 dt$$

$$= \frac{c\varrho_0}{8}\sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)^2\int_0^{2L/c}\left(-a_n\cos\left(\frac{n\pi ct}{L}\right)+b_n\sin\left(\frac{n\pi ct}{L}\right)\right)^2 dt$$

$$= \frac{cT_0}{8}\sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)^2\int_0^{2L/c}\left(a_n^2\cos^2\left(\frac{n\pi ct}{L}\right)-2a_nb_n\sin\left(\frac{n\pi ct}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)+b_n^2\sin^2\left(\frac{n\pi ct}{L}\right)\right)dt$$

$$= \frac{c\varrho_0}{8} \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right)^2 \int_0^{2L/c} \left(a_n^2 \, \frac{1+\cos\left(\frac{2n\pi ct}{L}\right)}{2} - a_n b_n \, \sin\left(\frac{2n\pi ct}{L}\right) + b_n^2 \, \frac{1-\cos\left(\frac{2n\pi ct}{L}\right)}{2} \right) dt$$

$$= \frac{c\varrho_0}{8} \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right)^2 \left[a_n^2 \left(\frac{t}{2} + \frac{\sin\left(\frac{2n\pi ct}{L}\right)}{4n\pi c/L} \right) + a_n b_n \, \frac{\cos\left(\frac{2n\pi ct}{L}\right)}{2n\pi c/L} + b_n^2 \left(\frac{t}{2} - \frac{\sin\left(\frac{2n\pi ct}{L}\right)}{4n\pi c/L} \right) \right]_0^{2L/c}$$

$$= \frac{c\varrho_0}{8} \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right)^2 \left(a_n^2 \left(\frac{2L/c}{2} + \frac{0-0}{4n\pi c/L} \right) + a_n b_n \, \frac{1-1}{2n\pi c/L} + b_n^2 \left(\frac{2L/c}{2} - \frac{0-0}{4n\pi c/L} \right) \right)$$

$$= \frac{\varrho_0 c^2 \pi^2 L}{8} \sum_{n=1}^{\infty} n^2 \left(a_n^2 + b_n^2 \right) .$$

But, $c^2 = \dfrac{T_0}{\varrho_0}$, so

$$\overline{KE} = \frac{\varrho_0 T_0/\varrho_0 \pi^2 L}{8} \sum_{n=1}^{\infty} n^2 \left(a_n^2 + b_n^2 \right) = \frac{T_0 \pi^2 L}{8} \sum_{n=1}^{\infty} n^2 \left(a_n^2 + b_n^2 \right) = \overline{PE},$$

as we were asked to explain.

10.6.1.3. The solution of the PDE has the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right).$$

At the end of this section are calculations that explain why it will be enough to explain why the sum

$$S = \varrho_0 c^2 \left(-a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right) \right)^2 + T_0 \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right)^2$$

is constant in time. We have

$$S = \varrho_0 \cdot c^2 \left(a_n^2 \sin^2\left(\frac{n\pi ct}{L}\right) - 2a_n b_n \sin\left(\frac{n\pi ct}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + b_n^2 \cos^2\left(\frac{n\pi ct}{L}\right) \right)^2$$

$$+T_0 \left(a_n^2 \cos^2\left(\frac{n\pi ct}{L}\right) + 2a_n b_n \sin\left(\frac{n\pi ct}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + b_n^2 \sin^2\left(\frac{n\pi ct}{L}\right) \right)^2 .$$

But, $c^2 = \dfrac{T_0}{\varrho_0}$, so

$$S = \varrho_0 \cdot \frac{T_0}{\varrho_0} \left(a_n^2 \sin^2\left(\frac{n\pi ct}{L}\right) - 2a_n b_n \sin\left(\frac{n\pi ct}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + b_n^2 \cos^2\left(\frac{n\pi ct}{L}\right) \right)^2$$

$$+T_0 \left(a_n^2 \cos^2\left(\frac{n\pi ct}{L}\right) + 2a_n b_n \sin\left(\frac{n\pi ct}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + b_n^2 \sin^2\left(\frac{n\pi ct}{L}\right) \right)^2 ,$$

so

$$S = T_0 \left(a_n^2 \sin^2\left(\frac{n\pi ct}{L}\right) \cancel{- 2a_n b_n \sin\left(\frac{n\pi ct}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)} + b_n^2 \cos^2\left(\frac{n\pi ct}{L}\right) \right)^2$$

$$+T_0\left(a_n^2\cos^2\left(\frac{n\pi ct}{L}\right)+\cancel{2a_nb_n\sin\left(\frac{n\pi ct}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)}+b_n^2\sin^2\left(\frac{n\pi ct}{L}\right)\right)^2$$

$$=T_0\left(a_n^2\left(\cos^2\left(\frac{n\pi ct}{L}\right)+\sin^2\left(\frac{n\pi ct}{L}\right)\right)+b_n^2\left(\sin^2\left(\frac{n\pi ct}{L}\right)+\cos^2\left(\frac{n\pi ct}{L}\right)\right)\right)$$

$$=T_0\left(a_n^2\,(1)+b_n^2\,(1)\right)$$

$$=T_0\left(a_n^2+b_n^2\right)$$

is constant. This explains why

$$E(t)\triangleq\int_0^L\frac{1}{2}\left(T_0\left(\frac{\partial u}{\partial x}\right)^2+\varrho_0\left(\frac{\partial u}{\partial t}\right)^2\right)dx=\sum_{n=1}^{\infty}T_0\left(a_n^2+b_n^2\right)$$

is constant.

Chapter Eleven

Section 11.1

11.1.1. This problem is the same as Example 11.1 except that here $L = \pi$, instead of $L = 5$, and the initial condition is different. So, we do not have to "reinvent the wheel." As in Example 11.1 in Section 11.1, the boundary conditions are those in the first group of Table 11.1, so the general solution of the PDE and the BCs is

$$T(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-\alpha\, n^2 t}\,.$$

The initial condition is satisfied by solving

$$\sin x - \frac{1}{2}\sin 3x = T(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)\,,\ 0 < x < \pi\,.$$

Because $T(x,0) = \sin x - \frac{1}{2}\sin 3x$ is its own Fourier sine series we do not need integrals to conclude that $b_1 = 1$, $b_3 = -\dfrac{1}{2}$, and all other $b_n = 0$.

The solution of the problem is

$$T(x,t) = e^{-\alpha t}\sin x - \frac{1}{2}e^{-9\alpha t}\sin 3x\,.$$

11.1.3. First find the equilibrium solution, $v = v(x)$, satisfying $\dfrac{\partial v}{\partial t} \equiv 0$ and $\left\{ \begin{array}{c} 0 = \alpha v''(x),\ 0 < x < \pi, \\ \\ v'(0) = v'(\pi) = 1 \end{array} \right\}$.

The general solution of the ODE $v'' = 0$ is

$$v(x) = c_1 + c_2 x,$$

where c_1, c_2 are arbitrary constants. Substitute that into the BCs to get

$$\left\{ \begin{array}{c} 1 = v'(0) = c_2 \\ 1 = v'(\pi) = c_2 \end{array} \right\},$$

which are redundantly solved by choosing $c_2 = 1$, hence $v(x) = c_1 + x$, where c_1 is arbitrary.

Define $w = w(x,t) = T(x,t) - v(x)$. Similar to work in Example 11.2 in Section 11.1, $w(x,t)$ should satisfy the homogeneous problem

$$(\star)\quad \left\{ \begin{array}{c} \dfrac{\partial w}{\partial t} = \alpha\,\dfrac{\partial^2 w}{\partial x^2},\ 0 < x < \pi,\ t > 0, \\ \\ \dfrac{\partial w}{\partial x}(0,t) = \dfrac{\partial w}{\partial x}(\pi,t) = 0,\ t > 0 \end{array} \right\}.$$

The boundary conditions correspond to the second group in Table 11.1 with $L = \pi$, so, similar to the work in Example 11.1 in Section 11.1, we look for solutions of the PDE in the form

$$w(x,t) = \cos(nx)\, G(t),$$

for $n = 0, 1, 2, ...$. [Recall that $X_0(x) = \cos(0 \cdot x) \equiv 1$ is an "honorary cosine function."]

Substitute w into the PDE to get

$$\cos(nx)\,\frac{dG}{dt} = -\alpha\, n^2 \cos(nx)\, G(t),$$

hence

$$\frac{dG}{dt} = -\alpha\, n^2 G(t)\,.$$

The general solution of $(\star)$ is

$$w(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\, e^{-\alpha\, n^2 t}\,.$$

The solution of the original problem's PDE and BCs is

$$T(x,t) = v(x) + w(x,t) = c_1 + x + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\, e^{-\alpha\, n^2 t},$$

where c_1 is an arbitrary constant.

But $c_1 + \frac{a_0}{2}$ is no more or less of an arbitrary constant then is $\frac{a_0}{2}$, so it is simpler to write the general solution of the PDE and the BCs as

$$T(x,t) = x + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\, e^{-\alpha\, n^2 t}.$$

The initial condition is satisfied by solving

$$2x = T(x,0) = x + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,\ 0 < x < \pi,$$

that is,

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\,,\ 0 < x < \pi\,.$$

Fourier analysis implies that this is done by finding the coefficients a_n, for $n = 0, 1, 2, ...$ In fact, this was done in Example 9.9 in Section 9.2:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi} = \pi$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx = \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = ... = \frac{2\big((-1)^n - 1\big)}{n^2\pi}.$$

But for $n = even$, $(-1)^n - 1 = 0$; for $n = odd = 2k-1$, $(-1)^n - 1 = -2$. The solution of the problem is

$$T(x,t) = x + \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos\big((2k-1)x\big)\, e^{-\alpha(2k-1)^2 t}.$$

11.1.5. The physical situation is modeled by

$$\left\{\begin{array}{c} \dfrac{\partial T}{\partial t} = \alpha\, \dfrac{\partial^2 T}{\partial x^2},\ 0 < x < 5,\ t > 0, \\ \\ \dfrac{\partial T}{\partial x}(0,t) = \dfrac{\partial T}{\partial x}(5,t) = 0,\ t > 0, \\ \\ T(x,0) = x,\ 0 < x < 5 \end{array}\right\},$$

where the thermal diffusivity of copper is $\alpha = 1.15 \times 10^{-4} m^2/s$ and x is measured in m.

The boundary conditions correspond to the second group in Table 11.1 with $L = 5$, so, similar to the work in Example 11.1 in Section 11.1, we look for solutions of the PDE in the form

$$T(x,t) = \cos\left(\frac{n\pi x}{5}\right) G(t),$$

for $n = 0, 1, 2, ...$. [Recall that $X_0(x) = \cos(0 \cdot x) \equiv 1$ is an "honorary cosine function."]

Substitute T into the PDE to get

$$\cos\left(\frac{n\pi x}{5}\right)\frac{dG}{dt} = -\alpha\left(\frac{n\pi}{5}\right)^2 \cos\left(\frac{n\pi x}{5}\right) G(t),$$

hence

$$\frac{dG}{dt} = -\alpha\left(\frac{n\pi}{5}\right)^2 G(t)\,.$$

The general solution of the PDE and the BCs is

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{5}\right) e^{-\alpha\left(\frac{n\pi}{5}\right)^2 t}.$$

The initial condition is satisfied by solving

$$x = T(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{5}\right),\ 0 < x < 5,$$

that is,

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{5}\right),\ 0 < x < 5\,.$$

Fourier analysis implies that this is done by finding the coefficients a_n, for $n = 0, 1, 2, ...$. In fact, this is similar to what was done in Example 9.9 in Section 9.2:

$$a_0 = \frac{2}{5}\int_0^5 x\,dx = \frac{2}{5}\left[\frac{1}{2}\,x^2\right]_0^5 = 5$$

and

$$a_n = \frac{2}{5}\int_0^5 x\cos\left(\frac{n\pi x}{5}\right)dx = \frac{2}{5}\Big[\frac{x\sin\left(\frac{n\pi x}{5}\right)}{n\pi/5} + \frac{\cos\left(\frac{n\pi x}{5}\right)}{(n\pi/5)^2}\Big]_0^5 = ... = \frac{2\big((-1)^n - 1\big)}{\pi^2 n^2/5}\,.$$

But for $n = even$, $(-1)^n - 1 = 0$; for $n = odd = 2k-1$, $(-1)^n - 1 = -2$. The solution of the problem is

$$T(x,t) = \frac{5}{2} - \frac{20}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\left(\frac{(2k-1)\pi x}{5}\right)e^{-1.15\times 10^{-4}\left(\frac{(2k-1)\pi}{5}\right)^2 t}\,.$$

11.1.7. The equilibrium solution, $v = v(x)$, satisfies $\dfrac{\partial v}{\partial t} \equiv 0$ and

$$\left\{\begin{array}{c} 0 = \alpha v''(x) - 1,\ 0 < x < 2, \\ \\ v(0) = v'(2) = 0 \end{array}\right\}.$$

The general solution of the ODE $v'' = \dfrac{1}{\alpha}$ is

$$v(x) = \frac{1}{2\alpha}\,x^2 + c_1 + c_2 x,$$

where c_1, c_2 are arbitrary constants. Substitute that into the BCs to get

$$\left\{\begin{array}{c} 0 = v(0) = c_1 \\ \\ 0 = v'(2) = \frac{1}{\alpha}\cdot 2 + c_2 \end{array}\right\}.$$

The equilibrium solution is

$$v(x) = \frac{1}{2\alpha}\,x^2 - \frac{2}{\alpha}\,x\,.$$

Define $w = w(x,t) = T(x,t) - v(x)$. Similar to work in Example 11.2 in Section 11.1, $w(x,t)$ should satisfy the homogeneous problem

$$\left\{\begin{array}{c} \dfrac{\partial w}{\partial t} = \alpha\,\dfrac{\partial^2 w}{\partial x^2},\ 0 < x < 2,\ t > 0, \\ \\ w(0,t) = \dfrac{\partial w}{\partial x}(2,t) = 0,\ t > 0 \end{array}\right\}.$$

The boundary conditions fit the third group of entries of Table 11.1, with $L = 2$. Define

$$X_n(x) \triangleq \sin\left(\frac{\big(n - \frac{1}{2}\big)\pi x}{2}\right).$$

Substitute product solutions of the form

$$w(x,t) = X_n(x)G(t)$$

into the PDE to get

$$X_n(x)\dot{G}(t) = \frac{\partial w}{\partial t} = \alpha\,\frac{\partial^2 w}{\partial x^2} = -\alpha\Big(\frac{\big(n - \frac{1}{2}\big)\pi}{2}\Big)^2 X_n(x)G(t),$$

hence

$$\dot{G}(t) = -\alpha\Big(\frac{\big(n - \frac{1}{2}\big)\pi}{2}\Big)^2 G(t)\,.$$

So, the product solutions of the homogeneous problem are of the form

$$w_n(x,t) = \sin\Big(\frac{(n-\frac{1}{2})\pi x}{2}\Big)e^{-\alpha\left(\frac{(n-\frac{1}{2})\pi}{2}\right)^2 t}.$$

The general solution of the PDE+BCs is $T = v + w$, that is,

$$T(x,t) = \frac{1}{2\alpha}(x^2-4x) + \sum_{n=1}^{\infty} c_n \sin\Big(\frac{(n-\frac{1}{2})\pi x}{2}\Big)e^{-\alpha\left(\frac{(n-\frac{1}{2})\pi}{2}\right)^2 t}.$$

The initial condition is satisfied by solving

$$0 = T(x,0) = \frac{1}{2\alpha}(x^2-4x) + \sum_{n=1}^{\infty} c_n \sin\Big(\frac{(n-\frac{1}{2})\pi x}{2}\Big),\ 0<x<2,$$

that is,

$$-\frac{1}{2\alpha}(x^2-4x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{(n-\frac{1}{2})\pi x}{2}\Big),\ 0<x<2\,.$$

We calculate the generalized Fourier coefficients

$$c_n = \frac{2}{2}\int_0^2 -\frac{1}{2\alpha}(x^2-4x)\sin\Big(\frac{(n-\frac{1}{2})\pi x}{2}\Big)dx = ... = \frac{8}{\pi^3\alpha}\cdot\frac{1}{\big(n-\frac{1}{2}\big)^3}\,.$$

The solution of the problem is

$$T(x,t) = \frac{1}{2\alpha}(x^2-4x) + \frac{8}{\pi^3\alpha}\sum_{n=1}^{\infty}\frac{1}{\big(n-\frac{1}{2}\big)^3}\sin\Big(\frac{(n-\frac{1}{2})\pi x}{2}\Big)e^{-\alpha\left(\frac{(n-\frac{1}{2})\pi}{2}\right)^2 t}.$$

11.1.9. The equilibrium solution, $v = v(x)$, satisfies $\frac{\partial v}{\partial t} \equiv 0$ and

$$\left\{\begin{array}{c} 0 = \alpha v''(x),\ 0<x<L, \\ \\ v(0)=0,\ v'(L)=10 \end{array}\right\}.$$

The general solution of the ODE $v''=0$ is

$$v(x) = c_1 + c_2 x,$$

where c_1, c_2 are arbitrary constants. Substitute that into the BCs to get

$$\left\{\begin{array}{c} 0 = v(0) = c_1 \\ 10 = v'(L) = c_2 \end{array}\right\}.$$

The equilibrium solution is

$$v(x) = 10\,x\,.$$

Define $w = w(x,t) = T(x,t) - v(x)$. Similar to work in Example 11.2 in Section 11.1, $w(x,t)$ should satisfy the homogeneous problem

$$\left\{\begin{array}{c} \frac{\partial w}{\partial t} = \alpha\frac{\partial^2 w}{\partial x^2},\ 0<x<L,\ t>0, \\ \\ w(0,t) = \frac{\partial w}{\partial x}(L,t) = 0,\ t>0 \end{array}\right\}.$$

The boundary conditions fit the third group of entries of Table 11.1. Define

$$X_n(x) \triangleq \sin\Big(\frac{(n-\frac{1}{2})\pi x}{L}\Big).$$

Substitute product solutions of the form

$$w(x,t) = X_n(x)G(t)$$

into the PDE to get

$$X_n(x)\dot{G}(t) = \frac{\partial w}{\partial t} = \alpha\,\frac{\partial^2 w}{\partial x^2} = -\alpha\Big(\frac{(n-\frac{1}{2})\pi}{L}\Big)^2 X_n(x)G(t),$$

hence

$$\dot{G}(t) = -\alpha\Big(\frac{(n-\frac{1}{2})\pi}{L}\Big)^2 G(t)\,.$$

So, the product solutions of the homogeneous problem are of the form

$$w_n(x,t) = \sin\Big(\frac{(n-\frac{1}{2})\pi x}{L}\Big)e^{-\alpha\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2 t}.$$

The general solution of the PDE+BCs is $T = v + w$, that is,

$$T(x,t) = 10\,x + \sum_{n=1}^{\infty} c_n \sin\Big(\frac{(n-\frac{1}{2})\pi x}{2}\Big)e^{-\alpha\left(\frac{(n-\frac{1}{2})\pi}{2}\right)^2 t}.$$

The initial condition is satisfied by solving

$$f(x) = T(x,0) = 10\,x + \sum_{n=1}^{\infty} c_n \sin\Big(\frac{(n-\frac{1}{2})\pi x}{L}\Big),\ 0 < x < L\,.$$

From the graph of $f(x)$ we have

$$f(x) = \left\{\begin{array}{rr} 0, & 0 \le x \le \frac{L}{2} \\ \\ -20\sin\left(\frac{2\pi x}{L}\right), & \frac{L}{2} \le x \le L \end{array}\right\}$$

So

$$f(x) - 10\,x = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{(n-\frac{1}{2})\pi x}{L}\Big),\ 0 < x < L\,.$$

Using $\mathsf{Mathematica}^{\mathrm{TM}}$, we calculated the generalized Fourier coefficients

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L (f(x) - 10\,x)\sin\Big(\frac{(n-\frac{1}{2})\pi x}{L}\Big)dx \\ &= -\frac{2}{L}\int_0^L 10\,x\,\sin\Big(\frac{(n-\frac{1}{2})\pi x}{L}\Big)dx - \int_{L/2}^L 20\sin\Big(\frac{2\pi x}{L}\Big)\sin\Big(\frac{(n-\frac{1}{2})\pi x}{L}\Big)dx \\ &= \ldots = \frac{80(-1)^n L}{\pi^2(n-\frac{1}{2})^2} + \frac{320\left(\sqrt{2}(-1)^n + \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)}{\pi(4n^2 - 4n - 15)}\,. \end{aligned}$$

The solution of the problem is

$$T(x,t) = 10x + \sum_{n=1}^{\infty}\left(\frac{80(-1)^n L}{\pi^2(n-\frac{1}{2})^2} + \frac{320\left(\sqrt{2}(-1)^n + \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)}{\pi(4n^2 - 4n - 15)}\right)\sin\Big(\frac{(n-\frac{1}{2})\pi x}{L}\Big)e^{-\alpha\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2 t}.$$

11.1.11. First find the equilibrium solution, $v = v(x)$, satisfying $\frac{\partial v}{\partial t} \equiv 0$ and $\left\{\begin{array}{c} 0 = v''(x) + \cos x,\ 0 < x < \pi, \\ \\ v'(0) = v'(\pi) = 0 \end{array}\right\}$.

The indefinite integral of the ODE $v'' = -\cos x$ gives

$$v'(x) = -\sin x + c_2,$$

where c_2 is an arbitrary constant. A second indefinite integration implies that the general solution of the ODE $0 = v''(x) + \cos x$ is

$$v(x) = \cos x + c_1 + c_2 x,$$

where c_1, c_2 are arbitrary constants. Substitute that into the BCs to get

$$\left\{\begin{array}{l} 0 = v'(0) = -0 + c_2 \\ 0 = v'(\pi) = -0 + c_2 \end{array}\right\},$$

which are redundantly solved by choosing $c_2 = 0$, hence $v(x) = -\cos x + c_1$, where c_1 is arbitrary.

Define $w = w(x,t) = T(x,t) - v(x)$. Similar to work in Example 11.2 in Section 11.1, $w(x,t)$ should satisfy the homogeneous problem

$$(\star)\quad \left\{ \begin{array}{l} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2},\ 0 < x < \pi,\ t > 0, \\ \\ \dfrac{\partial w}{\partial x}(0,t) = \dfrac{\partial w}{\partial x}(\pi,t) = 0,\ t > 0 \end{array} \right\}.$$

The boundary conditions correspond to the second group in Table 11.1 with $L = \pi$, so, similar to the work in Example 11.1 in Section 11.1, we look for solutions of the PDE in the form

$$w(x,t) = \cos(nx)\, G(t),$$

for $n = 0, 1, 2, \ldots$. [Recall that $X_0(x) = \cos(0 \cdot x) \equiv 1$ is an "honorary cosine function."]

Substitute w into the PDE to get

$$\cos(nx)\, \frac{dG}{dt} = -n^2 \cos(nx)\, G(t),$$

hence

$$\frac{dG}{dt} = -n^2 G(t)\,.$$

The general solution of $(\star)$ is

$$w(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\, e^{-n^2 t}\,.$$

The solution of the original problem's PDE + BCs is

$$T(x,t) = v(x) + w(x,t) = \cos x + c_1 + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\, e^{-n^2 t},$$

where c_1 is an arbitrary constant.

But $c_1 + \dfrac{a_0}{2}$ is no more or less of an arbitrary constant then is $\dfrac{a_0}{2}$, so it is simpler to write the general solution of the PDE and BCs as

$$T(x,t) = \cos x + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\, e^{-n^2 t}\,.$$

The initial condition is satisfied by solving

$$-x + \cos x = T(x,0) = \cos x + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\,,\ 0 < x < \pi,$$

that is,

$$-x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\,,\ 0 < x < \pi\,.$$

Fourier analysis implies that this is done by finding the coefficients a_n, for $n = 0, 1, 2, \ldots$. In fact, except for a minus sign factor, this was done in Example 9.9 in Section 9.2:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} -x\, dx = \frac{2}{\pi} \left[\frac{1}{2} - x^2 \right]_0^{\pi} = -\pi$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} -x \cos nx\, dx = -\frac{2}{\pi} \Big[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \Big]_0^{\pi} = \ldots = -\frac{2\big((-1)^n - 1\big)}{n^2 \pi}\,.$$

But for $n = even$, $(-1)^n - 1 = 0$; for $n = odd = 2k-1$, $(-1)^n - 1 = -2$. The solution of the problem is

$$T(x,t) = \cos(x) - \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2} \cos((2k-1)x) e^{-(2k-1)^2 t}\,.$$

11.1.13. The equilibrium solution is $v = v(x) \equiv 0$.

The boundary conditions correspond to the first group in Table 11.1 with $L = \pi$, so, similar to the work in Example 11.1 in Section 11.1, we look for solutions of the PDE in the form

$$T(x,t) = \sin(nx)\,G(t),$$

for $n = 1, 2, 3, \ldots$.

Substitute T into the PDE to get

$$\sin(nx)\,\frac{dG}{dt} = -\frac{\alpha\, n^2}{t+1}\sin(nx)\,G(t),$$

hence

$$\frac{dG}{dt} = -\frac{\alpha\, n^2}{t+1}\,G(t)\,.$$

This is a separable ODE, whose solution is found using

$$\ln|G| = \int \frac{dG}{G} = \int -\frac{\alpha\, n^2}{t+1}\,dt = -\alpha\, n^2\,\ln|t+1| + c = \ln|t+1|^{-\alpha\, n^2} + c,$$

where c is an arbitrary constant, hence

$$G = C|t+1|^{-\alpha\, n^2} = C(t+1)^{-\alpha\, n^2},$$

because $t + 1 > 0$ for $0 < t < \infty$.

The general solution of the PDE +BCs is

$$T(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\,(t+1)^{-\alpha\, n^2}\,.$$

The initial condition is satisfied by solving

$$f(x) = T(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)\,,\ 0 < x < \pi,$$

that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)\,,\ 0 < x < \pi\,.$$

We calculate the Fourier sine series coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx\,.$$

The solution of the problem is

$$T(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L}\int_0^{L} f(x)\sin nx\,dx\right)(t+1)^{-\alpha\, n^2}\sin nx\,.$$

11.1.15. The boundary conditions correspond to the first group in Table 11.1, so, similar to the work in Example 11.6 in Section 11.1, we look for the solution of the PDE in the form

$$T(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi x}{L}\right)\,.$$

Substitute that and the Fourier sines series for 1, in the form

$$1 = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right),$$

[we'll calculate the Fourier coefficients f_n later] into the PDE to get

$$\sum_{n=1}^{\infty} \dot{b}_n(t)\sin\left(\frac{n\pi x}{L}\right) = \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + e^{-t} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(\frac{n\pi x}{L}\right) + e^{-t}\sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right),$$

that is,

$$0 = \sum_{n=1}^{\infty} \left(\dot{b}_n(t) + \left(\frac{n\pi}{L}\right)^2 b_n(t) - f_n e^{-t}\right)\sin\left(\frac{n\pi x}{L}\right),$$

hence, for $n = 1, 2, 3...,$

$$(\star) \quad \dot{b}_n(t) + \left(\frac{n\pi}{L}\right)^2 b_n(t) = f_n e^{-t}.$$

For each n, this is a first order linear ODE. We can solve it using the method of integrating factor or the method of undetermined coefficients. Using the latter, the assumption $\frac{\pi}{L} > 1$ implies that the corresponding homogeneous ODE and the right hand side do not duplicate roots, so we can assume a particular solution in the form

$$b_{n,p} = A_n e^{-t}.$$

Substitute this into $(\star)$ to get

$$-A_n e^{-t} + \left(\frac{n\pi}{L}\right)^2 A_n e^{-t} = f_n e^{-t},$$

so

$$A_n = \frac{f_n}{\left(\frac{n\pi}{L}\right)^2 - 1}.$$

The solution of the homogeneous ODE corresponding to $(\star)$, is

$$b_{n,h} = C_n e^{-\left(\frac{n\pi}{L}\right)^2 t}.$$

So,

$$T(x,t) = \sum_{n=1}^{\infty} \big(b_{n,h}(t) + b_{n,p}(t)\big)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \left(C_n e^{-\left(\frac{n\pi}{L}\right)^2 t} + \frac{f_n}{\left(\frac{n\pi}{L}\right)^2 - 1}\, e^{-t}\right)\sin\left(\frac{n\pi x}{L}\right).$$

The next to next to last thing to do is to satisfy the initial condition

$$0 = T(x,0) = \sum_{n=1}^{\infty} \left(C_n + \frac{f_n}{\left(\frac{n\pi}{L}\right)^2 - 1}\right)\sin\left(\frac{n\pi x}{L}\right),\ 0 < x < L,$$

hence

$$C_n = -\frac{f_n}{\left(\frac{n\pi}{L}\right)^2 - 1}.$$

The next to last thing to do is a Fourier analysis:

$$1 = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$f_n = \frac{2}{L}\int_0^L 1\cdot\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\Big[\frac{\cos\frac{n\pi x}{L}}{-n\pi/L}\Big]_0^L = \frac{2\big(1-(-1)^n\big)}{n\pi}.$$

So, $f_n = 0$ for $n = even$.

The solution of the original problem is

$$T(x,t) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)\left(\left(\frac{(2k-1)\pi}{L}\right)^2 - 1\right)}\left(e^{-t} - e^{-\left(\frac{(2k-1)\pi}{L}\right)^2 t}\right)\sin\left(\frac{(2k-1)\pi x}{L}\right).$$

11.1.17. The boundary conditions correspond to the first group in Table 11.1, so, similar to the work in Example 11.6 in Section 11.1, we look for the solution of the PDE in the form

$$T(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi x}{L}\right).$$

Substitute that and the Fourier sines series for 1, in the form

$$1 = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right),$$

[we'll calculate the Fourier coefficients f_n later] into the PDE to get

$$\sum_{n=1}^{\infty} \dot{b}_n(t)\sin\left(\frac{n\pi x}{L}\right) = \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + g(t) = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(\frac{n\pi x}{L}\right) + g(t)\sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right),$$

that is,

$$0 = \sum_{n=1}^{\infty} \left(\dot{b}_n(t) + \left(\frac{n\pi}{L}\right)^2 b_n(t) - f_n g(t)\right)\sin\left(\frac{n\pi x}{L}\right),$$

hence, for $n = 1, 2, 3...$,

$$(\star) \quad \dot{b}_n(t) + \left(\frac{n\pi}{L}\right)^2 b_n(t) = f_n g(t).$$

For each n, this is a first order linear ODE. We can solve it using the method of integrating factor, $\mu(t)$: Multiply the ODE $(\star)$ through by

$$\mu(t) = \exp\left(\int \left(\frac{n\pi}{L}\right)^2 dt\right) = e^{\left(\frac{n\pi}{L}\right)^2 t}$$

to get

$$\dot{b}_n(t)e^{\left(\frac{n\pi}{L}\right)^2 t} + \left(\frac{n\pi}{L}\right)^2 e^{\left(\frac{n\pi}{L}\right)^2 t} b_n(t) = f_n e^{\left(\frac{n\pi}{L}\right)^2 t} g(t),$$

that is,

$$\frac{d}{dt}\left[e^{\left(\frac{n\pi}{L}\right)^2 t} b_n(t)\right] = f_n e^{\left(\frac{n\pi}{L}\right)^2 t} g(t),$$

Take the *definite* integral of both sides of

$$\frac{d}{ds}\left[e^{\left(\frac{n\pi}{L}\right)^2 s} b_n(s)\right] = f_n e^{\left(\frac{n\pi}{L}\right)^2 s} g(s),$$

to get

$$e^{\left(\frac{n\pi}{L}\right)^2 t} b_n(t) - b_n(0) = \int_0^t \frac{d}{ds}\left[e^{\left(\frac{n\pi}{L}\right)^2 s} b_n(s)\right] ds = \int_0^t f_n e^{\left(\frac{n\pi}{L}\right)^2 s} g(s)\, ds.$$

This can be rewritten as

$$b_n(t) = b_n(0)e^{-\left(\frac{n\pi}{L}\right)^2 t} + \int_0^t f_n e^{-\left(\frac{n\pi}{L}\right)^2 (t-s)} g(s)\, ds,$$

where $b_n(0)$ is a constant. So,

$$T(x,t) = \sum_{n=1}^{\infty} \big(b_{n,h}(t) + b_{n,p}(t)\big)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \left(b_n(0)e^{-\left(\frac{n\pi}{L}\right)^2 t} + \int_0^t f_n e^{-\left(\frac{n\pi}{L}\right)^2 (t-s)} g(s)\, ds\right)\sin\left(\frac{n\pi x}{L}\right).$$

The next to next to last thing to do is to satisfy the initial condition

$$0 = T(x,0) = \sum_{n=1}^{\infty} \left(b_n(0) + \int_0^0 f_n e^{-\left(\frac{n\pi}{L}\right)^2 (0-s)} g(s)\, ds\right)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n(0)\sin\left(\frac{n\pi x}{L}\right)$$

for $0 < x < L$, hence

$$b_n(0) = 0,\ n = 1, 2, 3, \ldots\ .$$

So,

$$T(x,t) = \sum_{n=1}^{\infty} f_n \left(\int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 (t-s)} g(s)\, ds \right) \sin\left(\frac{n\pi x}{L}\right).$$

The next to last thing to do is a Fourier analysis:

$$1 = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$f_n = \frac{2}{L}\int_0^L 1 \cdot \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\left[\frac{\cos\frac{n\pi x}{L}}{-n\pi/L}\right]_0^L = \frac{2\big(1-(-1)^n\big)}{n\pi}.$$

So, $f_n = 0$ for $n = even$.

The solution of the problem is

$$T(x,t) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\left(\frac{(2k-1)\pi x}{L}\right) \int_0^t e^{-\alpha\left(\frac{(2k-1)\pi}{L}\right)^2 (t-s)} g(s) ds.$$

11.1.19. (a) Work with the three cases of $\lambda < 0, \lambda = 0, \lambda > 0$ as in Example 9.14 in Section 9.3:

Case 1: If $\lambda = 0$, then the differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) = 0$, whose solutions are $X = c_1 + c_2 x$, for arbitrary constants c_1, c_2. In this case, $X'(x) = c_2$.

Applying the first BC gives $0 = X(-L) = c_1 - c_2 L$. Applying the second BC gives $0 = X(L) = c_1 + c_2 L$. Adding the two equations gives $0 = 0 + 0 = 2c_1$, which implies $c_1 = 0$. Substituting that into the first BC gives $0 = 0 - c_2 L$, hence $c_2 = 0$. So, both BCs are satisfied if, and only if, $c_1 = c_2 = 0$. When $\lambda = 0$, the ODE-BVP has only the trivial solution. So, $\lambda = 0$ is not an eigenvalue for this problem.

Case 2: If $\lambda > 0$, rewrite $\lambda = \omega^2$, where $\omega \triangleq \sqrt{\lambda} > 0$. The differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) + \omega^2 X(x) = 0$, whose solutions are $X = c_1 \cos \omega x + c_2 \sin \omega x$, for arbitrary constants c_1, c_2.

Applying the first BC gives

$$0 = X(-L) = c_1 \cos(-\omega L) + c_2 \sin(-\omega L) = c_1 \cos \omega L - c_2 \sin \omega L.$$

Applying the second BC gives

$$0 = X(L) = c_1 \cos \omega L + c_2 \sin \omega L.$$

Adding the two equations gives $0 = 0 + 0 = 2c_1 \cos \omega L$, which implies $c_1 \cos \omega L = 0$.

Instead, subtracting the two equations gives $0 = 0 - 0 = 2c_2 \sin \omega L$, which implies $c_2 \sin \omega L = 0$.

So, there is a non-trivial solution if and only if,

$$(1)\ c_1 \cos \omega L = 0 \quad \textbf{and} \quad (2)\ c_2 \sin \omega L = 0.$$

As a logical matter, in principle there are four possibilities:

$$(a)\ c_1 = 0 \quad \textbf{and} \quad c_2 = 0,$$

$$(b)\ c_1 = 0 \quad \textbf{and} \quad \sin \omega L = 0,$$

$$(c)\ \cos \omega L = 0 \quad \textbf{and} \quad c_2 = 0,$$

$$(d)\ \cos \omega L = 0 \quad \textbf{and} \quad \sin \omega L = 0.$$

Case (a) gives no eigenfunction, because eigenfunctions must not be identically zero. Case (d) is impossible, because $\cos^2 \omega L + \sin^2 \omega L = 1$.

Case (b) gives characteristic equation $\sin \omega L = 0$. As in Example 9.14 in Section 9.3, there are infinitely many eigenvalues: $\omega = \frac{n\pi}{L}$, any positive integer n, with corresponding eigenfunctions $X_n(x) = \sin \frac{n\pi x}{L}$.

Case (c) gives characteristic equation $\cos \omega L = 0$. As in problem 9.3.1, there are infinitely many eigenvalues: $\omega = \frac{\left(n-\frac{1}{2}\right)\pi}{L}$, any positive integer n, with corresponding eigenfunctions $X_n(x) = \cos \frac{\left(n-\frac{1}{2}\right)\pi x}{L}$.

Case 3: If $\lambda < 0$, rewrite $\lambda = -\omega^2$, where $\omega \triangleq \sqrt{-\lambda}$. The differential equation $X''(x) + \lambda X(x) = 0$ is $X''(x) - \omega^2 X(x) = 0$, whose solutions are $X = c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$, for arbitrary constants c_1, c_2.

Applying the first BC gives

$$\begin{aligned}0 = X(-L) &= c_1 \cosh(-\omega L) + c_2 \sinh(-\omega L)\\ &= c_1 \cosh \omega L - c_2 \sinh \omega L\,.\end{aligned}$$

Applying the second BC gives

$$0 = X(L) = c_1 \cosh \omega L + c_2 \sinh \omega L\,.$$

Adding the two equations gives $0 = 0 + 0 = 2c_1 \cosh \omega L$, which implies $c_1 = 0$ because $\omega > 0$ implies $\cosh \omega L > 0$.

Instead, subtracting the two equations gives $0 = 0 - 0 = 2c_2 \sinh \omega L$, which implies $c_2 = 0$ because $\omega > 0$ implies $\sinh \omega L > 0$.

So there is no eigenfunction if $\lambda < 0$. In summary, the eigenvalues/eigenfunctions are

$$(i)\ \lambda = \frac{2k\pi}{2L},\ X_{2k}(x) = \sin\left(\frac{2k\pi x}{2L}\right)$$

and

$$(ii)\ \lambda = \frac{(2k-1)\pi}{2L},\ X_{2k-1}(x) = \cos\left(\frac{(2k-1)\pi x}{2L}\right).$$

(b) Using the eigenfunctions found in part (a), the general solution of the PDE and BCs is

$$T(x,t) = \sum_{k=1}^{\infty} A_{2k-1} \cos\left(\frac{(2k-1)\pi x}{2L}\right) e^{-\alpha\left(\frac{(2k-1)\pi}{2L}\right)^2 t} + \sum_{k=1}^{\infty} A_{2k} \sin\left(\frac{2k\pi x}{2L}\right) e^{-\alpha\left(\frac{2k\pi}{2L}\right)^2 t}.$$

The initial condition to be satisfied is

$$100 = T(x,0) = \sum_{k=1}^{\infty} \left(A_{2k-1} \cos\left(\frac{(2k-1)\pi x}{2L}\right) + A_{2k} \sin\left(\frac{2k\pi x}{2L}\right)\right),$$

for $0 < x < L$. While this involves both sine and cosine functions, this is not a Fourier series because $\left(\frac{(2k-1)\pi}{2L}\right)$, the frequencies in the cosine functions, are not the same as the frequencies in the sine functions, $\left(\frac{2k\pi}{2L}\right)$.

Using the result of Theorem 9.6 in Section 9.3, the Fourier coefficients are found by, first,

$$A_{2k-1} = \frac{\int_0^L 100 \cos\left(\frac{(2k-1)\pi x}{2L}\right) dx}{\int_0^L \left|\cos\left(\frac{(2k-1)\pi x}{2L}\right)\right|^2 dx} = \left[\frac{200L}{(2k-1)\pi}\sin\left(\frac{(2k-1)\pi x}{2L}\right)\right]_0^L \div \int_0^L \frac{1}{2}\left(1 + \cos\left(\frac{2(2k-1)\pi x}{2L}\right)\right) dx$$

$$= \frac{200L}{(2k-1)\pi}\left(\sin\left(\frac{(2k-1)\pi}{2}\right) - 0\right) \div \frac{1}{2}\left[1 + \frac{\sin\left(\frac{2(2k-1)\pi x}{2L}\right)}{(2k-1)\pi/L}\right]_0^L$$

$$= \frac{200L}{(2k-1)\pi}(-1)^{k+1} \div \frac{L}{2} = \frac{400}{(2k-1)\pi}(-1)^{k+1},$$

and, second, by

$$A_{2k} = \frac{\int_0^L 100 \sin\left(\frac{k\pi x}{L}\right) dx}{\int_0^L \left|\sin\left(\frac{k\pi x}{L}\right)\right|^2 dx} = \left[-\frac{100L}{k\pi}\cos\left(\frac{k\pi x}{L}\right)\right]_0^L \div \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{2k\pi x}{L}\right)\right) dx$$

$$= -\frac{100L}{k\pi}\left((-1)^k - 1\right) \div \frac{1}{2}\left[1 - \frac{\sin\left(\frac{2k\pi x}{L}\right)}{2k\pi/L}\right]_0^L = \frac{100L}{k\pi}\left(1 - (-1)^k\right) \div \frac{L}{2} = \frac{200}{k\pi}\left(1 - (-1)^k\right).$$

But for $k = even$, $1 - (-1)^k = 0$; for $k = odd = 2\ell - 1$, $1 - (-1)^k = 2$. So, $A_{(2\cdot even)} = 0$ and

$$A_{2(2\ell-1)} = \frac{400}{(2\ell-1)\pi}.$$

The solution of the problem is

$$T(x,t) = \frac{400}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}\cos\left(\frac{(2k-1)\pi x}{2L}\right) e^{-\alpha\left(\frac{(2k-1)\pi}{2L}\right)^2 t} + \frac{400}{\pi}\sum_{\ell=1}^{\infty} \frac{1}{(2\ell-1)}\sin\left(\frac{(2\ell-1)\pi x}{L}\right) e^{-\alpha\left(\frac{(2\ell-1)\pi}{L}\right)^2 t}.$$

11.1.21. Define Condition ($\star$) to mean that all solutions of the PDE-BVP-IVP satisfy "$\lim_{t\to\infty} T(x,t) = 0$, for all x in $[0, L]$."

The equilibrium solution, $v = v(x)$, satisfies $\dfrac{\partial v}{\partial t} \equiv 0$ and

$$\left\{ \begin{array}{c} 0 = \alpha v''(x) + \beta\, v,\ 0 < x < L, \\ \\ v(0) = v(L) = 0 \end{array} \right\}.$$

Note that the constant α is positive and β is assumed to be constant. If we divide by α we get the ODE-BVP

$$\left\{ \begin{array}{c} 0 = v''(x) + \lambda\, v,\ 0 < x < L, \\ \\ v(0) = v(L) = 0 \end{array} \right\},$$

where $\lambda = \dfrac{\beta}{\alpha}$. But we know this as the eigenvalue problem of Example 9.14 in Section 9.3. Condition ($\star$) is true *only if* the equilibrium solution, $v(x)$ must be identically zero, that is, *only if* $\lambda = \dfrac{\beta}{\alpha}$ is not an eigenvalue, that is, *only if*

$$\beta \neq \alpha \cdot \left(\frac{m\pi}{L}\right)^2,\ \text{for } m = 1, 2, 3, \ldots$$

Define $w = w(x,t) = T(x,t) - v(x)$. Similar to work in Example 11.2 in Section11.1, $w(x,t)$ should satisfy the homogeneous problem

$$(\star)\quad \left\{ \begin{array}{l} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \beta w,\ 0 < x < \pi,\ t > 0, \\ \\ w(0,t) = w(\pi,t) = 0,\ t > 0 \end{array} \right\}.$$

The boundary conditions correspond to the first group in Table 11.1, so, similar to the work in Example 11.1 in Section 11.1, we look for solutions of the PDE in the form

$$w(x,t) = \sin\left(\frac{n\pi x}{L}\right) G(t),$$

for $n = 1, 2, 3, \ldots$.

Substitute w into the PDE to get

$$\sin\left(\frac{n\pi x}{L}\right)\frac{dG}{dt} = -\alpha\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) G(t) + \beta \sin\left(\frac{n\pi x}{L}\right) G(t)$$

hence

$$\frac{dG}{dt} = \left(\beta - \alpha\left(\frac{n\pi}{L}\right)^2\right) G(t)\,.$$

The general solution of ($\star$) is

$$w(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(\beta - \alpha\left(\frac{n\pi}{L}\right)^2\right)t}.$$

The solution of the original problem's PDE + BCs is

$$T(x,t) = v(x) + w(x,t) = v(x) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(\beta - \alpha\left(\frac{n\pi}{L}\right)^2\right)t},$$

where the b_n's are arbitrary constants that depend on the initial condition.

In order for all solutions to go to zero as $t \to \infty$, it is necessary and sufficient that $v(x) \equiv 0$ and $\beta - \alpha\left(\frac{n\pi}{L}\right)^2 < 0$ for all integers $n \geq 1$, that is, $\beta < \alpha\left(\frac{\pi}{L}\right)^2$.

The time constant is $\tau = \frac{1}{\alpha\left(\frac{\pi}{L}\right)^2 - \beta}$.

Section 11.2

11.2.1. This problem fits directly into the form of problem (11.21) through (11.24) in Section 11.2, in the special case of the string starting from rest and $L = 3$, so the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{3}\right) \sin\left(\frac{n\pi x}{3}\right).$$

The initial condition is satisfied by solving

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{3}\right),\ 0 < x < 3,$$

so

$$a_n = \frac{2}{3}\int_0^3 u(x,0)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\left(\int_0^1 x\,\sin\left(\frac{n\pi x}{3}\right)dx + \int_1^3 \frac{1}{2}\,(3-x)\,\sin\left(\frac{n\pi x}{3}\right)dx\right)$$

$$= \frac{2}{3}\left(\left[\frac{x\cos\frac{n\pi x}{3}}{-n\pi/3} + \frac{\sin\frac{n\pi x}{3}}{(n\pi/3)^2}\right]_0^1 + \frac{1}{2}\left[\frac{(3-x)\cos\frac{n\pi x}{3}}{-n\pi/3} - \frac{\sin\frac{n\pi x}{3}}{(n\pi/3)^2}\right]_1^3\right) = ... = \frac{9}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right).$$

The solution of the problem is

$$u(x,t) = \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi ct}{3}\right)\sin\left(\frac{n\pi x}{3}\right).$$

11.2.3. This problem fits directly into the form of problem (11.21) through (11.24) in Section 11.2, in the special case of the string starting from rest, except that the eigenfunctions must satisfy the boundary conditions $X(0) = X'(L) = 0$. The third group of entries of Table 11.1 imply that the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(n-\frac{1}{2})\pi ct}{L}\right) \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right).$$

The initial condition is satisfied by solving

$$\sin\left(\frac{\pi x}{2L}\right) - \frac{1}{3}\sin\left(\frac{5\pi x}{2L}\right) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right),$$

for $0 < x < L$, so $a_1 = 1$, $a_3 = -\frac{1}{3}$, and all other $a_n = 0$.

The solution of the problem is

$$u(x,t) = \cos\left(\frac{\pi ct}{2L}\right)\sin\left(\frac{\pi x}{2L}\right) - \frac{1}{3}\cos\left(\frac{5\pi ct}{2L}\right)\sin\left(\frac{5\pi x}{2L}\right).$$

11.2.5. This problem fits directly into the form of problem (11.21) through (11.24) in Section 11.2, in the special case of the string starting from rest, so the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

The initial condition is satisfied by solving

$$1 - \cos\left(\frac{\pi x}{L}\right) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right),\ 0 < x < L,$$

so

$$a_n = \frac{2}{L}\int_0^L u(x,0)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \left(1 - \cos\left(\frac{\pi x}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L \left(\sin\left(\frac{n\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{(n-1)\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{(n+1)\pi x}{L}\right)\right)dx\,.$$

For $n = 1$,

$$a_1 = \frac{2}{L}\int_0^L \left(\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right)\right) dx\,. = \frac{2}{L}\Big[\frac{\cos\frac{\pi x}{L}}{-\pi/L} - \frac{1}{2}\cdot\frac{\cos\frac{2\pi x}{L}}{-2\pi/L}\Big]_0^L = ... = \frac{4}{\pi}\,.$$

For $n \neq 1$,

$$a_n = \frac{2}{L}\Big[\frac{\cos\frac{n\pi x}{L}}{-n\pi/L} - \frac{1}{2}\cdot\frac{\cos\frac{(n-1)\pi x}{L}}{-(n-1)\pi/L} - \frac{1}{2}\cdot\frac{\cos\frac{(n+1)\pi x}{L}}{-(n+1)\pi/L}\Big]_0^L = \frac{2}{L}\Big(\frac{(-1)^n - 1}{-n\pi/L} - \frac{1}{2}\cdot\frac{(-1)^{n-1}-1}{-(n-1)\pi/L} - \frac{1}{2}\cdot\frac{(-1)^{n+1}-1}{-(n+1)\pi/L}\Big).$$

Note that $(-1)^{n-1} = (-1)^{n+1}$, so

$$a_n = \frac{2\big(1-(-1)^n\big)}{n\pi} - \frac{1-(-1)^{n+1}}{\pi}\cdot\Big(\frac{1}{n-1} + \frac{1}{n+1}\Big).$$

Case 1: If $n = even = 2k$ then $1-(-1)^n = 0$ and $1-(-1)^{n+1} = 2$, so

$$a_{2k} = -\frac{2}{\pi}\left(\frac{n+1}{(n-1)(n+1)} + \frac{n-1}{(n+1)(n-1)}\right) = -\frac{4\cdot 2k}{\pi\big((2k)^2-1\big)}\,.$$

Case 2: If $n = odd = 2\ell - 1$ then $1-(-1)^n = 2$ and $1-(-1)^{n+1} = 0$, so

$$a_{2\ell-1} = \frac{2\big(2\big)}{n\pi} = \frac{4}{(2\ell-1)\pi}\,.$$

The solution of the problem is

$$u(x,t) = \frac{4}{\pi}\cos\left(\frac{\pi ct}{L}\right)\sin\left(\frac{\pi x}{L}\right) - \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{k}{4k^2-1}\cos\left(\frac{2k\pi ct}{L}\right)\sin\left(\frac{2k\pi x}{L}\right)$$

$$+\frac{4}{\pi}\sum_{\ell=2}^{\infty}\frac{1}{2\ell-1}\cos\left(\frac{(2\ell-1)\pi ct}{L}\right)\sin\left(\frac{(2\ell-1)\pi x}{L}\right).$$

11.2.7. The general solution of the PDE and BCs is given in (11.26) in Section 11.2:

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$+\sum_{n=1}^{\infty}\left(\frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right),$$

where $f(x) = u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = g(x) = \cos\left(\frac{2\pi x}{L}\right)$, for $0 < x < L$.

For $n \neq 2$, we calculate that

$$\frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{n\pi c}\int_0^L \cos\left(\frac{2\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{n\pi c}\int_0^L\left(\sin\left(\frac{(n-2)\pi x}{L}\right) + \sin\left(\frac{(n+2)\pi x}{L}\right)\right)dx = \frac{1}{n\pi c}\Big[\frac{\cos\frac{(n-2)\pi x}{L}}{-(n-2)\pi/L} + \frac{\cos\frac{(n+2)\pi x}{L}}{-(n+2)\pi/L}\Big]_0^L$$

$$= \frac{L}{n\pi^2 c}\Big(\frac{(-1)^{n-2}-1}{-(n-2)} + \frac{(-1)^{n+2}-1}{-(n+2)}\Big) = \frac{L}{n\pi^2 c}\cdot(1-(-1)^n)\cdot\Big(\frac{1}{n-2} + \frac{1}{n+2}\Big)$$

$$= \frac{L}{n\pi^2 c}\cdot(1-(-1)^n)\cdot\frac{2n}{n^2-4}\,.$$

For $n = even > 2$, $a_n = 0$.

For $n = odd = 2k-1$,

$$a_{2k-1} = \frac{L}{(2k-1)\pi^2 c} \cdot 2 \cdot \frac{2(2k-1)}{(2k-1)^2-4} = \frac{4L}{\pi^2 c\big((2k-1)^2-4\big)}\,.$$

For $n = 2$, we calculate

$$\frac{2}{2\pi c}\int_0^L g(x)\sin\Big(\frac{2\pi x}{L}\Big)dx = \frac{2}{n\pi c}\int_0^L \cos\Big(\frac{2\pi x}{L}\Big)\sin\Big(\frac{2\pi x}{L}\Big)dx = \frac{1}{n\pi c}\left[\frac{1}{2}\cdot\frac{L}{2\pi}\cdot\sin^2\Big(\frac{2\pi x}{L}\Big)\right]_0^L$$

$$= \frac{1}{n\pi c}\cdot\frac{1}{2}\cdot\frac{L}{2\pi}\cdot\Big(\sin^2(2\pi) - \sin^2(0)\Big) = 0\,.$$

The solution of the problem is

$$u(x,t) = \frac{4L}{\pi^2 c}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-4}\sin\Big(\frac{(2k-1)\pi ct}{L}\Big)\sin\Big(\frac{(2k-1)\pi x}{L}\Big)\,.$$

11.2.9. As in Example 11.6 in Section 11.1, let's write $T(x,t) = v(x,t) + w(x,t)$, where $v(x,t)$ *only* satisfies the two inhomogeneous boundary conditions. Of course, an infinite number of functions $v(x,t)$ satisfy the two inhomogeneous boundary conditions. We might as well choose $v(x,t)$ to be as simple as possible, because of the basic principle to *keep things as simple as possible.* Also, we suspect that the simpler the function $v(x,t)$, the easier it will be to find $w(x,t)$.

As far as dependence on x is concerned, the simplest function v would have the form $v(x,t) = \beta(t) + \gamma(t)x$. Substitute this into the BCs to get

$$\left\{\begin{array}{c}\delta(t) = u(0,t) = \beta(t),\\ \\ \epsilon(t) = u(L,t) = \beta(t) + \gamma(t)L\end{array}\right\},$$

hence

$$v(x,t) = \delta(t) + \big(-\delta(t) + \epsilon(t)\big)\,\frac{x}{L}\,.$$

Because $\dfrac{\partial^2 v}{\partial x^2} \equiv 0$, the PDE to be satisfied by $w(x,t) = u(x,t) - v(x,t)$ is found by noting that

$$\ddot{\delta}(t) + \big(-\ddot{\delta}(t) + \ddot{\epsilon}(t)\big)\frac{x}{L} + \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2 w}{\partial x^2}\,.$$

So, the problem to be satisfied by $w(x,t)$ is

$$\left\{\begin{array}{l}\dfrac{\partial^2 w}{\partial t^2} = 4\dfrac{\partial^2 w}{\partial x^2} - \ddot{\delta}(t) - \big(-\ddot{\delta}(t) + \ddot{\epsilon}(t)\big)\dfrac{x}{L},\ 0 < x < L, t > 0,\\ \\ w(0,t) = w(L,t) = 0,\ t > 0,\\ \\ w(x,0) = f(x) - \delta(0) - \big(-\delta(0) + \epsilon(0)\big)\,\dfrac{x}{L},\ 0 < x < L,\\ \\ \dfrac{\partial w}{\partial t}(x,0) = g(x) - \dot{\delta}(0) - \big(-\dot{\delta}(0) + \dot{\epsilon(0)}\big)\,\dfrac{x}{L},\ 0 < x < L.\end{array}\right\}$$

Because of the homogeneous BCs, let's look for a solution in the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\Big(\frac{n\pi x}{L}\Big)\,.$$

Substitute that into the PDE for w to get

$$\sum_{n=1}^{\infty}\ddot{b}_n(t)\sin\Big(\frac{n\pi x}{L}\Big) = \sum_{n=1}^{\infty} -4\Big(\frac{n\pi}{L}\Big)^2 b_n(t)\sin\Big(\frac{n\pi x}{L}\Big) - \ddot{\delta}(t) - \big(-\ddot{\delta}(t) + \ddot{\epsilon}(t)\big)\frac{x}{L}\,.$$

We need to get everything in terms of Fourier sine series components $\sin\left(\frac{n\pi x}{L}\right)$, so we do Fourier sine series analyses:

$$1 = \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$h_n = \frac{2}{L}\int_0^L 1\cdot\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[\frac{\cos\frac{n\pi x}{L}}{-n\pi/L}\right]_0^L = \frac{2}{L}\left(\frac{(-1)^n - 1}{-n\pi/L}\right) = \frac{2\big(1-(-1)^n\big)}{n\pi},$$

and

$$\frac{x}{L} = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$k_n = \frac{2}{L}\int_0^L \frac{x}{L}\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L^2}\left[\frac{x\cos\frac{n\pi x}{L}}{-n\pi/L} + \frac{\sin\frac{n\pi x}{L}}{(n\pi/L)^2}\right]_0^L = \frac{2}{L^2}\left(\frac{L(-1)^n - 0}{-n\pi/L} + \frac{0-0}{(n\pi/L)^2}\right) = \frac{2(-1)^{n+1}}{n\pi}\,.$$

So, the PDE can be rewritten as

$$\sum_{n=1}^{\infty} \ddot{b}_n(t)\sin\left(\frac{n\pi x}{L}\right) = -4\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(\frac{n\pi x}{L}\right) - \ddot{\delta}(t)\sum_{n=1}^{\infty}\frac{2\big(1-(-1)^n\big)}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

$$-\big(-\ddot{\delta}(t)+\ddot{\epsilon}(t)\big)\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{L}\right),$$

that is,

$$0 = \sum_{n=1}^{\infty}\left(\ddot{b}_n(t) + 4\left(\frac{n\pi}{L}\right)^2 b_n(t) + \ddot{\delta}(t)\,\frac{2\big(1-(-1)^n\big)}{n\pi} + \big(-\ddot{\delta}(t)+\ddot{\epsilon}(t)\big)\,\frac{2(-1)^{n+1}}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Orthogonality implies that for $n = 1, 2, 3, ...,$

$$\ddot{b}_n + \left(\frac{2n\pi}{L}\right)^2 b_n = \frac{2}{n\pi}\left(-\big(1-(-1)^n\big)\ddot{\delta}(t) - (-1)^{n+1}\big(-\ddot{\delta}(t)+\ddot{\epsilon}(t)\big)\right).$$

For temporary convenience, define

$$\phi_n(t) \triangleq \frac{2}{n\pi}\left(-\big(1-(-1)^n\big)\delta(t) - (-1)^{n+1}\big(-\delta(t)+\epsilon(t)\big)\right)$$

hence

$$\phi_n(t) = \frac{2}{n\pi}\big(-\delta(t) + (-1)^n\epsilon(t)\big).$$

The ODE $\ddot{b}_n + \left(\frac{2n\pi}{L}\right)^2 b_n = \ddot{\phi}_n(t)$ can be solved using the method of variation of parameters or using Laplace transforms. [If the functions $\delta(t)$ and $\epsilon(t)$ were explicitly given then it might be possible to use the method of undetermined coefficients.]

Using the result of Example 4.33 in Section 4.5, with $\omega = \frac{2n\pi}{L}$, a particular solution is

$$b_{n,p}(t) = \int_0^t \frac{1}{2n\pi/L}\sin\left(\frac{2n\pi}{L}(t-s)\right)\ddot{\phi}_n(s)ds$$

$$= \frac{L}{2n\pi}\int_0^t\left(\sin\left(\frac{2n\pi t}{L}\right)\cos\left(\frac{2n\pi s}{L}\right) - \cos\left(\frac{2n\pi t}{L}\right)\sin\left(\frac{2n\pi s}{L}\right)\right)\ddot{\phi}_n(s)ds$$

$$= \frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)\int_0^t\cos\left(\frac{2n\pi s}{L}\right)\ddot{\phi}_n(s)ds - \frac{L}{2n\pi}\cos\left(\frac{2n\pi t}{L}\right)\int_0^t\sin\left(\frac{2n\pi s}{L}\right)\ddot{\phi}_n(s)ds\,.$$

Integration by parts yields

$$b_{n,p}(t) = \frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)\left(\cos\left(\frac{2n\pi s}{L}\right)\dot{\phi}_n(s)\Big|_0^t - \int_0^t\left(-\frac{2n\pi}{L}\right)\sin\left(\frac{2n\pi s}{L}\right)\dot{\phi}_n(s)ds\right)$$

$$
\begin{aligned}
&-\frac{L}{2n\pi}\cos\left(\frac{2n\pi t}{L}\right)\left(\sin\left(\frac{2n\pi s}{L}\right)\dot{\phi}(s)\Big|_0^t-\int_0^t\frac{2n\pi}{L}\cos\left(\frac{2n\pi s}{L}\right)\dot{\phi}_n(s)ds\right)\\
&=\frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)\left(-\dot{\phi}(0)+\cancel{\cos\left(\frac{2n\pi t}{L}\right)\dot{\phi}(t)}+\frac{2n\pi}{L}\int_0^t\sin\left(\frac{2n\pi s}{L}\right)\dot{\phi}_n(s)ds\right)\\
&\quad-\frac{L}{2n\pi}\cos\left(\frac{2n\pi t}{L}\right)\left(-0+\cancel{\sin\left(\frac{2n\pi t}{L}\right)\dot{\phi}(t)}-\frac{2n\pi}{L}\int_0^t\cos\left(\frac{2n\pi s}{L}\right)\dot{\phi}_n(s)ds\right)\\
&=-\dot{\phi}(0)\frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)+\sin\left(\frac{2n\pi t}{L}\right)\int_0^t\sin\left(\frac{2n\pi s}{L}\right)\dot{\phi}_n(s)ds+\cos\left(\frac{2n\pi t}{L}\right)\int_0^t\cos\left(\frac{2n\pi s}{L}\right)\dot{\phi}_n(s)ds\,.
\end{aligned}
$$

Further integration by parts yields

$$
\begin{aligned}
b_{n,p}(t)&=-\dot{\phi}(0)\frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)+\sin\left(\frac{2n\pi t}{L}\right)\left(\sin\left(\frac{2n\pi s}{L}\right)\phi_n(s)\Big|_0^t-\int_0^t\frac{2n\pi}{L}\cos\left(\frac{2n\pi s}{L}\right)\phi_n(s)ds\right)\\
&\qquad+\cos\left(\frac{2n\pi t}{L}\right)\left(\cos\left(\frac{2n\pi s}{L}\right)\phi_n(s)\Big|_0^t-\int_0^t\left(-\frac{2n\pi}{L}\right)\sin\left(\frac{2n\pi s}{L}\right)\phi_n(s)ds\right)\\
&=-\dot{\phi}_n(0)\frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)+\sin\left(\frac{2n\pi t}{L}\right)\left(-0+\sin\left(\frac{2n\pi t}{L}\right)\phi_n(t)-\frac{2n\pi}{L}\int_0^t\cos\left(\frac{2n\pi s}{L}\right)\phi_n(s)ds\right)\\
&\qquad+\cos\left(\frac{2n\pi t}{L}\right)\left(-\phi(0)+\cos\left(\frac{2n\pi t}{L}\right)\phi_n(t)+\frac{2n\pi}{L}\int_0^t\sin\left(\frac{2n\pi s}{L}\right)\phi_n(s)ds\right)\\
&=-\dot{\phi}_n(0)\frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)-\phi_n(0)\cos\left(\frac{2n\pi t}{L}\right)+\left(\sin^2\left(\frac{2n\pi t}{L}\right)+\cos^2\left(\frac{2n\pi t}{L}\right)\right)\phi_n(t)\\
&\qquad-\frac{2n\pi}{L}\left(\sin\left(\frac{2n\pi t}{L}\right)\int_0^t\cos\left(\frac{2n\pi s}{L}\right)\phi_n(s)ds-\cos\left(\frac{2n\pi t}{L}\right)\int_0^t\sin\left(\frac{2n\pi s}{L}\right)\phi_n(s)ds\right)\\
&=-\dot{\phi}_n(0)\frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)-\phi_n(0)\cos\left(\frac{2n\pi t}{L}\right)+\phi_n(t)-\frac{2n\pi}{L}\int_0^t\sin\left(\frac{2n\pi}{L}(t-s)\right)\phi_n(s)ds\,.
\end{aligned}
$$

The general solution for $b_n(t)$ is

$$
\begin{aligned}
b_n(t)&=b_{n,h}(t)+b_{n,p}(t)\\
&=\widetilde{a}_n\cos\left(\frac{2n\pi t}{L}\right)+\widetilde{b}_n\sin\left(\frac{2n\pi t}{L}\right)-\dot{\phi}(0)\frac{L}{2n\pi}\sin\left(\frac{2n\pi t}{L}\right)-\phi(0)\cos\left(\frac{2n\pi t}{L}\right)+\phi_n(t)\\
&\qquad-\frac{2n\pi}{L}\int_0^t\sin\left(\frac{2n\pi}{L}(t-s)\right)\phi_n(s)ds,
\end{aligned}
$$

where $\widetilde{a}_n,\widetilde{b}_n$ are arbitrary constants. By combining terms, this can be rewritten as

$$
b_n(t)=A_n\cos\left(\frac{2n\pi t}{L}\right)+B_n\sin\left(\frac{2n\pi t}{L}\right)+\phi_n(t)-\frac{2n\pi}{L}\int_0^t\sin\left(\frac{2n\pi}{L}(t-s)\right)\phi_n(s)ds,
$$

where A_n,B_n are arbitrary constants.

The general solution of the PDE and time varying BCs is

$$
u(x,t)=v(x,t)+w(x,t)=\delta(t)+\big(-\delta(t)+\epsilon(t)\big)\frac{x}{L}+\sum_{n=1}^{\infty}b_n(t)\sin\left(\frac{n\pi x}{L}\right),
$$

that is,

$$
\begin{aligned}
u(x,t)&=\delta(t)+\big(-\delta(t)+\epsilon(t)\big)\frac{x}{L}\\
&\quad+\sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{2n\pi t}{L}\right)+B_n\sin\left(\frac{2n\pi t}{L}\right)+\phi_n(t)-\frac{2n\pi}{L}\int_0^t\sin\left(\frac{2n\pi}{L}(t-s)\right)\phi_n(s)ds\right)\sin\left(\frac{n\pi x}{L}\right),
\end{aligned}
$$

where A_n,B_n's are constants. The first IC this needs to satisfy is

$$
f(x)=u(x,0)=\delta(0)+\big(-\delta(0)+\epsilon(0)\big)\frac{x}{L}+\sum_{n=1}^{\infty}\big(A_n+\phi(0)\big)\sin\left(\frac{n\pi x}{L}\right).
$$

Recalling that $\phi_n(0) = \frac{2}{n\pi}\big(-\delta(0) + (-1)^n\epsilon(0)\big)$, we see that

$$A_n + \frac{2}{n\pi}\big(-\delta(0)+(-1)^n\epsilon(0)\big) = \frac{2}{L}\int_0^L \Big(f(x) - \delta(0) - \big(-\delta(0)+\epsilon(0)\big)\,\frac{x}{L}\Big)\sin\Big(\frac{n\pi x}{L}\Big)dx$$

$$= \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)dx - \delta(0)\,\frac{2}{L}\int_0^L 1\cdot\sin\Big(\frac{n\pi x}{L}\Big)dx - \big(-\delta(0)+\epsilon(0)\big)\frac{2}{L}\int_0^L \frac{x}{L}\,\sin\Big(\frac{n\pi x}{L}\Big)dx$$

$$= \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)dx - \frac{2\big(1-(-1)^n\big)}{n\pi}\,\delta(0) - \frac{2(-1)^{n+1}}{n\pi}\big(-\delta(0)+\epsilon(0)\big)$$

$$= \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)dx + \frac{2}{n\pi}\Big(\big(-1+\cancel{(-1)^n}+\cancel{(-1)^{n+1}}\big)\delta(0) + (-1)^n\epsilon(0)\Big)$$

$$= \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)dx + \frac{2}{n\pi}\big(-\delta(0)+(-1)^n\epsilon(0)\big),$$

so

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)dx\,.$$

Using Leibniz's rule, we calculate

$$\frac{\partial u}{\partial t}(x,t) = \dot\delta(t) + \big(-\dot\delta(t)+\dot\epsilon(t)\big)\,\frac{x}{L}$$

$$+\sum_{n=1}^{\infty}\Big(-\frac{2n\pi}{L}\,A_n\sin\Big(\frac{2n\pi t}{L}\Big) + \frac{2n\pi}{L}\,B_n\cos\Big(\frac{2n\pi t}{L}\Big) + \dot\phi_n(t)\Big)\sin\Big(\frac{n\pi x}{L}\Big)$$

$$+\sum_{n=1}^{\infty}\Big(-\frac{2n\pi}{L}\sin\Big(\frac{2n\pi}{L}\,(t-s)\Big)\phi_n(s)\Big|_{s=t} - \int_0^t \frac{2n\pi}{L}\cos\Big(\frac{2n\pi}{L}\,(t-s)\Big)\phi_n(s)ds\Big)\sin\Big(\frac{n\pi x}{L}\Big),$$

so, the second IC requires

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \dot\delta(0) + \big(-\dot\delta(0)+\dot\epsilon(0)\big)\,\frac{x}{L} + \sum_{n=1}^{\infty}\Big(\frac{2n\pi}{L}\,B_n + \dot\phi_n(0)\Big)\sin\Big(\frac{n\pi x}{L}\Big) + 0.$$

Recalling that $\dot\phi_n(0) = \frac{2}{n\pi}\big(-\dot\delta(0) + (-1)^n\dot\epsilon(0)\big)$, we see that

$$\frac{2n\pi}{L}\,B_n + \frac{2}{n\pi}\big(-\dot\delta(0)+(-1)^n\dot\epsilon(0)\big) = \frac{2}{L}\int_0^L \Big(g(x) - \dot\delta(0) - \big(-\dot\delta(0)+\dot\epsilon(0)\big)\,\frac{x}{L}\Big)\sin\Big(\frac{n\pi x}{L}\Big)dx$$

$$= \frac{2}{L}\int_0^L g(x)\sin\Big(\frac{n\pi x}{L}\Big)dx - \dot\delta(0)\,\frac{2}{L}\int_0^L 1\cdot\sin\Big(\frac{n\pi x}{L}\Big)dx - \big(-\dot\delta(0)+\dot\epsilon(0)\big)\frac{2}{L}\int_0^L \frac{x}{L}\,\sin\Big(\frac{n\pi x}{L}\Big)dx$$

$$= \frac{2}{L}\int_0^L g(x)\sin\Big(\frac{n\pi x}{L}\Big)dx - \frac{2\big(1-(-1)^n\big)}{n\pi}\,\dot\delta(0) - \frac{2(-1)^{n+1}}{n\pi}\big(-\dot\delta(0)+\dot\epsilon(0)\big)$$

$$= \frac{2}{L}\int_0^L g(x)\sin\Big(\frac{n\pi x}{L}\Big)dx + \frac{2}{n\pi}\Big(\big(-1+\cancel{(-1)^n}+\cancel{(-1)^{n+1}}\big)\dot\delta(0) + (-1)^n\dot\epsilon(0)\Big),$$

so

$$B_n = \frac{2}{L}\int_0^L g(x)\sin\Big(\frac{n\pi x}{L}\Big)dx\,.$$

The solution of the problem is

$$u(x,t) = \delta(t) + \big(-\delta(t)+\epsilon(t)\big)\,\frac{x}{L} +$$

$$+\frac{2}{L}\sum_{n=1}^{\infty}\Big(\Big(\int_0^L f(x)\sin nx\,dx\Big)\cos\Big(\frac{2n\pi t}{L}\Big) + \Big(\int_0^L g(x)\sin nx\,dx\Big)\sin\Big(\frac{2n\pi t}{L}\Big) + \phi_n(t)\Big)\sin\Big(\frac{n\pi x}{L}\Big)$$

$$-\frac{4}{L}\sum_{n=1}^{\infty}\left(\int_0^t \sin\left(\frac{2n\pi}{L}(t-s)\right)\left(-\ddot{\delta}(s)+(-1)^n\ddot{\epsilon}(s)\right)ds\right)\sin\left(\frac{n\pi x}{L}\right).$$

11.2.11. The boundary conditions correspond to the first group in Table 11.1, so, similar to the work in Example 11.6 in Section 11.1, we look for product solutions of the PDE in the form

$$u(x,t)=\sin\left(\frac{n\pi x}{L}\right)G(t).$$

Substitute this into the PDE to get

$$\sin\left(\frac{n\pi x}{L}\right)\ddot{G}+2\sin\left(\frac{n\pi x}{L}\right)\dot{G}=\frac{\partial^2 u}{\partial t^2}+2\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}-u=-c^2\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right)G-\sin\left(\frac{n\pi x}{L}\right)G,$$

hence

$$\ddot{G}+2\dot{G}=-c^2\left(\frac{n\pi}{L}\right)^2G-G,$$

that is,

$$\ddot{G}+2\dot{G}+\left(1+\left(\frac{n\pi c}{L}\right)^2\right)G=0.$$

Substitute in solutions in the form $G(t)=e^{st}$ to get the requirement that

$$s^2+2s+1+\left(\frac{n\pi c}{L}\right)^2=0,$$

that is,

$$(s+1)^2+\left(\frac{n\pi c}{L}\right)^2=0,$$

whose solutions are

$$s=-1\pm i\,\frac{n\pi c}{L}.$$

So,

$$G(t)=c_1e^{-t}\cos\left(\frac{n\pi ct}{L}\right)+c_2e^{-t}\sin\left(\frac{n\pi ct}{L}\right),$$

where c_1, c_2 are arbitrary constants.

The solution of the problem is

$$u(x,t)=e^{-t}\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi ct}{L}\right)+b_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right),$$

whre a_n, b_n are arbitrary constants.
[Note: instead of substituting product solutions into the PDE, similar to Example 11.6 in Section 11.1 we could have plugged into the PDE a solution in the form

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n(t)\cos\left(\frac{n\pi ct}{L}\right)+b_n(t)\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right).$$

The work would have been more complicated and would require a little more thought, but still we could have arrived at the same final conclusion.]

11.2.13. $u\left(\frac{L}{3},t\right)\equiv 0$ and

$$u(x,t)=\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

imply

$$(\star)\ 0=u\left(\frac{L}{3},t\right)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi ct}{L}\right).$$

But $(\star)$ expresses a conclusion about a Fourier cosine series! Define $A_n = a_n \sin\left(\frac{n\pi}{3}\right)$, so $(\star)$ is

$$(\star\star)\ 0 = u\Big(\frac{L}{3}, t\Big) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi ct}{L}\right).$$

This is true for all $t \geq 0$, so, in particular it is true on the interval $0 \leq t \leq \frac{2L}{c}$. The function $g(t) \equiv 0$, defined on the interval $\Big[0, \frac{2L}{c}\Big]$, is its own Fourier cosine series. So, all of its Fourier coefficients must be zero, that is,

$$0 = A_n = a_n \sin\left(\frac{n\pi}{3}\right),\ n = 1, 2, 3, \dots\ .$$

But, for all n of the form $n = 3k$ for some integer k, $\sin\left(\frac{n\pi}{3}\right) = 0$, so $0 = a_{3k} \cdot 0$ gives no conclusion about a_{3k}. For $n \neq 3k$, $\sin\left(\frac{n\pi}{2}\right) \neq 0$, so

$$0 = A_n = a_n \sin\left(\frac{n\pi}{2}\right)$$

implies $a_n = 0$. I.e., $a_n = 0$ for all n of the form $n = 3k - 1$ or $n = 3k - 2$ for some integer $k \geq 1$.

11.2.15. (a) If $\lambda = 0$, $\frac{d^4X}{dx^4} + 0 = 0$ gives $X(x) = c_0 + c_1x + c_2x^2 + c_3x^3$, where c_0, c_1, c_2, c_3 are arbitrary constants. Substitute this and $X''(x) = 2c_2 + 6c_3x$ into the BCs to conclude, successively

$$\begin{gathered} 0 = X(0) = c_0; 0 = X''(0) = 2c_2, \Rightarrow X(x) = c_1x + c_3x^3. \\ \Rightarrow 0 = X''(L) = 6c_3L \Rightarrow c_3 = 0 \\ \Rightarrow X(x) = c_1x \Rightarrow 0 = X(L) = c_1L \Rightarrow c_1 = 0. \end{gathered}$$

so $X(x) \equiv 0$, so $\lambda = 0$ is not an eigenvalue.

(b) Suppose $\lambda > 0$ and $X(x)$ is an eigenfunction. Multiply through the differential equation $X'' + \lambda X = 0$ by $X(x)$, then integrate both sides of the ODE and use two integrations by parts on the first integral to get

$$\begin{aligned} 0 = \int_0^L X(x)X''''(x)\,dx + \int_0^L \lambda\big(X(x)\big)^2 dx &= \Big[X(x)X'''(x)\Big]_0^L - \int_0^L X'(x)X'''(x)\,dx + \int_0^L \lambda\big(X(x)\big)^2 dx \\ &= X(L)X'''(L) - X(0)X'''(0) - \Big[X'(x)X''(x)\Big]_0^L + \int_0^L X''(x)X''(x)\,dx + \int_0^L \lambda\big(X(x)\big)^2 dx \\ &= 0 - X'(L)X''(L) + X'(0)X''(L) + \int_0^L \big(X''(x)\big)^2\,dx + \int_0^L \lambda\big(X(x)\big)^2 dx \\ &= 0 + \int_0^L \big(X''(x)\big)^2\,dx + \int_0^L \lambda\big(X(x)\big)^2 dx\,. \end{aligned}$$

So, $\lambda > 0$ implies $\int_0^L \big(X(x)\big)^2 dx = 0$, which implies $X(x) \equiv 0$ on the interval $[0, L]$, contradicting $X(x)$ being an eigenfunction. So, $\lambda > 0$ cannot be an eigenvalue.

(c) Define $\omega = (-\lambda)^{1/4}$. The ODE $X'''' + \lambda X = 0$'s characteristic polynomial $s^4 + \lambda = s^4 - \omega^4 = (s^2 - \omega^2)(s^2 + \omega^2)$ has roots $s = \pm\omega, \pm i\omega$, so the ODE's general solution is $X(x) = c_1 \cosh \omega x + c_2 \sinh \omega x + c_3 \cos \omega x + c_4 \sin \omega x$. Noting that $X''(x) = \omega^2\,(c_1 \cosh \omega x + c_2 \sinh \omega x - c_3 \cos \omega x - c_4 \sin \omega x)$, the BCs imply

$$\begin{gathered} (1)\ 0 = X(0) = c_1 + c_3 \text{ and } 0 = X''(0) = \omega^2\,(c_1 - c_3)\,; \\ \text{the latter } \Rightarrow (2)\ 0 = c_1 - c_3. \end{gathered}$$

Add equations (1) and (2) to get $0 = 2c_1$; subtract (2) from (1) to get $0 = 2c_3$. So, $X(x) = c_2 \sinh \omega x + c_4 \sin \omega x$. Substitute this into the two remaining boundary conditions to get

$$\begin{gathered} (3)\ 0 = X(L) = c_2 \sinh \omega L + c_4 \sin \omega L, \\ \text{and} \\ (4)\ 0 = \omega^{-2}X''(L) = c_2 \sinh \omega L - c_4 \sin \omega L. \end{gathered}$$

Add equations (3) and (4) to get (5) $0 = 2c_2 \sinh \omega L$. Because $\omega > 0$ and $L > 0$ imply $\sinh \omega L > 0$, (5) implies $c_2 = 0$. Subtract (4) from (3) to get $0 = 2c_4 \sin \omega L$. So, there is an eigenfunction $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ only corresponding to eigenvalues $\lambda_n = -\left(\frac{n\pi}{L}\right)^4$.

11.2.17. Substitute $y(x,t) = \sum\limits_{n=1}^{\infty} G_n(t)\sin\left(\frac{n\pi x}{L}\right)$ into the PDE to get

$$(\star) \qquad \sum_{n=1}^{\infty} \ddot{G}_n(t)\sin\left(\frac{n\pi x}{L}\right) + 2\beta \sum_{n=1}^{\infty} \dot{G}_n(t)\sin\left(\frac{n\pi x}{L}\right) = \frac{\partial^2 y}{\partial t^2} + 2\beta\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2} + \cos\omega t$$

$$= \sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 G_n(t)\sin\left(\frac{n\pi x}{L}\right) + \cos\omega t \cdot \sum_{n=1}^{\infty} \frac{2\big(1-(-1)^n\big)}{n\pi}\sin\left(\frac{n\pi x}{L}\right).$$

Here, as in problems 11.1.15 through 11.1.17, we have used the Fourier sine series expansion

$$1 = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right).$$

From $(\star)$ it follows that for $n = 1, 2, 3, ..,$

$$\ddot{G}_n + 2\beta\dot{G}_n + \left(\frac{n\pi}{L}\right)^2 G_n = \frac{2\big(1-(-1)^n\big)}{n\pi}\, s\cos\omega t\,.$$

Because the constant $\beta > 0$, $y(x,t)$ has steady state oscillations in time, t.

Section 11.3

11.3.1. We need to solve
$$\left\{\begin{array}{c} \dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}=0,\ 0<x<2\pi,\ 0<y<\pi, \\[1em] T(x,0)=T(x,\pi)=0,\ 0<x<2\pi, \\[1em] \dfrac{\partial T}{\partial x}(0,y)=0,\ \dfrac{\partial T}{\partial x}(2\pi,y)=10,\ 0<y<\pi \end{array}\right\}.$$
The first pair of boundary conditions are homogeneous and belong to the first group of entries of Table 11.1 with $Y(0)=Y(\pi)=0$, for a function $Y(y)$. For $n=1,2,3,...$, when we substitute into the PDE a product solution in the form
$$T(x,y)=X(x)\sin(ny)$$
and then divide through by $\sin(ny)$ we get the ODE
$$X''-n^2X=0\,.$$
Using clairvoyance, the remaining two BCs, $\dfrac{\partial T}{\partial x}(0,y)=0$ and $\dfrac{\partial T}{\partial x}(2\pi,y)=10$ suggest writing $X(x)$ in the form
$$X(x)=c_1\cosh nx+c_2\cosh\big(n(2\pi-x)\big)\,.$$
So, the general solution of the PDE and the two homogeneous BCs can be written in the form
$$T(x,y)=\sum_{n=1}^{\infty}\Big(a_n\cosh nx+b_n\cosh\big(n(2\pi-x)\big)\Big)\sin ny,$$
hence
$$\frac{\partial T}{\partial x}(x,y)=\sum_{n=1}^{\infty}n\Big(a_n\sinh nx-b_n\sinh\big(n(2\pi-x)\big)\Big)\sin ny\,.$$
[The minus sign comes from the chain rule and the derivative of $n(2\pi-x)$ with respect to x.]

Substitute this into the two remaining BCs to get
$$0=\frac{\partial T}{\partial x}(0,y)=\sum_{n=1}^{\infty}n\Big(a_n\cdot 0-b_n\sinh 2n\pi\Big)\sin ny,$$
for $0<y<\pi$, hence $b_n=0$ for all n, and
$$10=\frac{\partial T}{\partial x}(2\pi,y)=\sum_{n=1}^{\infty}n\Big(a_n\sinh 2n\pi-b_n\cdot 0\Big)\sin ny,$$
for $0<y<\pi$, hence
$$n\,\sinh 2n\pi\cdot a_n=\frac{2}{\pi}\int_0^{\pi}10\sin ny\,dy=\frac{20}{\pi}\Big[\,\frac{\cos ny}{-n}\,\Big]_0^{\pi}=\frac{20}{\pi}\Big(\frac{(-1)^n-1}{-n}\Big)=\frac{20}{n\pi}\left(1-(-1)^n\right).$$
So, $a_{even}=0$.

The solution of the problem is $T(x,y)=\dfrac{40}{\pi}\displaystyle\sum_{k=1}^{\infty}\frac{\cosh\big((2k-1)x\big)}{(2k-1)^2\sinh(2(2k-1)\pi)}\sin\big((2k-1)y\big)$.

11.3.3. We need to solve
$$\left\{\begin{array}{c} \dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}=0,\ 0<x<2\pi,\ 0<y<\pi, \\[1em] T(0,y)=T(2\pi,y)=0,\ 0<y<\pi, \\[1em] T(x,0)=100,\ \dfrac{\partial T}{\partial y}(x,0)=-10,\ 0<x<2\pi \end{array}\right\}.$$

The first pair of boundary conditions are homogeneous and belong to the first group of entries of Table 11.1, that is, with $X(0) = X(2\pi) = 0$ for a function $X(x)$. When we substitute into the PDE a product solution in the form

$$T(x,y) = \sin\Big(\frac{nx}{2}\Big)Y(y)$$

and then divide through by $\sin\big(\frac{nx}{2}\big)$, we get the ODE

$$Y'' - \Big(\frac{n}{2}\Big)^2 Y = 0\,.$$

The remaining two BCs, $T(x,0) = 100$ and $\dfrac{\partial T}{\partial y}(x,0) = 0$, do not suggest much about how to write $Y(y)$;

$$Y(y) = c_1 \cosh\Big(\frac{ny}{2}\Big) + c_2 \sinh\Big(\frac{ny}{2}\Big)$$

is good enough. [In fact, these two BCs are more like initial conditions! That's what makes this problem strange.]

The general solution of the PDE and the two homogeneous BCs can be written in the form

$$T(x,y) = \sum_{n=1}^{\infty} \Big(a_n \cosh\Big(\frac{ny}{2}\Big) + b_n \sinh\Big(\frac{ny}{2}\Big)\Big) \sin\Big(\frac{nx}{2}\Big)\,.$$

Substitute this into the first of the remaining BCs to get

$$100 = T(x,0) = \sum_{n=1}^{\infty} a_n \sin\Big(\frac{nx}{2}\Big),$$

so

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} 100 \sin\Big(\frac{nx}{2}\Big)dx = \frac{100}{\pi}\Big[\frac{\cos\frac{nx}{2}}{-n/2}\Big]_0^{2\pi} = \frac{200}{n\pi}\big(1-(-1)^n\big)\,.$$

We calculate

$$\frac{\partial T}{\partial y}(x,y) = \sum_{n=1}^{\infty} \frac{n}{2}\Big(a_n \sinh\Big(\frac{ny}{2}\Big) + b_n \cosh\Big(\frac{ny}{2}\Big)\Big) \sin\Big(\frac{nx}{2}\Big)\,.$$

The last BC requires

$$-10 = \frac{\partial T}{\partial y}(x,0) = \sum_{n=1}^{\infty} \frac{n}{2}\, b_n \sin\Big(\frac{nx}{2}\Big),$$

hence

$$\frac{n}{2}\, b_n = \frac{2}{2\pi}\int_0^{2\pi} -10 \sin\Big(\frac{nx}{2}\Big)dx = ... = -\frac{20}{n\pi}\big(1-(-1)^n\big)\,.$$

The solution of the problem is

$$T(x,y) = \sum_{n=1}^{\infty} \frac{200}{n\pi}\big(1-(-1)^n\big)\cosh\Big(\frac{ny}{2}\Big)\sin\Big(\frac{nx}{2}\Big) - \sum_{n=1}^{\infty} \frac{40}{n^2\pi}\big(1-(-1)^n\big)\sinh\Big(\frac{ny}{2}\Big)\sin\Big(\frac{nx}{2}\Big)\,.$$

But, $1-(-1)^n = 0$ for $n = \mathit{even}$ and $1-(-1)^n = 2$ for $n = \mathit{odd} = 2k-1$, so the solution is

$$T(x,y) = \frac{80}{\pi}\sum_{k=1}^{\infty}\Big(\frac{5}{2k-1}\cosh\Big(\frac{(2k-1)y}{2}\Big) - \frac{1}{(2k-1)^2}\sinh\Big(\frac{(2k-1)y}{2}\Big)\Big)\sin\Big(\frac{(2k-1)x}{2}\Big)\,.$$

11.3.5. The third and fourth BCs are homogeneous and correspond to the third group of entries in Table 11.1 with $X(0) = X'(\pi) = 0$, so look for product solutions of the PDE in the form

$$T(x,y) = \sin\Big(\frac{(2n-1)x}{2}\Big)Y(y)\,.$$

Substitute that into the PDE and divide through by $\sin\big(\frac{(2n-1)x}{2}\big)$ to get

$$Y'' - \Big(\frac{2n-1}{2}\Big)^2 Y = 0\,.$$

The remaining two BCs are $T(x,0)=20$ and $T(x,1)=f(x)$, so clairvoyance suggests writing the solutions for $Y(y)$ in the form
$$Y(y)=c_1\sinh\Big(\frac{2n-1}{2}\,(1-y)\Big)+c_2\sinh\Big(\frac{2n-1}{2}\,y\Big).$$
The general solution of the PDE and the two homogeneous BCs can be written in the form

$$T(x,y)=\sum_{n=1}^{\infty}\left(a_n\sinh\Big(\frac{2n-1}{2}\,(1-y)\Big)+b_n\sinh\Big(\frac{2n-1}{2}\,y\Big)\right)\sin\Big(\frac{(2n-1)x}{2}\Big).$$

The first of the two remaining BCs requires

$$20=T(x,0)=\sum_{n=1}^{\infty}a_n\sinh\Big(\frac{2n-1}{2}\Big)\sin\Big(\frac{(2n-1)x}{2}\Big),$$

so
$$a_n\sinh\Big(\frac{2n-1}{2}\Big)=\frac{2}{\pi}\int_0^{\pi}20\sin\Big(\frac{(2n-1)x}{2}\Big)dx=\frac{40}{\pi}\Big[\frac{\cos\frac{(2n-1)x}{2}}{-(2n-1)/2}\Big]_0^{\pi}=...=\frac{80}{(2n-1)\pi}\quad.$$
The last BC requires

$$\sum_{n=3}^{10}\alpha_n\sin\left((n-\frac{1}{2})x\right)=f(x)=T(x,1)=\sum_{n=1}^{\infty}b_n\sinh\Big(\frac{2n-1}{2}\Big)\sin\Big(\frac{(2n-1)x}{2}\Big),$$

so
$$b_n\sinh\Big(\frac{2n-1}{2}\Big)=\left\{\begin{array}{cl}\alpha_n, & 3\le n\le 10\\ \\ 0, & n<3\text{ or }n>10\end{array}\right\}.$$
The solution of the problem is

$$T(x,y)=\sum_{n=3}^{10}\frac{\alpha_n}{\sinh\big(\frac{2n-1}{2}\big)}\sinh\left(\frac{(2n-1)y}{2}\right)\sin\left(\frac{(2n-1)x}{2}\right)$$
$$+\frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh\big(\frac{2n-1}{2}\big)}\sinh\left(\frac{(2n-1)(1-y)}{2}\right)\sin\left(\frac{(2n-1)x}{2}\right).$$

11.3.7. Method I: Because three of the four sides are kept at $20^\circ C.$, it makes sense to write the solution in the special form $T(x,y)=20+u(x,y)$. The problem that u needs to solve is

$$\left\{\begin{array}{c}\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}=0,\ 0<x<a,0<y<b,\\ \\ u(0,y)=-20,\ u(a,y)=0,0<y<b,\\ \\ u(x,0)=0,\ u(x,b)=0,\ 0<x<a\end{array}\right\}.$$

The last two BCs fit the first group of entries in Table 11.1 with $Y(0)=Y(b)=0$. Substitute into the PDE for u product solutions in the form
$$u(x,y)=X(x)\sin\Big(\frac{n\pi y}{b}\Big)$$
and then divide through by $\sin\big(\frac{n\pi y}{b}\big)$ to get

$$X''-\Big(\frac{n\pi}{b}\Big)^2X=0.$$

Clairvoyance for the remaining BCs, $u(0,y)=-20$ and $u(a,y)=0$ suggest writing the general solution of u's PDE and the two homogeneous BCs in the form

$$u=\sum_{n=1}^{\infty}\left(a_n\sinh\Big(\frac{n\pi}{b}(a-x)\Big)+b_n\sinh\Big(\frac{n\pi}{b}\,x\Big)\right)\sin\Big(\frac{n\pi y}{b}\Big).$$

The first of the remaining two BCs requires

$$-20 = u(0,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right),$$

so

$$a_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b (-20)\sin\left(\frac{n\pi y}{b}\right)dy = -\frac{40}{b}\left[\frac{\cos\frac{n\pi y}{b}}{-n\pi/b}\right]_0^b = -\frac{40}{n\pi}\left(1-(-1)^n\right).$$

The last of the BCs requires

$$0 = u(a,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right),$$

so $b_n = 0$ for all n.

The solution of the problem for u is

$$u = -\frac{80}{\pi}\sum_{k=1}^{\infty}\frac{\sinh\left(\frac{(2k-1)\pi(a-x)}{b}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi a}{b}\right)}\sin\left(\frac{(2k-1)\pi y}{b}\right).$$

The solution of the whole problem is

$$T(x,y) = 20 + u(x,y) = 20 - \frac{80}{\pi}\sum_{k=1}^{\infty}\frac{\sinh\left(\frac{(2k-1)\pi(a-x)}{b}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi a}{b}\right)}\sin\left(\frac{(2k-1)\pi y}{b}\right).$$

Method II: The whole, original problem is to find $T(x,y)$ satisfying

$$\left\{\begin{array}{c}\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0,\ 0<x<a,\ 0<y<b, \\[2ex] T(x,0) = 20,\ T(x,b) = 20,\ 0<x<a \\[2ex] T(0,y) = 0,\ T(a,y) = 20,\ 0<y<b\end{array}\right\}.$$

As in Example 11.12 in Section 11.3, write the solution of the whole problem as $T(x,y) = T_1(x,y) + T_2(x,y)$, where

$$\left\{\begin{array}{c}\dfrac{\partial^2 T_1}{\partial x^2} + \dfrac{\partial^2 T_1}{\partial y^2} = 0,\ 0<x<a,\ 0<y<b, \\[2ex] T_1(x,0) = T_1(x,b) = 0,\ 0<x<a \\[2ex] T_1(0,y) = 0,\ T_1(a,y) = 20,\ 0<y<b\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\dfrac{\partial^2 T_2}{\partial x^2} + \dfrac{\partial^2 T_2}{\partial y^2} = 0,\ 0<x<a,\ 0<y<b, \\[2ex] T_2(x,0) = T_2(x,b) = 20,\ 0<x<a \\[2ex] T_2(0,y) = T_2(a,y) = 0,\ 0<y<b\end{array}\right\}.$$

The problem for $T_1(x,y)$ has its first two BCs fitting the first group of entries of Table 11.1 with $Y(0) = Y(b) = 0$. Substitute into the PDE for T_1 product solutions in the form

$$T_1(x,y) = X(x)\sin\left(\frac{n\pi y}{b}\right)$$

and then divide through by $\sin\left(\frac{n\pi y}{b}\right)$ to get

$$X'' - \left(\frac{n\pi}{b}\right)^2 X = 0.$$

Clairvoyance for the remaining BCs, $T_1(0,y)=0$ and $T_1(a,y)=20$ suggest writing the general solution of the PDE and the two homogeneous BCs for T_1 as

$$T_1 = \sum_{n=1}^{\infty} \left(a_n \sinh\left(\frac{n\pi}{b}(a-x)\right) + b_n \sinh\left(\frac{n\pi}{b}\,x\right)\right)\sin\left(\frac{n\pi y}{b}\right).$$

The first of the remaining two BCs requires

$$0 = T_1(0,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right),$$

so $a_n = 0$ for all n.

The last BC requires

$$20 = T_1(a,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right),$$

so

$$b_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b 20\sin\left(\frac{n\pi y}{b}\right)dy = \frac{40}{b}\left[\frac{\cos\frac{n\pi y}{b}}{-n\pi/b}\right]_0^b = \frac{40}{n\pi}\left(1-(-1)^n\right).$$

The solution for $T_1(x,y)$ is

$$T_1(x,y) = \frac{80}{\pi}\sum_{k=1}^{\infty} \frac{\sinh\left(\frac{(2k-1)\pi x}{b}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi a}{b}\right)}\sin\left(\frac{(2k-1)\pi y}{b}\right).$$

For $T_2(x,y)$, the separated homogeneous BCs $T_2(0,y)=T_2(a,y)=0,\ 0<y<b$ suggest looking for product solutions in the form

$$T_2(x,y) = \sin\left(\frac{n\pi x}{a}\right)Y(y).$$

Substitute that into the the PDE and divide through by $\sin\left(\frac{n\pi x}{a}\right)$ to get

$$Y'' - \left(\frac{n\pi}{a}\right)^2 Y = 0.$$

The remaining BCs being $T_2(x,0)=T_2(x,b)=20,\ 0<x<a$ and clairvoyance suggest writing the general solution of the PDE and the two homogeneous BCs in the form

$$T_2(x,y) = \sum_{n=1}^{\infty}\left(a_n\sinh\left(\frac{n\pi}{a}(b-y)\right) + b_n\sinh\left(\frac{n\pi}{a}\,y\right)\right)\sin\left(\frac{n\pi x}{a}\right).$$

The first of the remaining two BCs requires

$$20 = T_2(x,0) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right),$$

so

$$a_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a 20\sin\left(\frac{n\pi x}{a}\right)dx = \frac{40}{a}\left[\frac{\cos\frac{n\pi x}{a}}{-n\pi/a}\right]_0^a = \frac{40}{n\pi}\left(1-(-1)^n\right).$$

The last BC requires

$$20 = T_2(x,b) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right),$$

so, similarly,

$$b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a 20\sin\left(\frac{n\pi x}{a}\right)dx = \frac{40}{n\pi}\left(1-(-1)^n\right).$$

The solution for $T_2(x,y)$ is

$$T_2(x,y) = \frac{80}{\pi}\sum_{k=1}^{\infty}\frac{\sinh\left(\frac{(2k-1)\pi(b-y)}{a}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi b}{a}\right)}\sin\left(\frac{(2k-1)\pi x}{a}\right) + \frac{80}{\pi}\sum_{k=1}^{\infty}\frac{\sinh\left(\frac{(2k-1)\pi y}{a}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi b}{a}\right)}\sin\left(\frac{(2k-1)\pi x}{a}\right).$$

Altogether, the solution of the original problem is

$$T(x,y) = T_1(x,y) + T_2(x,y)$$

$$= \frac{80}{\pi} \sum_{k=1}^{\infty} \left(\frac{\sinh\left(\frac{(2k-1)\pi x}{b}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi a}{b}\right)} \sin\left(\frac{(2k-1)\pi y}{b}\right) + \frac{\sinh\left(\frac{(2k-1)\pi(b-y)}{a}\right) + \sinh\left(\frac{(2k-1)\pi y}{a}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi b}{a}\right)} \sin\left(\frac{(2k-1)\pi x}{a}\right)\right).$$

11.3.9. For $n = 0, 1, 2, ...$, substitute product solutions $T(x,y) = \cos\left(\frac{n\pi x}{a}\right)Y(y)$ into the PDE $0 = \frac{\partial^2 T}{\partial x^2} + 3\frac{\partial^2 T}{\partial y^2}$ to get

$$0 = \frac{\partial^2 T}{\partial x^2} + 3\frac{\partial^2 T}{\partial y^2} = -\left(\frac{n\pi}{a}\right)^2 \cos\left(\frac{n\pi x}{a}\right)Y(y) + 3\cos\left(\frac{n\pi x}{a}\right)Y''(y)\,.$$

Divide through by $3\cos\left(\frac{n\pi x}{a}\right)$ to get

$$Y'' - \left(\frac{n\pi}{a\sqrt{3}}\right)^2 Y = 0\,.$$

The remaining two BCs are $T(x,0) = 0$ and $T(x,b) = \frac{a}{2} - |x - \frac{a}{2}|$, so clairvoyance suggests writing the solutions for $Y(y)$ in the form, for $n = 1, 2, 3, ...$,

$$Y_n(y) = c_1 \sinh\left(\frac{n\pi}{a\sqrt{3}}\,(b-y)\right) + c_2 \sinh\left(\frac{n\pi}{a\sqrt{3}}\,y\right).$$

For $n = 0$, clairvoyance suggests writing the solutions of $Y'' = 0$ in the form

$$Y_0(y) = c_1(b-y) + c_2 y\,.$$

The general solution of the PDE and the two homogeneous BCs can be written in the form

$$T(x,y) = \frac{a_0}{2}\,(b-y) + \frac{b_0}{2}\,y + \sum_{n=1}^{\infty} \left(a_n \sinh\left(\frac{n\pi}{a\sqrt{3}}\,(b-y)\right) + b_n \sinh\left(\frac{n\pi}{a\sqrt{3}}\,y\right)\right)\cos\left(\frac{n\pi x}{a}\right).$$

The first of the two remaining BCs requires

$$0 = T(x,0) = \frac{a_0}{2}\,b + \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi b}{a\sqrt{3}}\right)\cos\left(\frac{n\pi x}{a}\right),$$

so $a_n = 0$ for all $n \ge 0$.

The last BC requires

$$\frac{a}{2} - |x - \frac{a}{2}| = T(x,b) = \frac{b_0}{2}\,b + \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi b}{a\sqrt{3}}\right)\cos\left(\frac{n\pi x}{a}\right).$$

Note that

$$\frac{a}{2} - \left|x - \frac{a}{2}\right| = \left\{ \begin{array}{cc} \frac{a}{2} - \left(-\left(x - \frac{a}{2}\right)\right), & 0 < x < \frac{a}{2} \\ \\ \frac{a}{2} - \left(x - \frac{a}{2}\right), & \frac{a}{2} < x < a \end{array} \right\} = \left\{ \begin{array}{ll} x, & 0 < x < \frac{a}{2} \\ \\ a - x, & \frac{a}{2} < x < a \end{array} \right\}.$$

For $n = 0$,

$$b_0 \cdot b = \frac{2}{a}\int_0^a \left(\frac{a}{2} - \left|x - \frac{a}{2}\right|\right)dx = \frac{2}{a}\left(\int_0^{a/2} x\,dx + \int_0^{a/2}(a-x)dx\right) = \frac{2}{a}\left(\left[\frac{1}{2}\,x^2\right]_0^{a/2} + \left[ax - \frac{1}{2}\,x^2\right]_{a/2}^{a}\right)$$

$$= \frac{2}{a}\left(\frac{1}{8}\,a^2 + \frac{1}{2}\,a^2 - \frac{3}{8}\,a^2\right) = \frac{a}{2}\,.$$

For $n = 1, 2, 3, ...$,

$$b_n \sinh\left(\frac{n\pi b}{a\sqrt{3}}\right) = \frac{2}{a}\int_0^a \left(\frac{a}{2} - \left|x - \frac{a}{2}\right|\right)\cos\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}\left(\int_0^{a/2} x\cos\left(\frac{n\pi x}{a}\right)dx + \int_0^{a/2}(a-x)\cos\left(\frac{n\pi x}{a}\right)dx\right)$$

$$= \frac{2}{a}\Bigg(\Big[\, \frac{x\,\sin\frac{n\pi x}{a}}{n\pi/a} + \frac{\cos\frac{n\pi x}{a}}{(n\pi/a)^2}\,\Big]_0^{a/2} + \Big[\, \frac{(a-x)\,\sin\frac{n\pi x}{a}}{n\pi/a} - \frac{\cos\frac{n\pi x}{a}}{(n\pi/a)^2}\,\Big]_{a/2}^{a}\Bigg)$$

$$= \frac{2}{a}\Bigg(\Big(\, \cancel{\frac{\frac{a}{2}\,\sin\frac{n\pi}{2} - 0}{n\pi/a}} + \frac{\cos\frac{n\pi}{2} - 1}{(n\pi/a)^2}\,\Big) + \Big[\, \cancel{\frac{0 - \frac{a}{2}\,\sin\frac{n\pi}{2}}{n\pi/a}} - \frac{(-1)^n - \cos\frac{n\pi}{2}}{(n\pi/a)^2}\,\Big)\Bigg)$$

$$= \frac{2a}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - 1 - (-1)^n\right) \triangleq \frac{2a}{n^2\pi^2}\cdot\beta_n\,.$$

There are four cases:

(1) If $n = 4m$ for some positive integer m, then $\beta_n = 2\cos(2m\pi) - 1 - (-1)^{4m} = 2\cdot 1 - 1 - 1 = 0$;

(2) If $n = 4k-1$ for some positive integer k, then $\beta_n = 2\cos(2k\pi - \frac{\pi}{2}) - 1 - (-1)^{4k-1} = 2\cdot 0 - 1 - (-1) = 0$;

(3) If $n = 4\ell - 2$ for some positive integer ℓ, then $\beta_n = 2\cos(2\ell\pi - \pi) - 1 - (-1)^{4\ell-2} = 2\cdot(-1) - 1 - 1 = -4$;

(4) If $n = 4p-3$ for some positive integer p, then $\beta_n = 2\cos(2p\pi - \frac{3\pi}{2}) - 1 - (-1)^{4p-3} = 2\cdot 0 - 1 - (-1) = 0$.

So, for $n = 4\ell - 2 = 2(2\ell - 1)$,

$$b_{2(2\ell-1)}\sinh\left(\frac{2(2\ell-1)\pi b}{a\sqrt{3}}\right) = -\frac{8a}{\big(2(2\ell-1)\big)^2\pi^2},$$

and all other $b_n = 0$ for $n \ge 1$.

The solution of the problem is

$$T(x,y) = \frac{ay}{4b} - \frac{2a}{\pi^2}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell-1)^2\sinh\left(\frac{2(2\ell-1)\pi b}{a\sqrt{3}}\right)}\sinh\left(\frac{2(2\ell-1)\pi}{a\sqrt{3}}\,y\right)\cos\left(\frac{2(2\ell-1)\pi x}{a}\right).$$

11.3.11. The first pair of boundary conditions are homogeneous and belong to the second group of entries of Table 11.1 with $X'(0) = X'(\pi) = 0$. For $n = 0, 1, 2, \ldots$, when we substitute into the PDE a product solution in the form

$$T(x,y) = \cos(nx)Y(y)$$

to get

$$0 = 2\,\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -2n^2\cos(nx)Y(y) + \cos(nx)Y''(y)$$

and then divide through by $\cos(nx)$, we get the ODE

$$Y'' - 2n^2 Y = 0\,.$$

Recall that $X_0(x) \equiv 1$ is an "honorary cosine function."

For $n = 1, 2, 3, \ldots$, using clairvoyance, the remaining two BCs, $\frac{\partial T}{\partial y}(x,0) = 0$ and $T(x,1) = 1 - \frac{1}{3}\cos 2x + \frac{1}{5}\cos 5x$ suggest writing $Y(y)$ in the form

$$Y_n(y) = c_1\cosh(n\sqrt{2}\,y) + c_2\sinh\left(n\sqrt{2}\,(1-y)\right).$$

For $n = 0$, using clairvoyance, the remaining two BCs, $\frac{\partial T}{\partial y}(x,0) = 0$ and $T(x,1) = 1 - \frac{1}{3}\cos 2x + \frac{1}{5}\cos 5x$ suggest writing $Y(y)$ in the form

$$Y_0(y) = c_1 + c_2\,y\,.$$

So, the general solution of the PDE and the two homogeneous BCs can be written in the form

$$T(x,y) = \frac{a_0}{2} + \frac{b_0}{2}\,y + \sum_{n=1}^{\infty}\left(a_n\cosh(n\sqrt{2}\,y) + b_n\sinh\left(n\sqrt{2}\,(1-y)\right)\right)\cos(nx)\,.$$

It follows that

$$\frac{\partial T}{\partial y}(x,y) = \frac{b_0}{2} + \sum_{n=1}^{\infty} n\sqrt{2}\left(a_n\sinh(n\sqrt{2}\,y) - b_n\cosh\left(n\sqrt{2}\,(1-y)\right)\right)\cos(nx)\,.$$

Substitute this into the first of the two remaining BCs to get

$$0 = \frac{\partial T}{\partial y}(x,0) = \frac{b_0}{2} - \sum_{n=1}^{\infty} n\sqrt{2}\, b_n \cosh(n\sqrt{2}) \cos(nx),\ 0 < x < \pi,$$

which implies $b_n = 0$ for $n = 0, 1, 2, ...$. So,

$$T(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cosh(n\sqrt{2}\, y) \cos(nx)\,.$$

The last BC is

$$1 - \frac{1}{3}\cos 2x + \frac{1}{5}\cos 5x = T(x,1) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cosh(n\sqrt{2}\,) \cos(nx),\ 0 < x < \pi,$$

which is its own Fourier cosine series, so $a_0 = 2$, $a_2 \cdot \cosh(2\sqrt{2}\,) = -\frac{1}{3}$, $a_5 \cdot \cosh(5\sqrt{2}\,) = \frac{1}{5}$, and $a_n = 0$ for all other n.

The solution of the problem is

$$T(x,y) = 1 - \frac{\cosh(2\sqrt{2}\, y)}{3\cosh(2\sqrt{2})}\cos 2x + \frac{\cosh(5\sqrt{2}\, y)}{5\cosh(5\sqrt{2})}\cos 5x\,.$$

11.3.13. The first pair of boundary conditions are homogeneous and belong to the first group of entries of Table 11.1 with $X(0) = X(a) = 0$ for a function $X(x)$. For $n = 1, 2, 3, ...$, when we substitute into the PDE a product solution in the form

$$T(x,y) = \sin\left(\frac{n\pi x}{a}\right) Y(y)$$

and then divide through by $\sin\left(\frac{n\pi x}{a}\right)$ we get the ODE

$$Y'' - \left(\frac{n\pi}{a}\right)^2 Y = 0\,.$$

Using clairvoyance, the remaining two BCs, $T(x,0) = 0$ and $T(x,b) = f(x)$ suggest writing $Y(y)$ in the form

$$Y(y) = c_1 \sinh\left(\frac{n\pi}{a}(b-y)\right) + c_2 \sinh\left(\frac{n\pi}{a}\, y\right).$$

So, the general solution of the PDE and the two homogeneous BCs can be written in the form

$$T(x,y) = \sum_{n=1}^{\infty} \left(a_n \sinh\left(\frac{n\pi}{a}(b-y)\right) + b_n \sinh\left(\frac{n\pi}{a}\, y\right)\right) \sin\left(\frac{n\pi x}{a}\right).$$

The first of the remaining two BCs requires

$$0 = T(x,0) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right),$$

so $a_n = 0$ for all n.

The last BC requires

$$f(x) = T(x,b) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right),$$

so

$$b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx\,.$$

(a) The solution of the problem can be written in the form

$$T(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \left(\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx\right) \sin\left(\frac{n\pi x}{a}\right).$$

(b) Change the variable inside the integral to z, so the solution is written as

$$T(x,y)=\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\left(\int_0^a f(z)\sin\left(\frac{n\pi z}{a}\right)dz\right)\sin\left(\frac{n\pi x}{a}\right).$$

and then interchange the integration and summation, while ignoring theoretical mathematical issues of convergence, to get

$$\begin{aligned}T(x,y)&=\int_0^a \frac{2}{a}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}f(z)\sin\left(\frac{n\pi z}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dz\\ &=\int_0^a\left(\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi z}{a}\right)\right)f(z)\,dz=\int_0^a G(x,y,z)\,f(z)\,dz,\end{aligned}$$

where

$$G(x,y,z)\triangleq\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi z}{a}\right).$$

The latter is called a “Green’s function" or “propagator."

Section 11.4

11.4.1. BCs $X'_m(0) = X'_m(a) = 0$ give eigenfunctions $X_m(x) = \cos\left(\frac{m\pi x}{a}\right)$, for $m = 0, 1, ...$. BCs $Y_n(0) = Y_n(b) = 0$ give eigenfunctions $Y_n(y) = \sin\left(\frac{n\pi y}{b}\right)$, for $n = 1, 2, ...$. Substitute product functions $\phi_{m,n}(x, y) = X_m(x)Y_n(y) = \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$, for $m = 0, 1, ...; n = 1, 2, ...$, into the eigenvalue problem's PDE to get

$$\lambda\,\phi_{m,n}(x, y) = -\Delta\phi_{m,n}(x, y) = -Y_n(y)\frac{\partial^2 X_m}{\partial x^2} - X_m(x)\frac{\partial^2 Y_n}{\partial y^2} = \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)\phi_{m,n}(x, y)\,.$$

So, the eigenvalues are $\lambda_{m,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$, $m = 0, 1, ...; n = 1, 2, ...$, with corresponding eigenfunctions $\phi_{m,n}(x, y) = \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$. Note that for the eigenfunctions having $m = 0$, $\cos(0\cdot x) \equiv 1$, so $\phi_{0,n}(x, y) = \sin\left(\frac{n\pi y}{b}\right)$.

11.4.3. BCs $X_m(0) = X'_m(\pi) = 0$ give eigenfunctions $X_m(x) = \sin\left(\left(m - \frac{1}{2}\right)x\right)$, for $m = 1, 2, ...$. BCs $Y_n(0) = Y_n(2) = 0$ give eigenfunctions $Y_n(y) = \sin\left(\frac{n\pi y}{2}\right)$, for $n = 1, 2, ...$. Substitute product functions $\phi_{m,n}(x, y) = X_m(x)Y_n(y) = \sin\left(\left(m - \frac{1}{2}\right)x\right)\sin\left(\frac{n\pi y}{2}\right)$, for $m = 1, 2, ...; n = 1, 2, ...$, into the eigenvalue problem's PDE to get

$$\lambda\,\phi_{m,n}(x, y) = -\Delta\phi_{m,n}(x, y) = -Y_n(y)\frac{\partial^2 X_m}{\partial x^2} - X_m(x)\frac{\partial^2 Y_n}{\partial y^2} = \left(\left(m - \frac{1}{2}\right)^2 + \left(\frac{n\pi}{2}\right)^2\right)\phi_{m,n}(x, y)\,.$$

So, the eigenvalues are $\lambda_{m,n} = \left(m - \frac{1}{2}\right)^2 + \left(\frac{n\pi}{2}\right)^2$, $m = 1, 2, ...; n = 1, 2, ...$, with corresponding eigenfunctions $\phi_{m,n}(x, y) = \sin\left(\left(m - \frac{1}{2}\right)x\right)\sin\left(\frac{n\pi y}{2}\right)$.

Substitute into the PDE product solutions of the form $u(x, y, t) = \phi_{m,n}(x, y)G_{m,n}(t)$ to get

$$\phi_{m,n}(x, y)\dot{G}_{m,n}(t) = \frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \alpha\,G_{m,n}(t)\Delta\phi_{m,n} = -\alpha\,G_{m,n}(t)\lambda_{m,n}\,\phi_{m,n}(x, y)\,.$$

Because $\phi_{m,n}(x, y)$ is not identically zero, divide through by it to get

$$\dot{G}_{m,n}(t) = -\alpha\left(\left(m - \frac{1}{2}\right)^2 + \left(\frac{n\pi}{2}\right)^2\right)G_{m,n}(t)\,.$$

The general solution of the PDE and the four homogeneous boundary conditions is

$$T(x, y, t) = \sum_{\substack{n=1\\m=1}}^{\infty} b_{m,n}e^{-\alpha\left(\left(m-\frac{1}{2}\right)^2+\left(\frac{n\pi}{2}\right)^2\right)t}\sin\left(\left(m - \frac{1}{2}\right)x\right)\sin\left(\frac{n\pi y}{2}\right)\,.$$

Substitute this into the initial condition to get

$$\sin\left(\frac{x}{2}\right)(1 - \cos\pi y) = T(x, y, 0) = \sum_{\substack{n=1\\m=1}}^{\infty} b_{m,n}\sin\left(\left(m - \frac{1}{2}\right)x\right)\sin\left(\frac{n\pi y}{2}\right)$$

Because $\displaystyle\int_0^\pi \sin\left(\frac{x}{2}\right)\sin\left(\left(m - \frac{1}{2}\right)x\right)dx = 0$ for $m \geq 2$, this implies $b_{m,n} = 0$ for $m \geq 2$, as well as

$$(1 - \cos\pi y) = \sum_{n=1}^{\infty} b_{1,n}\sin\left(\frac{n\pi y}{2}\right)\,.$$

Fourier sine series analysis gives

$$b_{1,n} = \frac{2}{2}\int_0^2 (1 - \cos\pi y)\sin\left(\frac{n\pi y}{2}\right)dy = \int_0^2 \sin\left(\frac{n\pi y}{2}\right)dy - \int_0^2 \cos(\pi y)\sin\left(\frac{n\pi y}{2}\right)dy\,.$$

For $n = 2$ we have

$$b_{1,2} = \left[\frac{\cos(\pi y)}{-\pi}\right]_0^2 - \int_0^2 \cos(\pi y)\sin(\pi y)\,dy = \frac{1-1}{-\pi} - \left[\frac{1}{2\pi}\sin^2(\pi y)\right]_0^2 = 0\,.$$

For $n \neq 2$ we have

$$b_{1,n} = \left[\frac{\cos(n\pi y/2)}{-n\pi/2} \right]_0^2 - \int_0^2 \cos(\pi y) \sin(n\pi y/2)\, dy$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) - \frac{1}{2}\int_0^2 \left(\sin\left((\frac{n}{2}-1)\pi y \right) + \sin\left((\frac{n}{2}+1)\pi y \right) \right) dy$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) - \frac{1}{2}\left[\frac{\cos\left((\frac{n}{2}-1)\pi y\right)}{-(\frac{n}{2}-1)\pi} + \frac{\cos\left((\frac{n}{2}+1)\pi y\right)}{-(\frac{n}{2}+1)\pi} \right]_0^2$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) - \frac{1}{2}\left(\frac{\cos\left((n-2)\pi\right)-1}{-(\frac{n}{2}-1)\pi} + \frac{\cos\left((n+2)\pi\right)-1}{-(\frac{n}{2}+1)\pi} \right)$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) - \frac{1}{2}\left(\frac{(-1)^{n-2}-1}{-(\frac{n}{2}-1)\pi} + \frac{(-1)^{n+2}-1}{-(\frac{n}{2}+1)\pi} \right)$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) - \frac{1}{2\pi}\left(1-(-1)^n\right)\left(\frac{1}{\frac{n}{2}-1} + \frac{1}{\frac{n}{2}+1} \right) = \frac{1}{2\pi}\left(1-(-1)^n\right)\left(\frac{4}{n} - \frac{n}{\left(\frac{n}{2}\right)^2 - 1} \right)$$

$$= \frac{2}{\pi}\left(1-(-1)^n\right)\left(\frac{1}{n} - \frac{n}{n^2-4} \right) = \frac{2}{\pi}\left(1-(-1)^n\right) \cdot \frac{-4}{n\,(n^2-4)}\,.$$

The final conclusion is

$$T(x,y,t) = \sum_{n \neq 2}^{\infty} \frac{2}{\pi}\left(1-(-1)^n\right) \cdot \frac{-4}{n\,(n^2-4)} e^{-\alpha\left((\frac{\pi}{2})^2 + (\frac{n\pi}{2})^2 \right)t} \sin\left(\frac{x}{2}\right) \sin\left(\frac{n\pi y}{2}\right)$$

$$= -\frac{16}{\pi} \sin\left(\frac{x}{2}\right) \sum_{k=1}^{\infty} \frac{1}{(2k-1)\left((2k-1)^2-4\right)} e^{-\alpha\left((\frac{\pi}{2})^2 + \left(\frac{(2k-1)\pi}{2}\right)^2 \right)t} \sin\left(\frac{(2k-1)\pi y}{2} \right)\,.$$

11.4.5. BCs $X_m(0) = X_m(a) = 0$ give eigenfunctions $X_m(x) = \sin\left(\frac{m\pi x}{a}\right)$, for $m = 1, 2, \ldots$. BCs $Y_n(0) = Y_n(b) = 0$ give eigenfunctions $Y_n(y) = \sin\left(\frac{n\pi y}{b}\right)$, for $n = 1, 2, \ldots$. The sum of product functions is

$$T(x,y) = \sum_{\substack{n=1 \\ m=1}}^{\infty} b_{m,n}\phi_{m,n}(x,y) = \sum_{\substack{n=1 \\ m=1}}^{\infty} b_{m,n} X_m(x) Y_n(y) = \sum_{\substack{n=1 \\ m=1}}^{\infty} b_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right),$$

which we substitute into the PDE to get

$$(\star) \quad \frac{1}{\kappa} = \Delta T(x,y) = \Delta \sum_{\substack{n=1 \\ m=1}}^{\infty} b_{m,n} X_m(x) Y_n(y) = \sum_{\substack{n=1 \\ m=1}}^{\infty} -\lambda_{m,n}\, b_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right),$$

where

$$\lambda_{m,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2, \text{ for } m = 1, 2, \ldots; n = 1, 2, \ldots$$

$(\star)$ is a double Fourier expansion problem. The solution for the coefficients is given by

$$-\lambda_{m,n} b_{m,n} = \frac{4}{ab}\int_0^a \int_0^b \frac{1}{\kappa} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dy\, dx = \frac{4}{ab\kappa}\left(\int_0^a \sin\left(\frac{m\pi x}{a}\right) dx \right)\left(\int_0^b \sin\left(\frac{n\pi y}{a}\right) dy \right)$$

$$= \frac{4}{ab\kappa} \cdot \left[\frac{\cos\left(\frac{m\pi x}{a}\right)}{-m\pi/a} dx \right]_0^a \cdot \left[\frac{\cos\left(\frac{n\pi y}{a}\right)}{-n\pi/b} dy \right]_0^b = \frac{4}{ab\kappa} \cdot \frac{a}{m\pi} \cdot \frac{b}{n\pi} \cdot \left(1-(-1)^m\right)\left(1-(-1)^n\right)\,.$$

The final conclusion is that the solution is

$$T(x,y) = -\frac{16}{\kappa\,\pi^2} \sum_{\substack{k=1 \\ \ell=1}}^{\infty} \frac{1}{(2k-1)(2\ell-1)\left(\left(\frac{(2k-1)\pi}{a}\right)^2 + \left(\frac{(2\ell-1)\pi}{b}\right)^2 \right)} \sin\left(\frac{(2k-1)\pi x}{a} \right) \sin\left(\frac{(2\ell-1)\pi y}{b} \right)\,.$$

11.4.7. BCs $Y_m(0)=Y_m(b)=0$ give eigenfunctions $Y_m(y)=\sin\left(\frac{m\pi y}{b}\right)$, for $m=1,2,\dots$. BCs $Z_n'(0)=Z_n'(c)=0$ give eigenfunctions $Z_n(z)=\cos\left(\frac{n\pi z}{c}\right)$, for $n=0,1,2,\dots$. Substitute product functions $\phi_{m,n}(y,z)=Y_m(y)Z_n(z)=\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right)$, for $m=1,2,...;n=0,1,2,...$, into the eigenvalue problem's PDE to get

$$\lambda\,\phi_{m,n}(y,z)=-\Delta\phi_{m,n}(y,z)=-\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi_{m,n}(y,z)=-Z_n(z)\frac{\partial^2 Y_m}{\partial y^2}-Y_m(y)\frac{\partial^2 Z_n}{\partial z^2}$$

$$=\left(\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{c}\right)^2\right)\phi_{m,n}(y,z)\,.$$

So, the eigenvalues are $\lambda_{m,n}=\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{c}\right)^2$, $m=1,2,...;n=0,1,2,...$, with corresponding eigenfunctions $\phi_{m,n}(y,z)=Y_m(y)Z_n(z)=\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right)$.

Substitute into the PDE product solutions of the form $T(x,y,z)=X_{m,n}(x)\phi_{m,n}(y,z)$ to get

$$\phi_{m,n}(y,z)X''_{m,n}(x)=\frac{\partial T}{\partial x^2}=-\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=-X_{m,n}(x)\Delta\phi_{m,n}(y,z)=X_{m,n}(x)\lambda_{m,n}\,\phi_{m,n}(y,z)\,.$$

Because $\phi_{m,n}(y,z)$ is not identically zero, divide through by it to get $X''_{m,n}(x)=\lambda_{m,n}X_{m,n}(x)$, that is,

$$X''_{m,n}(x)+\lambda_{m,n}X_{m,n}(x)=0\,.$$

The remaining BCs are $T(0,y,z)=0, T(a,y,z)=f(y,z),\ 0<y<b,\ 0<z<c$, which suggests writing the general solution of the PDE+four homogeneous BCs in the form

$$T(x,y,z)=\sum_{m=1}^{\infty}\left(a_{m,0}\sinh\left(\frac{m\pi}{b}(a-x)\right)+b_{m,0}\sinh\left(\frac{m\pi x}{b}\right)\right)\sin\left(\frac{m\pi y}{b}\right)$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(a_{m,n}\sinh\left(\sqrt{\lambda_{m,n}}\,(a-x)\right)+b_{m,n}\sinh\left(\sqrt{\lambda_{m,n}}\;x\right)\right)\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right).$$

Substitute that into the two remaining BCs to get

$$0=T(0,y,z)=\sum_{m=1}^{\infty}a_{m,0}\sinh\left(\frac{m\pi a}{b}\right)\sin\left(\frac{m\pi y}{b}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}\sinh\left(\sqrt{\lambda_{m,n}}\;a\right)\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right),$$

hence $a_{m,n}=0$ for all m,n, and

$$f(y,z)=T(a,y,z)=\sum_{m=1}^{\infty}b_{0,n}\sinh\left(\frac{m\pi a}{b}\right)\sin\left(\frac{m\pi y}{b}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}b_{m,n}\sinh\left(\sqrt{\lambda_{m,n}}\;a\right)\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right),$$

hence for $m=1,2,...,$

$$b_{m,0}\sinh\frac{m\pi a}{b}=\frac{2}{bc}\int_0^b\int_0^c f(y,z)\sin\left(\frac{m\pi y}{b}\right)dz\,dy$$

and

$$b_{m,n}\sinh\left(\sqrt{\lambda_{m,n}}\,a\right)=\frac{4}{bc}\int_0^b\int_0^c f(y,z)\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right)dz\,dy\,.$$

With where $\lambda_{m,n}=\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{c}\right)^2$, the solution of the problem is

$$T(x,y,z)=\frac{2}{bc}\sum_{m=1}^{\infty}\frac{1}{\sinh\left(\frac{m\pi a}{b}\right)}\left(\int_0^b\int_0^c f(y,z)\sin\left(\frac{m\pi y}{b}\right)dz\,dy\right)\sinh\left(\frac{m\pi x}{b}\right)\sin\left(\frac{m\pi y}{b}\right)$$

$$+\frac{4}{bc}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\sinh\left(\sqrt{\lambda_{m,n}}\,a\right)}\left(\int_0^b\int_0^c f(y,z)\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right)dzdy\right)\cdot$$

$$\cdot \sinh\left(\sqrt{\lambda_{m,n}}\,x\right)\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right).$$

11.4.9. Write the solution in the form $u(x,y) = v(y) + w(x,y)$, where first we find $v(y)$, the solution of the problem $-1 + 6\cos 4y = v''(y)$, $0 < y < \frac{\pi}{2}$ and then let $w(x,y) = u(x,y) - v(y)$.

Solving for v, we integrate twice the ODE $-1 + 6\cos 4y = v''(y)$ to get

$$v(y) = c_1 + c_2 y - \frac{1}{2}\,y^2 - \frac{3}{8}\,\cos 4y,$$

where c_1, c_2 are arbitrary constants. Substituting v into the two BCs $v(0) = v\left(\frac{\pi}{2}\right) = 2$ gives

$$\left\{\begin{array}{l} 2 = v(0) = c_1 - \frac{3}{8} \\ \\ 2 = v\left(\frac{\pi}{2}\right) = c_1 + c_2\cdot\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{3}{8}\cdot 1 \end{array}\right\}$$

This yields $c_1 = \frac{17}{8}$ and $c_2 = \frac{\pi}{4}$, so

$$v(y) = \frac{17}{8} + \frac{\pi}{4}\,y - \frac{1}{2}\,y^2 - \frac{3}{8}\,\cos 4y.$$

The problem that $w(x,y) = u(x,y) - v(y)$ should satisfy is, after some detailed calculations,

$$\left\{\begin{array}{l} 0 = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2},\ 0 < x < \pi,\ 0 < y < \frac{\pi}{2} \\ \\ w(0,y) = w(\pi,y) = -\frac{17}{8} - \frac{\pi}{4}\,y + \frac{1}{2}\,y^2 + \frac{3}{8}\,\cos 4y,\ \ 0 < y < \frac{\pi}{2} \\ \\ w(x,0) = w(x,\frac{\pi}{2}) = 0,\ \ 0 < x < \pi \end{array}\right\}.$$

Similar to work in Example 11.9 but using also clairvoyance, the general solution of the homogeneous PDE and the two homogeneous BCs is

$$w(x,y) = \sum_{n=1}^{\infty} \sin(2ny)\left(a_n \sinh\left(2n\pi(\pi - x)\right) + b_n \sinh(2n\pi x)\right).$$

The third BC requires

$$-\frac{17}{8} - \frac{\pi}{4}\,y + \frac{1}{2}\,y^2 + \frac{3}{8}\,\cos 4y = w(0,y) = \sum_{n=1}^{\infty} \sin(2ny) a_n \sinh\left(2n\pi^2\right).$$

Orthogonality implies that

$$a_n \sinh\left(2n\pi^2\right) = \int_0^{\pi/2}\left(-\frac{17}{8} - \frac{\pi}{4}\,y + \frac{1}{2}\,y^2 + \frac{3}{8}\,\cos 4y\right)\sin(2ny)\,dy.$$

Using Mathematica™, we calculated separately that

$$a_2 = \int_0^{\pi/2}\left(-\frac{17}{8} - \frac{\pi}{4}\,y + \frac{1}{2}\,y^2 + \frac{3}{8}\,\cos 4y\right)\sin(4y)\,dy = ... = 0,$$

and, for $n \neq 2$, for $n = even$, $a_n = 0$; however, for $n = (2k-1)$,

$$a_{2k-1}\sinh\left(2(2k-1)\pi^2\right) = \frac{(-1)^{k-1}}{8(2k-1)^2}\left(\pi\cos\left(\frac{(2k-1)\pi}{2}\right) + \frac{2(-1)^{k-1}(4 + 33(2k-1)^2 - 7(2k-1)^4}{(2k-1)(-4 + (2k-1)^2)}\right).$$

Similar to work on the third BC, the last BC requires

$$-\frac{17}{8} - \frac{\pi}{4}\,y + \frac{1}{2}\,y^2 + \frac{3}{8}\,\cos 4y = w(2\pi,y) = \sum_{n=1}^{\infty} \sin(2ny) b_n \sinh\left(2n\pi^2\right).$$

Orthogonality implies that

$$b_n \sinh\left(2n\pi^2\right) = \int_0^{\pi/2}\left(-\frac{17}{8} - \frac{\pi}{4}\,y + \frac{1}{2}\,y^2 + \frac{3}{8}\,\cos 4y\right)\sin(2ny)\,dy,$$

the latter being the same integral as was calculated for work on the third BC.

The solution of the original problem is

$$
\begin{aligned}
u(x,y) &= v(y) + w(x,y)\\
&= \frac{17}{8} + \frac{\pi}{4}\,y - \frac{1}{2}\,y^2 - \frac{3}{8}\,\cos 4y\\
&+ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{8(2k-1)^2 \sinh\big(2(2k-1)\pi^2\big)} \left(\pi\cos\Big(\frac{(2k-1)\pi}{2}\Big) + \frac{2(-1)^{k-1}(4+33(2k-1)^2-7(2k-1)^4)}{(2k-1)(-4+(2k-1)^2)}\right)\cdot\\
&\qquad\cdot\Big(\sinh\big(2(2k-1)\pi(\pi-x)\big) + \sinh(2(2k-1)\pi x)\Big)\sin(2(2k-1)y)
\end{aligned}
$$

Section 11.5

11.5.1. From the work in Example 11.16 in Section 11.5, the general solution of the PDE and BCs (11.52) and (11.54) in Section 11.5 is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) r^n,$$

where

$$|\theta| = f(\theta) = u(a,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a^n a_n \cos n\theta + a^n b_n \sin n\theta). \ -\pi < \theta < \pi\,.$$

The function $f(\theta)$ is even, so $b_n = 0$ for $n = 1, 2, ...$ and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\theta|\, d\theta = \frac{2}{\pi}\int_0^{\pi} \theta\, d\theta = \frac{2}{\pi}\left[\frac{1}{2}\theta^2\right]_0^{\pi} = \pi$$

and

$$a^n a_n = \frac{2}{\pi}\int_0^{\pi} |\theta| \cos n\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi} \theta \cos n\theta\, d\theta = \frac{2}{\pi}\left[\frac{\theta \sin n\theta}{n} + \frac{\cos n\theta}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{0-0}{n} + \frac{(-1)^n - 1}{n^2}\right]$$

$$= -\frac{2}{n^2\pi}\left(1 - (-1)^n\right).$$

So, $a_n = 0$ for $n = even$. The solution to the whole problem is

$$u(r,\theta) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\left(\frac{r}{a}\right)^{2k-1}\cos\big((2k-1)\theta\big).$$

11.5.3. Because the potentials $V(a,\theta) = V_a$ and $V(b,\theta) = V_b$ do not depend on θ, our solution can be $V = V(r)$, depending only on r. So, Laplace's equation reduces to the ODE $0 = r\dfrac{d}{dr}\left[r\dfrac{dV}{dr}\right] = r^2V'' + rV'$. This Cauchy-Euler equation has general solution

$$V(r) = c_1 + c_2 \ln r\,.$$

The BCs give the system of equations

$$\begin{bmatrix} 1 & \ln a \\ 1 & \ln b \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} V_a \\ V_b \end{bmatrix},$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & \ln a \\ 1 & \ln b \end{bmatrix}^{-1}\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \frac{1}{\ln b - \ln a}\begin{bmatrix} \ln b & -\ln a \\ -1 & 1 \end{bmatrix}\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \frac{1}{\ln(b/a)}\begin{bmatrix} V_a \ln b - V_b \ln a \\ V_b - V_a \end{bmatrix}$$

The solution to the problem is

$$V(r,\theta) = \frac{1}{\ln(b/a)}\left(V_a \ln b - V_b \ln a + (V_b - V_a)\ln r\right).$$

This can be rewritten as

$$V(r,\theta) = \frac{1}{\ln(b/a)}\left(V_a(\ln b - \ln r) + V_b(\ln r - \ln a)\right) = \frac{1}{\ln(b/a)}\left(V_a \ln\left(\frac{b}{r}\right) + V_b \ln\left(\frac{r}{a}\right)\right).$$

11.5.5. (a) The ODE-BVP using the separated homogeneous BCs is

$$\left\{\begin{array}{c} \Theta''(\theta) + \lambda\Theta(\theta) = 0, \\ \\ \Theta\left(-\frac{\pi}{2}\right) = \Theta\left(\frac{\pi}{2}\right) = 0 \end{array}\right\}$$

has eigenvalues only for $\lambda = \omega^2 > 0$. The general solution of the ODE is

$$\Theta = c_1 \cos\omega\theta + c_2 \sin\omega\theta,$$

so the BCs require

$$\begin{bmatrix} \cos\left(-\omega\frac{\pi}{2}\right) & \sin\left(-\omega\frac{\pi}{2}\right) \\ \\ \cos\left(\omega\frac{\pi}{2}\right) & \sin\left(\omega\frac{\pi}{2}\right) \end{bmatrix}\begin{bmatrix} c_1 \\ \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}.$$

Because

$$\begin{vmatrix} \cos\left(-\frac{\pi\omega}{2}\right) & \sin\left(-\frac{\pi\omega}{2}\right) \\ \cos\left(\frac{\pi\omega}{2}\right) & \sin\left(\frac{\pi\omega}{2}\right) \end{vmatrix} = \sin\left(\frac{\pi\omega}{2}\right)\cos\left(-\frac{\pi\omega}{2}\right) - \cos\left(\frac{\pi\omega}{2}\right)\sin\left(-\frac{\pi\omega}{2}\right) = \sin\left(\frac{\pi\omega}{2} - \left(-\frac{\pi\omega}{2}\right)\right) = \sin\omega\pi,$$

the eigenvalues are $\lambda_n = \omega_n^2 = n^2$. Using the adjugate matrix method, as in Examples 9.32 and 9.33 in Section 9.6, the corresponding eigenfunctions are

$$\Theta_n(\theta) = \sin\left(n\,\frac{\pi}{2}\right)\cos n\theta - \cos\left(n\,\frac{\pi}{2}\right)\sin n\theta\,.$$

For $n = even = 2k$, eigenfunctions are

$$\Theta_{2k}(\theta) = \sin\left(2k\,\frac{\pi}{2}\right)\cos 2k\theta - \cos\left(2k\,\frac{\pi}{2}\right)\sin 2k\theta = 0\cdot\cos 2k\theta - (\pm 1)\sin 2k\theta\,.$$

For simplicity, we can use $\Theta_{2k}(\theta) = \sin 2k\theta$.

For $n = odd = 2k-1$, eigenfunctions are

$$\Theta_{2k-1}(\theta) = \sin\left((2k-1)\,\frac{\pi}{2}\right)\cos\big((2k-1)\theta\big) - \cos\left((2k-1)\,\frac{\pi}{2}\right)\sin\big((2k-1)\theta\big)$$
$$= (\pm 1)\cdot\cos\big((2k-1)\theta\big) - 0\cdot\sin\big((2k-1)\theta\big).$$

For simplicity, we can use $\Theta_{2k-1}(\theta) = \cos\big((2k-1)\theta\big)$.

Substitute the product solutions $u_n(r,\theta) = R_n(r)\Theta_n(\theta)$ into the PDE, in the form

$$(\star)\quad r\frac{\partial u}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{\partial^2 u}{\partial\theta^2} = 0.$$

This gives that $R_n(r) = 0$ should satisfy $r(rR_n')' - n^2R = 0$. This Cauchy-Euler equation has solutions $R_n(r) = c_1r^n + c_2r^{-n}$, but the physical BC $|u(0^+,\theta)| < \infty$ implies $c_2 = 0$. So, the general solution of the PDE and all of the given BCs is

$$u(r,\theta) = \sum_{n=1}^{\infty} c_nR_n(r)\Theta_n(\theta) = \sum_{k=1}^{\infty} a_kR_{2k}(r)\Theta_{2k}(\theta) + \sum_{k=1}^{\infty} b_kR_{2k-1}(r)\Theta_{2k-1}(\theta)$$
$$= \sum_{k=1}^{\infty}\left(a_kr^{2k}\sin(2k\theta) + b_kr^{2k-1}\cos\big((2k-1)\theta\big)\right).$$

(b) To satisfy the final BC, we need

$$|\theta| = u(a,\theta) = \sum_{k=1}^{\infty}\left(a_ka^{2k}\sin(2k\theta) + b_ka^{2k-1}\cos\big((2k-1)\theta\big)\right).$$

Using orthogonality of the eigenfunctions $\{\Theta_n(\theta)\}$ as in Theorem 9.15 in Section 9.6, along with oddness or evenness of integrands, we have that

$$a_ka^{2k} = \frac{\displaystyle\int_{-\pi/2}^{\pi/2} |\theta|\cdot\sin(2k\theta)\,d\theta}{\displaystyle\int_{-\pi/2}^{\pi/2} \big(\sin(2k\theta)\big)^2\,d\theta} = 0$$

and

$$b_ka^{2k-1} = \frac{\displaystyle\int_{-\pi/2}^{\pi/2} |\theta|\cdot\cos\big((2k-1)\theta\big)d\theta}{\displaystyle\int_{-\pi/2}^{\pi/2} \Big(\cos\big((2k-1)\theta\big)\Big)^2\,d\theta} = \frac{2\cdot\displaystyle\int_0^{\pi/2} \theta\cdot\cos\big((2k-1)\theta\big)d\theta}{\frac{1}{2}\,\pi}$$

$$= \frac{4}{\pi}\left[\frac{\theta\sin\big((2k-1)\theta\big)}{2k-1} + \frac{\cos\big((2k-1)\theta\big)}{(2k-1)^2}\right]_0^{\pi/2} = \frac{4}{\pi}\left(\frac{\frac{\pi}{2}\sin\big((k-\frac{1}{2})\pi\big) - 0}{2k-1} + \frac{0-1}{(2k-1)^2}\right)$$

$$= \frac{4}{\pi}\left(\frac{\frac{\pi}{2}(-1)^{k-1}}{2k-1} - \frac{1}{(2k-1)^2}\right).$$

The solution to the problem is

$$u(r,\theta) = \sum_{k=1}^{\infty} \left(\frac{2}{2k-1}\,(-1)^{k-1} - \frac{4}{\pi(2k-1)^2} \right) \cdot \left(\frac{r}{a}\right)^{2k-1} \cdot \cos\big((2k-1)\theta\big).$$

11.5.7. The equilibrium temperature has $\dfrac{\partial T}{\partial t} \equiv 0$, so the given PDE reduces to

$$(PDE)\ 0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\,\frac{\partial T}{\partial r}\right] + \frac{\partial^2 T}{\partial z^2},$$

along with the physical BC $|\,T(0^+,z)\,| < \infty$. Similar to work in Example 11.19 in Section 11.5, we will try product solutions $T(r,z) = R(r)Z(z)$, where $R(r)$ solves the ODE-BVP eigenvalue problem

$$\left\{ \begin{array}{c} \dfrac{1}{r}\dfrac{d}{dr}\left[r\,\dfrac{dR}{dr}\right] + \lambda R(r) = 0, \\ \\ |\,R(0^+)\,| < \infty,\ R'(a) = 0 \end{array} \right\}$$

The ODE can be rewritten as

$$r\,\frac{d}{dr}\left[r\,\frac{dR}{dr}\right] + r^2\lambda R(r) = 0, 0 < r < a,$$

that is,

$$(\star)\quad r^2R''(r) + rR'(r) + \lambda r^2 R(r) = 0, 0 < r < a,$$

which is a form of Bessel's equation of order 0. For $\lambda > 0$, the solutions of $(\star)$ are given by $R(r) = c_1 J_0(\sqrt{\lambda}\ r) + c_2 Y_0(\sqrt{\lambda}\ r)$, $0 < r < a$. But, $\lim_{r\to 0^+} |Y_0(\sqrt{\lambda}\ r)| = \infty$, so, the mathematical/physical BC implies

$$R(r) = c_1 J_0(\sqrt{\lambda}\ r),\ \ 0 < r < a\,.$$

The second, physical BC gives a characteristic equation $0 = \dfrac{d}{dr}\Big[\, J_0(\sqrt{\lambda}\,r)\,\Big]\Big|_{r=a}$. Using the hint, this gives $0 = \sqrt{\lambda}\,J_1(\sqrt{\lambda}\,a)$, that is,

$$(\star\star)\quad J_1(\sqrt{\lambda}\,a) = 0\,.$$

Let $\gamma_{n,1}$, $n = 1, 2, ...$ denote the successive zeros of the function $J_1(x)$. By $(\star\star)$, the eigenvalues of $(\star)$ are

$$\lambda_n = \left(\frac{\gamma_{n,1}}{a}\right)^2,\ n = 1, 2, \ldots$$

Substitute into the PDE product solutions of the form $T(r,z) = R_n(r)Z_n(z)$ to get

$$0 = Z_n(z)\cdot \frac{1}{r}\frac{d}{dr}\left[r\,\frac{dR_n}{dr}\right] + R_n(r)Z_n''(z) = -\lambda_n\, Z_n(z)R_n(r) + R_n(r)Z_n''(z),$$

where we denote $Z_n'(z) = \dfrac{dZ_n}{dz}$. Dividing through by $R_n(r)$ gives

$$Z_n''(z) - \lambda_n\, Z_n = 0\,.$$

Clairvoyance tells us to write the general solution of the PDE, homogeneous BCs, and the physical BC $|\,T(0^+,z)\,| < \infty$ as

$$T(r,t) = \sum_{n=1}^{\infty} J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)\left(a_n \sinh\left(\frac{\gamma_{n,1}}{a}\,(L-z)\right) + b_n \sinh\left(\frac{\gamma_{n,1}}{a}\,z\right)\right).$$

We substitute this into the BCs to get

$$f(r) = T(r,0) = \sum_{n=1}^{\infty} \sinh\left(\frac{\gamma_{n,1}\,L}{a}\right) a_n J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)$$

and

$$g(r) = T(r,0) = \sum_{n=1}^{\infty} \sinh\left(\frac{\gamma_{n,1}\,L}{a}\right) b_n J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)\ .$$

Orthogonality as mentioned in (9.71) in Section 9.6 is

$$\int_0^a J_0\left(\frac{\gamma_{m,1}\,r}{a}\right) J_0\left(\frac{\gamma_{n,1}\,r}{a}\right) r\,dr = \left\{\begin{array}{cc} 0, & n \neq m \\ M_m, & n = m \end{array}\right\},$$

where

$$M_m \triangleq \int_0^a \left(J_0\left(\frac{\gamma_{m,1}\,r}{a}\right)\right)^2 r\,dr\,.$$

So,

$$\sinh\left(\frac{\gamma_{n,1}\,L}{a}\right) a_n = M_n^{-1} \int_0^a f(r)\, J_0\left(\frac{\gamma_{n,1}\,r}{a}\right) r\,dr$$

and a similar result for b_n. The solution to the problem is

$$T(r,z) = \sum_{n=1}^{\infty} \frac{1}{M_n \sinh\left(\frac{\gamma_{n,1}\,L}{a}\right)} J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)\cdot$$

$$\cdot\left(\left(\int_0^a f(r)\, J_0\left(\frac{\gamma_{n,1}\,r}{a}\right) r\,dr\right)\sinh\left(\frac{\gamma_{n,1}}{a}(L-z)\right) + \left(\int_0^a g(r)\, J_0\left(\frac{\gamma_{n,1}\,r}{a}\right) r\,dr\right)\sinh\left(\frac{\gamma_{n,1}}{a}\,z\right)\right),$$

where $M_n \triangleq \displaystyle\int_0^a \left(J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)\right)^2 r\,dr\,.$

11.5.9. (a) The heat equation without sources or sinks but κ is not constant is a special case of (10.28) in Section 10.2:

$$\frac{\partial}{\partial t}[c\varrho u] = \boldsymbol{\nabla} \bullet (\kappa \boldsymbol{\nabla} u)\,.$$

An equilibrium temperature T has $\dfrac{\partial T}{\partial t} \equiv 0$. Substituting in $\kappa = c\varrho r$, where c and ϱ are constants gives, the PDE in polar coordinates

$$0 = \boldsymbol{\nabla} \bullet (r\boldsymbol{\nabla} T) = \boldsymbol{\nabla} \bullet \left(r\left(\frac{\partial T}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\,\hat{\mathbf{e}}_\theta\right)\right) = \boldsymbol{\nabla} \bullet \left(r\,\frac{\partial T}{\partial r}\,\hat{\mathbf{e}}_r + \frac{\partial T}{\partial \theta}\,\hat{\mathbf{e}}_\theta\right) = \frac{1}{r}\frac{\partial}{\partial r}\Big[r\left(r\,\frac{\partial T}{\partial r}\right)\Big] + \frac{1}{r}\frac{\partial}{\partial \theta}\Big[\frac{\partial T}{\partial \theta}\Big],$$

that is,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r^2\,\frac{\partial T}{\partial r}\right] + \frac{1}{r}\frac{\partial^2 T}{\partial \theta^2}\,.$$

(b) Rewrite the PDE in part (a) as $0 = \dfrac{\partial}{\partial r}\left[r^2\,\dfrac{\partial T}{\partial r}\right] + \dfrac{\partial^2 T}{\partial \theta^2}$. The periodic boundary conditions give us the usual ODE-BVP

$$\left\{\begin{array}{c} \Theta''(\theta) + \lambda\Theta(\theta) = 0, \\ \\ \Theta(\pi) = \Theta(-\pi),\ \Theta'(\pi) = \Theta'(-\pi) \end{array}\right\}$$

has eigenvalues and corresponding eigenfunctions

$$\lambda_0 = 0:\ \Theta_0(\theta) = 1$$

$$\lambda_n = n^2:\ \Theta_n(\theta):\ \{\cos n\theta, \sin n\theta\};$$

the latter means that both $\cos n\theta$ and $\sin n\theta$ are eigenfunctions corresponding to the same eigenvalue λ_n.

Corresponding to eigenvalues λ_0 and λ_n we get product solutions

$$T_0(r,\theta) = R_0(r),\ T_{n,c}(r,\theta) = R_{n,c}(r)\cos n\theta,\ \text{and}\ T_{n,s}(r,\theta) = R_{n,s}(r)\sin n\theta\,.$$

Substituting $T_0(r,\theta)$ into the PDE gives, for $n = 1, 2, ...,$

$$0 = \frac{d}{dr}\left[r^2\,\frac{dR_0}{dr}\right] + 0 = r^2\frac{d^2R_0}{dr^2} + 2r\,\frac{dR_0}{dr},$$

a Cauchy-Euler ODE. Substituting in $R_0(r) = r^m$ gives characteristic equation $0 = m(m-1) + 2m = m^2 + m = m(m+1)$, so the roots are $m = 0, -1$. This gives

$$R_0(r) = c_1 + c_2 r^{-1}\,.$$

Substituting $T_{n,c}(r,\theta)$ into the PDE gives

$$0=\cos n\theta\left(\frac{d}{dr}\left[r^2\,\frac{dR_{n,c}}{dr}\right]-n^2R_{n,c}\right)=\cos n\theta\left(r^2\,\frac{d^2R_{n,c}}{dr^2}+2r\,\frac{dR_{n,c}}{dr}-n^2R_{n,c}\right),$$

hence we want

$$0=r^2\frac{d^2R_{n,c}}{dr^2}+2r\,\frac{dR_{n,c}}{dr}-n^2R_{n,c},$$

a Cauchy-Euler ODE. Substituting in $R_{n,c}(r)=r^m$ gives characteristic equation $0=m(m-1)+2m-n^2=m^2+m-n^2$, so the roots are $m=\frac{1}{2}\left(-1\pm\sqrt{1+4n^2}\right)$. This gives

$$R_{n,c}(r)=c_1\,r^{\frac{1}{2}\left(-1+\sqrt{1+4n^2}\right)}+c_2\,r^{\frac{1}{2}\left(-1-\sqrt{1+4n^2}\right)}.$$

The work for $u_{n,s}$ is similar.

At this point, the boundedness of $|T(0^+,\theta)|$ implies that we should "throw out" the solutions with powers r^{-1} or $r^{\frac{1}{2}\left(-1-\sqrt{1+4n^2}\right)}$. So, the general solution of the PDE is

$$T(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)r^{-\frac{1}{2}+\sqrt{n^2+\frac{1}{4}}}.$$

The last BC to satisfy is

$$f(\theta)=T(a,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)a^{-\frac{1}{2}+\sqrt{n^2+\frac{1}{4}}}.$$

Fourier analysis yields

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\,d\theta,$$

$$a^{-\frac{1}{2}+\sqrt{n^2+\frac{1}{4}}}\;a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta,$$

and

$$a^{-\frac{1}{2}+\sqrt{n^2+\frac{1}{4}}}\;b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta\,.$$

The solution to the whole problem is

$$T(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\,d\theta+\frac{1}{\pi}\sum_{n=1}^{\infty}\Big(\big(\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta\big)\cos n\theta+\big(\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta\big)\sin n\theta\Big)\left(\frac{r}{a}\right)^{-\frac{1}{2}+\sqrt{n^2+\frac{1}{4}}}.$$

11.5.11. The ODE-BVP eigenvalue problem (11.63) in Section 11.5 implies

$$\left\{\begin{array}{c}\dfrac{1}{r}\dfrac{d}{dr}\left[r\,\dfrac{dR}{dr}\right]+\lambda R(r)=0,\\[2ex] |\,R(0^+)\,|<\infty,\;\;R(2)=0\end{array}\right\}$$

has eigenvalues $\lambda_n=\left(\frac{\gamma_{n,0}}{2}\right)^2$ and corresponding eigenfunctions $R_n(r)=J_0\left(\frac{\gamma_{n,0}\,r}{2}\right)$. Substituting into the PDE product solutions $T_n(r,z)=J_0\left(\frac{\gamma_{n,0}\,r}{2}\right)Z(z)$ implies

$$Z''-\left(\frac{\gamma_{n,0}}{2}\right)^2Z=0\,.$$

Using clairvoyance, the general solution of the PDE and the BCs along the rod can be written as

$$T(r,z)=\sum_{n=1}^{\infty}J_0\left(\frac{\gamma_{n,0}\,r}{2}\right)\left(a_n\sinh\left(\frac{\gamma_{n,0}}{2}(4-z)\right)+b_n\sinh\left(\frac{\gamma_{n,0}}{2}\,z\right)\right).$$

Substitute into the BCs on the ends of the rod to get

$$50=T(r,0)=\sum_{n=1}^{\infty}J_0\left(\frac{\gamma_{n,0}\,r}{2}\right)a_n\sinh\left(\frac{4\gamma_{n,0}}{2}\right)$$

and

$$0 = \sum_{n=1}^{\infty} J_0\Big(\frac{\gamma_{n,0}\, r}{2}\Big) b_n \sinh\Big(\frac{4\gamma_{n,0}}{2}\Big)\,.$$

The latter implies $b_n = 0$ for all n, and the former implies

$$a_n \sinh\Big(\frac{4\gamma_{n,0}}{2}\Big) = \int_0^2 50 J_0\Big(\frac{\gamma_{n,0}\, r}{2}\Big) r\, dr = 50 \int_0^{\gamma_{n,0}} J_0(x)\, \frac{2x}{\gamma_{n,0}}\, \frac{2}{\gamma_{n,0}}\, dx = \frac{200}{\gamma_{n,0}^2}\Big[xJ_1(x)\Big]_0^{\gamma_{n,0}} = \frac{200}{\gamma_{n,0}} J_1(\gamma_{n,0}),$$

using the given fact about $\int J_0(x)\, x\, dx$.

The solution of the whole problem is

$$T(r,z) = \sum_{n=1}^{\infty} \frac{200 J_1(\gamma_{n,0})}{\gamma_{n,0}} \cdot J_0\Big(\frac{\gamma_{n,0}\, r}{2}\Big) \cdot \frac{\sinh\big(\frac{\gamma_{n,0}}{2}(4-z)\big)}{\sinh\big(\frac{4\gamma_{n,0}}{2}\big)}\,.$$

11.5.13. As in Example 11.18 in Section 11.5 and (11.55) in Section 11.5, the general solution of the PDE is

$$u(r,\theta) = \frac{a_0 + c_0 \ln r}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta) r^{-n}\,.$$

So,

$$\frac{\partial u}{\partial r} = \frac{c_0}{2}\,\frac{1}{r} + \sum_{n=1}^{\infty} n(a_n \cos n\theta + b_n \sin n\theta) r^{n-1} - \sum_{n=1}^{\infty} n(c_n \cos n\theta + d_n \sin n\theta) r^{-n-1}\,.$$

The homogeneous, flux BC

$$0 = b\frac{\partial u}{\partial r}(b,\theta) - a\frac{\partial u}{\partial r}(a,\theta),\ -\pi < \theta < \pi$$

implies

$$0 = \frac{c_0}{2}\left(b\,\frac{1}{b} - a\,\frac{1}{a}\right) + \sum_{n=1}^{\infty} n(a_n \cos n\theta + b_n \sin n\theta)(b^n - a^n) - \sum_{n=1}^{\infty} n(c_n \cos n\theta + d_n \sin n\theta)(b^{-n} - a^{-n})$$

Fourier analysis implies for $n = 1, 2, ...,$

$$0 = a_n(b^n - a^n) - c_n(b^{-n} - a^{-n}),$$

hence

$$c_n = \frac{b^n - a^n}{b^{-n} - a^{-n}}\, a_n,$$

and

$$0 = b_n(b^n - a^n) - d_n(b^{-n} - a^{-n}),$$

hence

$$d_n = \frac{b^n - a^n}{b^{-n} - a^{-n}}\, b_n,$$

So, the general solution of the PDE and the flux BC is

$$u(r,\theta) = \frac{a_0 + c_0 \ln r}{2} + \sum_{n=1}^{\infty} a_n \left(r^n + \frac{b^n - a^n}{b^{-n} - a^{-n}}\, r^{-n}\right) \cos n\theta + \sum_{n=1}^{\infty} b_n \left(r^n + \frac{b^n - a^n}{b^{-n} - a^{-n}}\, r^{-n}\right) \sin n\theta\,.$$

The final BC requires

$$f(\theta) = u(a,\theta) = \frac{a_0 + c_0 \ln a}{2} + \sum_{n=1}^{\infty} a_n \left(a^n + \frac{b^n - a^n}{b^{-n} - a^{-n}}\, a^{-n}\right) \cos n\theta + \sum_{n=1}^{\infty} b_n \left(a^n + \frac{b^n - a^n}{b^{-n} - a^{-n}}\, a^{-n}\right) \sin n\theta\,.$$

Fourier analysis gives

$$a_0 + c_0 \ln a = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta,$$

hence

$$a_0 = -c_0 \ln a + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta,$$

and

$$a_n\left(a^n+\frac{b^n-a^n}{b^{-n}-a^{-n}}\,a^{-n}\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta,$$

and

$$b_n\left(a^n+\frac{b^n-a^n}{b^{-n}-a^{-n}}\,a^{-n}\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta\,.$$

We can rewrite

$$\left(a^n+\frac{b^n-a^n}{b^{-n}-a^{-n}}\,a^{-n}\right)=\frac{a^n(b^{-n}-a^{-n})+(b^n-a^n)a^{-n}}{b^{-n}-a^{-n}}=\frac{a^n(b^{-n}-a^{-n})+(b^n-a^n)a^{-n}}{b^{-n}-a^{-n}}\cdot\frac{a^nb^n}{a^nb^n}$$

$$=\frac{a^{2n}-a^nb^n+b^{2n}-a^nb^n}{a^n-b^n}=\frac{(a^n-b^n)^2}{a^n-b^n}=(a^n-b^n),$$

so, for example,

$$b_n=\frac{1}{\pi(a^n-b^n)}\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta\,.$$

Similarly, we can rewrite

$$\left(r^n+\frac{b^n-a^n}{b^{-n}-a^{-n}}\,r^{-n}\right)=\frac{r^n(b^{-n}-a^{-n})+(b^n-a^n)r^{-n}}{b^{-n}-a^{-n}}\cdot\frac{a^nb^nr^n}{a^nb^nr^n}=\frac{a^nr^{2n}-b^nr^{2n}+a^nb^n(b^n-a^n)}{(a^n-b^n)r^n}$$

$$=\frac{(a^n-b^n)(r^{2n}-a^nb^n)}{(a^n-b^n)r^n}=\frac{(r^{2n}-a^nb^n)}{r^n}=r^n-\frac{a^nb^n}{r^n}=b^n\left(\frac{r^n}{b^n}-\frac{a^n}{r^n}\right)=b^n\left(\left(\frac{r}{b}\right)^n-\left(\frac{a}{r}\right)^n\right)\,.$$

The solution of the whole problem is

$$u(r,\theta)=\frac{-c_0\ln a+\frac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(\theta)\,d\theta+c_0\ln r}{2}+$$

$$+\sum_{n=1}^{\infty}\frac{b^n}{\pi(a^n-b^n)}\cdot\left(\left(\frac{r}{b}\right)^n-\left(\frac{a}{r}\right)^n\right)\left(\left(\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta\right)\cos n\theta+\left(\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta\right)\sin n\theta\right)$$

that is,

$$u(r,\theta)=\frac{c_0\ln(r/a)+\frac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(\theta)\,d\theta}{2}$$

$$-\sum_{n=1}^{\infty}\frac{b^n}{\pi(b^n-a^n)}\cdot\left(\left(\frac{r}{b}\right)^n-\left(\frac{a}{r}\right)^n\right)\left(\left(\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta\right)\cos n\theta+\left(\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta\right)\sin n\theta\right),$$

where c_0 is an arbitrary constant.

Section 11.6

11.6.1. Define the open half cylinder $\mathcal{D} = \{\, (r,\theta,z) : 0 < \theta < \pi,\ 0 < r < a,\ 0 < z < H \,\}$. The physical cylinder is an open box (a.k.a. parallelepiped) in (r,θ,z) coordinates, so the BCs $T(r,0,z) = T(r,\pi,z) = 0$ and the BCs $|T(0^+,\theta,z| < \infty,\ T(a,\theta,z) = 0$, express pairs of "homogeneous" boundary conditions on parallel sides of that box. As in Example 11.20, this suggests separating both the $r-$ and $\theta-$dependence away from the $z-$dependence by finding eigenvalues of the (r,θ) *part* of the Laplacian:

$$\frac{1}{r}\frac{\partial}{\partial r}\Big[\, r\,\frac{\partial \phi}{\partial r}\,\Big] + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \lambda\phi = 0,$$

along with the boundary conditions

$$\left\{ \begin{array}{c} \phi(r,0) = \phi(r,\pi) = 0,\ 0 < r < a, \\ \\ |\,\phi(0^+,\theta)\,| < \infty \quad \text{and} \quad \phi(a,\theta) \equiv 0,\ 0 < \theta < \pi. \end{array} \right\}.$$

The first pair of BCs suggests looking for solutions in the form $\phi(r,\theta) = \sin n\theta R(r)$, for $n = 1,2,3,...$. We substitute this into the Laplacian eigenvalue PDE to get

$$\sin n\theta\,\frac{1}{r}\frac{d}{dr}\left[r\,\frac{dR_n}{dr}\right] + R_n(r)\frac{1}{r^2}\cdot(-n^2)\sin n\theta + \lambda \sin n\theta R_n(r) = 0,$$

hence $R_n(r)$ should satisfy the ODE

$$\frac{1}{r}\frac{d}{dr}\left[r\,\frac{dR_n}{dr}\right] - n^2\cdot\frac{1}{r^2}\,R_n + \lambda R_n = 0,$$

that is, the ODE

$$r^2 R_n'' + rR_n' + (-n^2 + \lambda r^2)R_n = 0,$$

plus the boundary conditions $|\,R_n(0^+)\,| < \infty$ and $R_n(a) = 0$. For $\lambda > 0$, the ODE is a form of Bessel's equation of order n. Similar to Example 11.20, the solutions are $R_n(r) = c_1 J_n(\sqrt{\lambda}\, r) + c_2 Y_n(\sqrt{\lambda}\, r), 0 < r < a$. When we apply the boundary condition $|\,R_n(0^+)\,| < \infty$ we conclude that $c_2 = 0$, because $|\,Y_n(0^+)\,| = \infty$. When we apply the last boundary condition by graphing $J_n(\sqrt{\lambda}\, a)$ versus $\lambda > 0$, we find there are infinitely many eigenvalues $\lambda_{m,n} = \frac{\gamma_{m,n}}{a}$, for $m = 1,2,....,\ n = 1,2,....$.

So, we get solutions

$$\phi_{m,n}(r,\theta) = J_n(\sqrt{\lambda_{m,n}}\; r)\sin n\theta,\ m = 1,2,...,\ n = 1,2,....$$

When we substitute product solutions $\phi_{m,n}(r,\theta)Z(z)$ into the original PDE, we get

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\,\frac{\partial}{\partial r}\phi_{m,n}(r,\theta)Z(z)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\big[\,\phi_{m,n}(r,\theta)Z(z)\,\big] + \frac{\partial^2}{\partial z^2}\big[\,\phi_{m,n}(r,\theta)Z(z)\,\big],$$

hence

$$0 = -\lambda_{m,n}\,\phi_{m,n}(r,\theta)Z(z) + \phi_{m,n}(r,\theta)Z''(z),$$

hence $Z(z)$ should satisfy the ODE

$$Z''(z) - \lambda_{m,n}Z(z) = 0\,.$$

Using clairvoyance regarding the remaining two boundary conditions, $T(r,\theta,0) = f(r,\theta),\ T(r,\theta,H) = 0$, for $n = 1,2,3,...$ we get product solutions

$$J_n(\sqrt{\lambda_{m,n}}\; r)\sinh(\sqrt{\lambda_{m,n}}(H-z))\sin n\theta \quad \text{and} \quad J_n(\sqrt{\lambda_{m,n}}\; r)\sinh(\sqrt{\lambda_{m,n}}\, z)\sin n\theta\,.$$

We can put all product solutions together to get the general solution of the PDE and the homogeneous BCs is in the form

$$T(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{m,n}}\; r)\sin n\theta\Big(a_{m,n}\sinh(\sqrt{\lambda_{m,n}}(H-z)) + b_{m,n}\sinh(\sqrt{\lambda_{m,n}}\, z)\Big)\,.$$

It is this general solution that we will substitute into the remaining two boundary conditions, $T(r,\theta,0) = f(r,\theta),\ T(r,\theta,H) = 0,\ 0 < \theta < \pi$. The first BC gives

$$f(r,\theta) = T(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sinh(\sqrt{\lambda_{m,n}}\, H)a_{m,n}J_n(\sqrt{\lambda_{m,n}}\; r)\sin n\theta,$$

As in Example 11.20, orthogonality of the eigenfunctions $\phi_{m,n}(r,\theta)$ implies that

$$a_{m,n} = \frac{2}{\pi \sinh(\sqrt{\lambda_{m,n}}\, H)} \cdot \frac{\displaystyle\int_0^\pi \int_0^a f(r,\theta)\sin(n\theta) J_n(\sqrt{\lambda_{m,n}}\ r) r\, dr\, d\theta}{\displaystyle\int_0^a (J_n(\sqrt{\lambda_{m,n}}\ r))^2 r\, dr}.$$

The second BC gives

$$0 = T(r,\theta,0) = \sum_{m=1}^\infty \sum_{n=1}^\infty \ \sinh(\sqrt{\lambda_{m,n}}\, H) b_{m,n} J_n(\sqrt{\lambda_{m,n}}\ r)\sin n\theta,$$

As in Example 11.20, orthogonality of the eigenfunctions $\phi_{m,n}(r,\theta)$ implies that $b_{m,n} = 0$, for all m, n.

The final conclusion is that the solution of the problem is

$$T(r,\theta,z) = \sum_{m=1}^\infty \sum_{n=1}^\infty \ \frac{2\sinh(\sqrt{\lambda_{m,n}}(H-z))}{\pi \sinh(\sqrt{\lambda_{m,n}}\, H)} \cdot \frac{\displaystyle\int_0^\pi \int_0^a f(r,\theta)\sin(n\theta) J_n(\sqrt{\lambda_{m,n}}\ r) r\, dr d\theta}{\displaystyle\int_0^a (J_n(\sqrt{\lambda_{m,n}}\ r))^2 r\, dr} J_n(\sqrt{\lambda_{m,n}}\ r)\sin(n\theta).$$

11.6.3. Define the open rectangle $\mathcal{D} = \{\,(r,\theta,z) : -\pi < \theta \le \pi,\ a < r < b\,\}$. The physical annulus is a rectangle in (r,θ,z) coordinates, so the BCs that $T(r,\theta,t)$ is periodic in θ with period 2π, and the BCs $T(a,\theta,t) = T(b,\theta,t) = 0$, express pairs of "homogeneous" boundary conditions on parallel sides of that rectangle.

As in Example 11.20, this suggests separating both the $r-$ and $\theta-$dependence away from the $t-$dependence by finding eigenvalues of the (r,θ) *part* of the Laplacian:

$$(\star) \quad \frac{1}{r}\frac{\partial}{\partial r}\Big[\, r\,\frac{\partial \phi}{\partial r}\,\Big] + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \lambda\phi = 0,$$

along with the boundary conditions

$$(\star\star) \quad \left\{ \begin{array}{c} \phi(r,-\pi^+) = \phi(r,\pi^-) \quad \text{and} \quad \dfrac{\partial \phi}{\partial \theta}(r,-\pi^+) = \dfrac{\partial \phi}{\partial \theta}(r,\pi^-), 0 < r < a, \\ \\ \phi(a,\theta) = \phi(b,\theta) \equiv 0, -\pi < \theta < \pi. \end{array} \right\}.$$

The first pair of BCs suggests looking for eigenfunctions in the forms

$$\phi_0(r,\theta) = R_0(r), \quad \phi_{n,c}(r,\theta) = R_{n,c}(r)\cos\theta, \quad \text{and} \quad \phi_{n,s}(r,\theta) = R_{n,s}(r)\sin\theta,\ n = 1,2,... .$$

Substituting the first form into $(\star)$ and multiplying though by r^2 gives the ODE

$$r^2 R_0 + r R_0 + \lambda r^2 R_0 = 0,\ a < r < b$$

whose solutions are $R_0 = c_1 J_0(\sqrt{\lambda}\ r) + c_2 Y_0(\sqrt{\lambda}\ r)$. Note that we need the Y_0 solution of Bessel's equation of order zero because $a > 0$. Substitute the solutions into the BCs to get

$$\left\{ \begin{array}{c} R_0(a) = c_1 J_0(\sqrt{\lambda}\ a) + c_2 Y_0(\sqrt{\lambda}\ a) = 0 \\ R_0(b) = c_1 J_0(\sqrt{\lambda}\ b) + c_2 Y_0(\sqrt{\lambda}\ b) = 0 \end{array} \right\},$$

which can be re-written as

$$\begin{bmatrix} J_0(\sqrt{\lambda}\ a) & Y_0(\sqrt{\lambda}\ a) \\ J_0(\sqrt{\lambda}\ b) & Y_0(\sqrt{\lambda}\ b) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

There is an eigenfunction if, and only if,

$$(\star\star\star) \quad 0 = \begin{vmatrix} J_0(\sqrt{\lambda}\ a) & Y_0(\sqrt{\lambda}\ a) \\ J_0(\sqrt{\lambda}\ b) & Y_0(\sqrt{\lambda}\ b) \end{vmatrix} = J_0(\sqrt{\lambda}\ a)Y_0(\sqrt{\lambda}\ b) - Y_0(\sqrt{\lambda}\ a)J_0(\sqrt{\lambda}\ b),$$

in which case the adjugate matrix method says that any nonzero column of the adjugate matrix,

$$\begin{bmatrix} Y_0(\sqrt{\lambda}\ b) & -Y_0(\sqrt{\lambda}\ a) \\ -J_0(\sqrt{\lambda}\ b) & J_0(\sqrt{\lambda}\ a) \end{bmatrix},$$

gives an eigenvector for $[c_1 \quad c_2]^T$.

We know from Theorem 9.6 in Section 9.3 that there are infinitely many eigenvalues $\lambda_{m,0}, m = 1, 2, ...$ for the ODE-BVP-Eigenvalue problem

$$\left\{ \begin{array}{c} r^2 R_0 + rR_0 + \lambda r^2 R_0 = 0,\ a < r < b \\ R_0(a) = R_0(b) = 0 \end{array} \right\}.$$

So, for $m = 1, 2, ...$, using the adjugate matrix method we get eigenfunctions

$$R_{m,0}(r) = -Y_0(\sqrt{\lambda_{m,0}}\ a)J_0(\sqrt{\lambda_{m,0}}\ r) + J_0(\sqrt{\lambda_{m,0}}\ a)Y_0(\sqrt{\lambda_{m,0}}\ r)\,.$$

Corresponding eigenfunctions of the Laplacian problem consisting of $(\star)$ and $(\star\star)$ are

$$\phi_{m,0}(r,\theta) = R_{m,0}(r) \cdot 1 = -Y_0(\sqrt{\lambda_{m,0}}\ a)J_0(\sqrt{\lambda_{m,0}}\ r) + J_0(\sqrt{\lambda_{m,0}}\ a)Y_0(\sqrt{\lambda_{m,0}}\ r)\,.$$

Substituting the second form of product solution, $\phi_{n,c}(r,\theta) = R(r)\cos\theta$, into $(\star)$ and multiplying though by r^2 gives the ODE

$$r^2 R + rR + (\lambda r^2 - n^2)R = 0,\ a < r < b$$

whose solutions are $R_{n,c} = c_1 J_n(\sqrt{\lambda}\ r) + c_2 Y_n(\sqrt{\lambda}\ r)$. Note that we need the Y_n solution of Bessel's equation of order zero because $a > 0$. Substitute the solutions into the BCs to get

$$\left\{ \begin{array}{c} R(a) = c_1 J_n(\sqrt{\lambda}\ a) + c_2 Y_n(\sqrt{\lambda}\ a) = 0 \\ R(b) = c_1 J_n(\sqrt{\lambda}\ b) + c_2 Y_n(\sqrt{\lambda}\ b) = 0 \end{array} \right\},$$

which can be re-written as

$$\begin{bmatrix} J_n(\sqrt{\lambda}\ a) & Y_n(\sqrt{\lambda}\ a) \\ J_n(\sqrt{\lambda}\ b) & Y_n(\sqrt{\lambda}\ b) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

There is an eigenfunction if, and only if,

$$(\star\star\star) \quad 0 = \begin{vmatrix} J_n(\sqrt{\lambda}\ a) & Y_n(\sqrt{\lambda}\ a) \\ J_n(\sqrt{\lambda}\ b) & Y_n(\sqrt{\lambda}\ b) \end{vmatrix} = J_n(\sqrt{\lambda}\ a)Y_n(\sqrt{\lambda}\ b) - Y_n(\sqrt{\lambda}\ a)J_n(\sqrt{\lambda}\ b),$$

in which case the adjugate matrix method says that any nonzero column of the adjugate matrix,

$$\begin{bmatrix} Y_n(\sqrt{\lambda}\ b) & -Y_n(\sqrt{\lambda}\ a) \\ -J_n(\sqrt{\lambda}\ b) & J_n(\sqrt{\lambda}\ a) \end{bmatrix},$$

gives an eigenvector for $[c_1 \quad c_2]^T$.

We know from Theorem 9.6 in Section 9.3 that there are infinitely many eigenvalues $\lambda_{m,n}, m = 1, 2, ...$ for the ODE-BVP-Eigenvalue problem

$$\left\{ \begin{array}{c} r^2 R + rR + (\lambda r^2 - n^2)R = 0,\ a < r < b \\ R(a) = R(b) = 0 \end{array} \right\}.$$

So, for $m = 1, 2, ...$, using the adjugate matrix method we get eigenfunctions

$$R_{m,n}(r) = -Y_n(\sqrt{\lambda_{m,n}}\ a)J_n(\sqrt{\lambda_{m,n}}\ r) + J_n(\sqrt{\lambda_{m,n}}\ a)Y_n(\sqrt{\lambda_{m,n}}\ r)\,.$$

The work for the third form of product solution, $\phi_{n,c}(r,\theta) = R(r)\sin\theta$, is analogous to that for the second form.

To summarize, the Laplacian eigenvalue problem consisting of $(\star)$ and $(\star\star)$ has additional eigenvalues $\lambda_{m,n}, m = 1, 2, ..., n = 1, 2, ...$ and corresponding eigenfunctions

$$\phi_{m,n,c}(r,\theta) = \Big(-Y_n(\sqrt{\lambda_{m,n}}\ a)J_n(\sqrt{\lambda_{m,n}}\ r) + J_n(\sqrt{\lambda_{m,n}}\ a)Y_n(\sqrt{\lambda_{m,n}}\ r)\Big)\cos n\theta,\ m = 1, 2, ...,\ n = 1, 2, ...$$

$$\phi_{m,n,s}(r,\theta) = \Big(-Y_n(\sqrt{\lambda_{m,n}}\ a)J_n(\sqrt{\lambda_{m,n}}\ r) + J_n(\sqrt{\lambda_{m,n}}\ a)Y_n(\sqrt{\lambda_{m,n}}\ r)\Big)\sin n\theta,\ m = 1, 2, ...,\ n = 1, 2, ...$$

For $n \geq 0$, $m \geq 1$, if we substitute $T(r,\theta,t) = G_{m,n}(t)R_{m,n}(r)\cos n\theta$ into the original PDE we get

$$\dot{G}_{m,n}(t)R_{m,n}(r)\cos n\theta = \frac{\partial T}{\partial t} = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right) = \alpha\left(-\lambda_{m,n}G_{m,n}(t)R_{m,n}(r)\cos n\theta\right),$$

hence

$$\dot{G}_{m,n}(t) = -\alpha\lambda_{m,n}G_{m,n}(t)\,.$$

The general solution of the PDE and the BCs is

$$T(r,\theta,t) = \sum_{m=1}^{\infty} \frac{a_{m,0}}{2} R_{m,0}(r) e^{-\alpha\lambda_{m,0}t} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{m,n}(r)\left(a_{m,n}\cos n\theta + b_{m,n}\sin n\theta\right) e^{-\alpha\lambda_{m,n}t}\,.$$

The initial condition (IC) requires

$$f(r,\theta)=T(r,\theta,0)= \sum_{m=1}^{\infty} \frac{a_{m,0}}{2} R_{m,0}(r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{m,n}(r)\left(a_{m,n}\cos n\theta + b_{m,n}\sin n\theta\right), a<r<b, -\pi<\theta<\pi\,.$$

Using orthogonality, we find that the coefficients are given by

$$a_{m,0} = \frac{1}{\pi N_{m,0}} \int_{-\pi}^{\pi}\int_0^a f(r,\theta) R_{m,0}(r)\, r\, dr\, d\theta$$

$$a_{m,n} = \frac{1}{\pi N_{m,n}} \int_{-\pi}^{\pi}\int_0^a f(r,\theta) R_{m,n}(r)\cos(n\theta) r\, dr\, d\theta$$

and

$$b_{m,n} = \frac{1}{\pi N_{m,n}} \int_{-\pi}^{\pi}\int_0^a f(r,\theta) R_{m,n}(r)\sin(n\theta) r\, dr\, d\theta,$$

where, for $n \ge 0$, $m \ge 1$,

$$N_{m,n} = \int_0^a \Big(R_{m,n}(r)\Big)^2 r\, dr \triangleq \int_0^a \Big(-Y_n(\sqrt{\lambda_{m,n}}\ a)J_n(\sqrt{\lambda_{m,n}}\ r) + J_n(\sqrt{\lambda_{m,n}}\ a)Y_n(\sqrt{\lambda_{m,n}}\ r)\Big)^2 r\, dr\,.$$

11.6.5. (a)
$$\left\{\begin{array}{l} 0 = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left[r\dfrac{\partial T}{\partial r}\right] + \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial\theta^2} + \dfrac{\partial^2 T}{\partial z^2},\ 0<\theta<\pi,\ 0\le r<a,\ 0<z<H \\[2ex] T(a,\theta,z)=0,\ 0<\theta\le\pi,\ 0<z<H \\[2ex] \dfrac{\partial T}{\partial\theta}(r,0,z) = \dfrac{\partial T}{\partial\theta}(r,\pi,z)=0,\ 0\le r<a,\ 0<z<H \\[2ex] \dfrac{\partial T}{\partial z}(r,\theta,0)=0,\ T(r,\theta,H)=f(r,\theta),\ 0<\theta<\pi,\ 0\le r<a \end{array}\right\}.$$

(b) Define the open half cylinder $\mathcal{D} = \{\,(r,\theta,z) : 0<\theta<\pi,\ 0<r<a,\ 0<z<H\,\}$. The physical cylinder is an open box (a.k.a. parallelepiped) in (r,θ,z) coordinates, so the BCs $\dfrac{\partial T}{\partial\theta}(r,0,z) = \dfrac{\partial T}{\partial\theta}(r,\pi,z)=0$ and the BCs $|T(0^+,\theta,z|<\infty$, $T(a,\theta,z)=0$, express pairs of "homogeneous" boundary conditions on parallel sides of that box. As in Example 11.20 in Section 11.6, this suggests separating both the $r-$ and $\theta-$dependence away from the $z-$dependence by finding eigenvalues of the (r,θ) *part* of the Laplacian:

$$\frac{1}{r}\frac{\partial}{\partial r}\Big[r\frac{\partial\phi}{\partial r}\Big] + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \lambda\phi = 0,$$

along with the boundary conditions

$$\left\{\begin{array}{c} \dfrac{\partial\phi}{\partial\theta}(r,0) = \dfrac{\partial\phi}{\partial\theta}(r,\pi)=0,\ 0<r<a, \\[2ex] |\,\phi(0^+,\theta)\,|<\infty \quad\text{and}\quad \phi(a,\theta)\equiv 0,\ 0<\theta<\pi. \end{array}\right\}.$$

The first pair of BCs suggests looking for solutions in the form $\phi(r,\theta) = \cos n\theta R(r)$, for $n=0,1,2,...$. We substitute this into the Laplacian eigenvalue PDE to get

$$\cos n\theta\,\frac{1}{r}\frac{d}{dr}\left[r\frac{dR_n}{dr}\right] + R_n(r)\frac{1}{r^2}\cdot(-n^2)\cos n\theta + \lambda\cos n\theta R_n(r) = 0,$$

hence $R_n(r)$ should satisfy the ODE

$$\frac{1}{r}\frac{d}{dr}\left[r\,\frac{dR_n}{dr}\right]-n^2\cdot\frac{1}{r^2}\,R_n+\lambda R_n=0,$$

that is, the ODE

$$r^2R_n''+rR_n'+(-n^2+\lambda r^2)R_n=0,$$

plus the boundary conditions $|\,R_n(0^+)\,|<\infty$ and $R_n(a)=0$. For $\lambda>0$, the ODE is a form of Bessel's equation of order n. Similar to Example 11.20 in Section 11.6, the solutions are $R_n(r)=c_1J_n(\sqrt{\lambda}\;r)+c_2Y_n(\sqrt{\lambda}\;r), 0<r<a$. When we apply the boundary condition $|\,R_n(0^+)\,|<\infty$ we conclude that $c_2=0$, because $|\,Y_n(0^+)\,|=\infty$. When we apply the last boundary condition by graphing $J_n(\sqrt{\lambda}\,a)$ versus $\lambda>0$, we find there are infinitely many eigenvalues $\lambda_{m,n}=\frac{\gamma_{m,n}}{a}$, for $m=1,2,....$.

So, we get solutions

$$\phi_{m,n}(r,\theta)=J_n(\sqrt{\lambda_{m,n}}\;r)\cos n\theta,\;\; m=1,2,...,\;\; n=1,2,....$$

When we substitute product solutions $\phi_{m,n}(r,\theta)Z(z)$ into the original PDE, we get

$$0=\frac{1}{r}\frac{\partial}{\partial r}\left[r\,\frac{\partial}{\partial r}\phi_{m,n}(r,\theta)Z(z)\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\big[\,\phi_{m,n}(r,\theta)Z(z)\,\big]+\frac{\partial^2}{\partial z^2}\big[\,\phi_{m,n}(r,\theta)Z(z)\,\big],$$

hence

$$0=-\lambda_{m,n}\,\phi_{m,n}(r,\theta)Z(z)+\phi_{m,n}(r,\theta)Z''(z),$$

hence $Z(z)$ should satisfy the ODE

$$Z''(z)-\lambda_{m,n}Z(z)=0\,.$$

Using clairvoyance concerning the remaining two BCs $\dfrac{\partial T}{\partial z}(r,\theta,0)=0,\;T(r,\theta,H)=f(r,\theta)$, for $n=0,1,2,...$ we get product solutions

$$J_n(\sqrt{\lambda_{m,n}}\;r)\sinh(\sqrt{\lambda_{m,n}}(H-z))\cos n\theta\quad\text{and}\quad J_n(\sqrt{\lambda_{m,n}}\;r)\cosh(\sqrt{\lambda_{m,n}}\,z)\cos n\theta\,.$$

We can put all product solutions together to get the general solution of the PDE and homogeneous BCs is in the form

$$\begin{aligned}T(r,\theta,z)=&\sum_{m=1}^{\infty}J_0(\sqrt{\lambda_{m,0}}\;r)\left(\frac{a_{m,0}}{2}\sinh(\sqrt{\lambda_{m,0}}(H-z))+\frac{b_{m,0}}{2}\cosh(\sqrt{\lambda_{m,0}}\,z)\right)\\&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}J_n(\sqrt{\lambda_{m,n}}\;r)\cos n\theta\left(a_{m,n}\sinh(\sqrt{\lambda_{m,n}}(H-z))+b_{m,n}\cosh(\sqrt{\lambda_{m,n}}\,z)\right)\,.\end{aligned}$$

It is this general solution that we will substitute into the remaining two boundary conditions, $\dfrac{\partial T}{\partial z}(r,\theta,0)=0,\;T(r,\theta,H)=f(r,\theta),\;\;0<\theta<\pi$. The first BC gives

$$0=\frac{\partial T}{\partial z}(r,\theta,0)=\sum_{m=1}^{\infty}-\cosh(\sqrt{\lambda_{m,0}}\,H)\frac{a_{m,0}}{2}J_0(\sqrt{\lambda_{m,0}}\;r)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}-\cosh(\sqrt{\lambda_{m,n}}\,H)a_{m,n}J_n(\sqrt{\lambda_{m,n}}\;r)\cos n\theta\,.$$

As in Example 11.20 in Section 11.6, orthogonality of the eigenfunctions $\phi_{m,n}(r,\theta)$ implies that $a_{m,n}=0$, for all $m\geq 1,\,n\geq 0$.

The second BC gives

$$f(r,\theta)=T(r,\theta,H)=\sum_{m=1}^{\infty}\cosh(\sqrt{\lambda_{m,0}}\,H)\frac{b_{m,0}}{2}J_0(\sqrt{\lambda_{m,0}}\;r)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cosh(\sqrt{\lambda_{m,n}}\,H)b_{m,n}J_n(\sqrt{\lambda_{m,n}}\;r)\cos n\theta\,.$$

Orthogonality of the eigenfunctions $\phi_{m,n}(r,\theta)$ implies that

$$b_{m,0}=\frac{1}{\cosh(\sqrt{\lambda_{m,0}}\,H)}\;\frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a f(r,\theta)J_0(\sqrt{\lambda_{m,0}}\;r)r\,dr\,d\theta}{\displaystyle\pi\int_0^a(J_0(\sqrt{\lambda_{m,0}}\;r))^2r\,dr}$$

and

$$b_{m,n} = \frac{1}{\cosh(\sqrt{\lambda_{m,n}}\, H)} \, \frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\cos n\theta J_n(\sqrt{\lambda_{m,n}}\; r) r\, dr\, d\theta}{\displaystyle \pi\int_0^a (J_n(\sqrt{\lambda_{m,n}}\; r))^2 r\, dr}$$

The final conclusion is that the solution of the problem is

$$T(r,\theta,z) = \sum_{m=1}^{\infty} \left(\frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_0(\sqrt{\lambda_{m,0}}\, r) r dr\, d\theta}{\displaystyle 2\pi\int_0^a (J_0(\sqrt{\lambda_{m,0}}\, r))^2 r\, dr} \right) \frac{\cosh(\sqrt{\lambda_{m,0}}\, z)}{\cosh(\sqrt{\lambda_{m,0}}\, H)} J_0(\sqrt{\lambda_{m,0}}\, r) +$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(\frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\cos n\theta J_n(\sqrt{\lambda_{m,n}}\, r) r\, dr\, d\theta}{\displaystyle \pi\int_0^a (J_n(\sqrt{\lambda_{m,n}}\, r))^2 r\, dr} \right) \frac{\cosh(\sqrt{\lambda_{m,n}}\, z)}{\cosh(\sqrt{\lambda_{m,n}}\, H)} J_n(\sqrt{\lambda_{m,n}}\, r)\cos n\theta\,.$$

11.6.7. Define the open rectangle $\mathcal{D} = \{\, (r,\theta) : -\pi < \theta < \pi,\ 0 < r < a \,\}$ in (r,θ) coordinates. The BCs $u(a,\theta,t) = 0$ and $|u(0^+,\theta,t)| < \infty$, as well as the periodic BCs,

$$u(r,-\pi^+) = u(r,\pi^-) \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r,-\pi^+) = \frac{\partial u}{\partial \theta}(r,\pi^-),\ 0 \le r < a,$$

express pairs of "homogeneous" boundary conditions on parallel sides of the rectangle. This suggests separating both the $r-$ and $\theta-$dependence away from the $t-$dependence by finding eigenvalues of

$$\frac{1}{r}\frac{\partial}{\partial r}\Big[r\,\frac{\partial \phi}{\partial r}\Big] + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \lambda\phi = 0,$$

along with the boundary conditions

$$\left\{ \begin{array}{c} \phi(r,-\pi^+) = \phi(r,\pi^-) \quad \text{and} \quad \dfrac{\partial \phi}{\partial \theta}(r,-\pi^+) = \dfrac{\partial \phi}{\partial \theta}(r,\pi^-),\ 0 < r < a, \\ \\ \phi(a,\theta) \equiv 0,\ |\phi(0^+,\theta)| < \infty,\ -\pi < \theta < \pi. \end{array} \right\}.$$

The first pair of BCs suggests looking for solutions in the form $\phi(r,\theta) = R(r)$, as well as $\phi(r,\theta) = \{\cos n\theta, \sin n\theta\}R(r)$, for $n = 1, 2, \ldots$. We substitute these into the Laplacian eigenvalue PDE to get, for $n = 0, 1, 2, \ldots$,

$$\frac{1}{r}\frac{d}{dr}\left[r\,\frac{dR_n}{dr}\right] + R_n(r)\frac{1}{r^2}\cdot(-n^2) + \lambda R_n(r) = 0,$$

that is,

$$r^2 R_n'' + rR_n' + (-n^2 + \lambda r^2)R_n = 0\,.$$

The solutions of the ODE are $R_n(r) = c_1 J_n(\sqrt{\lambda}\, r) + c_2 Y_n(\sqrt{\lambda}\, r)$, where c_1, c_2 are arbitrary constants. The BC $|\phi(0^+,\theta)| < \infty$ implies that $c_2 = 0$, so $R_n(r) = c_1 J_n(\sqrt{\lambda}\, r)$. Substitute this into the fourth homogeneous BC to get the characteristic equation

$$J_n(\sqrt{\lambda}\, a) = 0,$$

so the eigenvalues are $\lambda_{m,n} = \dfrac{\gamma_{m,n}}{a}$, for $m = 1, 2, \ldots$, $n = 0, 1, 2, \ldots$, and the corresponding eigenfunctions are

$$\phi_{m,n}(r,\theta) = J_n(\sqrt{\lambda_{m,n}}\, r)\{1, \cos(n\theta), \sin(n\theta)\},$$

in an abbreviated notation.

For $m \ge 1$, $n \ge 0$, if we substitute $u(r,\theta,t) = G_{m,n}(t)\phi_{m,n}(r,\theta)$ into the original PDE, we eventually get

$$\ddot{G}_{m,n}(t) = -c^2\lambda_{m,n}G_{m,n}(t)\,.$$

The general solution of the PDE and the BCs is

$$u(r,\theta,t) = \sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{m,0}}\, r)\left(\frac{a_{m,0}}{2}\, \cos(\sqrt{\lambda_{m,0}}\, ct) + \frac{b_{m,0}}{2}\, \sin(\sqrt{\lambda_{m,0}}\, ct)\right)$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{m,n}}\,r)\cos(n\theta)\left(a_{m,n}\cos(\sqrt{\lambda_{m,n}}\,ct)+b_{m,n}\sin(\sqrt{\lambda_{m,n}}\,ct)\right)$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{m,n}}\,r)\sin(n\theta)\left(c_{m,n}\cos(\sqrt{\lambda_{m,n}}\,ct)+d_{m,n}\sin(\sqrt{\lambda_{m,n}}\,ct)\right),$$

where $a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n}$ are constants to be chosen in order to satisfy the initial conditions.

The first initial condition (IC) requires, for $0 < r < a,\ -\pi < \theta < \pi$, that

$$0=u(r,\theta,0)=\sum_{m=1}^{\infty}\frac{a_{m,0}}{2}J_0(\sqrt{\lambda_{m,0}}\,r)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}J_n(\sqrt{\lambda_{m,n}}\,r)\cos n\theta+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{m,n}J_n(\sqrt{\lambda_{m,n}}\,r)\sin n\theta.$$

Using orthogonality, we find that $a_{m,n}=0$, $c_{m,n}=0$ for all appropriate m, n.

The second initial condition (IC) requires, for $0 < r < a,\ -\pi < \theta < \pi$, that

$$\left(1-\frac{r}{a}\right)\cos(2\theta)=\frac{\partial u}{\partial t}(r,\theta,0)=\sum_{m=1}^{\infty}\frac{b_{m,0}}{2}\sqrt{\lambda_{m,0}}\,c\,J_0(\sqrt{\lambda_{m,0}}\,r)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}b_{m,n}\sqrt{\lambda_{m,n}}\,c\,J_n(\sqrt{\lambda_{m,n}}\,r)\cos n\theta$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}d_{m,n}\sqrt{\lambda_{m,n}}\,c\,J_n(\sqrt{\lambda_{m,n}}\,r)\sin n\theta,\ 0<r<a,\ -\pi<\theta<\pi\,.$$

Define

$$N_{m,n}=\int_0^a\left(J_n(\sqrt{\lambda_{m,n}}\ r)\right)^2 r\,dr\,.$$

Using orthogonality, we find that the coefficients are given by

$$b_{m,0}=\frac{1}{\pi N_{m,0}\sqrt{\lambda_{m,0}}\,c}\int_{-\pi}^{\pi}\int_0^a\left(1-\frac{r}{a}\right)\cos(2\theta)J_0(\sqrt{\lambda_{m,0}}\,r)\,r\,dr\,d\theta$$

$$=\frac{1}{\pi N_{m,0}\sqrt{\lambda_{m,0}}\,c}\left(\int_{-\pi}^{\pi}\cos(2\theta)\,d\theta\right)\left(\int_0^a\left(1-\frac{r}{a}\right)J_0(\sqrt{\lambda_{m,0}}\,r)\,r\,dr\right)=0,$$

and

$$b_{m,n}=\frac{1}{\pi N_{m,0}\sqrt{\lambda_{m,0}}\,c}\int_{-\pi}^{\pi}\int_0^a\left(1-\frac{r}{a}\right)\cos(2\theta)\cos(n\theta)J_0(\sqrt{\lambda_{m,n}}\,r)\,dr\,d\theta$$

$$=\frac{1}{\pi N_{m,n}\sqrt{\lambda_{m,n}}\,c}\left(\int_{-\pi}^{\pi}\cos(2\theta)\ \cos(n\theta)d\theta\right)\left(\int_0^a\left(1-\frac{r}{a}\right)J_n(\sqrt{\lambda_{m,n}}\,r)\,r\,dr\right)$$

$$=\frac{1}{\pi N_{m,n}\sqrt{\lambda_{m,n}}\,c}\left\{\begin{array}{ll}\pi, & n=2\\ 0, & n\neq 2\end{array}\right\}\left(\int_0^a\left(1-\frac{r}{a}\right)J_n(\sqrt{\lambda_{m,n}}\,r)\,r\,dr\right).$$

So, for $n\neq 2$, $b_{m,n}=0$, for all $m\geq 1$.

Finally,

$$d_{m,n}=\frac{1}{\pi N_{m,n}\sqrt{\lambda_{m,n}}\,c}\int_{-\pi}^{\pi}\int_0^a\left(1-\frac{r}{a}\right)\cos(2\theta)\ \sin(n\theta)J_n(\sqrt{\lambda_{m,n}}\,r)\sin(n\theta)\,r\,dr,$$

$$=\frac{1}{\pi N_n\sqrt{\lambda_{m,n}}\,c}\left(\int_{-\pi}^{\pi}\cos(2\theta)\ \cos(n\theta)d\theta\right)\left(\int_0^a\left(1-\frac{r}{a}\right)J_n(\sqrt{\lambda_{m,n}}\,r)\,r\,dr\right)=0.$$

The solution of the whole problem is

$$u(r,\theta,t)=\frac{1}{c}\sum_{m=1}^{\infty}\frac{1}{N_{m,2}\sqrt{\lambda_{m,2}}}\left(\int_0^a\left(1-\frac{r}{a}\right)J_2(\sqrt{\lambda_{m,2}}\,r)\,r\,dr\right)J_2(\sqrt{\lambda_{m,2}}\,r)\cos(2\theta)\sin(\sqrt{\lambda_{m,2}}\,ct)\,.$$

11.6.9. (a) Using the sine addition formula, $\sin(\alpha\pm\beta)=\sin\alpha\,\cos\beta\pm\cos\alpha\,\sin\beta$, we get

$$\sin\left(\frac{\pi}{2}+\varphi\right)=\sin\frac{\pi}{2}\,\cos\varphi+\cos\frac{\pi}{2}\,\sin\varphi=1\cdot\cos\varphi+0\cdot\sin\varphi=\cos\varphi$$

while
$$\sin\left(\frac{\pi}{2}-\varphi\right)=\sin\frac{\pi}{2}\,\cos\varphi-\cos\frac{\pi}{2}\,\sin\varphi=1\cdot\cos\varphi-0\cdot\sin\varphi=\cos\varphi.$$
So, $\sin(\phi)$ is even about $\phi=\frac{\pi}{2}$.

(b) First, we will explain why a function $h(\phi)$ being odd about $\phi=\frac{\pi}{2}$, implies $\displaystyle\int_0^\pi h(\phi)\,d\phi=0$: Using the change of variables $u=\dfrac{\pi}{2}-\phi$, that is, $\dfrac{\pi}{2}-u=\phi$, we have
$$\int_0^\pi h(\phi)\,d\phi=\int_0^{\pi/2}h(\phi)\,d\phi+\int_{\pi/2}^{\pi}h(\phi)\,d\phi=\int_0^{\pi/2}h(\phi)d\phi+\int_0^{-\pi/2}h\left(\frac{\pi}{2}-u\right)(-du).$$
So, h being odd about $\phi=\frac{\pi}{2}$ implies
$$\int_0^\pi h(\phi)\,d\phi=\int_0^{\pi/2}h(\phi)d\phi+\int_0^{-\pi/2}-h\left(\frac{\pi}{2}+u\right)(-du)=\int_0^{\pi/2}h(\phi)d\phi+\int_0^{-\pi/2}h\left(\frac{\pi}{2}+u\right)du,$$
and then the change of variables $\phi=\dfrac{\pi}{2}+u$, that is, $\phi-\dfrac{\pi}{2}=u$, gives
$$\int_0^\pi h(\phi)\,d\phi=\int_0^{\pi/2}h(\phi)d\phi+\int_{\pi/2}^{0}h(\phi)d\phi=\int_{\pi/2}^{\pi/2}h(\phi)d\phi=0.$$

Second, using the cosine addition formula, $\cos(\alpha\pm\beta)=\cos\alpha\,\cos\beta\mp\sin\alpha\,\sin\beta$, we get
$$\cos\left(\frac{\pi}{2}+\varphi\right)=\cos\frac{\pi}{2}\,\cos\varphi-\sin\frac{\pi}{2}\,\sin\varphi=0\cdot\cos\varphi-1\cdot\sin\varphi=-\sin\varphi$$
while
$$\cos\left(\frac{\pi}{2}-\varphi\right)=\cos\frac{\pi}{2}\,\cos\varphi+\sin\frac{\pi}{2}\,\sin\varphi=0\cdot\cos\varphi+1\cdot\sin\varphi=\sin\varphi.$$
So, $\cos(\phi)$ is odd about $\phi=\frac{\pi}{2}$.

Finally, using the fact that for $n=odd$, the Legendre polynomial $P_n(x)$ is an odd polynomial in x, we see that for $n=odd$, $P_n(\cos\phi)$ is an odd function of ϕ about $\phi=\frac{\pi}{2}$.

We are assuming that $f(\phi)$ is even about $\phi=\frac{\pi}{2}$. Using the result of part (a) and the results of part (b) so far, we have that for all odd n, the function
$$h(\phi)\triangleq f(\phi)P_n(\cos\phi)\sin\phi=even\cdot odd\cdot even$$
is odd about $\frac{\pi}{2}$, hence
$$\int_0^\pi f(\phi)P_n(\cos\phi)\sin\phi\,d\phi=0.$$
We are assuming that $f(\phi)$ is even about $\phi=\frac{\pi}{2}$. It follows that for all odd n,
$$a_n=\frac{\displaystyle\int_0^\pi f(\phi)P_n(\cos\phi)\sin\phi\,d\phi}{\displaystyle\int_0^\pi\big(P_n(\cos\phi)\big)^2\sin\phi\,d\phi}=0.$$
So,
$$f(\phi)=a_0+\sum_{n=1}^\infty a_n\Phi_n(\phi).$$

11.6.11. From Example 11.22, the solution of the PDE on the solid sphere is
$$V(\rho,\phi)=a_0+\sum_{n=1}^\infty a_n\rho^nP_n(\cos\phi),$$
and so the boundary condition requires
$$5\,\left(1-\cos^2\phi\right)=V(3,\phi)=a_0+\sum_{n=1}^\infty a_n3^nP_n(\cos\phi),\ 0<\phi<\pi.$$

We know that $P_0(x) = 1$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. Using the substitution $x = \cos\phi$, we express

$$5\ (1 - x^2) = -\left(\frac{10}{3}\cdot\frac{1}{2}(3x^2) - 5\right) = -\left(\frac{10}{3}\cdot\frac{1}{2}(3x^2 - 1) + \frac{5}{3} - 5\right) = -\frac{10}{3}P_2(x) + \frac{10}{3}P_0(x),$$

so the IC we need to satisfy is

$$\frac{10}{3}P_0(\cos\phi) - \frac{10}{3}P_2(\cos\phi) = a_0 P_0(\cos\phi) + \sum_{n=1}^{\infty} a_n 3^n P_n(\cos\phi),\ 0 < \phi < \pi.$$

Orthogonality implies that we have $a_0 = \frac{10}{3}$, $a_2 = -3^{-2}\frac{10}{3} = -\frac{10}{27}$, and all other $a_n = 0$. So, the solution of the original problem is

$$V(\rho,\phi) = \frac{10}{3} - \frac{5}{27}\rho^2(3\cos^2\phi - 1).$$

Section 12.1

12.1.5.1. (a) For $N=3$, define $x_j = j\,\frac{\pi}{3}$, $t_m = m\triangle t$, $T_j^m = T(x_j, t_m)$ for $j=0,...,3$ and $m=0,1,2,...$, and $\epsilon = \dfrac{\alpha\,\triangle t}{(\pi/3)^2} = \dfrac{9}{2\pi^2}$ is given.

For $j=0$, the BC $T(0,t)=0$ implies $T_0^{m+1} = T(0,t_{m+1}) = 0$. For $j=1,2,3$, the PDE implies $(\star)$ $T_j^{m+1} = T_j^m + \epsilon\cdot\big(T_{j+1}^m - 2T_j^m + T_{j-1}^m\big)$. Using the central difference approximation of the first derivative, the BC at $x=\pi$ implies

$$\psi^m \triangleq \psi(t_m) = \frac{\partial T}{\partial x}(x_3, t_m) \approx \frac{T_4^m - T_2^m}{2\,\triangle x},$$

hence

$$T_4^m = \frac{2\pi}{3}\psi^m + T_2^m.$$

Substitute that into $(\star)$ for $j=3$ to get

$$T_3^{m+1} = T_3^m + \epsilon\cdot(T_4^m - 2T_3^m + T_2^m) = T_3^m + \epsilon\cdot\left(\frac{2\pi}{3}\,\psi^m + T_2^m - 2T_3^m + T_2^m\right)$$

$$= 2\epsilon T_2^m + (1-2\epsilon)T_3^m + \epsilon\cdot\frac{2\pi}{3}\,\psi^m.$$

Define $\mathbf{T}^m \triangleq [\,T_1^m \quad T_2^m \quad T_3^m\,]^T$. The system of replacement equations that take into account both the PDE and BCs is

$$\mathbf{T}^{m+1} = A(\epsilon)\mathbf{T}^m + \begin{bmatrix} 0 \\ 0 \\ \epsilon\cdot\frac{2\pi}{3}\,\psi^m \end{bmatrix},$$

where $\epsilon = \frac{9}{2\pi^2}$ gives

$$A\Big(\frac{9}{2\pi^2}\Big) = \begin{bmatrix} 1-\frac{9}{\pi^2} & \frac{9}{2\pi^2} & 0 \\ \frac{9}{2\pi^2} & 1-\frac{9}{\pi^2} & \frac{9}{2\pi^2} \\ 0 & \frac{9}{\pi^2} & 1-\frac{9}{\pi^2} \end{bmatrix}.$$

Mathematica says that the eigenvalues of A are $\lambda \approx 0.877830, -0.701611, 0.0881093$, all of which have magnitude less than one.

(b) The PDE has product solutions $\sin(nx)e^{-\alpha n^2 t}$, so the decay rate is $\alpha\cdot 1^2 = \alpha$. The discretization of the PDE has

$$\triangle t = \alpha^{-1}\Big(\frac{\pi}{3}\Big)^2\cdot\epsilon = \alpha^{-1}\Big(\frac{\pi}{3}\Big)^2\cdot\frac{9}{2\pi^2} = \frac{1}{2\alpha},$$

so it takes 2α time steps of the discretization to account for one unit of time.

So, the discretization of the PDE would have the same decay rate as the solutions of the PDE if, and only if, the discretization's largest eigenvalue, λ_1, has $e^{-\alpha} \approx |\lambda_1|^{2\alpha} \approx 0.877830^{2\alpha}$. Taking the natural logarithm of both sides gives $-\alpha \approx 2\alpha\ln(0.877830)$, that is, $-1 \approx 2\ln(0.877830) \approx -0.2606$. So, this discretization of the PDE does not give a decay rate that agrees with the decay rate of solutions of the PDE.

12.1.5.3. (a) For $N=3$, define $x_j = j\,\frac{\pi}{3}$, $t_m = m\,\triangle t$, $T_j^m = T(x_j, t_m)$ for $j=0,...,3$ and $m=0,1,2,...$, and $\epsilon = \frac{\alpha\triangle t}{(\pi/3)^2}$.

For $j=1,2,3$, the PDE implies $(\star)$ $T_j^{m+1} = T_j^m + \epsilon\cdot\big(T_{j+1}^m - 2T_j^m + T_{j-1}^m\big)$.

Using the central difference approximation of the first derivative, the BC at $x=0$ implies

$$\phi^m \triangleq \phi(t_m) = \frac{\partial T}{\partial x}(x_0, t_m) \approx \frac{T_1^m - T_{-1}^m}{2\,\triangle\,x},$$

hence

$$T_{-1}^m = -\frac{2\pi}{3}\phi^m + T_1^m.$$

Substitute that into $(\star)$ for $j=0$ to get

$$T_0^{m+1} = T_0^m + \epsilon\cdot\left(T_1^m - 2T_0^m + T_{-1}^m\right) = T_0^m + \epsilon\cdot\left(T_1^m - 2T_0^m - \frac{2\pi}{3}\phi^m + T_1^m\right)$$

$$= (1-2\epsilon)T_0^m + 2\epsilon T_1^m - \epsilon\cdot\frac{2\pi}{3}\,\phi^m.$$

Using the central difference approximation of the first derivative, the BC at $x=\pi$ implies

$$\psi^m \triangleq \psi(t_m) = \frac{\partial T}{\partial x}(x_3,t_m) \approx \frac{T_4^m - T_2^m}{2\,\triangle\, x},$$

hence

$$T_4^m = \frac{2\pi}{3}\psi^m + T_2^m.$$

Substitute that into $(\star)$ for $j=3$ to get

$$T_3^{m+1} = T_3^m + \epsilon\cdot(T_4^m - 2T_3^m + T_2^m) = T_3^m + \epsilon\cdot\left(\frac{2\pi}{3}\,\psi^m + T_2^m - 2T_3^m + T_2^m\right)$$

$$= 2\epsilon T_2^m + (1-2\epsilon)T_3^m + \epsilon\cdot\frac{2\pi}{3}\,\psi^m.$$

Define $\mathbf{T}^m \triangleq [\,T_0^m \quad T_1^m \quad T_2^m \quad T_3^m\,]^T$. The system of replacement equations that take into account both the PDE and BCs is

$$\mathbf{T}^{m+1} = A(\epsilon)\mathbf{T}^m + \begin{bmatrix} -\epsilon\cdot\frac{2\pi}{3}\,\phi^m \\ 0 \\ 0 \\ \epsilon\cdot\frac{2\pi}{3}\,\psi^m \end{bmatrix},$$

where $\epsilon = \frac{9}{2\pi^2}$ gives

$$A\Big(\frac{9}{2\pi^2}\Big) = \begin{bmatrix} 1-\frac{9}{\pi^2} & \frac{9}{\pi^2} & 0 & 0 \\ \frac{9}{2\pi^2} & 1-\frac{9}{\pi^2} & \frac{9}{2\pi^2} & 0 \\ 0 & \frac{9}{2\pi^2} & 1-\frac{9}{\pi^2} & \frac{9}{2\pi^2} \\ 0 & 0 & \frac{9}{\pi^2} & 1-\frac{9}{\pi^2} \end{bmatrix}.$$

Mathematica says that the eigenvalues of A are $\lambda \approx 1., -0.823781, 0.544055, -0.367836$. Three of the eigenvalues have magnitude less than one, which corresponds to those solutions of the PDE with corresponding *homogeneous* BCs that are transient in time. The eigenvalue $\lambda = 1$ corresponds to a particular solution $T_P(x,t)$ of the form $T_P(x,t) = xf(t) + (\pi - x)g(t) + w(x,t)$ that satisfies the BCs $\frac{\partial T}{\partial x}(0,t) = \phi(t)$, $\frac{\partial T}{\partial x}(\pi,t) = \psi(t),\ t>0$ and has $w(x,t)\to 0$ as $t\to\infty$.

(b) Concerning the decaying solutions of the PDE, it has product solutions $\cos(nx)e^{-\alpha n^2 t}$, so the decay rate is $\alpha\cdot 1^2 = \alpha$. The discretization of the PDE has

$$\triangle t = \alpha^{-1}\Big(\frac{\pi}{3}\Big)^2\cdot\epsilon = \alpha^{-1}\Big(\frac{\pi}{3}\Big)^2\cdot\frac{9}{2\pi^2} = \frac{1}{2\alpha},$$

so it takes 2α time steps of the discretization to account for one unit of time.

So, the discretization of the PDE would have the same decay rate as the decaying solutions of the PDE if, and only if, the discretization's largest eigenvalue, λ_1, has $e^{-\alpha} \approx |\lambda_1|^{2\alpha} \approx 0.823781^{2\alpha}$. Taking the natural logarithm of both sides gives $-\alpha \approx 2\alpha\ln(0.823781)$, that is, $-1 \approx 2\ln(0.877830) \approx -0.3877$. So, this discretization of the PDE does not give a decay rate that agrees with the decay rate of solutions of the PDE.

12.1.5.5. Yes, there are some non-zero initial temperature distributions, $\mathbf{T}^0$, for which $||\mathbf{T}^m|| \to 0$, as $m\to\infty$. Just use as $\mathbf{T}^0$ any eigenvector of $A(\epsilon)$ corresponding to an eigenvalue whose magnitude is less than one.

Ex: For Example 12.2, which has $\triangle x = \frac{\pi}{6}$, let $\mathbf{T}^0 = \begin{bmatrix} T_1^0 \\ T_2^0 \\ T_3^0 \\ T_4^0 \\ T_5^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Let $\triangle t = \dfrac{1}{4}$. Then we get the numerical solution in the table below.

Other initial distributions that give solutions, for problem 12.1.5.5, with $||\mathbf{T}^m|| \to 0,$ as $m \to \infty$, include $\mathbf{T}^0 = \begin{bmatrix} 1 \\ \sqrt{3} \\ 2 \\ \sqrt{3} \\ 1 \end{bmatrix}$ and $\mathbf{T}^0 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ which are other eigenvectors of $A(\epsilon)$ corresponding to eigenvalues whose magnitude is less than one. Also, any linear combination of the above three eigenvectors will give an example.

Table 1: Problem 12.1.5.5: An example of finite differences for a heat equation with $\triangle t = \frac{1}{4}$

i	t_i	T_1^m	T_2^m	T_3^m	T_4^m	T_5^m
0	0	1	0	−1	0	1
1	0.25	−0.823781306	0	0.823781306	0	−0.823781306
2	0.50	0.678615639	0	−0.678615639	0	0.678615639
3	0.75	−0.559030877	0	0.559030877	0	−0.559030877
4	1.00	0.460519186	0	−0.460519186	0	0.460519186
5	1.25	−0.379367096	0	0.379367096	0	−0.379367096
6	1.50	0.312515522	0	−0.312515522	0	0.312515522
7	1.75	−0.257444445	0	0.257444445	0	−0.257444445
8	2.00	0.212077921	0	−0.212077921	0	0.212077921
...	...	...	...	...	...	...
16	4.00	0.044977044	0	−0.044977044	0	0.044977044
...	...	...	...	...	...	...
24	6.00	0.009538638	0	−0.009538638	0	0.009538638
...	...	...	...	...	...	...
32	8.00	0.002022935	0	−0.002022935	0	0.002022935

Section 12.2

12.2.2.1. (a) Recall that $Q_n = \dfrac{1+\epsilon\left(-1+\cos\frac{n\pi}{N}\right)}{1-\epsilon\left(-1+\cos\frac{n\pi}{N}\right)}$. Suppose $\epsilon > 0$. Because $-1 < \cos\frac{n\pi}{N} < 1$ for $n = 1, ..., N-1$, we have a succession of implied inequalities concerning the denominator of Q_n:

$$-2 < -1+\cos\frac{n\pi}{N} < 0 \quad\Longrightarrow\quad 2 > -\left(-1+\cos\frac{n\pi}{N}\right) > 0.$$

Using the assumption that $\epsilon > 0$, this implies

$$2\epsilon > -\epsilon\left(-1+\cos\frac{n\pi}{N}\right) > 0$$

which implies

$$1+\big(2\epsilon\big) > 1+\left(-\epsilon\left(-1+\cos\frac{n\pi}{N}\right)\right) > 1+\big(0\big)$$

which implies

$$(1)\ \ 1+2\epsilon > 1-\epsilon\left(-1+\cos\frac{n\pi}{N}\right)$$

and

$$(2)\ \ 1-\epsilon\left(-1+\cos\frac{n\pi}{N}\right) > 1.$$

Also, concerning the numerator of Q_n,

$$-1 < \cos\frac{n\pi}{N} < 1 \quad\Longrightarrow\quad -2 < -1+\cos\frac{n\pi}{N} < 0$$

Using the assumption that $\epsilon > 0$, this implies

$$-2\epsilon < \epsilon\left(-1+\cos\frac{n\pi}{N}\right) < 0$$

which implies

$$1+\big(-2\epsilon\big) < 1+\left(\epsilon\left(-1+\cos\frac{n\pi}{N}\right)\right) < 1+\big(0\big)$$

which implies

$$(3)\ \ 1-2\epsilon < 1+\epsilon\left(-1+\cos\frac{n\pi}{N}\right)$$

and

$$(4)\ \ 1+\epsilon\left(-1+\cos\frac{n\pi}{N}\right) < 1.$$

Combining (2) and (4) implies $Q_n < 1$.

To explain why $Q_n > -1$, suppose instead $Q_n \le -1$, that is,

$$(\star)\ \ \frac{1+\epsilon\left(-1+\cos\frac{n\pi}{N}\right)}{1-\epsilon\left(-1+\cos\frac{n\pi}{N}\right)} \le -1.$$

Because (2) implies that the denominator is positive, we can multiply through $(\star)$ by the denominator of Q_n without changing the direction of the inequality. This gives

$$1+\epsilon\big(-1+\cos\frac{n\pi}{N}\big) \le -\left(1-\epsilon\big(-1+\cos\frac{n\pi}{N}\big)\right),$$

which can be rewritten as

$$1-\epsilon+\epsilon\cos\frac{n\pi}{N} \le -1-\epsilon+\epsilon\cos\frac{n\pi}{N},$$

hence $1 \le -1$, giving a contradiction.

(b) Because the amplification factor Q_n has magnitude less than one, the Crank-Nicholson method is stable for all $\epsilon > 0$.

12.2.2.3. If $\epsilon < 0$, define $\beta = -\epsilon > 0$. Then

$$Q_n = \frac{1+\epsilon\left(-1+\cos\frac{n\pi}{N}\right)}{1-\epsilon\left(-1+\cos\frac{n\pi}{N}\right)} = \frac{1-\beta\left(-1+\cos\frac{n\pi}{N}\right)}{1+\beta\left(-1+\cos\frac{n\pi}{N}\right)} \triangleq \frac{1}{R_n}\,.$$

Because $\beta > 0$ and

$$R_n = \frac{1+\beta\left(-1+\cos\frac{n\pi}{N}\right)}{1-\beta\left(-1+\cos\frac{n\pi}{N}\right)},$$

problem 12.2.2.1(a) implies that

$$-1 < R_n < 1,$$

for $n = 1, ..., N-1$. It follows that $|Q_n| > 1$, hence the Crank-Nicholson method is unstable.

Section 12.3

Throughout Section 12.3, define $u_{ij} \triangleq u(x_i, y_j)$, where $x_i \triangleq i \triangle x$, $y_j \triangleq j \triangle y$.

12.3.2.1.

$$\begin{array}{cccc} & 0 & 0 & \\ 0 & u_{12} & u_{22} & 1 \\ 0 & u_{11} & u_{21} & 1 \\ & 0 & 0 & \end{array}$$

Symmetry about $y = \frac{1}{2}$ suggests trying solution with $u_{12} = u_{11}$ and $u_{22} = u_{21}$, so we get

$$\begin{array}{cccc} & 0 & 0 & \\ 0 & u_{11} & u_{21} & 1 \\ 0 & u_{11} & u_{21} & 1 \\ & 0 & 0 & \end{array}$$

The PDE and five point Laplacian give

$$@(x_1, y_1): \quad u_{11} + 0 + 0 + u_{21} \quad -4u_{11} = 0$$

$$@(x_2, y_1): \quad 0 + u_{11} + u_{21} + 1 \quad -4u_{21} = 0.$$

So,

$$\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Approximate solution is

$$\begin{array}{cccc} & 0 & 0 & \\ 0 & \frac{1}{8} & \frac{3}{8} & 1 \\ 0 & \frac{1}{8} & \frac{3}{8} & 1 \\ & 0 & 0 & \end{array}$$

that is, $u(\frac{1}{3}, \frac{1}{3}) = u(\frac{1}{3}, \frac{2}{3}) \approx \frac{1}{8}$, $u(\frac{2}{3}, \frac{1}{3}) = u(\frac{2}{3}, \frac{2}{3}) \approx \frac{3}{8}$

12.3.2.3.

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & u_{13} & u_{23} & u_{33} & 0 \\ 0 & u_{12} & u_{22} & u_{32} & 0 \\ 0 & u_{11} & u_{21} & u_{31} & 0 \\ 0 & 0 & 0 & 0 & \end{array}$$

Symmetry about the line $y = 1 - x$ gives the simpler problem

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & u_{13} & u_{12} & u_{11} & 0 \\ 0 & u_{12} & u_{22} & u_{21} & 0 \\ 0 & u_{11} & u_{21} & u_{31} & 0 \\ 0 & 0 & 0 & 0 & \end{array}$$

The PDE and five point Laplacian give

$$\begin{array}{lcl}
@(x_1, y_1): & u_{12} + 0 + 0 + u_{21} & -4u_{11} = -\frac{1}{16} \\
@(x_1, y_2): & u_{13} + 0 + u_{11} + u_{22} & -4u_{12} = -\frac{1}{16} \\
@(x_1, y_3): & 0 + 0 + u_{12} + u_{12} & -4u_{13} = -\frac{1}{16} \\
@(x_2, y_1): & u_{22} + u_{11} + 0 + u_{31} & -4u_{21} = -\frac{1}{16} \\
@(x_2, y_2): & u_{12} + u_{12} + u_{21} + u_{21} & -4u_{22} = -\frac{1}{16} \\
@(x_3, y_1): & u_{21} + u_{21} + 0 + 0 & -4u_{31} = -\frac{1}{16}.
\end{array}$$

So,

$$\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{31} \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 1 \\ 0 & 2 & 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 2 & 0 & -4 \end{bmatrix}^{-1} \left(-\frac{1}{16}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{256} \begin{bmatrix} 11 \\ 14 \\ 11 \\ 14 \\ 18 \\ 11 \end{bmatrix}.$$

Approximate solution is

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11}{256} & \frac{14}{256} & \frac{11}{256} & 0 \\ 0 & \frac{14}{256} & \frac{126}{256} & \frac{14}{256} & 0 \\ 0 & \frac{11}{256} & \frac{14}{256} & \frac{11}{256} & 0 \\ 0 & 0 & 0 & 0 & \end{array}$$

that is, $u(\frac{1}{4}, \frac{1}{4}) = u(\frac{1}{4}, \frac{3}{4}) = u(\frac{3}{4}, \frac{3}{4}) = u(\frac{3}{4}, \frac{1}{4}) \approx \frac{11}{256}$, $u(\frac{1}{4}, \frac{2}{4}) = u(\frac{2}{4}, \frac{3}{4}) = u(\frac{2}{4}, \frac{1}{4}) = u(\frac{3}{4}, \frac{2}{4}) \approx \frac{14}{256}$, $u(\frac{2}{4}, \frac{2}{4}) \approx \frac{18}{256}$

12.3.2.5.

$$\begin{array}{cccc} & u_{13} & u_{23} & \\ 4 & u_{12} & u_{22} & 4 \\ 4 & u_{11} & u_{21} & 4 \\ & 0 & 0 & \end{array}$$

Symmetry about the line $x = \frac{\pi}{2}$ (note: there is not symmetry about the line $y = \frac{\pi}{2}$ because the BC at $y = 0$ is not the same as the BC at $y = \pi$) gives simpler problem

$$\begin{array}{cccc} & u_{13} & u_{13} & \\ 4 & u_{12} & u_{12} & 4 \\ 4 & u_{11} & u_{11} & 4 \\ & 0 & 0 & \end{array}$$

The PDE and five point Laplacian give

$$\begin{aligned} &(1) \quad @(x_1, y_1): \quad u_{12} + 4 + 0 + u_{11} \quad -4u_{11} = -\tfrac{\pi^2}{9} \\ &(2) \quad @(x_1, y_2): \quad u_{13} + 4 + u_{11} + u_{12} \quad -4u_{12} = -\tfrac{\pi^2}{9}\,. \end{aligned}$$

Method I: Forward difference of BC at $y = \pi$ gives

$$(3) \quad u_{13} - u_{12} = \frac{\pi}{3}.$$

So, (1), (2), and (3), give the system

$$\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\pi^2}{9} - 4 \\ -\frac{\pi^2}{9} - 4 \\ \frac{\pi}{3} \end{bmatrix} \approx \begin{bmatrix} 3.267413137 \\ 4.705616700 \\ 5.752814251 \end{bmatrix}.$$

Approximate solution is

$$\begin{array}{cccc} & 5.752814251 & 5.752814251 & \\ 4 & 4.705616700 & 4.705616700 & 4 \\ 4 & 3.267413137 & 3.267413137 & 4 \\ & 0 & 0 & \end{array}$$

that is, $u(\frac{\pi}{3}, \frac{\pi}{3}) = u(\frac{2\pi}{3}, \frac{\pi}{3}) \approx 3.267413147$, $u(\frac{\pi}{3}, \frac{2\pi}{3}) = u(\frac{2\pi}{3}, \frac{2\pi}{3}) \approx 4.705616700$, $u(\frac{\pi}{3}, \pi) = u(\frac{2\pi}{3}, \pi) \approx 5.752814251$

Method II: Using central difference approximation of the BC at $y = \pi$, we start with

$$\begin{array}{cccc} & u_{14} & u_{24} & \\ 4 & u_{13} & u_{23} & 4 \\ 4 & u_{12} & u_{22} & 4 \\ 4 & u_{11} & u_{21} & 4 \\ & 0 & 0 & \end{array}$$

To solve for the fictitious value $u(\frac{\pi}{3}, \frac{4\pi}{3}) \approx u_{14}$ we include it in the five point Laplacian approximation for $\nabla^2 u(\frac{\pi}{3}, \pi)$; but, then we will need a value for $u(0, 1)$. We are taking it to be 4, equal to the values of $u(0, y)$

for $0 < y < \pi$. Symmetry about the line $x = \frac{\pi}{2}$ (note: there is not symmetry about the line $y = \frac{\pi}{2}$ because the BC at $y = 0$ is not the same as the BC at $y = \pi$) gives simpler problem

$$\begin{array}{cccc} & u_{14} & u_{14} & \\ 4 & u_{13} & u_{13} & 4 \\ 4 & u_{12} & u_{12} & 4 \\ 4 & u_{11} & u_{11} & 4 \\ & 0 & 0 & \end{array}$$

The PDE and five point Laplacian give

$$\begin{aligned} @(x_1,y_1): &\quad u_{12}+4+0+u_{11} \quad -4u_{11} = -\tfrac{\pi^2}{9} \\ @(x_1,y_2): &\quad u_{13}+4+u_{11}+u_{12} \quad -4u_{12} = -\tfrac{\pi^2}{9} \\ @(x_1,y_3): &\quad u_{14}+4+u_{12}+u_{13} \quad -4u_{13} = -\tfrac{\pi^2}{9}\,. \end{aligned}$$

Central difference of BC at $y = \pi$ gives

$$u_{14} - u_{12} = 2 \cdot \frac{\pi}{3}.$$

So,

$$\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\pi^2}{9}-4 \\ -\frac{\pi^2}{9}-4 \\ -\frac{\pi^2}{9}-4 \\ 2\cdot\frac{\pi}{3} \end{bmatrix} \approx \begin{bmatrix} 3.230958051 \\ 4.596251443 \\ 5.461173567 \\ 6.690646545 \end{bmatrix}.$$

Approximate solution is

$$\begin{array}{cccc} 4 & 5.461173567 & 5.461173567 & 4 \\ 4 & 4.596251443 & 4.596251443 & 4 \\ 4 & 3.230958051 & 3.230958051 & 4 \\ & 0 & 0 & \end{array}$$

that is, $u(\frac{\pi}{3},\frac{\pi}{3}) = u(\frac{2\pi}{3},\frac{\pi}{3}) \approx 3.230958051$, $u(\frac{\pi}{3},\frac{2\pi}{3}) = u(\frac{2\pi}{3},\frac{2\pi}{3}) \approx 4.596251443$,
$u(\frac{\pi}{3},\pi) = u(\frac{2\pi}{3},\pi) \approx 5.461173567$.

12.3.2.7.

$$\begin{array}{cccc} & u_{14} & u_{24} & \\ 0 & u_{13} & u_{23} & 0 \\ 0 & u_{12} & u_{22} & 0 \\ 0 & u_{11} & u_{21} & 0 \\ 0 & u_{10} & u_{20} & 0 \\ & u_{1,-1} & u_{2,-1} & \end{array}$$

To solve for the fictitious values $u(\frac{\pi}{3}, \frac{4\pi}{3}) \approx u_{14}$ and $u(\frac{\pi}{3}, -\frac{\pi}{3}) \approx u_{1,-1}$, we include them in the five point Laplacian approximations for $\nabla^2 u(\frac{\pi}{3}, \pi)$ and $\nabla^2 u(\frac{\pi}{3}, 0)$, respectively; but, then we will need values for $u(0,1)$ and $u(0,0)$, resp. We are taking them it to be 0, equal to the values of $u(0,y)$ and $u(\pi, y)$ for $0 < y < \pi$. Symmetry about the lines $x = \frac{\pi}{2}$ (note: there is not symmetry about the line $y = \frac{\pi}{2}$ because the BC at $y = 0$ is not the same as the BC at $y = \pi$) gives a simpler problem,

$$
\begin{array}{cccc}
 & u_{14} & u_{14} & \\
0 & u_{13} & u_{13} & 0 \\
0 & u_{12} & u_{12} & 0 \\
0 & u_{11} & u_{11} & 0 \\
0 & u_{10} & u_{10} & 0 \\
 & u_{1,-1} & u_{1,-1} &
\end{array}
$$

The PDE and five point Laplacian give

$$
\begin{aligned}
@(x_1, y_0): &\quad u_{11} + 0 + u_{1,-1} + u_{10} \quad -4u_{10} = 0 \\
@(x_1, y_1): &\quad u_{12} + 0 + u_{10} + u_{11} \quad -4u_{11} = 0 \\
@(x_1, y_2): &\quad u_{13} + 0 + u_{11} + u_{12} \quad -4u_{12} = 0 \\
@(x_1, y_3): &\quad u_{14} + 0 + u_{12} + u_{13} \quad -4u_{13} = 0.
\end{aligned}
$$

Central difference of BC at $y = 0$ gives

$$u_{11} - u_{1,-1} = -5 \cdot 2\,\frac{\pi}{3}.$$

Central difference approximation of BC at $y = \pi$ gives

$$3 \cdot u_{13} + \frac{u_{14} - u_{12}}{2 \cdot \frac{\pi}{3}} = 6,$$

that is,

$$2\pi u_{13} + u_{14} - u_{12} = 4\pi.$$

So, the system is

$$
\begin{bmatrix} u_{1,-1} \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{bmatrix}
=
\begin{bmatrix}
1 & -3 & 1 & 0 & 0 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & 1 & -3 & 1 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2\pi & 1
\end{bmatrix}^{-1}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10\pi/3 \\ 4\pi \end{bmatrix}
\approx
\begin{bmatrix} 12.48634779 \\ 4.833573356 \\ 2.014372277 \\ 1.209543477 \\ 1.614258153 \\ 3.633230982 \end{bmatrix}.
$$

Approximate solution is

0	1.614258153	1.614258153	0
0	1.209543477	1.209543477	0
0	2.014372277	2.014372277	0
0	4.833573356	4.833573356	0

that is, $u(\frac{\pi}{3},0)=u(\frac{2\pi}{3},0)\approx 4.833573356$, $u(\frac{\pi}{3},\frac{\pi}{3})=u(\frac{2\pi}{3},\frac{\pi}{3})\approx 2.014372277$, $u(\frac{\pi}{3},\frac{2\pi}{3})=u(\frac{2\pi}{3},\frac{2\pi}{3})\approx 1.209543477$, $u(\frac{\pi}{3},\pi)=u(\frac{2\pi}{3},\pi)\approx 1.614258153$.

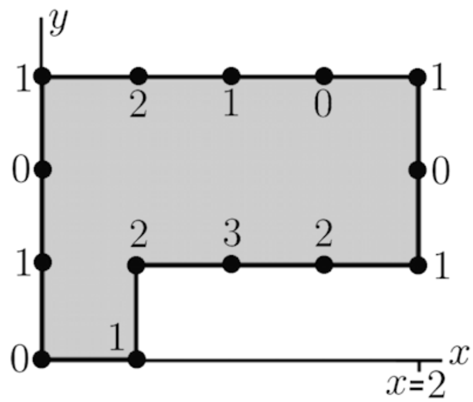


Figure 1: Problem 12.3.2

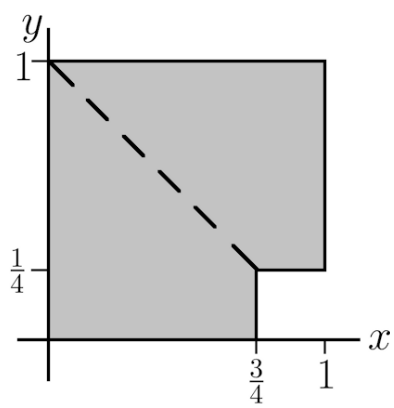


Figure 2: Problem 12.3.3

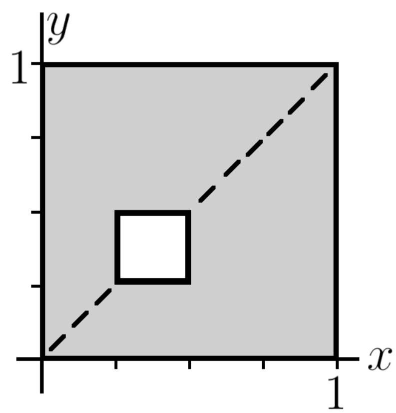


Figure 3: Problem 12.3.4

Section 12.4

12.4.3.1. Let $\epsilon \triangleq \dfrac{c\,\triangle t}{h}$. Our approximations are

$$\frac{\partial w_1}{\partial t} = \varrho_0 \frac{\partial u_1}{\partial t} = -P'(\varrho_0)\cdot \frac{\partial \varrho_1}{\partial x} = -c^2 \frac{\partial \varrho_1}{\partial x} = -c\,\frac{\partial w_2}{\partial x}$$

and

$$\frac{\partial w_2}{\partial t} = c\,\frac{\partial \varrho_1}{\partial t} = -c\varrho_0 \frac{\partial u_1}{\partial x} = -c\,\frac{\partial w_1}{\partial x},$$

so

$$\frac{\partial}{\partial t}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = -c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

12.4.3.3. Note that the statement of the problem has been corrected in the Errata page on the website.

Let $\epsilon \triangleq \dfrac{c\,\triangle t}{h}$. Substitute into (12.31) both the forward time difference approximation (12.32), $\dfrac{\partial w}{\partial t}(x_j,t_m) \approx \dfrac{w_j^{m+1}-w_j^m}{\triangle t}$, and the backward space difference approximation (12.38), $\dfrac{\partial w}{\partial x}(x_j,t_m) \approx \dfrac{w_j^m - w_{j-1}^m}{h}$, to get

$$\frac{w_j^{m+1}-w_j^m}{\triangle t} + c\,\frac{w_j^m - w_{j-1}^m}{h} = 0.$$

This can be re-written as

$$w_j^{m+1} = (1-\epsilon)w_j^m + \epsilon w_{j-1}^m.$$

Substitute in (12.35), that is, $w_j^m = Q^m e^{i\alpha j}$, and divide through by $Q^m e^{i\alpha j}$ to get

$$Q = 1-\epsilon+\epsilon e^{-i\alpha} = 1-\epsilon+\epsilon\cos\alpha - i\epsilon\sin\alpha,$$

hence

$$|Q|^2 = (1-\epsilon+\epsilon\cos\alpha)^2 + \epsilon^2\sin^2\alpha = ... = 1 + 2\epsilon(-1+\cos\alpha) + 2\epsilon^2(1-\cos\alpha)$$

that is,

$$|Q|^2 = 1 - 2\epsilon(1-\epsilon)(1-\cos\alpha).$$

If $\epsilon(1-\epsilon) \ge 0$, then the fact that $\cos\alpha \le 1$ implies that

$$|Q|^2 \le 1 - 2\epsilon(1-\epsilon) \le 1,$$

hence the approximation scheme is stable. So, for $0 \le \epsilon \le 1$, the approximation scheme is stable.

On the other hand, if $\epsilon(1-\epsilon) < 0$ then we may choose α so that $\cos\alpha \ne 1$, hence $|Q|^2 > 1$. So, for $\epsilon < 0$ and for $\epsilon > 1$, the approximation scheme is unstable.

12.4.3.5. Let $\epsilon \triangleq \dfrac{c\,\triangle t}{h}$. Substitute into (12.31) the backward time difference approximation and the forward space difference approximation to get

$$\frac{w_j^m - w_j^{m-1}}{\triangle\, t} + c\, \frac{w_{j+1}^m - w_j^m}{h} = 0.$$

Substitute in (12.35), that is, $w_j^m = Q^m e^{i\alpha j}$, and divide through by $Q^{m-1} e^{i\alpha j}$ to get

$$\frac{Q-1}{\triangle\, t} + c\, \frac{Q(e^{i\alpha}-1)}{h} = 0,$$

hence

$$\left(\frac{1}{\triangle\, t} + \frac{c}{h}(e^{i\alpha}-1) \right) Q = \frac{1}{\triangle\, t},$$

hence

$$R \triangleq \frac{1}{Q}$$

satisfies

$$R = \triangle\, t \left(\frac{1}{\triangle\, t} + \frac{c}{h}(e^{i\alpha}-1) \right) = 1 + \epsilon(e^{i\alpha}-1) = 1 - \epsilon + \epsilon\cos\alpha + i\epsilon\sin\alpha.$$

So,

$$|R|^2 = (1-\epsilon+\epsilon\cos\alpha)^2 + \epsilon^2\sin^2\alpha = ... = 1 - 2\epsilon(1-\epsilon)(1-\cos\alpha).$$

(a) Choose any α other than an integer multiple of 2π to see that $1-\cos\alpha > 0$, hence $|R|^2 > 1$ as long as $c < 0$, hence $|Q| < 1$ as long as $c < 0$. So, there is numerical stability for $c < 0$.

(b) Assume $c > 0$. The same formula for $|R|^2$ found above part (b) and the fact that $1 - \cos\alpha \geq 0$ implies that $|R|^2 < 1$ if $|\epsilon| < 1$ and α is not an integer multiple of 2π. So, $c > 0$ and $|\epsilon| < 1$ implies numerical instability.

Strangely, if $c > 0$ and $\epsilon > 1$ then there is numerical stability because $|R|^2 \geq 1$ implies $|Q| \leq 1$. ($|Q| = 1$ only if α is an integer multiple of 2π, in which case $Q = 1$ and $w_j^m \equiv 1$, the constant solution, which is not deficient.)

Section 12.5

12.5.4.1. The differential operator A is defined by

$$Ay = -y''(x) + x^2 y,$$

$f(x) = 1 + x$, and the usual inner product is given by $\langle u, v\rangle \triangleq \int_0^1 u(x)v(x)dx$. Using the same three functions $\phi_j(x)$ as in Example 12.7, we get

$$\mathsf{A} \triangleq [\langle \phi_i,\ A\phi_j\rangle]_{\substack{1\le i\le 3\\ 1\le j\le 3}} = \begin{bmatrix} \frac{1}{6} - \frac{1}{4\pi^2} + \frac{\pi^2}{2} & -\frac{8}{9\pi^2} & \frac{3}{16\pi^2} \\ -\frac{8}{9\pi^2} & \frac{1}{6} - \frac{1}{16\pi^2} + 2\pi^2 & -\frac{24}{25\pi^2} \\ \frac{3}{16\pi^2} & -\frac{24}{25\pi^2} & \frac{1}{6} - \frac{1}{36\pi^2} + \frac{9\pi^2}{2} \end{bmatrix}.$$

As in Example 12.7,

$$\mathbf{f} = [\,\langle\phi_1, f\rangle \quad \langle\phi_2, f\rangle \quad \langle\phi_3, f\rangle\,]^T = \begin{bmatrix} \frac{3}{\pi} & -\frac{1}{2\pi} & \frac{1}{\pi}\end{bmatrix}^T.$$

So,

$$\mathbf{c} = [\,c_1 \quad c_2 \quad c_3\,]^T = \mathsf{A}^{-1}\mathbf{f} \approx [\,0.187969 \quad -0.00711276 \quad 0.00704504\,]^T,$$

and the approximate solution is

$$y(x) \approx 0.187969\sin(\pi x) - 0.00711276\sin(2\pi x) + 0.00704504\sin(3\pi x).$$

In the figure we graphed the approximate solution as a red, dashed curve; Mathematica$^{\text{TM}}$produced a numerical approximate solution of the ODE-BVP using NDSolve and then we graphed it as the solid, blue curve. We see that the approximate solution found by Galerkin's method works extremely well.

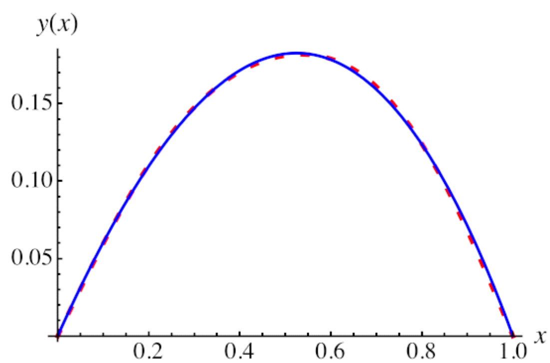


Figure 4: Answer for problem 12.5.4.1

12.5.4.3. It makes sense for the functions $\phi_j(x)$ to satisfy the boundary conditions. Let's use $\phi_1(x) = \cos\left(\frac{\pi x}{2}\right)$, $\phi_2(x) = \cos\left(\frac{3\pi x}{2}\right)$, $\phi_3(x) = \cos\left(\frac{5\pi x}{2}\right)$. The approximate solution is to be of the form

$$y(x) = c_1\cos\left(\frac{\pi x}{2}\right) + c_2\cos\left(\frac{3\pi x}{2}\right) + c_3\cos\left(\frac{5\pi x}{2}\right).$$

As in Example 12.7, the differential operator A is defined by

$$Ay = -y''(x) + xy, \quad \text{for functions satisfying BCs } y'(0) = y(1) = 0,$$

$f(x) = 1+x$, and the inner product is given by $\langle u, v\rangle \triangleq \int_0^1 u(x)v(x)dx$. Using Mathematica we calculated the matrix

$$\mathsf{A} \triangleq [\langle \phi_i,\ A\phi_j\rangle]_{\substack{1\le i\le 3\\ 1\le j\le 3}} = \begin{bmatrix} \frac{1}{4} - \frac{1}{\pi^2} + \frac{\pi^2}{8} & -\frac{1}{\pi^2} & -\frac{1}{9\pi^2} \\ -\frac{1}{\pi^2} & \frac{1}{4} - \frac{1}{9\pi^2} + \frac{9\pi^2}{8} & -\frac{1}{\pi^2} \\ -\frac{1}{9\pi^2} & -\frac{1}{\pi^2} & \frac{1}{4} - \frac{1}{25\pi^2} + \frac{25\pi^2}{8} \end{bmatrix}$$

and the vector

$$\mathbf{f} = [\,\langle \phi_1, f\rangle \quad \langle \phi_2, f\rangle \quad \langle \phi_3, f\rangle\,]^T = \begin{bmatrix} \dfrac{4(\pi-1)}{\pi^2} & -\dfrac{4(3\pi+1)}{9\pi^2} & \dfrac{4(5\pi-1)}{25\pi^2} \end{bmatrix}^T.$$

So,

$$\mathbf{c} = [\,c_1 \quad c_2 \quad c_3\,]^T = \mathsf{A}^{-1}\mathbf{f} \approx [\,0.625314 \quad -0.0357342 \qquad 0.00777959\,]^T,$$

and the approximate solution is

$$y(x) \approx 0.625314\cos\left(\frac{\pi x}{2}\right) - 0.0357342\cos\left(\frac{3\pi x}{2}\right) + 0.00777959\cos\left(\frac{5\pi x}{2}\right).$$

In the figure we graphed the approximate solution as a red, dashed curve; Mathematica™ produced a numerical approximate solution of the ODE-BVP using NDSolve and then we graphed it as the solid, blue curve. We see that the approximate solution found by Galerkin's method works extremely well.

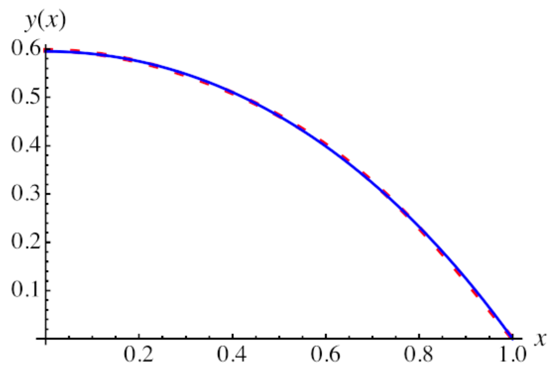


Figure 5: Answer for problem 12.5.4.3

12.5.4.5. The differential operator A is defined by

$$Au = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}, \quad \text{for functions satisfying the four BCs,}$$

$f(x,y) = -(1+xy)$, and the usual inner product is given by $\langle u, v\rangle \triangleq \int_0^1 \int_0^1 u(x,y)v(x,y)dx\,dy$. As in Example 12.8, we find three test functions that satisfy the boundary conditions: Let

$$\phi_1(x,y) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right), \quad \phi_2(x,y) = \cos\left(\frac{3\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right), \quad \phi_3(x,y) = x^2y^2(1-x)(1-y).$$

The approximate solution is to be of the form

$$u(x,y) = c_1\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right) + c_2\cos\left(\frac{3\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right) + c_3x^2y^2(1-x)(1-y).$$

Using Mathematica we calculated the matrix

$$\mathsf{A} \triangleq [\langle \phi_i,\ A\phi_j\rangle]_{\substack{1\le i\le 3\\ 1\le j\le 3}} = \begin{bmatrix} -\frac{\pi^2}{8} & 0 & -\frac{512(-3+\pi)^2}{\pi^6} \\ 0 & -\frac{5\pi^2}{8} & \frac{2560(-3+\pi)(1+\pi)}{27\pi^6} \\ -\frac{512(-3+\pi)^2}{\pi^6} & \frac{2560(-3+\pi)(1+\pi)}{27\pi^6} & -\frac{4}{1575} \end{bmatrix}$$

and the vector

$$\mathbf{f} = [\,\langle \phi_1, f\rangle \quad \langle \phi_2, f\rangle \quad \langle \phi_3, f\rangle\,]^T = \begin{bmatrix} -\dfrac{8(2-2\pi+\pi^2)}{\pi^4} & \dfrac{8(-2-2\pi+3\pi^2)}{9\pi^4} & -\dfrac{17}{1800} \end{bmatrix}^T.$$

So,

$$\mathbf{c} = [\, c_1 \; c_2 \; c_3 \,]^T = \mathsf{A}^{-1}\mathbf{f} \approx [\, 0.35531161853 \qquad -0.013588155903 \qquad 1.9155528075 \,]^T .$$

So, the approximate solution is

$$u(x,y) = 0.35531161853 \cos\left(\tfrac{\pi x}{2}\right) \cos\left(\tfrac{\pi y}{2}\right) - 0.013588155903 \cos\left(\tfrac{3\pi x}{2}\right) \cos\left(\tfrac{\pi y}{2}\right)$$
$$+1.9155528075 x^2 y^2 (1-x)(1-y).$$

In the figure we graphed the absolute error, $|u(x,y) - v(x,y)|$, where $u(x,y)$ is the approximate solution and $v(x,y)$ is a truncated version of the exact solution that we obtained through the method of separation of variables, as in Section 11.4.4. Even though we used 900 functions in the truncated exact solution and only 3 functions in the the solution using Galerkin's method, the error appears to be only about 6%.

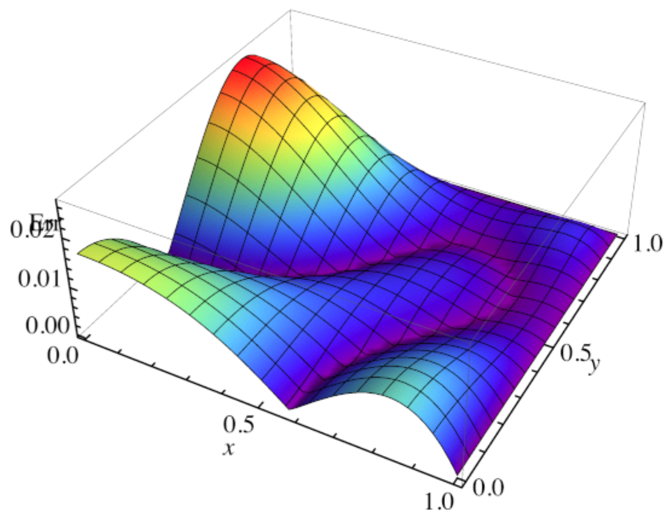


Figure 6: Answer for problem 12.5.4.5

Section 13.1.3

13.1.3.1. Let x be the width of the printed region in inches. The area of the printed region is 40 in^2, so the length of the printed region is $\dfrac{40}{x}$ in. Because of the margins, the dimensions of the paper are $(x+2)$ in and $\left(\dfrac{40}{x}+4\right)$ in, so the area of the paper is

$$A = f(x) \triangleq (x+2)\left(\frac{40}{x}+4\right) = 48 + 4x + \frac{80}{x},$$

where $0 < x < \infty$. We will try to use Theorem 13.3 in Section 13.1 to find the global minimum of $f(x)$ on the domain $0 < x < \infty$. We have

$$f'(x) = 4 - \frac{80}{x^2} = \frac{4(x^2-20)}{x^2}\,,$$

so $f'(x) < 0$ for $0 < x < \sqrt{20}$, $f'(\sqrt{20}) = 0$, and $f'(x) > 0$ for $\sqrt{20} < x < \infty$. By Theorem 13.3 $\big($applied to the function $-f(x)\big)$, the minimum area occurs when $x = \sqrt{20}$. So, the paper of minimum area has width$=(x+2) = \sqrt{20}+2 = 2(1+\sqrt{5})$ in and length$=\dfrac{40}{\sqrt{20}}+4 = 2\sqrt{20}+4 = 4(1+\sqrt{5})$ in.

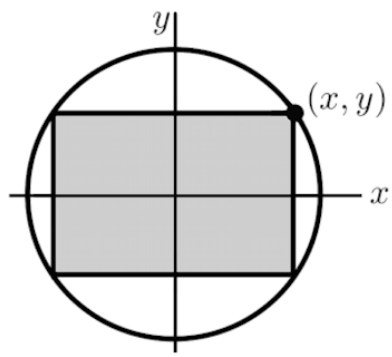


Figure 1: Example 13.1.3.3

13.1.3.3. Using the quantity x defined implicitly in the figure, the width of the cross section of the beam is $2x$, in m, and the height of the cross section is $2y$, in m. Because the point (x,y) lies on the circular boundary of the log, $x^2+y^2 = \left(\frac{1}{2}\right)^2$. So, the strength of the beam is

$$f(x) = k(cross\ section\ width)(square\ of\ cross\ section\ height) = k2x(2y)^2 = 8kx\left(\frac{1}{4}-x^2\right) = k(2x-8x^3),$$

where k is a constant of proportionality and x is in the interval $\mathcal{I} \triangleq \left[0,\ \dfrac{1}{2}\right]$.

We calculate

$$f'(x) = k(2-24x^3) = 2k(1-12x^2),$$

so the only critical points in the interval $\left[0,\ \dfrac{1}{2}\right]$ is at $x = \sqrt{\dfrac{1}{12}} = \dfrac{1}{2\sqrt{3}}$. We have that $f'(x) > 0$ for $0 \le x < \dfrac{1}{2\sqrt{3}}$ and $f'(x) < 0$ for $\dfrac{1}{2\sqrt{3}} < x \le \dfrac{1}{2}$. By Theorem 13.3 in Section 13.1, the global maximum of $f(x)$ on the interval $\mathcal{I}$ is at $x^\star = \dfrac{1}{2\sqrt{3}}$. So, the strongest beam has cross section that has width $2x = 2\cdot\dfrac{1}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ m and height $2y = 2\sqrt{\dfrac{1}{4}-(x^\star)^2} = ... = \dfrac{2}{\sqrt{6}} = \sqrt{\dfrac{2}{3}}$ m.

13.1.3.5. Define $f(t) \triangleq \dot{N}(t)$, so $f(t)$ measures the rate at which people become aware that they have the illness, and measure t in days. So, the hospital should be ready for the greatest influx of patients at the time when $f(t)$ is at a global maximum over the interval $0 < t < \infty$.

We calculate

$$f(t) = \dot{N}(t) = \frac{d}{dt}\Big[2000(1+999e^{-2t})^{-1}\Big] = \frac{(-1)\cdot 2000\cdot 999\cdot(-2)e^{-2t}}{(1+999e^{-2t})^2} = 3996000\, e^{-2t}(1+999e^{-2t})^{-2}$$

so,

$$\dot{f}(t) = \frac{d}{dt}\Big[f(t)\Big] = 3996000\Big(-2e^{-2t}(1+999e^{-2t})^{-2} + e^{-2t}\cdot(-2)\cdot(-2)\cdot 999e^{-2t}(1+999e^{-2t})^{-3}\Big)$$

$$= 7992000\, e^{-2t}(1+999e^{-2t})^{-3}\Big(-(1+999e^{-2t}) + 1998e^{-2t}\Big)$$

$$= 7992000\, e^{-2t}(1+999e^{-2t})^{-3}\Big(-1+999e^{-2t}\Big).$$

The only critical number t satisfies $0 = -1 + 999e^{-2t}$, that is, $\frac{1}{999} = e^{-2t}$, that is, $t^\star = -\frac{1}{2}\ln\left(\frac{1}{999}\right) \approx 3.4533773889$.

Because

$$\Big(-1+999e^{-2t}\Big) = \left\{\begin{array}{ll} -, & 0<t<t^\star \\ 0, & t=t^\star \\ +, & t>t^\star \end{array}\right\} \quad\Rightarrow\quad \dot{f}(t) = \left\{\begin{array}{ll} -, & 0<t<t^\star \\ 0, & t=t^\star \\ +, & t>t^\star \end{array}\right\},$$

by Theorem 13.3 in Section 13.1, the hospital should be ready for the greatest influx of patients at $t = t^\star$, that is, at about 3 days and 11 hours after the beginning of the epdiemic.

13.1.3.7. Let the sides of the right triangle have lengths x, y, and $\sqrt{x^2+y^2}$. The area of a right triangle is one half the base length times the height, so $1000\ \mathrm{m}^2 = \dfrac{1}{2}\,xy$ implies that $y = \dfrac{2000}{x}$ m. The perimeter fencing has total length

$$f(x) \triangleq x + y + \sqrt{x^2+y^2} = x + \frac{2000}{x} + \sqrt{x^2 + \frac{4\cdot 10^6}{x^2}}.$$

We calculate that for all $x > 0$ there exists

$$f'(x) = 1 - \frac{2000}{x^2} + \frac{1}{2}\cdot\Big(x^2 + \frac{4\cdot 10^6}{x^2}\Big)^{-1/2}\cdot\Big(2x - \frac{8\cdot 10^6}{x^3}\Big)$$

$$= \Big(x^2 + \frac{10^6}{x^2}\Big)^{-1/2}\left(\Big(1-\frac{2000}{x^2}\Big)\cdot\Big(x^2+\frac{4\cdot 10^6}{x^2}\Big)^{1/2} + x\Big(1-\frac{4\cdot 10^6}{x^4}\Big)\right).$$

The critical numbers x satisfy

$$\Big(1-\frac{2000}{x^2}\Big)\cdot\Big(x^2+\frac{4\cdot 10^6}{x^2}\Big)^{1/2} = -x\Big(1-\frac{4\cdot 10^6}{x^4}\Big),$$

hence satisfy

$$\Big(1-\frac{2000}{x^2}\Big)^2\cdot\Big(x^2+\frac{4\cdot 10^6}{x^2}\Big) = x^2\Big(1-\frac{4\cdot 10^6}{x^4}\Big)^2,$$

that is

$$\Big(1-\frac{4000}{x^2}+\frac{4\cdot 10^6}{x^4}\Big)\cdot\Big(x^2+\frac{4\cdot 10^6}{x^2}\Big) = x^2\Big(1-\frac{8\cdot 10^6}{x^4}+\frac{16\cdot 10^{12}}{x^8}\Big),$$

that is

$$\cancel{x^2} - 4000 + \frac{4\cdot 10^6}{x^2} + \frac{4\cdot 10^6}{x^2} - \frac{16\cdot 10^9}{x^4} + \cancel{\frac{16\cdot 10^{12}}{x^6}} = \cancel{x^2} - \frac{8\cdot 10^6}{x^2} + \cancel{\frac{16\cdot 10^{12}}{x^6}}.$$

This can be rewritten as
$$0=-4000+\frac{16\cdot 10^6}{x^2}-\frac{16\cdot 10^9}{x^4},$$
or, equivalently, after multiplying through by $-\dfrac{x^4}{4000}$, as
$$0=x^4-4000x^2+4\cdot 10^6=(x^2-2000)^2.$$
So, the only critical number is at $x^\star=\sqrt{2000}=20\sqrt{5}$. [Note that we squared both sides of an equation to get to this point, so we do need to check that we have not created a "spurious" solution. In fact, $f'(20\sqrt{5})=0$.] Because the function $f(x)$ is continuously differentiable for all $x>0$ and $\lim_{x\to 0^+}f(x)=\infty$ and $\lim_{x\to\infty}f(x)=\infty$, we conclude that the global minimum of the function $f(x)$ is at $x=x^\star=20\sqrt{5}$. The sides of the best triangle have lengths $x=x^\star=20\sqrt{5}$,
$$y=\frac{2000}{x^\star}=\frac{2000}{20\sqrt{5}}=20\sqrt{5},$$
and
$$\sqrt{(x^\star)^2+\frac{4\cdot 10^6}{(x^\star)^2}}=\sqrt{2000+\frac{4\cdot 10^6}{2000}}=\sqrt{4000}=20\sqrt{10},$$
in m.

13.1.3.9. We will explain the "if, and only if, " in two parts. In part (a) we will explain why $f''(x)\ge 0$ for all x in $[a,b]$ implies that f is convex on $[a,b]$. In part (b) we will explain why f being convex on $[a,b]$ implies that $f''(x)\ge 0$ for all x in $[a,b]$.

(a) Suppose $f''(x)\ge 0$ for all x in $[a,b]$. Choose any $x<y$ in $[a,b]$ and any λ with $0<\lambda<1$. Define $z\triangleq \lambda x+(1-\lambda)y$. Note that $x<z<y$.

By the Mean Value Theorem of Calculus I, there exists ξ with $x\le\xi\le z$ and $\dfrac{f(z)-f(x)}{z-x}=f'(\xi)$; likewise, there exists η with $z\le\eta\le y$ and $\dfrac{f(y)-f(z)}{y-z}=f'(\eta)$. Because $f''\ge 0$ on $[a,b]$, $\xi\le z$ implies $f'(\xi)\le f'(z)$; likewise, $z\le\eta$ implies $f'(z)\le f'(\eta)$. Putting this all together, we get
$$(\star)\quad \frac{f(z)-f(x)}{z-x}=f'(\xi)\le f'(z)\le f'(\eta)=\frac{f(y)-f(z)}{y-z}.$$
Because $z-x>0$ and $y-z>0$, we can multiply both extreme sides of $(\star)$ by $(z-x)(y-z)$ without disturbing the direction of the inequality. This gives
$$(y-z)\big(f(z)-f(x)\big)\le (z-x)\big(f(y)-f(z)\big),$$
hence
$$(\star\star)\quad (y-z)f(z)+(z-x)f(z)\le (y-z)f(x)+(z-x)f(y).$$
But, $y-z=y-\big(\lambda x+(1-\lambda)y\big)=\lambda(y-x)$ and $z-x=\big(\lambda x+(1-\lambda)y\big)-x=(1-\lambda)(y-x)$ substituted into $(\star\star)$ imply
$$(y-x)f(z)\le\lambda(y-x)f(x)+(1-\lambda)(y-x)f(y).$$
Because $y-x>0$, we can divide through by $(y-x)$ without disturbing the direction of the inequality to get
$$f\big(\lambda x+(1-\lambda)y\big)=f(z)\le\lambda f(x)+(1-\lambda)f(y).$$
Since this was true for all such x,y,λ, we conclude that f is convex on $[a,b]$.

(b) Choose any x in the interval (a,b). By Theorem 8.12(b) in Section 8.6,
$$f''(x)=\lim_{h\to 0}\ \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}\,.$$

[See a note later about how to get this result directly without having to use Theorem 8.12(b) in Section 8.6.]

Now, $f(x)$ being convex implies that, using $\lambda = \frac{1}{2}$,

$$f(x+h) = f\Big(\frac{1}{2}x + \frac{1}{2}(x+2h)\Big) \le \frac{1}{2}f(x) + \frac{1}{2}f(x+2h),$$

hence

$$2f(x+h) \le f(x+2h) + f(x),$$

hence

$$f(x+2h) - 2f(x+h) + f(x) \ge 0.$$

It follows that

$$f''(x) = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \ge 0.$$

If $x_0 = a$ or $x_0 = b$, then $f(x)$ being twice continuously differentiable function on the interval $[a,b]$ implies that $f''(x_0) = \lim_{x\to x_0} f''(x) \ge 0$.

Note: If you don't want to use Theorem 8.12(b) in Section 8.6, argue this way: By definition,

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{\lim_{\delta\to 0} \frac{f(x+h+\delta)-f(x+h)}{\delta} - \lim_{\delta\to 0} \frac{f(x+\delta)-f(x)}{\delta}}{h}$$

$$= \lim_{h\to 0 \text{ and } \delta\to 0} \frac{f(x+h+\delta) - f(x+h) - f(x+\delta) + f(x)}{h\cdot\delta}.$$

Since we are given that $f(x)$ is twice continuously differentiable, the double limiting process should exist and be the same no matter how $h \to 0$ and $\delta \to 0$, so the limit should agree with the single limit in which $\delta = h \to 0$. So,

$$f''(x) = \lim_{h\to 0} \frac{f(x+h+h) - f(x+h) - f(x+h) + f(x)}{h\cdot h} = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}.$$

13.1.3.11. Suppose that there are points x and y, possibly equal, in $S \triangleq \{x \text{ is in } \mathcal{I} : f(x) \le M\}$, a subset of the convex set $\mathcal{I}$. Then $f(x) \le M$ and $f(y) \le M$. Also, for $0 < t < 1$, $\big((1-t)x + ty\big)$ is in $\mathcal{I}$ and, because the function f is convex on the convex set $\mathcal{I}$,

$$f\big((1-t)x + ty\big) \le (1-t)f(x) + tf(y) \le (1-t)M + tM = M,$$

hence $\big((1-t)x + ty\big)$ is in S. This being true for all x and y in S and $0 < t < 1$, we have that S is convex, unless it is empty.

13.1.3.13. We will explain the "if, and only if, " in two parts. In (a) we will explain why the existence of a positive γ for which $f''(x) \ge \gamma$ for all x in $[a,b]$ implies that f is strictly convex on $[a,b]$. In (b) we will explain why f being strictly convex on $[a,b]$ implies that that there exists a positive γ for which $f''(x) \ge \gamma$ for all x in $[a,b]$.

(a) Suppose there exists a positive γ for which $f''(x) \ge \gamma$ for all x in $[a,b]$. Choose any $x < y$ in $[a,b]$ and any λ with $0 < \lambda < 1$. Define $z \triangleq \lambda x + (1-\lambda)y$. Note that $x < z < y$.

By the Mean Value Theorem of Calculus I, there exists ξ with $x < \xi < z$ and $\dfrac{f(z) - f(x)}{z - x} = f'(\xi)$; likewise, there exists η with $z < \eta < y$ and $\dfrac{f(y) - f(z)}{y - z} = f'(\eta)$. Because $f'' \ge \gamma$ on $[a,b]$, $\xi < z$ implies $f'(\xi) < f'(z)$; likewise, $z < \eta$ implies $f'(z) < f'(\eta)$. Putting this all together, we get

$$(\star) \quad \frac{f(z) - f(x)}{z - x} = f'(\xi) < f'(z) < f'(\eta) = \frac{f(y) - f(z)}{y - z}.$$

Because $z - x > 0$ and $y - z > 0$, we can multiply both extreme sides of $(\star)$ by $(z-x)(y-z)$ without disturbing the direction of the inequality. This gives

$$(y-z)\big(f(z)-f(x)\big) < (z-x)\big(f(y)-f(z)\big),$$

hence

$$(\star\star) \quad (y-z)f(z) + (z-x)f(z) < (y-z)f(x) + (z-x)f(y).$$

But, $y - z = y - \big(\lambda x + (1-\lambda)y\big) = \lambda(y-x)$ and $z - x = \big(\lambda x + (1-\lambda)y\big) - x = (1-\lambda)(y-x)$ substituted into $(\star\star)$ imply

$$(y-x)f(z) < \lambda(y-x)f(x) + (1-\lambda)(y-x)f(y).$$

Because $y - x > 0$, we can divide through by $(y-x)$ without disturbing the direction of the inequality to get

$$f\big(\lambda x + (1-\lambda)y\big) = f(z) < \lambda f(x) + (1-\lambda)f(y).$$

Since this was true for all such x, y, λ, we conclude that f is strictly convex on $[\,a, b\,]$.

(b) Choose any x in the interval (a, b). By Theorem 8.12(b) in Section 8.6,

$$f''(x) = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}.$$

[See a note later about how to get this result directly without having to use Theorem 8.12(b) in Section 8.6.]

Now, $f(x)$ being strictly convex implies that, using $t = \frac{1}{2}$,

$$f(x+h) = f\Big(\frac{1}{2}x + \frac{1}{2}(x+2h)\Big) < \frac{1}{2}f(x) + \frac{1}{2}f(x+2h),$$

hence

$$2f(x+h) < f(x+2h) + f(x),$$

hence

$$f(x+2h) - 2f(x+h) + f(x) > 0.$$

Consider the function $g(h) \triangleq f(x+2h) - 2f(x+h) + f(x)$. Choose η to be small enough that $x \pm 2\eta$ are in the interval $[a, b]$. Note that $g(h)$ is continuous and positive on the interval $[-\eta, \eta]$. It follows that on $[-\eta, \eta]$, the minimum value of $g(h)$ exists and equals a number $\alpha > 0$. Define $\gamma = \eta^{-2}\alpha$. Then for all h in the interval $[-\eta, \eta]$,

$$\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = \frac{g(h)}{h^2} \geq \frac{\alpha}{h^2} \geq \frac{\alpha}{\eta^2} = \gamma.$$

It follows that

$$f''(x) = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \geq \gamma.$$

If $x_0 = a$ or $x_0 = b$, then $f(x)$ being twice continuously differentiable function on the interval $[a, b]$ implies that $f''(x_0) = \lim_{x\to x_0} f''(x) \geq \gamma$.

Note: If you don't want to use Theorem 8.12(b) in Section 8.6, argue this way: By definition,

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{\lim_{\delta\to 0} \frac{f(x+h+\delta) - f(x+h)}{\delta} - \lim_{\delta\to 0} \frac{f(x+\delta) - f(x)}{\delta}}{h}$$

$$= \lim_{h\to 0 \text{ and } \delta\to 0} \frac{f(x+h+\delta) - f(x+h) - f(x+\delta) + f(x)}{h\cdot\delta}.$$

Since we are given that $f(x)$ is twice continuously differentiable, the double limiting process should exist and be the same no matter how $h \to 0$ and $\delta \to 0$, so the limit should agree with the single limit in which $\delta = h \to 0$. So,

$$f''(x) = \lim_{h\to 0} \frac{f(x+h+h) - f(x+h) - f(x+h) + f(x)}{h\cdot h} = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}.$$

Section 13.2.3

13.2.3.1. (a) $\boldsymbol{\nabla} f(\mathbf{x})$ exists at all $\mathbf{x}=(x,y)$, so we need only to find all (x,y) for which both $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$, that is,

$$\left\{\begin{array}{rrl}\dfrac{\partial f}{\partial x}= & 2x+y^2 & =0\\[2ex] \dfrac{\partial f}{\partial y}= & 2xy+6y & =0\end{array}\right\},$$

that is,

$$(\star)\quad\left\{\begin{array}{rl}2x+y^2 & =0\\[2ex] 2y(x+3) & =0\end{array}\right\}.$$

The second equation is true when (1) $y=0$ or (2) $x=-3$.

Substitute (1) into the first equation to get $2x+0=0$, hence $x=0$. Alternatively, substitute (2) into the first equation to get $2(-3)+y^2=0$, hence $y=\pm\sqrt{6}$.

So, there are exactly three critical points: $(x,y)=(0,0)$, $(-3,\sqrt{6})$, and $(-3,-\sqrt{6})$.

(b) The Hessian matrix is

$$D^2f(x,y)=\begin{bmatrix}2 & 2y\\ 2y & 2(x+3)\end{bmatrix},$$

so

$$\det D^2f(0,0)=\begin{vmatrix}2 & 0\\ 0 & 6\end{vmatrix}=12>0,$$

and $a_{11}>0$. It follows that the matrix A is positive definite, so $(0,0)$ is a local minimum point.

Also,

$$\det D^2f(-3,\pm\sqrt{6})=\begin{vmatrix}2 & \pm2\sqrt{6}\\ \pm2\sqrt{6} & 0\end{vmatrix}=-24<0,$$

so both $(-3,\sqrt{6})$, and $(-3,-\sqrt{6})$ are saddle points.

13.2.3.3. (a) $\boldsymbol{\nabla} f(\mathbf{x})$ exists at all $\mathbf{x}=(x,y)$, so we need only to find all (x,y) for which both $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$, that is,

$$\left\{\begin{array}{rrl}\dfrac{\partial f}{\partial x}= & 6xy-6x & =0\\[2ex] \dfrac{\partial f}{\partial y}= & 3y^2+3x^2-6y & =0\end{array}\right\},$$

that is,

$$(\star)\quad\left\{\begin{array}{rl}6x(y-1) & =0\\[2ex] 3(x^2+y^2-2y) & =0\end{array}\right\}.$$

The first equation is true when (1) $x=0$ or (2) $y=1$.

Substitute (1) into the second equation to get $3(0+y^2-2y)=0$, hence $y=0$ or $y=2$. Alternatively, substitute (2) into the second equation to get $3(x^2+1-2)=0$, hence $x=-1$ or $x=1$.

So, there are exactly four critical points: $(x,y)=(0,0)$, $(0,2)$, $(-1,1)$, and $(1,1)$.

(b) The Hessian matrix is

$$D^2f(x,y)=\begin{bmatrix}6(y-1) & 6x\\ 6x & 6(y-1)\end{bmatrix},$$

so

$$\det D^2f(0,0)=\begin{vmatrix}-6 & 0\\ 0 & -6\end{vmatrix}=36>0,$$

and $a_{11} < 0$. It follows that the matrix $-A$ is positive definite, so the matrix A is negative definite, so $(0,0)$ is a local maximum point.

Also,

$$\det D^2 f(0,2) = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0,$$

and $a_{11} > 0$. It follows that the matrix A is positive definite, hence $(0,2)$ is a local minimum point.

Also,

$$\det D^2 f(\pm 1, 1) = \begin{vmatrix} 0 & \pm 6 \\ \pm 6 & 0 \end{vmatrix} = -36 < 0,$$

so both $(-1,1)$, and $(1,1)$ are saddle points.

13.2.3.5. We will use the method of Lagrange multipliers, with $f(x,y) \triangleq 4 + x - x^2 - 2y^2$ and $g(x,y) = x^2 + y^2 - 2$. At a maximizer $\mathbf{x}^\star = (x,y)$ that satisfies the constraint $g(x,y) = 0$, there is a Lagrange multiplier λ such that

$$(1-2x)\hat{\boldsymbol{\imath}} - 4y\,\hat{\boldsymbol{\jmath}} = \boldsymbol{\nabla} f(\mathbf{x}^\star) = \lambda \boldsymbol{\nabla} g(\mathbf{x}^\star) = \lambda\big(2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}}\big),$$

as long as the technical requirement, $\mathbf{0} \neq \boldsymbol{\nabla} g(\mathbf{x}^\star) = 2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}}$, that is, $\mathbf{x}^\star \neq \mathbf{0}$, is true. So,

$$\left\{ \begin{array}{rl} (1-2x) & = 2\lambda x \\ \\ -4y & = 2\lambda y \end{array} \right\},$$

that is,

$$(\star) \qquad \left\{ \begin{array}{rl} 2(1+\lambda)x & = 1 \\ \\ 2(2+\lambda)y & = 0 \end{array} \right\},$$

The second equation is true when (1) $y = 0$ or (2) $\lambda = -2$.

Substitute (1) into the first equation to get $x = \dfrac{1}{2(1+\lambda)}$, as long as $\lambda \neq -1$. Alternatively, substitute (2) into the first equation to get $2(-1)x = 1$, hence $x = -\dfrac{1}{2}$.

So far, the candidates for finding a maximizer are $(x,y) = \Big(\dfrac{1}{2(1+\lambda)}\,,\ 0\Big)$, for $\lambda \neq -1$, and $(x,y) = \Big(-\dfrac{1}{2}\,,\ y\Big)$, for any y. Which, if any, of these candidates satisfies the constraint $0 = g(x,y) = x^2 + y^2 - 2$?

The first choice, $(x,y) = \Big(\dfrac{1}{2(1+\lambda)}\,,\ 0\Big)$, satisfies $0 = x^2 + y^2 - 2$ only at the points $(x,y) = (\pm\sqrt{2},\ 0)$.

The second choice, $(x,y) = \Big(-\dfrac{1}{2},\ y\Big)$, satisfies $0 = x^2 + y^2 - 2$ only if $0 = \frac{1}{4} + y^2 - 2$, that is, only at the points $(x,y) = \Big(-\dfrac{1}{2}\,,\ \pm\dfrac{\sqrt{7}}{2}\Big)$.

To summarize, the only possible maximizers are $(x,y) = (\pm\sqrt{2},\ 0)$ and $\Big(-\dfrac{1}{2}\,,\ \pm\dfrac{\sqrt{7}}{2}\Big)$. We calculate

$$f(\pm\sqrt{2},\ 0) = 4 \pm \sqrt{2} - 2 - 2\cdot 0 = 2 \pm \sqrt{2}$$

and

$$f\Big(-\frac{1}{2}\,,\ \pm\frac{\sqrt{7}}{2}\Big) = 4 - \frac{1}{2} - \frac{1}{4} - 2\cdot\frac{7}{4} = -\frac{1}{4}\,.$$

So, the absolute maximum value $2 + \sqrt{2}$ is achieved at $(x,y) = (\sqrt{2}, 0)$.

13.2.3.7. A maximizer or minimizer can occur either strictly inside the disk $x^2 + y^2 \leq 4$, where we will use Fermat's Theorem, that is, Theorem 13.6 in Section 13.2, or on the boundary $x^2 + y^2 = 4$, where we will use the method of Lagrange multipliers.

Define $f(x,y) \triangleq 2x^2+y^2$ and $g(x,y)=x^2+y^2-4$.

Inside the disk, $\mathbf{0}=\boldsymbol{\nabla} f(\mathbf{x}^\star)=4x\,\hat{\boldsymbol{\imath}}+2y\,\hat{\boldsymbol{\jmath}}$ only at $(x,y)=(0,0)$.

On the other hand, at a maximizer $\mathbf{x}^\star=(x,y)$ on the boundary of the disk, there is a Lagrange multiplier λ such that

$$4x\,\hat{\boldsymbol{\imath}}+2y\,\hat{\boldsymbol{\jmath}}=\boldsymbol{\nabla} f(\mathbf{x}^\star)=\lambda\boldsymbol{\nabla} g(\mathbf{x}^\star)=\lambda\big(2x\,\hat{\boldsymbol{\imath}}+2y\,\hat{\boldsymbol{\jmath}}\big),$$

as long as the technical requirement, $\mathbf{0}\neq\boldsymbol{\nabla} g(\mathbf{x}^\star)=2x\,\hat{\boldsymbol{\imath}}+2y\,\hat{\boldsymbol{\jmath}}$, that is, $\mathbf{x}^\star\neq\mathbf{0}$, is true. So,

$$\left\{\begin{array}{ll} 4x & =2\lambda x \\ \\ 2y & =2\lambda y \end{array}\right\},$$

that is,

$$(\star)\quad \left\{\begin{array}{ll} 2(2-\lambda)x & =0 \\ \\ 2(1-\lambda)y & =0 \end{array}\right\},$$

The first equation is true when (1) $x=0$ or (2) $\lambda=2$. The second equation is true when (3) $y=0$ or (4) $\lambda=1$.

In principle, there are four possibilities:

$$\begin{array}{lllllll}
(1) & \text{and} & (3): & x=0 & \text{and} & y=0: & (x,y)=(0,0)\\
\\
(1) & \text{and} & (4): & x=0 & \text{and} & \lambda=1: & (x,y)=(0,y)\\
\\
(2) & \text{and} & (3): & \lambda=2 & \text{and} & y=0: & (x,y)=(x,0)\\
\\
(2) & \text{and} & (4): & \lambda=2 & \text{and} & \lambda=1: & \textit{impossible}
\end{array}$$

So, there are candidates for finding a maximizer are $(x,y)=(0,\ y)$, for any y, and $(x,y)=(x,\ 0)$, for any x. Note that the case of $(x,y)=(0,0)$ is subsumed by the candidates already listed.

Which, if any, of these candidates satisfies the constraint $0=g(x,y)=x^2+y^2-4$?

The first choice, $(x,y)=(0,y)$, satisfies $0=x^2+y^2-4$ only at the points $(x,y)=(0,\pm2)$.

The second choice, $(x,y)=(x,0)$, satisfies $0=x^2+y^2-4$ only at the points $(x,y)=(2,\ 0)$.

To summarize, the only possible extrema are at $(x,y)=(0,0)$, from Fermat's Theorem, and at $(x,y)=(0,\pm2)$ and $(\pm2,\ 0)$, from the method of Lagrange multipliers. We calculate

$$f(0,0)=0,$$

$$f(0,\ \pm2)=2\cdot0+4=4,$$

and

$$f(\pm2,\ 0)=2\cdot4+0=8,$$

So, the absolute maximum value 8 is achieved at $(x,y)=(\pm2,\ 0)$, and the absolute minimum value 0 is achieved at $(x,y)=(0,0)$.

13.2.3.9. [Note that the problem has been corrected to include the assumption that A is real.] If the real matrix is $A=[\,a_{ij}\,]$, the characteristic equation is

$$|\ A-\lambda I\ |=\begin{vmatrix} a_{11}-\lambda & a_{12}\\ a_{12} & a_{22}-\lambda\end{vmatrix}=(\lambda_1-\lambda)(\lambda_2-\lambda)=\lambda^2-(a_{11}+a_{22})\lambda+|A|,$$

so the quadratic equation gives us that the eigenvalues are

$$\lambda=\frac{a_{11}+a_{22}\pm\sqrt{(a_{11}+a_{22})^2-4\,|A|}}{2}.$$

Because we are assuming that $|A| < 0$,

$$(a_{11}+a_{22})^2 - 4\,|A| \ge -4\,|A| > 0,$$

hence the eigenvalues are real and distinct. More is true: Because $|A| < 0$,

$$(a_{11}+a_{22})^2 - 4\,|A| > (a_{11}+a_{22})^2 \ge 0,$$

hence

$$\sqrt{(a_{11}+a_{22})^2 - 4\,|A|} > \sqrt{(a_{11}+a_{22})^2} = |\,a_{11}+a_{22}\,|.$$

It follows that the larger eigenvalue is

$$\lambda_1 \triangleq \frac{a_{11}+a_{22}+\sqrt{(a_{11}+a_{22})^2 - 4\,|A|}}{2} > \frac{a_{11}+a_{22}+|\,a_{11}+a_{22}\,|}{2} > 0,$$

and the smaller eigenvalue is

$$\lambda_2 \triangleq \frac{a_{11}+a_{22}-\sqrt{(a_{11}+a_{22})^2 - 4\,|A|}}{2} < \frac{a_{11}+a_{22}-|\,a_{11}+a_{22}\,|}{2} < 0.$$

So, yes, $|A| < 0$ implies that the matrix $|A|$ has one negative and one positive eigenvalue.

13.2.3.11. $f(\mathbf{x}) = (A\mathbf{x}-\mathbf{b})^T W^T (A\mathbf{x}-\mathbf{b}) = ... = \mathbf{x}^T A^T W^T A\mathbf{x} - 2\mathbf{b}^T W^T A\mathbf{x} + \mathbf{b}^T W^T \mathbf{b}$.
Using the result of problem 6.7.6.33, that is, $\nabla \mathbf{x}^T A\mathbf{x} = \big(A + A^T\big)\mathbf{x}$, we have

$$\nabla\big(\mathbf{x}^T A^T W^T A\mathbf{x}\big) = \Big(A^T W^T A + \big(A^T W^T A\big)^T\Big)\,\mathbf{x} = \big(A^T W^T A + (A^T W A)\big)\,\mathbf{x} = 2A^T W^T A\mathbf{x},$$

because W is symmetric.

Using the result of problem 13.2.3.17, $\nabla\big(\mathbf{b}^T W^T A\mathbf{x}\big) = \big(\mathbf{b}^T W^T A\big)^T = A^T W\mathbf{b} = A^T W^T \mathbf{b}$. So, a minimizer of $f(\mathbf{x})$ must satisfy

$$\mathbf{0} = 2A^T W^T A\mathbf{x} - 2A^T W^T \mathbf{b} + \mathbf{0},$$

hence must satisfy the generalized normal equations $A^T\, W^T A\,\mathbf{x} = A^T\, W^T \mathbf{b}$.

13.2.3.13. Example 13.5 in Section 13.2 concluded that $(x,y) = (\frac{3}{4}, -\frac{3}{16})$ is the point on the curve $y = x^2 - x$ closest to the curve $Y = \frac{1}{2}\,X - 6$. Example 13.5 in Section 13.2 gave $2(Y-y) = 2\mu = 2\big(-2(X-x)\big)$, hence $2\left(Y + \frac{3}{16}\right) = -4\left(X - \frac{3}{4}\right)$, that is, $Y = -2X + \frac{21}{16}$. Substitute into the latter equation that $Y = \frac{1}{2}\,X - 6$, hence $-6 + \frac{1}{2}\,X = Y = -2X + \frac{21}{16}$. It follows that $X = \frac{117}{40}$, hence $Y = -\frac{363}{80}$, as we desired.

13.2.3.15. Any critical point (w,h,ℓ) of $f(w,h,\ell) \triangleq hw\ell$ satisfies

$$\mathbf{0} = \nabla f = h\ell\,\hat{\boldsymbol{w}} + w\ell\,\hat{\boldsymbol{h}} + hw\,\hat{\boldsymbol{\ell}},$$

that is

$$(\star)\quad \left\{\begin{array}{cl} h\ell & = 0 \\ w\ell & = 0 \\ hw & = 0 \end{array}\right\}.$$

The first equation implies either (1) $h = 0$ or (2) $\ell = 0$, the second equation implies either (3) $w = 0$ or (4) $\ell = 0$, and the third equation implies that either (5) $h = 0$ or (6) $w = 0$.

In order to satisfy the three equations simultaneously, we need to have one of what are, in principle, eight possibilities:

$$(1) \quad \text{and} \quad (3) \quad \text{and} \quad (5): \quad h=0 \quad \text{and} \quad w=0 \quad \text{and} \quad h=0: \ (w,h,\ell)=(0,0,\ell)$$

$$(1) \quad \text{and} \quad (3) \quad \text{and} \quad (6): \quad h=0 \quad \text{and} \quad w=0 \quad \text{and} \quad w=0: \ (w,h,\ell)=(0,0,\ell)$$

$$(1) \quad \text{and} \quad (4) \quad \text{and} \quad (5): \quad h=0 \quad \text{and} \quad \ell=0 \quad \text{and} \quad h=0: \ (w,h,\ell)=(w,0,0)$$

$$(1) \quad \text{and} \quad (4) \quad \text{and} \quad (6): \quad h=0 \quad \text{and} \quad \ell=0 \quad \text{and} \quad w=0: \ (w,h,\ell)=(0,0,0)$$

$$(2) \quad \text{and} \quad (3) \quad \text{and} \quad (5): \quad \ell=0 \quad \text{and} \quad w=0 \quad \text{and} \quad h=0: \ (w,h,\ell)=(0,0,0)$$

$$(2) \quad \text{and} \quad (3) \quad \text{and} \quad (6): \quad \ell=0 \quad \text{and} \quad w=0 \quad \text{and} \quad w=0: \ (w,h,\ell)=(0,h,0)$$

$$(2) \quad \text{and} \quad (4) \quad \text{and} \quad (5): \quad \ell=0 \quad \text{and} \quad \ell=0 \quad \text{and} \quad h=0: \ (w,h,\ell)=(w,0,0)$$

$$(2) \quad \text{and} \quad (4) \quad \text{and} \quad (6): \quad \ell=0 \quad \text{and} \quad \ell=0 \quad \text{and} \quad w=0: \ (w,h,\ell)=(0,h,0)$$

So, the critical points of the function $f(h,w,\ell)$ lie only on the lines $h=w=0$, $w=\ell=0$, and $\ell=h=0$.

13.2.3.17. $\boldsymbol{\nabla}\big(\mathbf{c}^T\mathbf{x}\big) = \boldsymbol{\nabla}\big(c_1x_1+...+c_nx_n\big) = [c_1 \quad c_2 \quad ... \quad c_n]^T = \mathbf{c}$.

Section 13.3.3

13.3.3.1. $\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & -7 \\ 3 & -4 & 0 & 1 & 11 \end{array}\right] \underset{-3R_1+R_2\to R_2}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & -7 \\ 0 & -1 & -3 & 1 & 32 \end{array}\right]$

$$\underset{\substack{-R_2\to R_2\\ R_2+R_1\to R_1}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 4 & -1 & -39 \\ 0 & 1 & 3 & -1 & -32 \end{array}\right]$$

The solutions of the linear system of algebraic equations are given by

$$(\star) \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -39-4c_1+c_2 \\ -32-3c_1+c_2 \\ c_1 \\ c_2 \end{bmatrix},$$

where $x_3 = c_1$ and $x_4 = c_2$ are arbitrary constants.

Basic feasible solutions are $\mathbf{x} = [\, x_1 \; \vdots \; ... \; \vdots \; x_4 \,]^T$ that fit the form $(\star)$, have all of $x_1, ..., x_4 \geq 0$, and have at least two of the $x_i = 0$. In principle, there are six possible ways to choose two of the four $x_i = 0$:

Case 1: $x_1 = x_2 = 0$ imply

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 39 \\ 32 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 39 \\ 32 \end{bmatrix} = \begin{bmatrix} -7 \\ 11 \end{bmatrix},$$

hence $x_3 = c_1 = -7 < 0$, hence there is no basic feasible solution in this case.

Case 2: $x_1 = x_3 = 0$ implies $c_1 = 0$, hence $0 = x_1 = -39 + c_2$ would imply $c_2 = 39$, hence $x_2 = -32 + 39 = 7$, hence the unique basic solution in this case is given by $\mathbf{x} = [\, x_1 \; \vdots \; ... \; \vdots \; x_4 \,]^T = [\, 0 \; \vdots \; 7 \; \vdots \; 0 \; \vdots \; 39 \,]^T$.

Case 3: $x_1 = x_4 = 0$ implies $c_2 = 0$, hence $0 = x_1 = -39 - 4c_1$ would imply $x_3 = c_1 = -39/4 < 0$, hence there is no basic feasible solution in this case.

Case 4: $x_2 = x_3 = 0$ implies $c_1 = 0$, hence $0 = x_2 = -32 + c_2$ would imply $c_2 = 32$, hence $x_1 = -7 < 0$, hence there is no basic feasible solution in this case.

Case 5: $x_2 = x_4 = 0$ implies $c_2 = 0$, hence $0 = x_2 = -32 - 3c_1$ would imply $x_3 = c_1 = -32/3 < 0$, hence there is no basic feasible solution in this case.

Case 6: $x_3 = x_4 = 0$ implies $c_1 = c_2 = 0$, hence $0 = x_1 = -39 < 0$, hence there is no basic feasible solution in this case.

[Alternatively, we could have broken the problem into the six cases from the beginning of the work.]

To summarize, there is exactly one basic feasible solution: $\mathbf{x} = [\, x_1 \; \vdots \; ... \; \vdots \; x_4 \,]^T = [\, 0 \; \vdots \; 7 \; \vdots \; 0 \; \vdots \; 39 \,]^T$.

13.3.3.3. $\left\{\begin{array}{rrrrl} x_1 & +2x_2 & -2x_3 & +x_4 & = -6 \\ -2x_1 & -4x_2 & +5x_3 & & = \;\; 17 \\ x_1 & +2x_2 & -x_3 & & = \;\; 0 \end{array}\right\}$

$$\left[\begin{array}{rrrr|r} 1 & 2 & -2 & 1 & -6 \\ -2 & -4 & 5 & 0 & 17 \\ 1 & 2 & -1 & 0 & 0 \end{array}\right] \underset{\substack{2R_1+R_2\to R_2\\ -R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -2 & 1 & -6 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -1 & 6 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 5 & | & 4 \\ 0 & 0 & 1 & 2 & | & 5 \\ 0 & 0 & 0 & -3 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & | & 17/3 \\ 0 & 0 & 1 & 0 & | & 17/3 \\ 0 & 0 & 0 & 1 & | & -1/3 \end{bmatrix}$$

$$\begin{array}{c} -R_2 + R_3 \to R_3 \\ 2R_2 + R_1 \to R_1 \end{array} \qquad \begin{array}{c} -\frac{1}{3} R_3 \to R_3 \\ -2R_3 + R_2 \to R_2 \\ -5R_3 + R_1 \to R_1 \end{array}$$

The solutions of the linear system of algebraic equations are given by

$$(\star) \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{17}{3} - 2c_1 \\ c_1 \\ \frac{17}{3} \\ -\frac{1}{3} \end{bmatrix},$$

where $x_2 = c_1$ is an arbitrary constant.

Because $x_4 = -\frac{1}{3} < 0$, there is no basic feasible solution.

13.3.3.5. (a) $\left\{ \begin{array}{rrrrcr} x_1 & +x_2 & +x_3 & & = & 2 \\ -2x_1 & -x_2 & & +x_4 & = & -3 \end{array} \right\}$ is the LP in standard form.

(b) $\begin{bmatrix} 1 & 1 & 1 & 0 & | & 2 \\ -2 & -1 & 0 & 1 & | & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & | & 2 \\ 0 & 1 & 2 & 1 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -1 & | & 1 \\ 0 & 1 & 2 & 1 & | & 1 \end{bmatrix}$

$$2R_1 + R_2 \to R_2 \qquad -R_2 + R_1 \to R_1$$

The solutions of the linear system of algebraic equations are given by

$$(\star) \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 + c_1 + c_2 \\ 1 - 2c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix},$$

where $x_3 = c_1$ and $x_4 = c_2$ are arbitrary constants.

Basic feasible solutions are $\mathbf{x} = [\,x_1 \ \vdots\ ... \ \vdots\ x_4\,]^T$ that fit the form $(\star)$, have all of $x_1, ..., x_4 \geq 0$, and have at least two of the $x_i = 0$. In principle, there are six possible ways to choose two of the four $x_i = 0$:

Case 1: $x_1 = x_2 = 0$ imply

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

hence $x_4 = c_2 = -3 < 0$, hence there is no basic feasible solution in this case.

Case 2: $x_1 = x_3 = 0$ implies $c_1 = 0$, hence $0 = x_1 = 1 + c_2$ would imply $x_4 = c_2 = -1$, hence there is no basic feasible solution in this case.

Case 3: $x_1 = x_4 = 0$ implies $c_2 = 0$, hence $0 = x_1 = 1 + c_1$ would imply $x_3 = c_1 = -1$, hence there is no basic feasible solution in this case.

Case 4: $x_2 = x_3 = 0$ implies $c_1 = 0$, hence $0 = x_2 = 1 - 2 \cdot 0 - c_2$ would imply $x_4 = c_2 = 1$, hence $x_1 = 1 + 0 + 1 = 2$, hence the unique basic solution in this case is given by $\mathbf{x} = [\,x_1 \ \vdots\ ... \ \vdots\ x_4\,]^T = [\,2 \ \vdots\ 0 \ \vdots\ 0 \ \vdots\ 1\,]^T$.

Case 5: $x_2 = x_4 = 0$ implies $c_2 = 0$, hence $0 = x_2 = 1 - 2c_1$ would imply $x_3 = c_1 = 1/2$, hence $x_1 = 1 + (1/2) = 3/2$, hence the unique basic solution in this case is given by $\mathbf{x} = [\,x_1 \;\vdots\; ... \;\vdots\; x_4\,]^T = [\,\frac{3}{2} \;\vdots\; 0 \;\vdots\; \frac{1}{2} \;\vdots\; 0\,]^T$.

Case 6: $x_3 = x_4 = 0$ implies $c_1 = c_2 = 0$, hence $0 = x_1 = 1$ and $x_2 = 1$, hence the unique basic solution in this case is given by $\mathbf{x} = [\,x_1 \;\vdots\; ... \;\vdots\; x_4\,]^T = [\,1 \;\vdots\; 1 \;\vdots\; 0 \;\vdots\; 0\,]^T$.

[Alternatively, we could have broken the problem into the six cases from the beginning of the work.]

To summarize, there are exactly three basic feasible solutions:

$$\mathbf{x} = [\,x_1 \;\vdots\; ... \;\vdots\; x_4\,]^T = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

13.3.3.7. Let x_1, x_2, x_3 be the *proportions* of wheat bran, oat flour, and rice flour used in the mixture.

(a) Directly the LP problem is

$$\left\{ \begin{array}{ll} \text{Minimize} & -15.55x_1 - 14.66x_2 - 7.23x_3 \\ & \\ \text{Subject to} & 216x_1 + 404x_2 + 363x_3 \ge 300 \\ & 42.8x_1 + 6.5x_2 + 4.6x_3 \ge 10 \\ & 2.212x_1 + 3.329x_2 + 0.996x_3 \ge 2.5 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, ..., x_3 \ge 0 \end{array} \right\}.$$

In standard form, this is

$$\left\{ \begin{array}{ll} \text{Minimize} & -15.55x_1 - 14.66x_2 - 7.23x_3 \\ & \\ \text{Subject to} & -216x_1 - 404x_2 - 363x_3 + x_4 = -300 \\ & -42.8x_1 - 6.5x_2 - 4.6x_3 + x_5 = -10 \\ & -2.212x_1 - 3.329x_2 - 0.996x_3 + x_6 = -2.5 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, ..., x_6 \ge 0 \end{array} \right\}$$

(b) $\left[\begin{array}{cccccc|c} -216 & -404 & -363 & 1 & 0 & 0 & -300 \\ -42.8 & -6.5 & -4.6 & 0 & 1 & 0 & -10 \\ -2.212 & -3.329 & -0.996 & 0 & 0 & 1 & -2.5 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$ does not *directly* give a feasible solution because, for example, $x_6 = -2 < 0$. The fourth row is fine the way it is because $x_5 = 70$ would be feasible.

We could start by using x_2 as a basic variable in the fourth row, and that will clear up some, but not all, of the infeasiblity issues.

$$\underset{\substack{3.329R_4 + R_3 \to R_3 \\ 6.5R_4 + R_2 \to R_2 \\ 404R_4 + R_1 \to R_1}}{\sim} \quad \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} 188 & 0 & 41 & \underline{1} & 0 & 0 & 104 \\ -36.3 & 0 & 1.9 & 0 & \underline{1} & 0 & -3.5 \\ 1.117 & 0 & 2.333 & 0 & 0 & \underline{1} & 0.829 \\ \underline{1} & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \end{array}$$

$$
\begin{array}{c}
\sim \\
\\
-\frac{1}{36.3}R_2 \to R_2 \\
-188R_2 + R_1 \to R_1 \\
-1.117R_2 + R_3 \to R_3 \\
-R_2 + R_4 \to R_4
\end{array}
\quad
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \\
\end{array}
$$

$$
\left[\begin{array}{cccccccc}
0 & 0 & 50.84 & \underline{1} & 5.179 & 0 & | & 85.87 \\
\underline{1} & 0 & -0.05234 & 0 & -0.02755 & 0 & | & 0.09642 \\
0 & 0 & 2.391 & 0 & 0.03077 & \underline{1} & | & 0.7213 \\
0 & \underline{1} & 1.052 & 0 & 0.02755 & 0 & | & 0.9036
\end{array}\right]
$$

This implies that $[\,x_1\ ...\ x_6\,]^T = [\,0.09642 \quad 0.9036 \quad 0 \quad 85.87 \quad 0 \quad 0.7213\,]^T$ is an example of a basic feasible solution of the LP problem in standard form.

Ex. $[\,x_1\ ...\ x_6\,]^T = [\,0.5 \quad 0.5 \quad 0 \quad 10 \quad 14.65 \quad 0.2705\,]^T$ is another basic feasible solution of the LP problem in standard form

Section 13.4

13.4.2.1. Use slack variable x_3 to put this problem in standard form (13.22) in Section 13.3, that is,

$$\left\{ \begin{array}{ll} \text{Minimize} & x_1 + x_2 \\ \\ \text{Subject to} & -x_1 - 2x_2 + x_3 = -3 \\ & x_1, x_2, x_3 \geq 0 \end{array} \right\}.$$

Next, we will find a basic feasible solution. In matrix-tableau form, the problem is

$$\begin{array}{cccc} 1 & 1 & 0 & f \\ x_1 & x_2 & x_3 & \\ \end{array} \qquad \qquad \begin{array}{cccc} 1 & 1 & 0 & f \\ x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} -1 & -2 & 1 & -3 \end{array} \right] \underset{-R_1 \to R_1}{\sim} \left[\begin{array}{ccc|c} \underline{1} & 2 & -1 & 3 \end{array} \right]$$

The underlined 1 in the first row correspond to the only basic variable, x_1. After that, the tableau is in Table 13.1, whose bottom row contains the unit costs reduction information $z_j - c_j$, which we are about to calculate. Also, later we will explain why the "2" is circled.

Table 1: After finding the first basic, feasible solution

1	1	0		f
x_1	x_2	x_3	$\mathbf{y}$	$3 = 1 \cdot 3$
1	2	-1	3	
	z_2	z_3		

So far, we have a basic feasible solution $(x_1, x_2, x_3) = (3, 0, 0)$. The next thing to do is to decide whether we should pivot by choosing a variable to enter the list of basic variables, and *after* that, by choosing a variable to move out.

We use the "maximum unit reduced cost" criterion for choosing which variable, if any, to move in. The unit reduced cost of x_2 is

$$z_2 - c_2 = \left[\begin{array}{c} c_1 \end{array} \right] \bullet \left[\begin{array}{c} \alpha_1 \end{array} \right] - c_2 = \left[\begin{array}{c} 1 \end{array} \right] \bullet \left[\begin{array}{c} 2 \end{array} \right] - 1 = 2 - 1 = 1.$$

Note that the $\boldsymbol{\alpha}$ column vector used to calculate z_2 sits under the x_2 variable in Table 13.1.

Similarly, the unit reduced cost of x_3 is

$$z_3 - c_3 = \left[\begin{array}{c} 1 \end{array} \right] \bullet \left[\begin{array}{c} -1 \end{array} \right] - 0 = -1 - 0 = -1.$$

The maximum reduced unit cost is 1, and so we choose x_2 to move into the list of basic variables.

The choice to move in x_2 will, in principle, affect the choice of which variable to move out. But, in this problem, there is only one basic variable, so we move x_1 out of the list of basic variables. Circle the "pivot position" 2 in Table 13.1 and now do row operation $0.5R_1 \to R_1$ to get the tableau in Table 13.2. After

Table 2: After doing a row operation on Table 13.1

1	1	0
x_1	x_2	x_3
0.5	1	-0.5

that, we permute the columns, specifically by exchanging the columns corresponding to variables x_1 and x_2,

Table 3: After finding the second basic solution

1	1	0		f
x_2	x_1	x_3	$\mathbf{y}$	$1.5 = 1 \cdot 1.5$
1	0.5	-0.5	1.5	
	$z_1 - c_1$	-0.5		

to put the tableau into standard form for using unit costs reduction to discuss the next round of possibly swapping variables:

$$z_3 - c_3 = \begin{bmatrix} 1 \end{bmatrix} \bullet \begin{bmatrix} -0.5 \end{bmatrix} - 0 = -0.5 - 0 = -0.5$$

[We did not bother to calculate $z_1 - c_1$ because we just moved the variable x_1 out of the set of basic variables in favor of moving x_2 in.] The only unit cost reduction is negative, so we have arrived at a minimizer! The solution is $(x_1, x_2) = (0, 1.5)$. The slack variable value of $x_3 = 0$ is not part of the solution to the original problem but does indicate how much "wiggle room" is left in the inequalities at the optimum solution.

13.4.2.3. No slack variable is needed and the problem is already in standard form (13.22) in Section 13.3, that is,

$$\left\{ \begin{array}{ll} \text{Minimize} & 3x_1 + x_2 + 2x_3 \\ & \\ \text{Subject to} & -x_1 - x_2 - x_3 \ + x_4 \ = -10 \\ & \ 2x_1 + x_2 + x_3 \quad + x_5 = \ 40 \\ & \ 3x_1 + x_2 + x_3 \qquad\quad = \ 50 \\ & x_1, ..., x_5 \geq 0. \end{array} \right\}.$$

First, we will find a basic feasible solution. First, we will do row operations to get a column $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$: In matrix-tableau form, the problem is

$$\begin{array}{c} \begin{array}{cccccc} 3 & 1 & 2 & 0 & 0 & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} -1 & -1 & -1 & 1 & 0 & -10 \\ 2 & 1 & 1 & 0 & 1 & 40 \\ 3 & 1 & 1 & 0 & 0 & 50 \end{array} \right] \end{array} \underset{\substack{-R_3 + R_2 \to R_2 \\ R_3 + R_1 \to R_1}}{\sim} \begin{array}{c} \begin{array}{cccccc} 3 & 1 & 2 & 0 & 0 & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 0 & 0 & 1 & 0 & 40 \\ -1 & 0 & 0 & 0 & 1 & -10 \\ 3 & 1 & 1 & 0 & 0 & 50 \end{array} \right] \end{array}$$

but this does not *directly* give a feasible solution because $x_5 = -10 < 0$. Next, do another row operation in order to use x_1 as a basic variable. [Why choose x_1? Because in the second equation, $... - x_1 ... = -10$ looks promising for finding a positive value of x_1.]

$$\underset{\substack{-R_2 \to R_2 \\ -2R_2 + R_1 \to R_1 \\ -3R_2 + R_3 \to R_3}}{\sim} \begin{array}{c} \begin{array}{cccccc} 3 & 1 & 2 & 0 & 0 & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & 2 & 20 \\ 1 & 0 & 0 & 0 & -1 & 10 \\ 0 & 1 & 1 & 0 & 3 & 20 \end{array} \right] \end{array}$$

The underlined 1's in the three rows correspond to the basic variables, x_4, x_1, x_2. After permuting the variables we get the tableau form shown in Table 5, whose bottom row contains the unit costs reduction information $z_j - c_j$, which we are about to calculate.

So far, we have a basic feasible solution $(x_1, x_2, x_3, x_4, x_5) = (10, 20, 0, 20, 0)$. The next thing to do is to decide whether we should pivot by choosing a variable to enter the list of basic variables, and *after* that, by choosing a variable to move out.

Table 4: After finding the first basic, feasible solution

0	3	1	2	0		f
x_4	x_1	x_2	x_3	x_5	$\mathbf{y}$	$50 = 0 \cdot 20 + 3 \cdot 10 + 1 \cdot 20$
1	0	0	0	2	20	
0	1	0	0	-1	10	
0	0	1	1	3	20	
			0	0		

We use the "maximum unit reduced cost" criterion for choosing which variable, if any, to move in. The unit reduced costs of x_1 and x_2 are

$$z_3 - c_3 = \begin{bmatrix} c_4 \\ c_1 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} - c_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2 = 1 - 2 = -1.$$

and

$$z_5 - c_5 = \begin{bmatrix} c_4 \\ c_1 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_4 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} - c_5 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - 0 = 0 - 0 = 0,$$

respectively. The only unit cost reductions are nonpositive, so we have arrived at a minimizer! The solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 25, 15, 15)$. The minimum value is 50.

By the way, the maximum of the unit cost reductions being 0 implies that there is a cost free way to find another minimizer or minimizers that produce the same minimum value of 50. In fact, another minimizer is given by $(x_1, x_2, x_3, x_4, x_5) = (\frac{50}{3}, 0, 0, \frac{20}{3}, \frac{20}{3})$.

13.4.2.5. Use slack variables x_4 and x_5 to put this problem in standard form (13.22) in Section 13.3, that is,

$$\left\{ \begin{array}{llllllll} \text{Minimize} & 2x_1 + 3x_2 + x_3 & & & & & & \\ & & & & & & & \\ \text{Subject to} & 2x_1 & +x_2 & +x_3 & & & = & 60 \\ & 3x_1 & +x_2 & +x_3 & +x_4 & & = & 80 \\ & -2x_1 & -x_2 & +x_3 & & +x_5 & = & -40 \\ & x_1, ..., x_5 \geq 0. & & & & & & \end{array} \right\}$$

Next, we will find a basic feasible solution. First, we will do row operations to get a column $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: In matrix-tableau form, the problem is

$$\begin{array}{c} \begin{array}{cccccc} 2 & 3 & 1 & 0 & 0 & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{rrrrr|r} 2 & 1 & 1 & 0 & 0 & 60 \\ 3 & 1 & 1 & 1 & 0 & 80 \\ -2 & -1 & 1 & 0 & 1 & -40 \end{array} \right] \end{array} \underset{\substack{-R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3}}{\sim} \begin{array}{c} \begin{array}{cccccc} 2 & 3 & 1 & 0 & 0 & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left[\begin{array}{rrrrr|r} 2 & 1 & 1 & 0 & 0 & 60 \\ 1 & 0 & 0 & 1 & 0 & 20 \\ -4 & -2 & 0 & 0 & 1 & -100 \end{array} \right] \end{array},$$

but this does not *directly* give a feasible solution because $x_5 = -100 < 0$. Next, do another row operation in order to use x_2 as a basic variable. [Why choose x_2? Because in the third equation, ... $- 2x_2$... $= -100$ looks promising for finding a positive value of x_2.]

$$\sim \quad \begin{array}{c} \\ \\ \\ \\ \\ -\frac{1}{2}R_3 \to R_3 \\ -R_3 + R_1 \to R_1 \end{array} \quad \begin{array}{cccccc} 2 & 3 & 1 & 0 & 0 & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array} \left[\begin{array}{ccccc|c} 0 & 0 & \underline{1} & 0 & 0.5 & 10 \\ 1 & 0 & 0 & \underline{1} & 0 & 20 \\ 2 & \underline{1} & 0 & 0 & -0.5 & 50 \end{array}\right],$$

The underlined 1's in the three rows correspond to the basic variables, x_3, x_4, x_2. After permuting the variables we get the tableau form shown in Table 6, whose bottom row contains the unit costs reduction information $z_j - c_j$, which we are about to calculate. Also, later we will explain why the "1" is circled.

Table 5: After finding the first basic, feasible solution

1	0	3	2	0		f
x_3	x_4	x_2	x_1	x_5	$\mathbf{y}$	$160 = 1 \cdot 10 + 0 \cdot 20 + 3 \cdot 50$
1	0	0	0	0.5	10	
0	1	0	1	0	20	
0	0	1	2	−0.5	50	
			1	−1		

So far, we have a basic feasible solution $(x_1, x_2, x_3, x_4, x_5) = (0, 50, 10, 20, 0)$. The next thing to do is to decide whether we should pivot by choosing a variable to enter the list of basic variables, and *after* that, by choosing a variable to move out.

We use the "maximum unit reduced cost" criterion for choosing which variable, if any, to move in. The unit reduced costs of x_1 and x_5 are

$$z_1 - c_1 = \begin{bmatrix} c_3 \\ c_4 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_3 \\ \alpha_4 \\ \alpha_2 \end{bmatrix} - c_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 5 = 6 - 5 = 1.$$

and

$$z_5 - c_5 = \begin{bmatrix} c_3 \\ c_4 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_3 \\ \alpha_4 \\ \alpha_2 \end{bmatrix} - c_5 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 0.5 \\ 0 \\ -0.5 \end{bmatrix} - 0 = -1 - 0 = -1,$$

respectively. Only one unit cost reduction is positive, so the only possible variable to move in is x_1. To decide which variable to move out, we calculate the minimum positive reduction, using $*$ to denote quantities not calculated because of $\alpha \le 0$, where the α's were those used to calculate z_1 when we decided we might move in the variable x_1:

$$\theta = \min_{\alpha_{i_\ell} > 0} \frac{y_{i_\ell}}{\alpha_{i_\ell}} = \min\left\{\frac{y_3}{\alpha_3}, \frac{y_4}{\alpha_4}, \frac{y_2}{\alpha_2}\right\} = \min\left\{*, \ \frac{20}{1}, \ \frac{50}{2}\right\} = 20,$$

which is achieved at index $L = 4$. So, to improve on our basic feasible solution we increase x_1 from 0 to $\theta = 20$ and reduce x_4 to 0; at the same time, the other basic variables x_{i_k} change from y_{i_k} to $y_{i_k} - \theta\alpha_{i_k}$.

Because we are moving x_4 out of the set of basic variables, circle the pivot position 1 in the tableau in Table 13.7, and do row operation $-2R_2 + R_3 \to R_3$; after that, permute the columns to get the tableau in Table 13.8.

The unit reduced costs of x_4 and x_5 are,

$$z_4 - c_4 = \begin{bmatrix} c_3 \\ c_1 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_3 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} - c_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} - 0 = -4 - 0 = -4$$

Table 6: After finding the second basic solution

1	2	3	2	0		f
x_3	x_1	x_2	x_4	x_5	$\mathbf{y}$	$80 = 1 \cdot 10 + 2 \cdot 20 + 3 \cdot 10$
1	0	0	0	0.5	10	
0	1	0	1	0	20	
0	0	1	-2	-0.5	10	
			4	-1		

and

$$z_5 - c_5 = \begin{bmatrix} c_3 \\ c_1 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_3 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} - c_5 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 0.5 \\ 0 \\ -0.5 \end{bmatrix} - 0 = -1 - 0 = -1,$$

respectively. All of the unit cost reductions are negative, so we have arrived at a minimizer! The solution is $(x_1, x_2, x_3) = (20, 10, 10)$. The minimum value is 80.

The slack variables values of $x_4 = 0$ and $x_5 = 0$ are not part of the solution to the original problem but do indicate that there is no "wiggle room" left in the inequalities at the optimum solution.

13.4.2.7. Use slack variables x_4, x_5, and x_6 to put this problem in standard form (13.22) in Section 13.3, that is,

$$\left\{ \begin{array}{lllllllll} \text{Minimize} & x_1 + x_2 + 2x_3 & & & & & & & \\ & & & & & & & & \\ \text{Subject to} & x_1 & +3x_2 & +4x_3 & +x_4 & & & = & 70 \\ & 3x_1 & -x_2 & +x_3 & & +x_5 & & = & 110 \\ & -x_1 & -x_2 & -x_3 & & & +x_6 & = & -40 \\ & x_1, ..., x_6 \geq 0. & & & & & & & \end{array} \right\}$$

Next, we will find a basic feasible solution. In matrix-tableau form, the problem is

$$\begin{array}{c} \begin{array}{cccccccc} 1 & 1 & 2 & 0 & 0 & 0 & & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \end{array} \\ \left[\begin{array}{rrrrrrcr} 1 & 3 & 4 & 1 & 0 & 0 & | & 70 \\ 3 & -1 & 1 & 0 & 1 & 0 & | & 110 \\ -1 & -1 & -1 & 0 & 0 & 1 & | & -40 \end{array} \right] \end{array}$$

but this does not *directly* give a feasible solution because $x_6 = -40 < 0$. Unfortunately, it seems that any attempt to put a pivot position in the third row leads to another basic variable becoming negative, that is, not feasible. So, we seem to need to not take the direct choice of x_4, x_5, x_6 being the basic variables but, instead, "start from scratch."

$$\underset{\substack{\frac{1}{3}R_1 \to R_1 \\ R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3}}{\sim} \quad \begin{array}{c} \begin{array}{cccccccc} 1 & 1 & 2 & 0 & 0 & 0 & & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \end{array} \\ \left[\begin{array}{rrrrrrcr} 1/3 & 1 & 4/3 & 1/3 & 0 & 0 & | & 70/3 \\ 10/3 & 0 & 7/3 & 1/3 & 1 & 0 & | & 400/3 \\ -2/3 & 0 & 1/3 & 1/3 & 0 & 1 & | & -50/3 \end{array} \right] \end{array}$$

$$
\begin{array}{r}
\\
\\
\sim \\
\\
-1.5R_3 \to R_3 \\
-\frac{10}{3}R_3 + R_2 \to R_2 \\
-\frac{1}{3}R_3 + R_1 \to R_1
\end{array}
\quad
\begin{array}{cccccccc}
1 & 1 & 2 & 0 & 0 & 0 & & f \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \\
\end{array}
$$

$$
\begin{bmatrix}
0 & \underline{1} & 3/2 & 1/2 & 0 & 1/2 & | & 15 \\
0 & 0 & 4 & 2 & \underline{1} & 5 & | & 50 \\
\underline{1} & 0 & -1/2 & -1/2 & 0 & -3/2 & | & 25
\end{bmatrix}
$$

The underlined 1's in the three rows correspond to the basic variables, x_3, x_4, x_1. After permuting the variables we get the tableau form shown in Table 9, whose bottom row contains the unit costs reduction information $z_j - c_j$, which we are about to calculate.

Table 7: After finding the first basic, feasible solution

1	0	1	2	0	0		f
x_2	x_5	x_1	x_3	x_4	x_6	**y**	$40 = 1 \cdot 15 + 0 \cdot 50 + 1 \cdot 25$
1	0	0	3/2	1/2	1/2	15	
0	1	0	4	2	5	50	
0	0	1	−1/2	−1/2	−3/2	25	
			−1	0	−1		

So far, we have a basic feasible solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (25, 15, 0, 0, 50, 0)$. The next thing to do is to decide whether we should pivot by choosing a variable to enter the list of basic variables, and *after* that, by choosing a variable to move out.

We use the "maximum unit reduced cost" criterion for choosing which variable, if any, to move in. The unit reduced costs of x_3, x_4, and x_6 are

$$
z_3 - c_3 = \begin{bmatrix} c_2 \\ c_5 \\ c_1 \end{bmatrix} \bullet \begin{bmatrix} \alpha_2 \\ \alpha_5 \\ \alpha_1 \end{bmatrix} - c_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 3/2 \\ 4 \\ -1/2 \end{bmatrix} - 2 = 1 - 2 = -1,
$$

$$
z_4 - c_4 = \begin{bmatrix} c_2 \\ c_5 \\ c_1 \end{bmatrix} \bullet \begin{bmatrix} \alpha_2 \\ \alpha_5 \\ \alpha_1 \end{bmatrix} - c_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1/2 \\ 2 \\ -1/2 \end{bmatrix} - 0 = 0 - 0 = 0,
$$

and

$$
z_6 - c_6 = \begin{bmatrix} c_2 \\ c_5 \\ c_1 \end{bmatrix} \bullet \begin{bmatrix} \alpha_2 \\ \alpha_5 \\ \alpha_1 \end{bmatrix} - c_6 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1/2 \\ 5 \\ -3/2 \end{bmatrix} - 0 = -1 - 0 = -1,
$$

respectively. All of the unit cost reductions are nonpositive, so we have arrived at a minimizer! The solution is $(x_1, x_2, x_3) = (25, 15, 0)$. The minimum value is 40.

By the way, the maximum of the unit cost reductions being 0 implies that there is a cost free way to find another minimizer or minimizers that produce the same minimum value of 50. In fact, another minimizer is given by $(x_1, x_2, x_3) = (\frac{37}{2}, \frac{5}{2}, 10)$.

The slack variables values of $x_4 = 0$, $x_5 = 50$, and $x_6 = 0$ are not part of the solution to the original problem but does indicate how much "wiggle room" is left in the inequalities at the optimum solution.

13.4.2.9. *Method 1*: The problem is

$$\left\{\begin{array}{ll} \text{Maximize} & \lambda \\ & \\ \text{Subject to} & 4M_1 + 2M_2 \qquad\quad = \lambda P\ell \\ & \qquad\quad 2M_2 + 4M_3 = 2\lambda P\ell \\ & \\ & -M_{P\ell} \le M_j \le M_{P\ell},\ j = 1,2,3 \end{array}\right\}$$

Substitute $\lambda P\ell = \mu$ and $x_j = M_j/M_{P\ell}$, $j = 1,2,3$, so that the problem can be rewritten as

$$\left\{\begin{array}{ll} \text{Maximize} & \frac{1}{P\ell}\,\mu \\ & \\ \text{Subject to} & 4x_1 + 2x_2 \qquad\quad - \frac{1}{M_{P\ell}}\,\mu = 0 \\ & \\ & \qquad\quad 2x_2 + 4x_3 - \frac{2}{M_{P\ell}}\,\mu = 0 \\ & \\ & -1 \le x_j \le 1,\ j = 1,2,3 \end{array}\right\}$$

Instead of using the Simplex Procedure, we will first solve the system of two constraint equalities for $(\mu, x_1) = \big(f(x_2,x_3), g(x_2,x_3)\big)$ in terms of (x_2,x_3) and, after that, maximize $\mu = f(x_2,x_3)$ over the domain $\{(x_2,x_3) : -1 \le x_2 \le 1, -1 \le x_3 \le 1, -1 \le g(x_2,x_3) \le 1\}$. This work will have a somewhat ad hoc nature. [Later, we will begin, but not complete, the work to implement the Simplex Procedure: After making a change of variables in order to replace constraints such as $-1 \le x_1 \le 1$ by $0 \le y_1 \le 2$ so that the feasible region has the usual non-negativity constraint in our standard linear programming format.]

The system

$$\left\{\begin{array}{l} 4x_1 + 2x_2 \qquad\quad - \frac{1}{M_{P\ell}}\,\mu = 0 \\ \\ \qquad\quad 2x_2 + 4x_3 - \frac{2}{M_{P\ell}}\,\mu = 0 \end{array}\right\}$$

can be rewritten as

$$\left\{\begin{array}{lllll} \mu & -4M_{P\ell}\,x_1 & -2M_{P\ell}\,x_2 & & = 0 \\ & & & & \\ 2\mu & & -2M_{P\ell}\,x_2 & -4M_{P\ell}\,x_3 & = 0 \end{array}\right\}.$$

Row reduction gives

$$\left[\begin{array}{cccc|c} 1 & -4M_{P\ell} & -2M_{P\ell} & 0 & 0 \\ 2 & 0 & -2M_{P\ell} & -4M_{P\ell} & 0 \end{array}\right] \underset{\substack{-2R_1 + R_2 \to R_2 \\ (8M_{P\ell})^{-1}R_2 \to R_2 \\ 4M_{P\ell}R_2 + R_1 \to R_1}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -M_{P\ell} & -2M_{P\ell} & 0 \\ 0 & 1 & 0.25 & -0.5 & 0 \end{array}\right],$$

so $\mu = f(x_2,x_3) \triangleq M_{P\ell}x_2 + 2M_{P\ell}x_3$ is the objective function we want to maximize, subject to the constraints that $-1 \le x_2 \le 1$, $-1 \le x_3 \le 1$. Defining $x_1 = g(x_2,x_3) \triangleq -0.25x_2 + 0.5x_3$, we must also satisfy the constraint that $-1 \le x_1 = g(x_2,x_3) \le 1$.

We can do this in an elementary way because we are in luck in this problem: Choosing $x_2 = x_3 = 1$ certainly gives the largest value that $\mu = M_{P\ell}x_2 + 2M_{P\ell}x_3$ could *possibly* be, namely $\mu_{max} = 3M_{P\ell}$ can be. It happens that $x_2 = x_3 = 1$ gives $x_1 = g(x_2,x_3) = -0.25x_2 + 0.5x_3 = 0.25$, and that *is* in the interval $-1 \le x_1 \le 1$.

So, the maximum compressive force is $\lambda \cdot P = \dfrac{1}{\ell}\,\mu_{max} = \dfrac{3M_{P\ell}}{\ell}$, and it is achieved when the bending moments are

$$(M_1, M_2, M_3) = M_{P\ell} \cdot (x_1, x_2, x_3) = M_{P\ell} \cdot (0.25,\ 1,\ 1).$$

Method 2: The problem is

$$\left\{\begin{array}{ll} \text{Minimize} & -\lambda \\ & \\ \text{Subject to} & 4M_1 + 2M_2 \qquad\quad = \lambda P\ell \\ & \qquad\quad 2M_2 + 4M_3 = 2\lambda P\ell \\ & \\ & -M_{P\ell} \le M_j \le M_{P\ell},\ j = 1,2,3 \end{array}\right\}$$

Substitute $\lambda P\ell = \mu$ and $x_j = M_j/M_{P\ell}$, $j = 1,2,3$, so that the problem can be rewritten as

$$\left\{\begin{array}{ll} \text{Minimize} & -\frac{1}{P\ell}\,\mu \\ & \\ \text{Subject to} & 4x_1 + 2x_2 \qquad\quad - \frac{1}{M_{P\ell}}\,\mu = 0 \\ & \\ & \qquad\quad 2x_2 + 4x_3 - \frac{2}{M_{P\ell}}\,\mu = 0 \\ & \\ & -1 \le x_j \le 1,\ j = 1,2,3 \end{array}\right\}$$

Next, to keep the usual definition of feasibility to include non-negativity of a variable, define $y_j = x_j + 1$, so that the inequalities $-1 \le x_j \le 1$ can be rewritten as $0 \le y_j \le 2$, $j = 1,2,3$. In terms of the new variables, the problem can be rewritten as

$$\left\{\begin{array}{ll} \text{Minimize} & -\frac{1}{P\ell}\,\mu \\ & \\ \text{Subject to} & 4y_1 + 2y_2 \qquad\quad - \frac{1}{M_{P\ell}}\,\mu = 6 \\ & \\ & \qquad\quad 2y_2 \ + 4y_3 - \frac{2}{M_{P\ell}}\,\mu = 6 \\ & \qquad\quad y_1 \le 2 \\ & \qquad\quad y_2 \le 2 \\ & \qquad\quad y_3 \le 2 \\ & \\ & \mu, y_j \ge 0,\ j = 1,2,3 \end{array}\right\}$$

Introduce slack variables to put the problem into standard form:

$$\left\{\begin{array}{ll} \text{Minimize} & -\frac{1}{P\ell}\,\mu \\ & \\ \text{Subject to} & 4y_1 + 2y_2 \qquad\quad - \frac{1}{M_{P\ell}}\,\mu = 6 \\ & \\ & \qquad\quad 2y_2 \ + 4y_3 - \frac{2}{M_{P\ell}}\,\mu = 6 \\ & \qquad\quad y_1 + y_4 = 2 \\ & \qquad\quad y_2 + y_5 = 2 \\ & \qquad\quad y_3 + y_6 = 2 \\ & \\ & \mu, y_j \ge 0,\ j = 1,2,...,6 \end{array}\right\}$$

At this point, we could use the rest of the Simplex Procedure: First, find a basic feasible solution, and then use unit cost reductions to decide whether to move in a new basic variable and move out an old basic variable. In this way, we would travel among the extreme points of a polytope in $\mathbb{R}^7$. Clearly, this would be more difficult than the ad hoc work we did in Method 1.

13.4.2.11. The problem is

$$\left\{\begin{array}{ll} \text{Maximize} & \lambda \\ & \\ \text{Subject to} & 4M_1 + 2M_2 \qquad\qquad = 5\lambda P\ell \\ & \qquad\qquad 2M_2 + 4M_3 = \lambda P\ell \\ & \\ & -M_{P\ell} \le M_j \le M_{P\ell},\ j = 1,2,3 \end{array}\right\}$$

Substitute $\lambda P\ell = \mu$ and $x_j = M_j/M_{P\ell}$, $j = 1,2,3$, so that the problem can be rewritten as

$$\left\{\begin{array}{ll} \text{Maximize} & \frac{1}{P\ell}\,\mu \\ & \\ \text{Subject to} & 4x_1 + 2x_2 \qquad\qquad - \frac{5}{M_{P\ell}}\,\mu = 0 \\ & \\ & \qquad\qquad 2x_2 + 4x_3 - \frac{1}{M_{P\ell}}\,\mu = 0 \\ & \\ & -1 \le x_j \le 1,\ j = 1,2,3 \end{array}\right\}$$

Instead of using the Simplex Procedure, we will first solve the system of two constraint equalities for $(\mu, x_1) = \big(f(x_2,x_3), g(x_2,x_3)\big)$ in terms of (x_2,x_3) and, after that, maximize $\mu = f(x_2,x_3)$ over the domain $\{(x_2,x_3) : -1 \le x_2 \le 1, -1 \le x_3 \le 1, -1 \le g(x_2,x_3) \le 1\}$. This work will have a somewhat ad hoc nature. [Later, we will begin, but not complete, the work to implement the Simplex Procedure after making a change of variables in order to replace constraints such as $-1 \le x_1 \le 1$ by $0 \le y_1 \le 2$ so that the feasible region has the usual non-negativity constraint in our standard linear programming format.]

The system

$$\left\{\begin{array}{l} 4x_1 + 2x_2 \qquad\qquad - \frac{5}{M_{P\ell}}\,\mu = 0 \\ \\ \qquad\qquad 2x_2 + 4x_3 - \frac{1}{M_{P\ell}}\,\mu = 0 \end{array}\right\}$$

can be rewritten as

$$\left\{\begin{array}{lllll} 5\mu & -4M_{P\ell}\,x_1 & -2M_{P\ell}\,x_2 & & = 0 \\ & & & & \\ \mu & & -2M_{P\ell}\,x_2 & -\,4M_{P\ell}\,x_3 & = 0 \end{array}\right\}.$$

Row reduction gives

$$\left[\begin{array}{cccc|c} 5 & -4M_{P\ell} & -2M_{P\ell} & 0 & 0 \\ 1 & 0 & -2M_{P\ell} & -4M_{P\ell} & 0 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_2 \\ -5R_1 + R_2 \to R_2 \\ -(4M_{P\ell})^{-1}R_2 \to R_2}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2M_{P\ell} & -4M_{P\ell} & 0 \\ 0 & 1 & -2 & -5 & 0 \end{array}\right],$$

so $\mu = f(x_2,x_3) \triangleq 2M_{P\ell}x_2 + 4M_{P\ell}x_3$ is the objective function we want to maximize, subject to the constraints that $-1 \le x_2 \le 1$, $-1 \le x_3 \le 1$. Defining $x_1 = g(x_2,x_3) \triangleq 2x_2 + 5x_3$, we must also satisfy the constraint that $-1 \le x_1 = g(x_2,x_3) \le 1$. [In problems 13.4.2.9 and 13.4.2.10 the last constraint was satisfied more easily than in problem 13.4.2.11.]

So, we have reduced the problem to

$$(\star)\quad \left\{\begin{array}{ll} \text{Maximize} & 2M_{P\ell}x_2 + 4M_{P\ell}x_3 \\ & \\ \text{Subject to} & -1 \le 2x_2 + 5x_3 \le 1 \\ & \qquad\quad -1 \le x_2 \le 1 \\ & \qquad\quad -1 \le x_3 \le 1 \\ & \end{array}\right\}.$$

We could solve this using the method of Section 13.2. Instead, we will note that our new problem is to maximize $f(x_2,x_3) \triangleq 2M_{P\ell}x_2 + 4M_{P\ell}x_3$ over the domain

$$\mathcal{D} \triangleq \{(x_2,x_3):\ 1 \le 2x_2+5x_3 \le 1,\ -1 \le x_2 \le 1,\ -1 \le x_3 \le 1\}.$$

Problem (13.19), which introduced LP problems in Section 13.3, is similar to our new problem ($\star$). Thinking geometrically, the line $k = 2M_{P\ell}x_2 + 4M_{P\ell}x_3$ has k decreased until the line just touches the domain $\mathcal{D}$, which is shown in the figure. The maximum is achieved at $x_2 = 1$, hence $2\cdot 1 + 5x_3 = 1$ implies $x_3 = -0.2$, where $k = 2M_{P\ell}\cdot 1 + 4M_{P\ell}\cdot(-0.2) = 1.2M_{P\ell}$ and $x_1 = 1$.

So, the maximum compressive force is $\lambda \cdot P = \dfrac{1}{\ell}\,\mu_{max} = \dfrac{1.2M_{P\ell}}{\ell}$, and it is achieved when the bending moments are

$$(M_1,M_2,M_3) = M_{P\ell}\cdot(x_1,x_2,x_3) = M_{P\ell}\cdot(1,\ 1,-0.2).$$

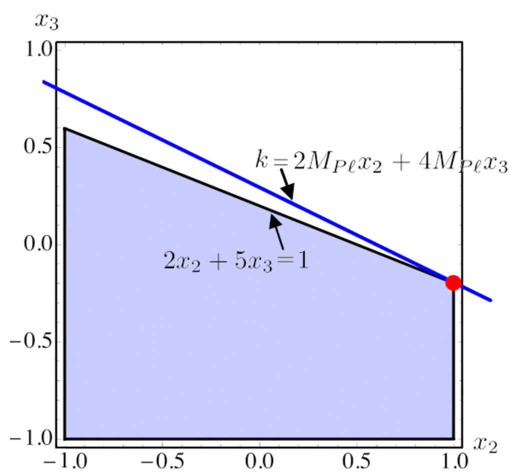


Figure 2: Answer key for problem 13.4.2.11

13.4.2.13. In standard form, the LP problem of problem 13.3.3.7 is

$$\left\{\begin{array}{ll} \text{Minimize} & -15.55x_1 - 14.66x_2 - 7.23x_3 \\ & \\ \text{Subject to} & -216x_1 - 404x_2 - 363x_3 + x_4 = -300 \\ & -42.8x_1 - 6.5x_2 - 4.6x_3 + x_5 = -10 \\ & -2.212x_1 - 3.329x_2 - 0.996x_3 + x_6 = -2.5 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, ..., x_6 \ge 0 \end{array}\right\}.$$

Next, we will find a basic feasible solution. In matrix-tableau form, the problem is

$$\begin{array}{c} \begin{array}{cccccccc} -15.55 & -14.66 & -7.23 & 0 & 0 & 0 & & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \end{array} \\ \left[\begin{array}{cccccc|c} -216 & -404 & -363 & 1 & 0 & 0 & -300 \\ -42.8 & -6.5 & -4.6 & 0 & 1 & 0 & -10 \\ -2.212 & -3.329 & -0.996 & 0 & 0 & 1 & -2.5 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

but this does not *directly* give a feasible solution because, for example, $x_6 = -2 < 0$. The fourth row is fine the way it is because $x_5 = 70$ would be feasible.

We could start by using x_2 as a basic variable in the fourth row, and that will clear up some, but not all, of the infeasiblity issues.

$$\sim \quad \begin{array}{cccccccc} -15.55 & -14.66 & -7.23 & 0 & 0 & 0 & & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \end{array}$$

$$\begin{bmatrix} 188 & 0 & 41 & \underline{1} & 0 & 0 & | & 104 \\ -36.3 & 0 & 1.9 & 0 & \underline{1} & 0 & | & -3.5 \\ 1.117 & 0 & 2.333 & 0 & 0 & \underline{1} & | & 0.829 \\ \underline{1} & 1 & 1 & 0 & 0 & 0 & | & 1 \end{bmatrix}$$

$3.329R_4 + R_3 \to R_3$
$6.5R_4 + R_2 \to R_2$
$404R_4 + R_1 \to R_1$

$$\sim \quad \begin{array}{cccccccc} -15.55 & -14.66 & -7.23 & 0 & 0 & 0 & & f \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \end{array}$$

$$\begin{bmatrix} 0 & 0 & 50.84 & \underline{1} & 5.179 & 0 & | & 85.87 \\ \underline{1} & 0 & -0.05234 & 0 & -0.02755 & 0 & | & 0.09642 \\ 0 & 0 & 2.391 & 0 & 0.03077 & \underline{1} & | & 0.7213 \\ 0 & \underline{1} & 1.052 & 0 & 0.02755 & 0 & | & 0.9036 \end{bmatrix}$$

$-\frac{1}{36.3}R_2 \to R_2$
$-188R_2 + R_1 \to R_1$
$-1.117R_2 + R_3 \to R_3$
$-R_2 + R_4 \to R_4$

The underlined 1's in the three rows correspond to the basic variables, x_4, x_5, x_6, x_1. After permuting the variables we get the tableau form shown in Table 13, whose bottom row contains the unit costs reduction information $z_j - c_j$, which we are about to calculate. Also, later we will explain why the "1" is circled.

Table 8: After finding the first basic, feasible solution

0	−15.55	0	−14.66	−7.23	0		f
x_4	x_1	x_6	x_2	x_3	x_5	**y**	$-14.75 = 0\cdot(85.87) + (-15.55)\cdot(0.09642)$
							$+0\cdot(0.7213) - 14.66\cdot(0.9036)$
1	0	0	0	50.84	5.179	85.87	
0	1	0	0	−0.05234	−0.02755	0.09642	
0	0	1	0	2.391	0.03077	0.7213	
0	0	0	1	1.052	0.02755	0.9036	
				−7.378	0.0245		

So far, we have a basic feasible solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (0.09642, 0.9036, 0, 87.87, 0, 0.7213)$. The next thing to do is to decide whether we should pivot by choosing a variable to enter the list of basic variables, and *after* that, by choosing a variable to move out.

The next thing to do is to decide whether we should pivot by choosing a variable to enter the list of basic variables, and *after* that, by choosing a variable to move out.

We use the "maximum unit reduced cost" criterion for choosing which variable, if any, to move in. The unit reduced costs of x_3 and x_5 are

$$z_3 - c_3 = \begin{bmatrix} c_4 \\ c_1 \\ c_6 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_4 \\ \alpha_4 \\ \alpha_6 \\ \alpha_2 \end{bmatrix} - c_3 = \begin{bmatrix} 0 \\ -15.55 \\ 0 \\ -14.66 \end{bmatrix} \bullet \begin{bmatrix} 50.84 \\ -0.05234 \\ 2.391 \\ 1.052 \end{bmatrix} - 14.61 - (-7.23) = -14.61 + 7.23 = -7.378,$$

and

$$z_5 - c_5 = \begin{bmatrix} c_4 \\ c_1 \\ c_6 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_4 \\ \alpha_4 \\ \alpha_6 \\ \alpha_2 \end{bmatrix} - c_5 = \begin{bmatrix} 0 \\ -15.55 \\ 0 \\ -14.66 \end{bmatrix} \bullet \begin{bmatrix} 5.179 \\ -0.02755 \\ 0.03077 \\ 0.02755 \end{bmatrix} - 0 = 0.0245 - 0 = 0.0245,$$

respectively. Only one unit cost reduction is positive, so we choose to move into the set of basic variables x_5.

To decide which variable to move out, we calculate the minimum positive reduction, using $*$ to denote quantities not calculated because of $\alpha \le 0$, where the α's were those used to calculate z_5 when we decided we might move in the variable x_5:

$$\theta = \min_{\alpha_{i_\ell}>0} \frac{y_{i_\ell}}{\alpha_{i_\ell}} = \min\left\{\frac{y_4}{\alpha_4},\ \frac{y_1}{\alpha_1},\ \frac{y_6}{\alpha_6},\ \frac{y_2}{\alpha_2}\right\} = \min\left\{\frac{85.87}{5.179},\ *,\ \frac{0.7213}{0.03077},\ \frac{0.9036}{0.02755}\right\}$$

$$= \min\{16.56,\ *,\ 23.44,\ 32.80\} = 16.56,$$

which is achieved at index $L = 4$. So, to improve on our basic feasible solution we increase x_5 from 0 to $\theta = 16.56$ and reduce x_4 to 0; at the same time, the other basic variables x_{i_k} change from y_{i_k} to $y_{i_k} - \theta\alpha_{i_k}$.

Circle the pivot position 5.179 in the tableau in Table 13.15, and do row operations $\frac{1}{5.179} R_1 \to R_1$, $0.02755R_1 + R_2 \to R_2$, $-0.03077R_1 + R_3 \to R_3$, and $-0.02755R_1 + R_4 \to R_4$; after that, permute the columns to get the tableau in Table 13.16.

Table 9: After finding the second basic, feasible solution

0	−15.55	0	−14.66	−7.23	0		f
x_5	x_1	x_6	x_2	x_3	x_4	$\mathbf{y}$	$-15.15 = 0\cdot(16.58)+(-15.55)\cdot(0.5532)$ $+0\cdot(0.2111) - 14.66\cdot(0.4468)$
1	0	0	0	9.816	0.1931	16.58	
0	1	0	0	0.2181	0.005319	0.5532	
0	0	1	0	2.089	−0.005941	0.2111	
0	0	0	1	0.7819	−0.005319	0.4468	
				−7.624	−0.004734		

So far, we have a basic feasible solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 1, 0, 4.660, 4.300, 1.329)$. The next thing to do is to decide whether we should pivot by choosing a variable to enter the list of basic variables, and *after* that, by choosing a variable to move out.

The next thing to do is to decide whether we should pivot by choosing a variable to enter the list of basic variables, and *after* that, by choosing a variable to move out.

We use the "maximum unit reduced cost" criterion for choosing which variable, if any, to move in. The unit reduced costs of x_3 and x_4 are

$$z_3 - c_3 = \begin{bmatrix} c_5 \\ c_1 \\ c_6 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_5 \\ \alpha_1 \\ \alpha_6 \\ \alpha_2 \end{bmatrix} - c_3 = \begin{bmatrix} 0 \\ -15.55 \\ 0 \\ -14.66 \end{bmatrix} \bullet \begin{bmatrix} 9.816 \\ 0.2181 \\ 2.089 \\ 0.7819 \end{bmatrix} - 14.85 - (-7.23) = -14.85 + 7.23 = -7.624,$$

and

$$z_4 - c_4 = \begin{bmatrix} c_5 \\ c_1 \\ c_6 \\ c_2 \end{bmatrix} \bullet \begin{bmatrix} \alpha_5 \\ \alpha_1 \\ \alpha_6 \\ \alpha_2 \end{bmatrix} - c_4 = \begin{bmatrix} 0 \\ -15.55 \\ 0 \\ -14.66 \end{bmatrix} \bullet \begin{bmatrix} 0.1931 \\ 0.005319 \\ -0.005941 \\ -0.005319 \end{bmatrix} = -0.004734 - 0 = -0.004734,$$

respectively. Both unit cost reductions are negative, so we have arrived at a minimizer! The solution is $(x_1, x_2, x_3) = (0.5532, 0.4468, 0)$. The minimum value is -15.15.

In terms of the original nutrition problem, we have that the maximum, 15.15 g protein content of 100 g of mixture, is when we use 55.32 g of wheat bran, 44.68 g of oat flour, and 0 g of rice flour. This result is not at all obvious!

The slack variables values of $x_4 = 0$, $x_5 = 16.58$, and $x_6 = 0.211$ are not part of the solution to the original problem but does indicate how much “wiggle room" is left in the inequalities at the optimum solution.

Section 13.5

13.5.3.1. $(X-x)^2+(Y-y)^2$ is the square of the distance between a point (x,y) in the region $x^2-2x-y\le 0$ and a point on (X,Y) in the region $8-X+Y\le 0$. We consider X,Y,x,y to be the variables of the problem, which we state as

$$\left\{\begin{array}{ll}\text{Minimize} & (X-x)^2+(Y-y)^2\\ & \\ \text{Subject to} & x^2-2x-y\le 0\\ & \;\;8-X+Y\le 0\end{array}\right\}.$$

Define $f(X,Y,x,y)\triangleq (X-x)^2+(Y-y)^2$, $f_1(X,Y,x,y)\triangleq x^2-2x-y$, $f_2(X,Y,x,y)\triangleq 8-X+Y$ and let $\lambda_1,\lambda_2\ge 0$ be the Lagrange multipliers.

Note that f, f_1, and f_2 are convex functions everywhere, as we can see from using either Theorem 13.13 in Section 13.5 or Corollary 13.3 in Section 13.5.

As in Examples 13.5 and 13.6 in Section 13.2, as well as Example 13.12 in Section 13.5, the gradient is

$$\boldsymbol{\nabla}=\hat{\boldsymbol{I}}\frac{\partial}{\partial X}+\hat{\boldsymbol{J}}\frac{\partial}{\partial Y}+\hat{\boldsymbol{\imath}}\frac{\partial}{\partial x}+\hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y}.$$

The stationarity condition, (13.32) in Section 13.5, is

$$(\star)\quad \begin{bmatrix}2(X-x)\\2(Y-y)\\-2(X-x)\\-2(Y-y)\end{bmatrix}=\boldsymbol{\nabla}f(X,Y,x,y)=-\begin{bmatrix}\frac{\partial f_1}{\partial X}&\frac{\partial f_1}{\partial Y}&\frac{\partial f_1}{\partial x}&\frac{\partial f_1}{\partial y}\\ & & & \\ \frac{\partial f_2}{\partial X}&\frac{\partial f_2}{\partial Y}&\frac{\partial f_2}{\partial x}&\frac{\partial f_2}{\partial y}\end{bmatrix}^T\boldsymbol{\lambda}=-\lambda_1\begin{bmatrix}0\\0\\2x-2\\-1\end{bmatrix}-\lambda_2\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}.$$

Equation $(\star)$ implies

$$(1)\quad \lambda_1(2x-2)=2(X-x)=\lambda_2\quad\text{and}\quad(2)\quad\lambda_1=-2(Y-y)=\lambda_2,$$

hence

$$0=\lambda_2-\lambda_2=\lambda_1(2x-2)-\lambda_1=\lambda_1\big((2x-2)-1\big).$$

It follows that

$$(\star\star)\quad\text{either }\lambda_1=0\quad\text{or}\quad x=\frac{3}{2}.$$

The complementarity conditions are

$$\lambda_1(x^2-2x-y)=\lambda_2(8-X+Y)=0.$$

In effect, the complementarity conditions say that either the multiplier λ_1 is zero or the point (x,y) lies on the boundary of the region $x^2-2x-y\le 0$, and either the multiplier λ_2 is zero or the point (X,Y) lies on the boundary of the region $8-X+Y\le 0$.

Combine the first complementarity condition and $(\star\star)$ to imply that

$$\lambda_1=0\quad\text{or}\quad y=x^2-2x=\left(\frac{3}{2}\right)^2-2\cdot\frac{3}{2}=-\frac{3}{4}.$$

So, either $\lambda_1=0$ or $(x,y)=\left(\frac{3}{2},-\frac{3}{4}\right)$.

Suppose $\lambda_1=0$. Then (1) and (2) imply $X=x$, $Y=y$, and $\lambda_2=0$. The second complementarity condition is then satisfied because $\lambda_2=0$. The only facts we have left to work with are the feasibility requirements in (13.33) in Section 13.5, specifically $\mathbf{f}(\mathbf{x}^\star)\le\mathbf{0}$, that is, that $x^2-2x-y\le 0$ and $8-X+Y\le 0$. It follows that

$$-8+X\ge Y\quad\text{and}\quad y\ge x^2-2x$$

But $X=x$ and $Y=y$, so $-8+x\ge y\ge x^2-2x$, hence $0\ge x^2-3x+8=\left(x-\frac{3}{2}\right)^2+\frac{23}{4}\ge\frac{23}{4}$, which is impossible. So, we conclude that $\lambda_1\ne 0$.

So far, we have concluded that $(x,y) = \left(\dfrac{3}{2}, -\dfrac{3}{4}\right)$ and $\lambda_1 > 0$. To find (X,Y), note that (1) and (2), along with $\lambda_1 > 0$ and the fact that $2x - 2 = 2 \cdot \frac{3}{2} - 2 \neq 0$, together imply

$$2Y + \frac{3}{2} = 2\left(Y + \frac{3}{4}\right) = 2(Y-y) = -\lambda_1 = -\frac{2(X-x)}{(2x-2)} = -\frac{2(X-\frac{3}{2})}{(2\cdot\frac{3}{2}-2)} = -\frac{(2X-3)}{1} = -2X+3$$

hence

$$Y = -X + \frac{3}{4}.$$

The second complementarity condition is that either $\lambda_2 = 0$ or $8 - X + Y = 0$. If $\lambda_2 = 0$ then, again, (1) and (2) imply $X = x$ and $Y = y$, leading eventually to a contradiction as in the above argument. If $8 - X + Y = 0$ then we use the fact that $Y = -X + \frac{3}{4}$, hence

$$0 = 8 - X + Y = 8 - X + \left(-X + \frac{3}{4}\right) = 8 - 2X + \frac{3}{4},$$

hence $X = \frac{35}{8}$ and thus $Y = -\frac{35}{8} + \frac{3}{4} = -\frac{29}{8}$. The closest approach of the two regions is where $(x,y) = \left(\frac{3}{2}, -\frac{3}{4}\right)$, $(X,Y) = \left(\frac{35}{8}, -\frac{29}{8}\right)$, and the minimum distance is

$$\sqrt{\left(\frac{35}{8} - \frac{3}{2}\right)^2 + \left(-\frac{29}{8} + \frac{3}{4}\right)^2} = \sqrt{\frac{23^2}{8^2} + \frac{23^2}{8^2}} = \frac{23\sqrt{2}}{8}.$$

The results are depicted in the figure.

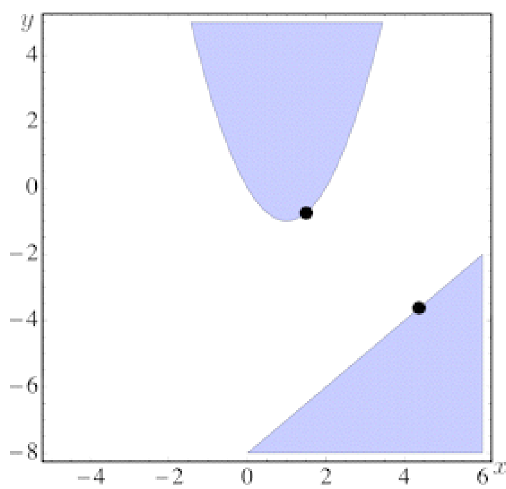


Figure 3: Answer key for problem 13.5.3.1

13.5.3.3. The objective function, $f(x,y) = -2\ln x - 3\ln y$, is convex on the convex set $C \triangleq \{(x,y) : x > 0, y > 0\}$ by problems 13.1.3.10 and 13.1.3.12(a). The constraint function $f_1(x,y) = 2x + y - 4$ is also convex on C, by a result in Corollary 13.3 in Section 13.5, so we can use the Kuhn-Tucker conditions to solve the CP problem in problem 13.5.3.3. The feasible region is shown in Figure 13.11. Let $\lambda \geq 0$ be the Lagrange multiplier.

The feasibility conditions are $2x + y - 4 \leq 0, x > 0, y > 0$. The complementarity condition is

$$\lambda \cdot (2x + y - 4) = 0.$$

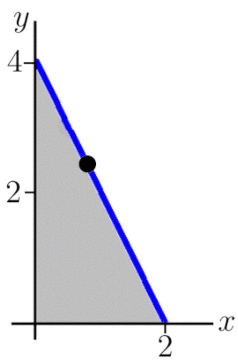


Figure 4: Answer Key for problem 13.5.3.3

The stationarity condition is

$$(\star)\quad \begin{bmatrix} -\frac{2}{x} \\ \\ -\frac{3}{y} \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}.$$

But, $(\star)$ implies $\lambda \neq 0$, because $-\frac{1}{x}$ cannot be zero for a feasible (x, y). So, the complementarity condition implies

$$(\star\star)\quad 2x + y - 4 = 0.$$

$(\star)$ also implies

$$y = \frac{3}{\lambda} \quad \text{and} \quad x = \frac{1}{\lambda}.$$

Substitute those into $(\star\star)$ to get

$$4 = 2\left(\frac{1}{\lambda}\right) + \left(\frac{3}{\lambda}\right) = \frac{5}{\lambda},$$

hence $\lambda = \frac{5}{4}$, hence $x = \frac{4}{5}$ and $y = \frac{12}{5}$. The minimum value of the objective function with these constraints is $f\left(\frac{4}{5}, \frac{12}{5}\right) = -2\ln\left(\frac{4}{5}\right) - 3\ln\left(\frac{12}{5}\right) = \ln\left(\frac{5^5}{4^2 12^3}\right)$.

13.5.3.5. We are given that C is convex, which by definition means that $(1-t)\mathbf{x} + t\mathbf{y}$ must be in C whenever $\mathbf{x}$ and $\mathbf{y}$ are in C and $0 \leq t \leq 1$.

For $p = 3$, define $\mu_{p-1} = \mu_2 \triangleq \lambda_1 + \lambda_2$, $\alpha_1 \triangleq \lambda_1/\mu_2$, and $\alpha_2 \triangleq \lambda_2/\mu_2$. Then

$$\mathbf{z} \triangleq \mu_2^{-1}(\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2) = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$$

is in C because C is convex and $\alpha_1 + \alpha_2 = 1$.

Then we see that $\lambda_1\mathbf{x}_1 + ... + \lambda_3\mathbf{x}_3$ is in C because

(1) $\lambda_1\mathbf{x}_1 + ... + \lambda_3\mathbf{x}_3 = (\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2) + \lambda_3\mathbf{x}_3 = \mu_2\mathbf{z} + \lambda_3\mathbf{x}_3$,

(2) $\mu_2 + \lambda_3 = (\lambda_1 + \lambda_2) + \lambda_3 = 1$, and

(3) $\mathbf{z}$ and $\mathbf{x}_3$ are in C.

This shows us how to use an inductive process to explain why $\lambda_1\mathbf{x}_1 + ... + \lambda_p\mathbf{x}_p$ must be in C if $\mathbf{x}_j$, $j = 1, ..., p$ are in C and the non-negative real numbers λ_j satisfy $\lambda_1 + ... + \lambda_p = 1$.

Continuing in this way, for any $p > 3$, we have $(\star)$ $\mathbf{z} \triangleq \mu_{p-1}^{-1}(\lambda_1\mathbf{x}_1 + ... + \lambda_{p-1}\mathbf{x}_{p-1})$ is in C. Because C is convex and $\mu_{p-1} + \lambda_p = (\lambda_1 + ... + \lambda_{p-1}) + \lambda_p = 1$, we have that

$$\lambda_1\mathbf{x}_1 + ... + \lambda_p\mathbf{x}_p = \mu_{p-1}\big(\mu_{p-1}^{-1}(\lambda_1\mathbf{x}_1 + ... + \lambda_{p-1}\mathbf{x}_{p-1})\big) + \lambda_p\mathbf{x}_p = \mu_{p-1}\mathbf{z} + \lambda_p\mathbf{x}_p$$

is in C.

13.5.3.7. Suppose that there are points $\mathbf{x}$ and $\mathbf{y}$, possibly equal, in $S \triangleq \{\mathbf{x}$ is in $C : f(\mathbf{x}) \le M\}$, a subset of the convex set C. Then $f(\mathbf{x}) \le M$ and $f(\mathbf{y}) \le M$. Also, for $0 < t < 1$, because the function f is convex on the convex set C, $\big((1-t)\mathbf{x} + t\mathbf{y}\big)$ is in C and

$$f\big((1-t)\mathbf{x} + t\mathbf{y}\big) \le (1-t)f(\mathbf{x}) + tf(\mathbf{y}) \le (1-t)M + tM = M,$$

hence $\big((1-t)\mathbf{x} + t\mathbf{y}\big)$ is in S. This being true for all $\mathbf{x}$ and $\mathbf{y}$ in S and $0 < t < 1$, we have that S is convex, unless it is empty.

13.5.3.9. Choose any $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$ and any λ with $0 < \lambda < 1$. Then

$$f\big(\mathbf{g}(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})\big) = f\big(A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + \mathbf{b}\big) = f\big(\lambda A\mathbf{x} + (1-\lambda)A\mathbf{y} + \lambda\mathbf{b} + (1-\lambda)\mathbf{b}\big)$$

$$= f\big(\lambda(A\mathbf{x} + \mathbf{b}) + (1-\lambda)(A\mathbf{y} + \mathbf{b})\big) \le \lambda f(A\mathbf{x} + \mathbf{b}) + (1-\lambda)f(A\mathbf{y} + \mathbf{b}) = \lambda f\big(g(\mathbf{x})\big) + (1-\lambda)f\big(g(\mathbf{y})\big),$$

hence $f\big(g(x)\big)$ is convex on C.

13.5.3.11. Suppose C_1 and C_2 are convex subsets of $\mathbb{R}^m$. Suppose $\mathbf{x}$ and $\mathbf{y}$ are any two points in $C_1 \cap C_2 \triangleq \{\mathbf{x} : \mathbf{x}$ is in both C_1 and $C_2\}$, and suppose t is any real number with $0 < t < 1$. Because $\mathbf{x}$ is in $C_1 \cap C_2$, $\mathbf{x}$ is in both C_1 and C_2, and similarly $\mathbf{y}$ is in both C_1 and C_2.

Because C_1 is convex, $(1-t)\mathbf{x} + t\mathbf{y}$ is in C_1, and because C_2 is convex, $(1-t)\mathbf{x} + t\mathbf{y}$ is in C_2. So, $(1-t)\mathbf{x} + t\mathbf{y}$ is in both C_1 and C_2, hence $(1-t)\mathbf{x} + t\mathbf{y}$ is in $C_1 \cap C_2$. This being true for all $\mathbf{x}$ and $\mathbf{y}$ in $C_1 \cap C_2$ and any real number t with $0 < t < 1$, we conclude that $C_1 \cap C_2$ is convex.

13.5.3.13. Suppose, to the contrary, that there is an $\mathbf{x}^\star$ that is a local minimizer for f but is *not* a global minimizer for f on C. $\mathbf{x}^\star$ being a local minimizer implies there is an $\delta > 0$ such that $f(\mathbf{x}^\star) \le f(\mathbf{x})$ for all $\mathbf{x}$ in C with $|\mathbf{x} - \mathbf{x}^\star| < \delta$. $\mathbf{x}^\star$ not being a global minimizer implies there is a $\widetilde{\mathbf{x}}$ such that $f(\widetilde{\mathbf{x}}) < f(\mathbf{x}^\star)$, which implies that $\big|\mathbf{x}^\star - \widetilde{\mathbf{x}}\big| \ge \delta$. Define $t = 0.9\delta/\big|\mathbf{x}^\star - \widetilde{\mathbf{x}}\big|$. We see that $0 < t < 1$. Note for future reference that $\big|\mathbf{x}^\star - \widetilde{\mathbf{x}}\big| = 0.9\delta/t$.

Define $\mathbf{x} \triangleq t\widetilde{\mathbf{x}} + (1-t)\mathbf{x}^\star$. By the convexity of f,

$$(\star) \quad f(\mathbf{x}) = f\big(t\widetilde{\mathbf{x}} + (1-t)\mathbf{x}^\star\big) \le tf\big(\widetilde{\mathbf{x}}\big) + (1-t)f\big(\mathbf{x}^\star\big) < tf\big(\mathbf{x}^\star\big) + (1-t)f\big(\mathbf{x}^\star\big) = f\big(\mathbf{x}^\star\big).$$

But, $\mathbf{x} = t\widetilde{\mathbf{x}} + (1-t)\mathbf{x}^\star$ satisfies

$$|\mathbf{x}^\star - \mathbf{x}| = \big|\mathbf{x}^\star - t\widetilde{\mathbf{x}} - (1-t)\mathbf{x}^\star\big| = \big|t\big(\mathbf{x}^\star - \widetilde{\mathbf{x}}\big)\big| = t \cdot \big|\mathbf{x}^\star - \widetilde{\mathbf{x}}\big| = t \cdot (0.9\delta/t) = 0.9\delta;$$

by the local minimizer assumption this implies $f(\mathbf{x}) \ge f(\mathbf{x}^\star)$, contradicting $(\star)$. So, the local minimizer must be the global minimizer on the interval C.

13.5.3.15. Define $f(\mathbf{x}) \triangleq \mathbf{x}^T A\mathbf{x}$, where A is an $n \times n$ real, constant, positive definite matrix. Suppose $\mathbf{x}$ and $\mathbf{y}$ are any two vectors in $\mathbb{R}^n$ and $0 < t < 1$.

Concerning $f(\mathbf{x})$, we have

$$f\big((1-t)\mathbf{x} + t\mathbf{y}\big) - (1-t)f(\mathbf{x}) - tf(\mathbf{y}) = \big((1-t)\mathbf{x} + t\mathbf{y}\big)^T A\big((1-t)\mathbf{x} + t\mathbf{y}\big) - (1-t)\mathbf{x}^T A\mathbf{x} - t\mathbf{y}^T A\mathbf{y}$$

$$= (1-t)^2\mathbf{x}^T A\mathbf{x} + t(1-t)\mathbf{x}^T(A + A^T)\mathbf{y} + t^2\mathbf{y}^T A\mathbf{y} - (1-t)\mathbf{x}^T A\mathbf{x} - t\mathbf{y}^T A\mathbf{y}$$

$$= \big((1-t)^2 - (1-t)\big)\mathbf{x}^T A\mathbf{x} + t(1-t)\mathbf{x}^T(A+A^T)\mathbf{y} + (t^2-t)\mathbf{y}^T A\mathbf{y}$$

$$= -t(1-t)\mathbf{x}^T A\mathbf{x} + t(1-t)\mathbf{x}^T(A+A^T)\mathbf{y} - t(1-t)\mathbf{y}^T A\mathbf{y}$$

$$= -t(1-t)(\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T(A+A^T)\mathbf{y} + \mathbf{y}^T A\mathbf{y})$$

$$= -t(1-t)(\mathbf{x}-\mathbf{y})^T A(\mathbf{x}-\mathbf{y}) \triangleq \alpha.$$

Because A is real and positive definite, for $\mathbf{z} \triangleq \mathbf{x}-\mathbf{y}$ we have $\mathbf{z}^T A\mathbf{z} > 0$, so $t > 0$ and $1-t > 0$ imply

$$\alpha = -t(1-t)\mathbf{z}^T A\mathbf{z} < 0.$$

So,

$$f\big((1-t)\mathbf{x}+t\mathbf{y}\big) - (1-t)f(\mathbf{x}) - tf(\mathbf{y}) = \big((1-t)\mathbf{x}+t\mathbf{y}\big)^T A\big((1-t)\mathbf{x}+t\mathbf{y}\big) - (1-t)\mathbf{x}^T A\mathbf{x} - t\mathbf{y}^T A\mathbf{y} < 0,$$

hence

$$f\big((1-t)\mathbf{x}+t\mathbf{y}\big) < (1-t)f(\mathbf{x}) + tf(\mathbf{y}).$$

This being true for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and any real number t with $0 < t < 1$, we conclude that $f(\mathbf{x})$ is strictly convex on $\mathbb{R}^n$.

Section 13.6

13.6.3.1. Similarly to the work for the maximization problem system (13.52), define $f(\mathbf{x}) \triangleq \mathbf{x}^T A\mathbf{x}$, $f_1(\mathbf{x}) \triangleq \mathbf{x}^T\mathbf{x} - 1$, $f_2(\mathbf{x}) \triangleq \mathbf{x}^T\mathbf{x}^{(1)}$,..., $f_{k+1}(\mathbf{x}) \triangleq \mathbf{x}^T\mathbf{x}^{(k)}$. Symmetry of A implies that we have $\boldsymbol{\nabla} f = 2A\mathbf{x}, \boldsymbol{\nabla} f_1 = 2\mathbf{x}, \boldsymbol{\nabla} f_2 = \mathbf{x}^{(1)}, ..., \boldsymbol{\nabla} f_{k+1} = \mathbf{x}^{(k)}$. There exists multipliers $\mu_1, ..., \mu_{k+1}$ and global minimizer $\mathbf{x}^\star$ for which stationarity holds, that is,

$$(\star)\ 2A\mathbf{x}^\star = 2\mu_1\mathbf{x}^\star + \mu_2\mathbf{x}^{(1)} + ... + \mu_{k+1}\mathbf{x}^{(k)}$$

and

$$(\star\star)\ ||\mathbf{x}^\star||^2 - 1 = 0 = \left(\mathbf{x}^\star\right)^T\mathbf{x}^{(1)} = ... = \left(\mathbf{x}^\star\right)^T\mathbf{x}^{(k)}.$$

Take the dot product of $(\star)$ with $\mathbf{x}^\star$ and use $(\star\star)$ to get

$$2\left(\mathbf{x}^\star\right)^T A\mathbf{x}^\star = 2\mu_1||\mathbf{x}^\star||^2 + \mu_2\left(\mathbf{x}^\star\right)^T\mathbf{x}^{(1)} + ... + \mu_{k+1}\left(\mathbf{x}^\star\right)^T\mathbf{x}^{(k)} = 2\mu_1 \cdot 1 + \mu_2 \cdot 0 + ... + \mu_{k+1} \cdot 0 = 2\mu_1.$$

This shows that μ_1 is the global maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the $k+1$ constraints.

On the other hand, the set of eigenvectors $\mathbf{x}^{(1)}$,...,$\mathbf{x}^{(k)}$ is orthonormal, so when we take the dot product of $(\star)$ with each of the unit vectors $\mathbf{x}^{(1)}$,...,$\mathbf{x}^{(k)}$ and use $(\star\star)$ we get

$$2\left(\mathbf{x}^{(1)}\right)^T A\mathbf{x}^\star = 2\mu_1\left(\mathbf{x}^{(1)}\right)^T\mathbf{x}^\star + \mu_2||\mathbf{x}^{(1)}||^2 + ... + \mu_{k+1}||\mathbf{x}^{(k)}||^2,$$

so

$$(\star\star\star)\ 2\left(\mathbf{x}^{(1)}\right)^T A\mathbf{x}^\star = 2\mu_1 \cdot 0 + \mu_2 \cdot 1 = \mu_2, \quad ... \quad , 2\left(\mathbf{x}^{(k)}\right)^T A\mathbf{x}^\star = \mu_{k+1}.$$

But, symmetry of A implies

$$\left(\mathbf{x}^{(1)}\right)^T A\mathbf{x}^\star = \left(\mathbf{x}^{(1)}\right)^T A^T\mathbf{x}^\star = \left(A\mathbf{x}^{(1)}\right)^T\mathbf{x}^\star = \left(\lambda_1\mathbf{x}^{(1)}\right)^T\mathbf{x}^\star = \lambda_1\left(\mathbf{x}^{(1)}\right)^T\mathbf{x}^\star = \lambda_1 \cdot 0,$$

so $(\star\star\star)$ implies $\mu_2 = 0$, and similarly,..., $\mu_{k+1} = 0$. It follows that stationarity, $(\star)$, actually implies

$$2A\mathbf{x}^\star = 2\mu_1\mathbf{x}^\star + 0 \cdot \mathbf{x}^{(1)} + ... + 0 \cdot \mathbf{x}^{(k)} = 2\mu_1\mathbf{x}^\star,$$

hence $\lambda_{k+1} \triangleq \mu_1$ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}^\star$.

The last thing to notice is that $\lambda_{k+1} \le \lambda_k$ follows from

$$\lambda_k = \max\{\mathcal{R}_A(\mathbf{x}) : \mathbf{x} \text{ satisfying } ||\,\mathbf{x}\,|| = 1,\ 0 = \left(\mathbf{x}^\star\right)^T\mathbf{x}^{(1)} = ... = \left(\mathbf{x}^\star\right)^T\mathbf{x}^{(k-1)}\}$$

$$\ge \max\{\mathcal{R}_A(\mathbf{x}) : \mathbf{x} \text{ satisfying } ||\,\mathbf{x}\,|| = 1,\ 0 = \left(\mathbf{x}^\star\right)^T\mathbf{x}^{(1)} = ... = \left(\mathbf{x}^\star\right)^T\mathbf{x}^{(k)}\} = \lambda_{k-1}.$$

13.6.3.3. Let $||\mathbf{x}|| = [\,x \quad y \quad z\,]^T$ be an unspecified nonzero vector. We calculate

$$\mathcal{R}_A(\mathbf{x}) = \frac{\mathbf{x}^T A\mathbf{x}}{||\mathbf{x}||^2} = \frac{\mathbf{x}^T A\mathbf{x}}{x^2+y^2+z^2} = \frac{1}{x^2+y^2+z^2} \cdot [\,x \quad y \quad z\,]\begin{bmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \frac{2x^2 + 2\sqrt{3}xy - z^2}{x^2+y^2+z^2} \triangleq f(x,y,z)$$

Mathematica$^{\text{TM}}$on $f(x,y,z)$ over the unit cube, as in Example 2.37 in Section 2.9, gives maximum value of $\lambda_1 = 3$ and minimum value of $\lambda_3 = -1$.

Corresponding to eigenvalue $\lambda_1 = 3$, Mathematica's FindMaximum command also gives us an eigenvector $\mathbf{x}^{(1)} \triangleq \left[\sqrt{3} \quad 1 \quad 0\right]^T$. Using Mathematica again, we found an approximate solution of the problem

$$\left\{\begin{array}{ll} \text{Maximize} & \mathbf{x}^T A\mathbf{x} \\ \\ \text{Subject to} & \mathbf{x}^T\mathbf{x} - 1 = 0 \\ \\ & \quad \mathbf{x}^T\mathbf{x}^{(1)} = 0 \end{array}\right\}.$$

which produces a second greatest eigenvalue, hence middle eigenvalue, $\lambda_2 \approx -1$. Here are the Mathematica commands and their output.

$$f[x\_, y\_, z\_] := \frac{2x^2 + 2\sqrt{3}\,xy - z^2}{x^2 + y^2 + z^2}$$

$\mathsf{FindMaximum}[\{\mathsf{f}[\mathsf{x}, \mathsf{y}, \mathsf{z}], 1 \geq \mathsf{x} \geq -1\&\&1 \geq \mathsf{y} \geq -1\&\&1 \geq \mathsf{z} \geq -1\&\&\sqrt{3}\mathsf{x} + \mathsf{y} == 0\}, \{\mathsf{x}, \mathsf{y}, \mathsf{z}\}]$

$\{-1., \{\mathsf{x} \to 0.29263, \mathsf{y} \to -0.506849, \mathsf{z} \to -0.0886408\}$

Chapter Fourteen

Section 14.1.2

14.1.2.1. It makes sense to look for an approximate solution that also satisfies the boundary conditions, for example, in the form

$$y(x) = c_1 x^2(\pi - x) + c_2 \cos\frac{x}{2}.$$

Mathematica$^{\text{TM}}$gave

$$f(c_1,c_2) \triangleq \int_0^\pi \left(\big(3+\cos x\big)\Big(c_1(2\pi x - 3x^2) - \frac{1}{2}\,c_2 \sin\frac{x}{2}\Big)^2 - 2\Big(c_1 x^2(\pi - x) + c_2 \cos\frac{x}{2}\Big)0.1x \right) dx$$

$$= -\frac{1}{100}\pi^5 c_1 + \frac{2}{5}(2-\pi)c_2 + \frac{2}{5}\left(180 - 20\pi^2 + \pi^4\right)c_1^2 + \frac{32}{9}(-34+11\pi)c_1c_2 + \frac{5\pi}{16}\,c_2^2.$$

The approximate $\min_{c_1,c_2} f(c_1,c_2)$is achieved at $c_1 \approx 0.0130544671$, $c_2 \approx 0.219383918$. So, using only two terms, an approximate solution of the ODE-BVP is given by

$$y(x) \approx 0.0130544671x^2(\pi - x) + 0.219383918\cos\frac{x}{2}.$$

[By the way, the problem did not ask for a graph of the approximate solution, but here it is if you're curious. As in Example 14.5 in Section 14.1, the Rayleigh-Ritz method's approximate solution is given as the dashed, red graph and Mathematica's approximate solution $y(x)$ is the solid, blue graph.]

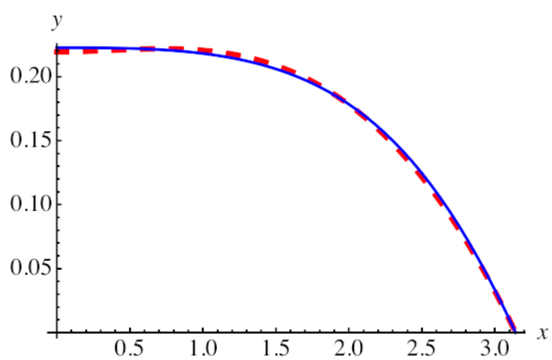


Figure 1: Answer key for problem 14.1.2.1

14.1.2.3. It makes sense to look for an approximate solution that also satisfies the boundary conditions.
Ex. 1: For an approximate solution in the form $y(x) = c_1(1-r) + c_2 r(1-r)$, Mathematica$^{\text{TM}}$gave

$$f(c_1,c_2) \triangleq \int_0^1 \Big(r\,\big(-c_1 + c_2(1-2r)\big)^2 - 4r^2\big(c_1(1-r) + c_2 r(1-r)\big)^2 - 2\big(c_1(1-r) + c_2 r(1-r)\big)r^3 \Big)\,dr$$

$$= -\frac{1}{10}\,c_1 + \frac{11}{30}\,c_1^2 - \frac{1}{15}\,c_2 + \frac{1}{5}\,c_1c_2 + \frac{9}{70}\,c_2^2$$

The approximate $\min_{c_1,c_2} f(c_1,c_2)$is achieved at $c_1 \approx 0.08333333442105048$, $c_2 \approx 0.19444444183629322$. So, using only two terms, an approximate solution of the ODE-BVP is given by

$$u(r) \approx 0.08333333442105048(1-r) + 0.19444444183629322r(1-r)$$

[By the way, the problem did not ask for a graph of the approximate solution, but here it is if you're curious. As in Example 14.5 in Section 14.1, the Rayleigh-Ritz method's approximate solution is given as the dashed, red graph.]

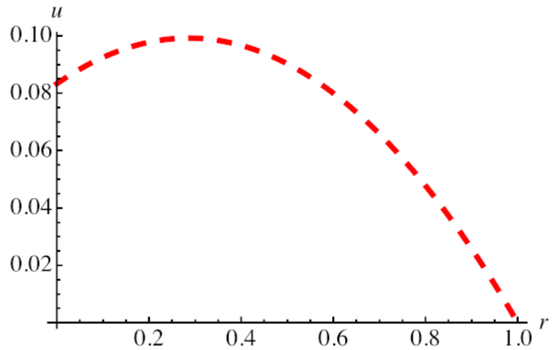


Figure 2: Answer key for first example of a solution to problem 14.1.2.3

Ex. 2: For an approximate solution in the form $y(x) = d_1(1-r) + d_2(1-r)^2$, Mathematica™gave

$$f(d_1, d_2) \triangleq \int_0^1 \Big(r\left(-d_1 + d_2 2(1-r)\right)^2 - 4r^2\big(d_1(1-r) + d_2(1-r)^2\big)^2 - 2\big(d_1(1-r) + d_2(1-r)^2\big)r^3\Big)\, dr$$

$$= -\frac{1}{10}\, d_1 + \frac{11}{30}\, d_1^2 - \frac{1}{30}\, d_2 + \frac{8}{15}\, d_1 d_2 + \frac{31}{105}\, d_2^2$$

The approximate $\min_{d_1,d_2} f(d_1, d_2)$is achieved at $d_1 \approx 0.2777777777871004$, $d_2 \approx -0.19444444445493192$. So, using only two terms, an approximate solution of the ODE-BVP is given by

$$u(r) \approx 0.2777777777871004(1-r) - 0.19444444445493192(1-r)^2$$

[By the way, the problem did not ask for a graph of the approximate solution, but here it is if you're curious. As in Example 14.5 in Section 14.1, the Rayleigh-Ritz method's approximate solution is given as the dashed, red graph.]

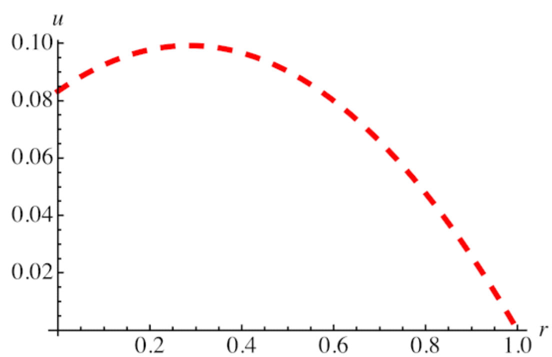


Figure 3: Answer key for second example of a solution to problem 14.1.2.3

Ex. 3: For an approximate solution in the form $y(x) = e_1(1-r) + e_2 \cos\left(\frac{\pi r}{2}\right)$, Mathematica™gave

$$f(e_1, e_2) \triangleq \int_0^1 \left(r\left(-e_1 - e_2\, \frac{\pi}{2}\, \sin\left(\frac{\pi r}{2}\right)\right)^2 - 4r^2\left(e_1(1-r) + e_2 \cos\left(\frac{\pi r}{2}\right)\right)^2 - 2\left(e_1(1-r) + e_2 \cos\left(\frac{\pi r}{2}\right)\right) r^3\right) dr$$

$$= -\frac{1}{10}\, e_1 + \frac{11}{30}\, e_1^2 + \frac{4}{\pi^4}\,(-48 + 24\pi - \pi^3)e_2 + \frac{4}{\pi^4}\,(192 - 64\pi + \pi^3)e_1e_2 + \frac{1}{48\pi^2}\,(192 - 20\pi^2 + 3\pi^4)e_2^2$$

The approximate $\min_{e_1,e_2} f(e_1, e_2)$is achieved at $e_1 \approx -0.16338185122419183$, $e_2 \approx 0.24393320411062294$. Using only two terms, an approximate solution of the ODE-BVP is given by

$$u(r) \approx -0.16338185122419183(1-r) + 0.24393320411062294 \cos\left(\frac{\pi r}{2}\right)$$

[By the way, the problem did not ask for a graph of the approximate solution, but here it is if you're curious. As in Example 14.5 in Section 14.1, the Rayleigh-Ritz method's approximate solution is given as the dashed, red graph.]

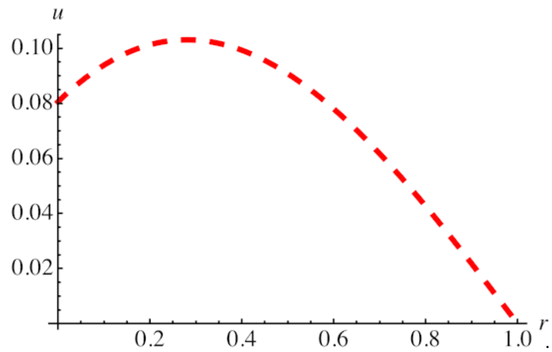


Figure 4: Answer key for third example of a solution to problem 14.1.2.3

The three examples of approximate solutions agree very much with each other. The only slight disagreement is that the third approximate solution has slightly larger values of $|y''(x)|$.

Section 14.2.5

14.2.5.1. This problem fits the form of Theorem 14.1 in Section 14.2, with $F(x,y,y') = -p(x)\big(y'\big)^2 - 2yf(x)$, $y_a = y(0) = 0$ and $y_b = y(L) = 0$, so the minimizer must satisfy the Euler-Lagrange equation

$$0 \equiv \frac{\partial F}{\partial y} - \frac{d}{dx}\Big[\frac{\partial F}{\partial y'}\Big] = -2f(x) - \frac{d}{dx}\Big[-2p(x)y'\Big],$$

that is,

$$(p(x)y')' = f(x),\ 0 < x < L.$$

The latter is an ODE satisfied by all solutions of the calculus of variations problem.

14.2.5.3. This problem fits the form of Corollary 14.1 in Section 14.2, with $F\Big(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\Big) = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2 - 2uf(x,y)$, so the minimizer must satisfy the Euler-Lagrange equation

$$0 \equiv \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\Big[\frac{\partial F}{\partial(\partial u/\partial x)}\Big] - \frac{\partial}{\partial y}\Big[\frac{\partial F}{\partial(\partial u/\partial y)}\Big] = -2f(x,y) - \frac{\partial}{\partial x}\Big[2\frac{\partial u}{\partial x}\Big] - \frac{\partial}{\partial y}\Big[2\frac{\partial u}{\partial y}\Big],\ (x,y) \text{ in } \mathcal{D}.$$

that is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x,y),\ (x,y) \text{ in } \mathcal{D}.$$

The latter is a PDE satisfied by all solutions of the calculus of variations problem.

14.2.5.5. This problem fits the form of Theorem 14.1 in Section 14.2, with $F(x,y,y') = -x\big(y'\big)^2 + xy^2$, $y(1) = y_a = 0$ and $y(3) = y_b = -1$, so the minimizer must satisfy the Euler-Lagrange equation

$$0 \equiv \frac{\partial F}{\partial y} - \frac{d}{dx}\Big[\frac{\partial F}{\partial y'}\Big] = -2xy - \frac{d}{dx}\Big[-2xy'\Big],$$

that is,

$$(xy')' + xy = 0,\ 1 < x < 3,$$

which can be rewritten as the ODE

$$x^2y'' + xy' + x^2y = 0,\ 1 < x < 3,$$

which is Bessel's equation of order zero.

So, the solution of the minimization problem must satisfy the ODE-BVP

$$(\star)\quad \left\{\begin{array}{c} x^2y'' + xy' + x^2y = 0,\ 1 < x < 3, \\ \\ y(1) = 0,\ y(3) = -1. \end{array}\right\}$$

Bessel's equation of order zero has solutions, $J_0(x)$ and $Y_0(x)$, that are known with as much accuracy as the trigonometric functions $\sin x$ and $\cos x$ are known. The general solution of ODE $y'' + \frac{1}{x}y' + y = 0$ is $y(x) = c_1J_0(x) + c_2Y_0(x)$, where c_1, c_2 are arbitrary constants. Substituting this into the boundary conditions gives

$$\left\{\begin{array}{rr} J_0(1)c_1 + Y_0(1)c_2 = & 0 \\ J_0(3)c_1 + Y_0(3)c_2 = & -1 \end{array}\right\},$$

a system of two algebraic equations in unknowns c_1, c_2. Using tabulated values of $J_0(1), Y_0(1), J_0(3), Y_0(3)$, we arrive at the solution of $(\star)$:

$$y(x) \approx 0.2662994102\, J_0(x) - 2.469811754\, Y_0(x).$$

[By the way, this ODE-BVP also was studied using numerical, approximate solutions in Example 8.19 in Section 8.8 and Example 8.21 in Section 8.9.]

14.2.5.7. We are allowed to vary $\delta y = \delta y(x)$, $\delta y' = \delta y'(x)$, $\delta v = \delta v(x)$, and $\delta v' = \delta v'(x)$ arbitrarily, except for requiring that δy and δv be continuously differentiable on (a,b), $\delta y(a) = \delta y(b) = \delta v(a) = \delta v(b) = 0$, $\delta y'$ and $\delta v'$ be piecewise continuous on (a,b). The second group of assumptions comes from the need for $y_0(x) + \delta y(x)$ to satisfy the boundary conditions $y(a) = y_a, y(b) = y_b$, $v(a) = v_a, v(b) = v_b$. In addition, consistency, that is, $(\delta y)' = \delta(y')$ and $(\delta v)' = \delta(v')$, implies that $\int_a^b \delta y'(x)dx = \big[\,\delta y(x)\,\big]_a^b = \delta y(b) - \delta y(a) = 0$ and $\int_a^b \delta v'(x)dx = \big[\,\delta v(x)\,\big]_a^b = 0$.

Analogous to linear approximation in $\mathbb{R}^5$, we have

$$\delta F(x,y,y') = \frac{\partial F}{\partial y}(x,y,y',v,v')\cdot\delta y + \frac{\partial F}{\partial y'}(x,y,y',v,v')\cdot\delta y' + \frac{\partial F}{\partial v}(x,y,y',v,v')\cdot\delta v + \frac{\partial F}{\partial v'}(x,y,y',v,v')\cdot\delta v'.$$

There is no partial derivative of F with respect to an x term because x is not being varied in the minimization process. So,

$$\begin{aligned}\delta J &= \int_a^b \delta F(x,y,y',v,v')dx \\ &= \int_a^b \left(\frac{\partial F}{\partial y}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta y(x) + \frac{\partial F}{\partial y'}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta y'(x)\right)dx \\ &\quad+\int_a^b \left(\frac{\partial F}{\partial v}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta v(x) + \frac{\partial F}{\partial v'}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta v'(x)\right)dx.\end{aligned}$$

Concerning the second and fourth terms, integration by parts gives

$$\begin{aligned}&\int_a^b \frac{\partial F}{\partial y'}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta y'(x)\,dx = \\ &= \left[\frac{\partial F}{\partial y'}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta y(x)\right]_a^b - \int_a^b\left(\frac{d}{dx}\Big[\frac{\partial F}{\partial y'}\big(x,y(x),y'(x),v(x),v'(x)\big)\Big]\right)\cdot\delta y(x)\,dx.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_a^b \frac{\partial F}{\partial v'}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta v'(x)\,dx = \\ &= \left[\frac{\partial F}{\partial v'}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta v(x)\right]_a^b - \int_a^b\left(\frac{d}{dx}\Big[\frac{\partial F}{\partial v'}\big(x,y(x),y'(x),v(x),v'(x)\big)\Big]\right)\cdot\delta v(x)\,dx.\end{aligned}$$

But, $\delta y(a) = \delta y(b) = 0$ and $\delta v(a) = \delta v(b) = 0$, so

$$\begin{aligned}\delta J &= \int_a^b \frac{\partial F}{\partial y}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta y(x)\,dx - \int_a^b\left(\frac{d}{dx}\Big[\frac{\partial F}{\partial y'}\big(x,y(x),y'(x),v(x),v'(x)\big)\Big]\right)\cdot\delta y(x)\,dx \\ &\quad+\int_a^b \frac{\partial F}{\partial v}\big(x,y(x),y'(x),v(x),v'(x)\big)\cdot\delta v(x)\,dx - \int_a^b\left(\frac{d}{dx}\Big[\frac{\partial F}{\partial v'}\big(x,y(x),y'(x),v(x),v'(x)\big)\Big]\right)\cdot\delta v(x)\,dx.\end{aligned}$$

Written briefly,

$$\delta J = \int_a^b\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\Big[\frac{\partial F}{\partial y'}\Big]\right)\delta y\,dx + \int_a^b\left(\frac{\partial F}{\partial v} - \frac{d}{dx}\Big[\frac{\partial F}{\partial v'}\Big]\right)\delta v\,dx.$$

14.2.5.9. Define $J[\,y\,] \triangleq \displaystyle\int_0^L \left(\frac{1}{2}\,EI(y'')^2 + yf(x)\right)dx$, where $y(x)$ is an admissible function, namely one that (1) is once continuously differentiable on the interval $(0,L)$, (2) has second derivative that is piecewise continuous on $(0,L)$, and (3) satisfies the BCs $y(0) = y(L) = 0$.

We are allowed to vary $\delta y = \delta y(x)$, $\delta y' = \delta y'(x)$, and $\delta y'' = \delta y''(x)$ arbitrarily, except for requiring that δy and $\delta y'$ be continuously differentiable on $(0, L)$, and that $\delta y(0) = \delta y(L) = 0$ and $\delta y''$ be piecewise continuous on $(0, L)$. The second group of assumptions comes from the need for $y_0(x) + \delta y(x)$ to satisfy the boundary conditions $y(0) = 0$ and $y(L) = 0$. In addition, consistency, that is, $(\delta y)' = \delta(y')$ and $(\delta y')' = \delta(y'')$, implies $\int_0^L \delta y'(x)dx = \big[\,\delta y(x)\,\big]_0^L = \delta y(L) - \delta y(0) = 0$.

The work is similar to that for Example 14.9 in Section 14.2.

(a) Suppose $y(x)$ is the minimizer of (14.2) in Section 14.1, that is, the minimization in problem 14.2.5.9. We calculate the variation

$$\delta J = \int_0^L \delta\left(\frac{1}{2}\,EI(y'')^2 + yf(x)\right)dx = \int_0^L \big(EIy_0''(x)\,\delta y''(x) + f(x)\,\delta y(x)\big)\,dx.$$

At a minimizer $y_0(x)$, stationarity implies that

$$0 = \delta J = \int_0^L \big(EIy_0''(x)\,\delta y''(x) + f(x)\,\delta y(x)\big)\,dx.$$

(b) Next, using $(\delta y')' = \delta(y'')$ and $(\delta y)' = \delta(y')$, integrate by parts, twice successively to get

$$0 = \delta J = \Big[\,EIy_0''(x)\,\delta y'(x)\Big]_0^L - \int_0^L EIy_0'''(x)\,\delta y'(x)\,dx + \int_0^L f(x)\,\delta y(x)\,dx,$$

hence

$$(\star)\quad 0 = \delta J = \Big[\,EIy_0''(x)\,\delta y'(x)\Big]_0^L - \Big[\,EIy_0'''(x)\,\delta y(x)\Big]_0^L + \int_0^L EIy_0''''(x)\,\delta y(x)\,dx + \int_0^L f(x)\,\delta y(x)\,dx.$$

Note that $\delta y(0) = \delta y(L) = 0$ implies that

$$\Big[\,EIy_0'''(x)\,\delta y(x)\Big]_0^L = EIy_0'''(L)\,\delta y(L) - E(y_0'''(0)\,\delta y(0) = EIy_0'''(L)\;\cdot 0 - EIy_0'''(0)\;\cdot 0 = 0.$$

Substitute this into $(\star)$ and combine the two integrals to get

$$(\star\star)\quad 0 = EIy_0''(L)\,\delta y'(L) - EIy_0''(0)\,\delta y'(0) + \int_0^L \big(EIy_0''''(x) + f(x)\big)\,\delta y(x)\,dx.$$

(c) In $(\star\star)$, we are allowed to vary independently (1) the function $\delta y(x)$ for $0 < x < L$, (2) the number $\delta y'(L)$, and (3) the number $\delta y'(0)$. It follows that $(\star\star)$ implies

$$y_0''(L) = 0, \quad y_0''(0) = 0,$$

and

$$\big(y_0''''(x) + f(x)\big) \equiv 0, 0 < x < L.$$

So, the natural boundary conditions $y''(0) = y''(L) = 0$ appear as a consequence of the minimization.

Putting everything together, the minimizer $y_0(x)$ satisfies the fourth order ODE-BVP

$$(\star)\quad \left\{\begin{array}{c} EIy'''' + f(x) = 0 \\ \\ y(0) = y(L) = 0, \quad y''(0) = y''(L) = 0 \end{array}\right\}.$$

14.2.5.11. We are allowed to vary $\delta\mathbf{y} = \delta\mathbf{y}(x)$ and $\delta\mathbf{y}' = \delta\mathbf{y}'(x)$ arbitrarily, except for requiring that $\delta\mathbf{y}$ be continuously differentiable on (a, b), $\delta\mathbf{y}(a) = \delta\mathbf{y}(b) = 0$, and that $\delta\mathbf{y}'$ be piecewise continuous on (a, b) and $\int_a^b \delta\mathbf{y}'(x)dx = 0$. The second assumption comes from the need for $\mathbf{y}_0(x) + \delta\mathbf{y}(x)$ to satisfy the boundary conditions $\mathbf{y}(a) = \mathbf{y}_a$, $\mathbf{y}(b) = \mathbf{y}_b$. In addition, consistency, that is, $(\delta\mathbf{y})' = \delta(\mathbf{y}')$, implies that $\int_a^b \delta\mathbf{y}'(x)dx = \big[\,\delta\mathbf{y}(x)\,\big]_a^b = \delta\mathbf{y}(b) - \delta\mathbf{y}(a) = 0$.

Analogous to linear approximation in $\mathbb{R}^3$, we have

$$\delta F(x, \mathbf{y}, \mathbf{y}') = \big(\boldsymbol{\nabla}_{\mathbf{y}} F\big(x, \mathbf{y}, \mathbf{y}'\big) \bullet \delta \mathbf{y} + \big(\boldsymbol{\nabla}_{\mathbf{y}'} F\big(x, \mathbf{y}, \mathbf{y}'\big) \bullet \delta \mathbf{y}'.$$

There is no partial derivative of F with respect to an x term because x is not being varied in the minimization process. So,

$$\delta J = \int_a^b \delta F\big(x, \mathbf{y}, \mathbf{y}'\big) dx = \int_a^b \Big(\big(\boldsymbol{\nabla}_{\mathbf{y}} F\big(x, \mathbf{y}, \mathbf{y}'\big) \bullet \delta \mathbf{y} + \big(\boldsymbol{\nabla}_{\mathbf{y}'} F\big(x, \mathbf{y}, \mathbf{y}'\big) \bullet \delta \mathbf{y}'\Big)\, dx.$$

Concerning the last term, integration by parts gives

$$\int_a^b \big(\boldsymbol{\nabla}_{\mathbf{y}'} F\big(x, \mathbf{y}, \mathbf{y}'\big) \bullet \delta \mathbf{y}'\, dx = \Big[\big(\boldsymbol{\nabla}_{\mathbf{y}'} F\big(x, \mathbf{y}, \mathbf{y}'\big) \bullet \delta \mathbf{y}\Big]_a^b - \int_a^b \left(\frac{d}{dx}\Big[\,\big(\boldsymbol{\nabla}_{\mathbf{y}'} F\big(x, \mathbf{y}, \mathbf{y}'\big)\,\Big]\right) \bullet \delta \mathbf{y}\, dx.$$

But, $\delta \mathbf{y}(a) = \delta \mathbf{y}(b) = 0$, so

$$\delta J = \int_a^b \big(\boldsymbol{\nabla}_{\mathbf{y}} F\big(x, \mathbf{y}, \mathbf{y}'\big) \bullet \delta \mathbf{y}\, dx - \int_a^b \left(\frac{d}{dx}\Big[\,\big(\boldsymbol{\nabla}_{\mathbf{y}'} F\big(x, \mathbf{y}, \mathbf{y}'\big)\,\Big]\right) \bullet \delta \mathbf{y}\, dx.$$

Stationarity at a global minimum gives

$$0 = \delta J = \int_a^b \big(\boldsymbol{\nabla}_{\mathbf{y}} F\big(x, \mathbf{y}, \mathbf{y}'\big) \bullet \delta \mathbf{y}\, dx - \int_a^b \left(\frac{d}{dx}\Big[\,\big(\boldsymbol{\nabla}_{\mathbf{y}'} F\big(x, \mathbf{y}, \mathbf{y}'\big)\,\Big]\right) \bullet \delta \mathbf{y}\, dx.$$

Because $\delta y(x)$ is arbitrary except for the requirements that it be continuously differentiable and satisfy $\delta y(a) = \delta y(b) = 0$, we conclude that necessary is that

$$\boldsymbol{\nabla}_{\mathbf{y}} F\big(x, \mathbf{y}, \mathbf{y}'\big) - \frac{d}{dx}\Big[\,\big(\boldsymbol{\nabla}_{\mathbf{y}'} F\big(x, \mathbf{y}, \mathbf{y}'\big)\,\Big] \equiv 0,\ \ a < x < b.$$

In more explicit notation, this says that if $\mathbf{y} = [\,y_1(x) \quad ... \quad y_n(x)\,]$, the necessary conditions are that, for $i = 1, ..., n$,

$$\frac{\partial F}{\partial y_i}\big(x, \mathbf{y}, \mathbf{y}'\big) - \frac{d}{dx}\Big[\,\frac{\partial F}{\partial y_i'}\big(x, \mathbf{y}, \mathbf{y}'\big)\,\Big] \equiv 0,\ \ a < x < b.$$

Section 14.3.2

14.3.2.1. This problem partially fits the form of Theorem 14.2 in Section 14.3, with $F(x,u,u') = \big(u'\big)^2 - q(x)u^2$, $G(x,u,u') = u^2$, and $u(0) = 0$, but the second BC, $u'(L) = 0$ does not fit the Theorem. In order to adapt Theorem 14.2 to problem 14.3.2.1 we only need to calculate as in the derivation of (14.18) in Section 14.2.

Define $J[\,u\,] \triangleq \int_0^L F\big(x, u(x), u'(x)\big)dx$, where $F(x,u,u') \triangleq \big((u')^2 - q(x)u^2\big)$ and $u(x)$ is an admissible function that satisfies the BCs $u(0) = u'(L) = 0$. We are allowed to vary $\delta u = \delta u(x)$ and $\delta u' = \delta u'(x)$ arbitrarily, except for requiring that δu be continuously differentiable on $(0, L)$, $\delta u(0) = 0$, $\delta u'$ be piecewise continuous on (a, b), $(\delta u')(L) = 0$, and $(\delta u)' = \delta(u')$. The second assumption comes from the need for $u_0(x) + \delta u(x)$ to satisfy the boundary condition $u(0) = 0$, the next to last assumption follows from the need for $u_0(x)+\delta u(x)$ to satisfy the boundary condition $u'(L) = 0$, and the last assumption is called "consistency."

Analogous to linear approximation in $\mathbb{R}^3$, we have

$$\delta F(x,u,u') = \frac{\partial F}{\partial u}(x,u,u')\cdot\delta u + \frac{\partial F}{\partial u'}(x,u,u')\cdot\delta u'.$$

There is no partial derivative of F with respect to an x term because x is not being varied in the minimization process. So,

$$\delta J = \int_0^L \delta F(x,u,u')dx = \int_0^L \left(\frac{\partial F}{\partial u}\big(x,u(x),u'(x)\big)\cdot\delta u(x) + \frac{\partial F}{\partial u'}\big(x,u(x),u'(x)\big)\cdot\delta u'(x)\right)dx.$$

Concerning the last term, integration by parts gives

$$\int_0^L \frac{\partial F}{\partial u'}\big(x,u(x),u'(x)\big)\cdot\delta u'(x)\,dx =$$

$$= \left[\frac{\partial F}{\partial u'}\big(x,u(x),u'(x)\big)\cdot\delta u(x)\right]_0^L - \int_0^L \left(\frac{d}{dx}\Big[\frac{\partial F}{\partial u'}\big(x,u(x),u'(x)\big)\Big]\right)\cdot\delta u(x)\,dx.$$

But, $\delta u(0) = 0$, so $\frac{\partial F}{\partial u'}\big(0,u(0),u'(0)\big)\cdot\delta u(0) = 0$. As to the other term on the boundary, in problem 14.3.2.1, $F(x,u,u') \triangleq \big((u')^2 - q(x)u^2\big)$, so

$$\frac{\partial F}{\partial u'}\big(L,u(L),u'(L)\big)\cdot\delta u(L) = 2u'(L)\cdot\delta u(L) = 2\cdot 0\cdot\delta u(L) = 0.$$

Putting everything together,

$$\delta J = \int_0^L \frac{\partial F}{\partial u}\big(x,u(x),u'(x)\big)\cdot\delta u(x)\,dx - \int_0^L \left(\frac{d}{dx}\Big[\frac{\partial F}{\partial u'}\big(x,u(x),u'(x)\big)\Big]\right)\cdot\delta u(x)\,dx.$$

Having adapted the calculation of δJ to problem 14.3.2.1, we can use the conclusion of Theorem 14.2 in Section 14.3 to see that there must be a Lagrange multiplier λ such that the minimizer, $u(x)$, must satisfy the equation

$$0 \equiv \frac{\partial(F-\lambda G)}{\partial u} - \frac{d}{dx}\Big[\frac{\partial(F-\lambda G)}{\partial u'}\Big] = -2q(x)u - 2\lambda u - \frac{d}{dx}\Big[2(u' + \lambda\cdot 0\Big],\ 0 < x < L$$

that is,

$$u'' + (q(x)+\lambda)u = 0,\ 0 < x < L.$$

The latter is an ODE satisfied by all solutions of the calculus of variations problem. The desired ODE-BVP is

$$\left\{\begin{array}{c} u'' + (q(x)+\lambda)u,\ 0 < x < L, \\ \\ u(0) = 0,\ u'(L) = 0. \end{array}\right\}$$

14.3.2.3. This problem fits the form of Corollary 14.1 in Section 14.3, with $F\Big(x,y,u,\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}\Big)=\Big(\dfrac{\partial u}{\partial x}\Big)^2+\Big(\dfrac{\partial u}{\partial y}\Big)^2-2uf(x,y)$ and $G(x,u,u')=u^2$, so the minimizer must satisfy the Euler-Lagrange equation

$$0\equiv\frac{\partial(F-\lambda G)}{\partial u}-\frac{\partial}{\partial x}\Big[\frac{\partial(F-\lambda G)}{\partial(\partial u/\partial x)}\Big]-\frac{\partial}{\partial y}\Big[\frac{\partial(F-\lambda G)}{\partial(\partial u/\partial y)}\Big]=-2f(x,y)u-2\lambda u-\frac{\partial}{\partial x}\Big[2\,\frac{\partial u}{\partial x}\Big]-\frac{\partial}{\partial y}\Big[2\,\frac{\partial u}{\partial y}\Big],$$

for (x,y) in $\mathcal{D}$, that is,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+(f(x,y)+\lambda)u=0,\ (x,y)\text{ in }\mathcal{D}.$$

The latter is a PDE satisfied by all solutions of the calculus of variations problem. The desired PDE-BVP is

$$\left\{\begin{array}{c}\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+(f(x,y)+\lambda)u=0,\ (x,y)\text{ in }\mathcal{D},\\ \\ u\equiv g(x,y)\text{ on }\partial\mathcal{D}.\end{array}\right\}$$

14.3.2.5. Equation (14.30) in Section 14.3 is

$$\boldsymbol{\nabla}_{\boldsymbol{\varepsilon}}(f-\lambda g)\Big|_{(\varepsilon_1,\varepsilon_2)=(0,0)}=\mathbf{0},$$

where

$$f(\varepsilon_1,\varepsilon_2)\triangleq J\big[\,y(\varepsilon_1,\varepsilon_2)\,\big]=\int_a^b F\big(x,y_0(x)+\varepsilon_1\phi_1(x)+\varepsilon_2\phi_2(x),y_0'(x)+\varepsilon_1\phi_1'(x)+\varepsilon_2\phi_2'(x)\big)dx$$

and

$$g(\varepsilon_1,\varepsilon_2)\triangleq K\big[\,y(\varepsilon_1,\varepsilon_2)\,\big]=\int_a^b G\big(x,y_0(x)+\varepsilon_1\phi_1(x)+\varepsilon_2\phi_2(x),y_0'(x)+\varepsilon_1\phi_1'(x)+\varepsilon_2\phi_2'(x)\big)dx.$$

So,

$$\frac{\partial f}{\partial\varepsilon_1}=\frac{\partial}{\partial\varepsilon_1}\int_a^b F\big(x,y_0(x)+\varepsilon_1\phi_1(x)+\varepsilon_2\phi_2(x),y_0'(x)+\varepsilon_1\phi_1'(x)+\varepsilon_2\phi_2'(x)\big)dx$$

$$=\int_a^b\Big(\frac{\partial F}{\partial y}\big(x,y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2,y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\phi_1(x)+\frac{\partial F}{\partial y'}\big(x,y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2,y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\phi_1'(x)\Big)dx,$$

which, after use of integration by parts, equals

$$=\int_a^b\frac{\partial F}{\partial y}\big(x,y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2,y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\phi_1(x)\,dx+\Big[\,\frac{\partial F}{\partial y'}\big(x,y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2,y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\phi_1(x)\,\Big]_a^b$$

$$-\int_a^b\frac{d}{dx}\Big[\,\frac{\partial F}{\partial y'}\big(x,y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2,y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\,\Big]\phi_1(x)\,dx$$

$$=\int_a^b\frac{\partial F}{\partial y}\big(x,y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2,y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\phi_1(x)\,dx$$

$$+\frac{\partial F}{\partial y'}\big(b,y_0(b)+\varepsilon_1\phi_1(b)+\varepsilon_2\phi_2(b),y_0'(b)+\varepsilon_1\phi_1(b)'+\varepsilon_2\phi_2(b)'\big)\phi_1(b)$$

$$-\frac{\partial F}{\partial y'}\big(a,y_0(a)+\varepsilon_1\phi_1(a)+\varepsilon_2\phi_2(a),y_0'(a)+\varepsilon_1\phi_1(a)'+\varepsilon_2\phi_2(a)'\big)\phi_1(a)$$

$$-\int_a^b\frac{d}{dx}\Big[\,\frac{\partial F}{\partial y'}\big(x,y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2,y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\,\Big]\phi_1(x)\,dx$$

$$= \int_a^b \frac{\partial F}{\partial y}\big(x, y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2, y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\phi_1(x)\,dx - \int_a^b \frac{d}{dx}\Big[\,\frac{\partial F}{\partial y'}\big(x, y_0+\varepsilon_1\phi_1+\varepsilon_2\phi_2, y_0'+\varepsilon_1\phi_1'+\varepsilon_2\phi_2'\big)\,\Big]\phi_1(x)\,dx,$$

after noting that $\phi(a) = \phi(b) = 0$.

Similarly,

$$\frac{\partial g}{\partial \varepsilon_1} = \int_a^b \frac{\partial F}{\partial y}\big(x, y_0 + \varepsilon_1\phi_1 + \varepsilon_2\phi_2, y_0' + \varepsilon_1\phi_1' + \varepsilon_2\phi_2'\big)\phi_1(x)\,dx$$

$$- \int_a^b \frac{d}{dx}\Big[\,\frac{\partial F}{\partial y'}\big(x, y_0 + \varepsilon_1\phi_1 + \varepsilon_2\phi_2, y_0' + \varepsilon_1\phi_1' + \varepsilon_2\phi_2'\big)\,\Big]\phi_1(x)\,dx.$$

So, the first component of (14.30), that is,

$$\frac{\partial f}{\partial \varepsilon_1} - \lambda\,\frac{\partial g}{\partial \varepsilon_1}\Big|_{(\varepsilon_1,\varepsilon_2)=(0,0)} = 0,$$

is actually that

$$0 = \int_a^b \Big(\frac{\partial(F-\lambda G)}{\partial y}\big(x, y_0 + \varepsilon_1\phi_1 + \varepsilon_2\phi_2, y_0' + \varepsilon_1\phi_1' + \varepsilon_2\phi_2'\big)\phi_1(x)$$

$$- \frac{d}{dx}\Big[\,\frac{\partial(F-\lambda G)}{\partial y'}\big(x, y_0 + \varepsilon_1\phi_1 + \varepsilon_2\phi_2, y_0' + \varepsilon_1\phi_1' + \varepsilon_2\phi_2'\big)\,\Big]\Big)\phi_1(x)dx\Big|_{(\varepsilon_1,\varepsilon_2)=(0,0)}.$$

This is (14.31) in Section 14.3, that is,

$$0 = \int_a^b \left(\frac{\partial(F-\lambda G)}{\partial y}\big(x, y_0(x), {y'}_0(x)\big) - \frac{d}{dx}\Big[\frac{\partial(F-\lambda G)}{\partial y'}\big(x, y_0(x), {y'}_0(x)\big)\Big]\right)\phi_1(x)\,dx.$$

The explanation for why (14.30) implies (14.32) is the same, except for replacing ε_1 by ε_2 and replacing $\phi_1(x)$ by $\phi_2(x)$.

Section 14.4.4

14.4.4.1. Define the operator $A = -\Delta$ on $\left\{ u \text{ in } H^2(\mathcal{D}) : \dfrac{\partial u}{\partial n} \equiv 0 \text{ on } \partial\mathcal{D} \right\}$ by

$$Au \triangleq -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = -\boldsymbol{\nabla} \bullet \boldsymbol{\nabla} u.$$

Using the Divergence Theorem and the result of Corollary 6.1 in Section 6.7,

$$\langle Au, u \rangle = -\iint_{\mathcal{D}} \big(\boldsymbol{\nabla} \bullet \boldsymbol{\nabla} u(x,y)\big) u(x,y)\, dxdy = -\oint_{\partial\mathcal{D}} \widehat{\mathbf{n}} \bullet \big(u \boldsymbol{\nabla} u\big) ds + \iint_{\mathcal{D}} \big(\boldsymbol{\nabla} u\big) \bullet \big(\boldsymbol{\nabla} u\big) dA$$

$$= -\oint_{\partial\mathcal{D}} u \cdot \big(\widehat{\mathbf{n}} \bullet \boldsymbol{\nabla} u\big) ds + \iint_{\mathcal{D}} \big(\boldsymbol{\nabla} u\big) \bullet \big(\boldsymbol{\nabla} u\big) dA = 0 + \iint_{\mathcal{D}} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) dA,$$

after using the boundary condition $\dfrac{\partial u}{\partial n} \equiv 0$ on $\partial\mathcal{D}$. So, a calculus of variations problem satisfied by the solution is given by

$$\left\{ \begin{array}{ll} \text{Minimize} & \displaystyle\iint_{\mathcal{D}} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) dA \\ \text{Subject to} & \displaystyle\iint_{\mathcal{D}} |u(x,y)|^2 \, dx\, dy = 1 \text{ and} \\ & \displaystyle\frac{\partial u}{\partial n} \equiv 0 \text{ on } \partial\mathcal{D} \end{array} \right\}.$$

14.4.4.3. Define the operator $A = -\Delta - f(x,y)$ on $H_0^2(\mathcal{D})$ by $\quad Au \triangleq -\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right) - f(x,y)u.$

Using the Divergence Theorem and the result of Corollary 6.1 in Section 6.7,

$$\langle Au, u \rangle = -\iint_{\mathcal{D}} \big(\boldsymbol{\nabla} \bullet \boldsymbol{\nabla} u + f(x,y)u\big) u(x,y) dxdy$$

$$= -\oint_{\partial\mathcal{D}} \widehat{\mathbf{n}} \bullet \big(u \boldsymbol{\nabla} u\big) ds + \iint_{\mathcal{D}} \big(\boldsymbol{\nabla} u\big) \bullet \big(\boldsymbol{\nabla} u\big) dA - \iint_{\mathcal{D}} f(x,y) \big(u(x,y)\big)^2 dxdy$$

$$= 0 + \iint_{\mathcal{D}} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 - f(x,y)u^2 \right) dA,$$

after using the boundary condition $u \equiv 0$ on $\partial\mathcal{D}$. So, a calculus of variations problem satisfied by the solution is given by

$$\left\{ \begin{array}{ll} \text{Minimize} & \displaystyle\iint_{\mathcal{D}} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 - f(x,y)u^2 \right) dA \\ \text{Subject to} & \displaystyle\iint_{\mathcal{D}} \sigma(x,y) |u(x,y)|^2 \, dx\, dy = 1 \\ & \text{and } u \equiv 0 \text{ on } \partial\mathcal{D} \end{array} \right\}$$

14.4.4.5. Example 14.15 in Section 14.4, with $p(x) \equiv 1$, $\sigma(x) \equiv 1$, $\epsilon_0 = 1$, $\epsilon_1 = 0$, $\gamma_0 = 1$, and $\gamma_1 = 0$, implies that the minimizer, $u(x)$, satisfies the Sturm-Liouville problem

$$\left\{ \begin{array}{c} X''(x) + \big(\lambda + q(x)\big) X(x) = 0, \;\; 0 < x < L, \\ X(0) = X(L) = 0 \end{array} \right\}$$

Section 14.5.4

14.5.4.1. Define $J[\,y\,] \triangleq \int_0^1 ((y')^2 + x^2y^2 - 2xy)dx$. Let $h = 0.25$, $x_j = jh$ for $j = -1, 0, 1, 2, 3, 4$,

$$\phi(t) \triangleq \left\{ \begin{array}{ll} t+1, & -1 \le t \le 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & |t| > 1 \end{array} \right\},$$

and uniform tent basis functions $T_j(x) = \phi\left(\frac{x-x_j}{h}\right)$, $j = 0, 1, 2, 3$. So, using only three terms, an approximate solution of the ODE-BVP is given by

$$y(x) \approx 0.127332639564\, T_1(x) + 0.178564177107\, T_2(x) + 0.141182350726\, T_3(x).$$

Here are the details of the Mathematica work used to find this approximate solution: After defining $\mathsf{h} = 0.25$, the basic tent function $T_1(x)$ was, as in Example 14.16, defined by

$$\mathsf{T1[t\_] := Piecewise}\Big[\{\{\frac{\mathsf{t}}{\mathsf{h}}, 0 < \mathsf{t} \le \mathsf{h}\}, \{1 - \frac{\mathsf{t}-\mathsf{h}}{\mathsf{h}}, \mathsf{h} < \mathsf{t} \le 2\mathsf{h}\}, \{0, -1 < \mathsf{t} \le 0\}, \{0, 2\mathsf{h} < \mathsf{t} \le 1\}\}\Big].$$

Noting that $T_2(x) = T_1(x-h)$, $T_3(x) = T_1(x-2h)$, we defined

$$\mathsf{y[x\_, c1\_, c2\_, c3\_] := c1 * T1[x] + c2 * T1[x-h] + c3 * T1[x-2h]}$$

and the functional by

$$\mathsf{J[c1\_, c2\_, c3\_] := Evaluate}\Big[\int_0^1 \big((\mathsf{D[y[x, c1, c2, c3], x]})^2 + \mathsf{x} * (\mathsf{y[x, c1, c2, c3]})^2 - 2 * (1 + \mathsf{x}) * \mathsf{y[x, c1, c2, c3]}\big)\mathsf{dx}\Big].$$

Then we used

$$\mathsf{FindMinimum[J[c1, c2, c3], \{\{c1, .1\}, \{c2, 0.1\}, \{c3, 0.1\}}$$

to get output

$$\{-0.168520, \{\mathsf{c1} \to 0.127332639564, \mathsf{c2} \to 0.178564177107, \mathsf{c3} \to 0.141182350726\}\}.$$

In a figure we show the approximate solution of the ODE-BVP as a dashed, red graph and Mathematica's approximate solution $y(x)$ as a solid, blue graph.

[Note: This is an example of a boundary value problem modeling a cantilever beam clamped at the end $x = L$, as in problem 9.7.1.7.]

[By the way, the problem did not ask for a graph of the approximate solution, but here it is if you're curious. As in Example 14.16 in Section 14.5, the Rayleigh-Ritz method's approximate solution is given as the dashed, red graph and Mathematica's approximate solution $y(x)$ is the solid, blue graph.]

14.5.4.3. Define $J[\,y\,] \triangleq \int_0^1 ((y')^2 + xy^2 - 2(1+x)y)dx$. Let $h = 0.5$, $x_j = jh$ for $j = -1, 0, 1, 2, 3$, $\psi(t)$ is the cubic spline, and the cubic uniform basis spline functions are $C_j(x) \triangleq \psi\left(\frac{x-x_j}{h}\right)$, $j = -1, 0, 1, 2, 3$. Using only five terms, an approximate solution of the ODE-BVP is given by

$$y(x) \approx -0.707679286807 C_{-1}(x) + 0.0445065089233 C_0(x) + 0.529653251113 C_1(x)$$

$$+0.697245134775 C_2(x) + 0.529653251113 C_3(x).$$

Here are the details of the Mathematica work used to find this approximate solution: After defining $\mathsf{h} = 0.5$, the cubic spline $\mathsf{B3[x]} \triangleq \psi(x)$ was, as in Example 14.17, defined by

$$\mathsf{BB[x\_] := Piecewise}\left[\left\{\left\{\frac{3\mathsf{x}^3 - 6\mathsf{x}^2 + 4}{6}, 0 < \mathsf{x} \le 1\right\}, \left\{\frac{(2-\mathsf{x})^3}{6}, 1 < \mathsf{x} < 2\right\}\right\}\right]$$

$$\mathsf{B3[x\_] := Piecewise[\{\{BB[x], 0 < x < 2\}, \{BB[-x], -2 < x < 0\}, \{0, x \le -2\}, \{0, x \ge 2\}\}]}$$

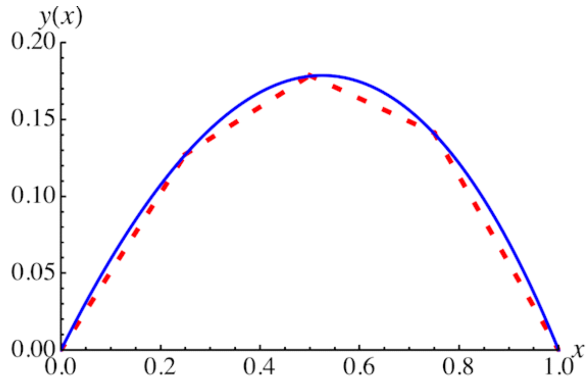


Figure 5: Answer key for problem 14.5.4.1

After that we used Mathematica$^{\text{TM}}$commands

$$\mathsf{h} = \tfrac{1}{2}; \mathsf{CM1[x\_] := B3}\left[\tfrac{\mathsf{x+h}}{\mathsf{h}}\right]; \mathsf{C0[x\_] := B3}\left[\tfrac{\mathsf{x}}{\mathsf{h}}\right]; \mathsf{C1[x\_] := B3}\left[\tfrac{\mathsf{x-h}}{\mathsf{h}}\right]; \mathsf{C2[x\_] := B3}\left[\tfrac{\mathsf{x-2h}}{\mathsf{h}}\right]; \mathsf{C3[x\_] := B3}\left[\tfrac{\mathsf{x-3h}}{\mathsf{h}}\right];$$

$$\mathsf{y[x\_, cm1\_, c0\_, c1\_, c2\_, c3\_, c4\_] := cm1\, CM1[x] + c0\, C0[x] + c1\, C1[x] + c2\, C2[x] + c3\, C3[x]}$$

$$\mathsf{f[cm1\_, c0\_, c1\_, c2\_, c3\_] := Evaluate}\Big[\int_0^1 \big((\mathsf{D[y[x, cm1, c0, c1, c2, c3], x]})^2$$

$$+\mathsf{x} * (\mathsf{y[x, cm1, c0, c1, c2, c3]})^2 - 2(1+\mathsf{x}) * \mathsf{y[x, cm1, c0, c1, c2, c3]}\big)\mathsf{dx}\Big]$$

$$\mathsf{FindMinimum[\{f[cm1, c0, c1, c2, c3], \{-c1 + c3 == 0, cm1 + 4c0 + c1 == 0\}\},}$$

$$\mathsf{\{\{cm1, .1\}, \{c0, .1\}, \{c1, .1\}, \{c2, 0.1\}, \{c3, .1\}\}]}$$

to get output

$$\{-0.69055, \{\mathsf{cm1} \to -0.707679286807, \mathsf{c0} \to 0.529653251113, \mathsf{c1} \to 0.529653251113,$$
$$\mathsf{c2} \to 0.697245134775, \mathsf{c3} \to 0.529653251113\}\}.$$

In a figure we show the approximate solution of the ODE-BVP as a dashed, red graph and also Mathematica's approximate solution $y(x)$ in the solid, blue graph.

[By the way, the problem did not ask for a graph of the approximate solution, but here it is if you're curious. As in Example 14.17 in Section 14.5, the Rayleigh-Ritz method's approximate solution is given as a dashed, red graph and Mathematica's approximate solution $y(x)$ as a solid, blue graph.]

14.5.4.5. (a) With $F(x,y,y') \triangleq \frac{1}{2}(y')^2 + \frac{1}{2}y^2 - \frac{1}{4}y^4 - xy$, the Euler Lagrange equation for $(\star\star)$ is

$$0 \equiv \frac{\partial F}{\partial y} - \frac{d}{dx}\Big[\frac{\partial F}{\partial y'}\Big] = y - y^3 - x - \frac{d}{dx}\big[\,y'\,\big],$$

that is, $-y'' + y - y^3 = x$, the ODE in the desired ODE-BVP.

(b) Let $h = 0.25$, $x_j = jh$ for $j = -1, 0, 1, 2, 3, 4$, $\phi(t) \triangleq \left\{\begin{array}{ll} t+1, & -1 \le t \le 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & |t| > 1 \end{array}\right\}$, and uniform tent basis functions $T_j(x) = \phi\left(\frac{x-x_j}{h}\right)$, $j = 0, 1, 2, 3$. Using only three terms, an approximate solution of the ODE-BVP is given by

$$y(x) \approx 0.035220059648\, T_1(x) + 0.056871998478\, T_2(x) + 0.050527720768\, T_3(x).$$

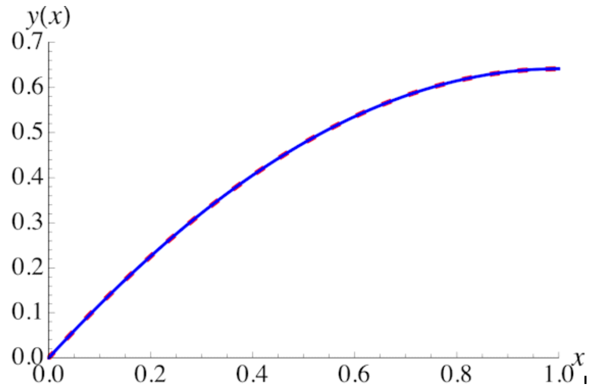


Figure 6: Answer key for problem 14.5.4.3

Here are the details of the Mathematica work used to find this approximate solution: After defining $\mathsf{h} = 0.2$, the basic tent function $T_1(x)$ was, as in Example 14.16, defined by

$$\mathsf{T1[t\_] := Piecewise}\Big[\{\{\frac{\mathsf{t}}{\mathsf{h}}, 0 < \mathsf{t} \le \mathsf{h}\}, \{1 - \frac{\mathsf{t-h}}{\mathsf{h}}, \mathsf{h} < \mathsf{t} \le 2\mathsf{h}\}, \{0, -1 < \mathsf{t} \le 0\}, \{0, 2\mathsf{h} < \mathsf{t} \le 1\}\}\Big].$$

Noting that $T_2(x) = T_1(x-h)$, $T_3(x) = T_1(x-2h)$, $T_4(x) = T_1(x-3h)$, we defined

$$\mathsf{y[x\_, c1\_, c2\_, c3\_, c4\_] := c1 * T1[x] + c2 * T1[x-h] + c3 * T1[x-2h] + c4 * T1[x-3h]}$$

and the functional by

$$\mathsf{J[c1\_, c2\_, c3\_]}$$

$$:= \mathsf{Evaluate}\Big[\int_0^1 \big((\frac{1}{2}\,\mathsf{D[y[x, c1, c2, c3], x]})^2 + \frac{1}{2}\,(\mathsf{y[x, c1, c2, c3]})^2 - \frac{1}{4}\,(\mathsf{y[x, c1, c2, c3]})^4 - \mathsf{x\,y[x, c1, c2, c3]}\big)\mathsf{dx}\Big].$$

Then we used

$$\mathsf{FindMinimum[J[c1, c2, c3, c4], \{\{c1, .05\}, \{c2, 0.5\}, \{c3, 0.05\}}$$

to get output

$$\{-0.00939116, \{\mathsf{c1} \to 0.035220059648, \mathsf{c2} \to 0.056871998478, \mathsf{c3} \to 0.050527720768\}\}.$$

[By the way, the problem did not ask for a graph of the approximate solution, but here it is if you're curious. As in Example 14.16 in Section 14.5, the Rayleigh-Ritz method's approximate solution is given as a dashed, red graph and Mathematica's approximate solution $y(x)$ as a solid, blue graph.]

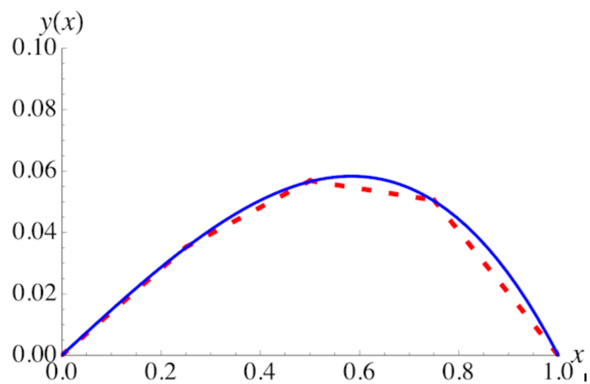


Figure 7: Answer key for problem 14.5.4.5(b)

Chapter Fifteen

Section 15.1.4

15.1.4.1. For all $z = x + iy$, $w = u + iv$, $LHS = \overline{wz} = \overline{(x+iy)(u+iv)} = \overline{(xu-yv)+i(xv+yu)} = (xu-yv)-i(xv+yu)$ vs. $RHS = \overline{w}\,\overline{z} = (x-iy)(u-iv) = (xu-yv)-i(xv+yu) = LHS$.

15.1.4.3. For all z_1, z_2, let $w = z_1\overline{z}_2$. Using the result of problem 15.1.4.1, we calculate $\overline{w} = \overline{z_1\overline{z}_2} = \overline{z_1}\;\overline{\overline{z}_2} = \overline{z_1}\,z_2$. So, $z_1\overline{z}_2 - z_2\overline{z}_1 = w - \overline{w}$, which is $i2$ times the imaginary part of w and thus is imaginary.

15.1.4.5. First, note that $z = 0$ cannot be a solution. Next, if $z \neq 0$ solves the equation, then multiplying both sides by z gives $|z|^2 = z\,\overline{z} = z\left(2 - \dfrac{1}{z}\right) = 2z - 1$, hence $2z = |z|^2 + 1$ must be real. It follows that $|z|^2 = z^2$, so the equation can be satisfied only if $0 = z^2 - 2z + 1 = (z-1)^2$. The only solution of the latter equation is $z = 1$. It *is* a solution as we can check by evaluating the left and right hand sides of the original equation when $z = 1$: $LHS = \overline{z} = \overline{1} = 1$ and $RHS = 2 - \frac{1}{1} = 1 = LHS$.

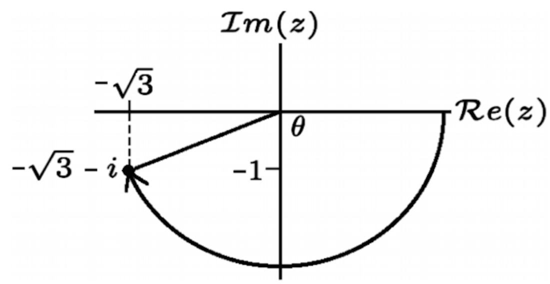


Figure 1: picture of $-\sqrt{3} - i$ used in problem 15.1.4.7

15.1.4.7. (a) using an appropriate picture of $(-\sqrt{3} - i)$ in the complex plane,

$$(-\sqrt{3} - i)^2 = \left(2e^{-i5\pi/6}\right) = 2^2 e^{2\cdot(-i5\pi/6)} = 4e^{-i5\pi/3} = 4e^{i\pi/3}$$

is its polar exponential form, and this equals $4\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) = 2 + i2\sqrt{3}$.

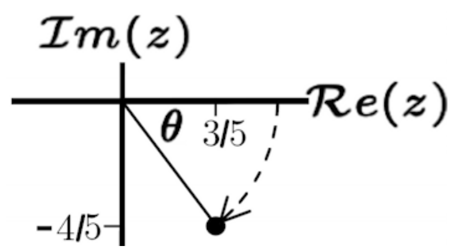


Figure 2: picture of $\frac{3}{5} - i\,\frac{4}{5}$ used in problem 15.1.4.7b

(b) using an appropriate picture of $(\cos\theta + i\sin\theta) = \left(\frac{3}{5} - i\,\frac{4}{5}\right)$ in the complex plane,

$$\left(\frac{3}{5} - i\,\frac{4}{5}\right)^2 = \left(e^{-i\ \arctan(4/3)}\right)^2 = e^{-i\,2\arctan(4/3)}$$

is its polar exponential form.

Using $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\,\cos\theta$, where $\theta = -\arctan(4/3)$, we have

$$e^{-i\,2\arctan(4/3)} = e^{i2\theta} = \cos 2\theta + i\sin 2\theta = \left(2\Big(\frac{3}{5}\Big)^2 - 1\right) + i2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{7+i24}{25} = -\frac{7}{25} + i\,\frac{24}{25}.$$

(c) using an appropriate pictures of $(-\sqrt{3}-i)$ and $(1+i)$ in the complex plane,

$$\frac{-\sqrt{3}-i}{1+i} = \frac{2e^{-i5\pi/6}}{\sqrt{2}e^{i\pi/4}} = \frac{2}{\sqrt{2}}\,e^{(-i5\pi/6)-(i\pi/4)} = \sqrt{2}e^{-i13\pi/12} = \sqrt{2}e^{i11\pi/12}$$

is its polar exponential form. On the other hand,

$$\frac{-\sqrt{3}-i}{1+i} = \frac{(-\sqrt{3}-i)(1-i)}{|1+i|^2} = \frac{-\sqrt{3}-1+i(-1+\sqrt{3})}{4} = \frac{-1-\sqrt{3}}{4} + i\,\frac{-1+\sqrt{3}}{4}.$$

15.1.4.9. (a) $\sin 3\theta = \mathcal{I}m\left(e^{i3\theta}\right) = \mathcal{I}m\left((\cos\theta + i\sin\theta)^3\right) = \mathcal{I}m\left(\cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\right)$

$$= 3\cos^2\theta\sin\theta - \sin^3\theta.$$

(b) $\cos 4\theta = \mathcal{R}e\left(e^{i4\theta}\right) = \mathcal{R}e\left((\cos\theta + i\sin\theta)^4\right)$

$$= \mathcal{R}e\big(\cos^4\theta + i4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - i4\cos\theta\sin^3\theta + \sin^4\theta\big)$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

15.1.4.11. The cube roots are $z_0 = \rho^{1/3}e^{i\alpha/3}$, $z_1 = \rho^{1/3}e^{i(2\pi+\alpha)/3}$, $z_2 = \rho^{1/3}e^{i(4\pi+\alpha)/3}$. The angle between the vector $\overline{Oz_1}$ and $\overline{Oz_0}$ is $2\pi/3$, the angle between $\overline{Oz_2}$ and $\overline{Oz_1}$ is $2\pi/3$, and the angle between $\overline{Oz_0}$ and $\overline{Oz_2}$ is $2\pi/3$, so z_0, z_1, z_2 are the vertices of a triangle with internal angles all being $120°$, that is, an equilateral triangle.

Because the triangle is equilateral, the length of each of the sides is equal to the (length of the vector $\overline{z_0z_1}$), which is

$$||\overline{z_0z_1}|| = dist(z_0,z_1) = \rho^{1/3}\cdot|\,e^{i(2\pi+\alpha)/3} - e^{i\alpha/3}| = \rho^{1/3}\cdot|\,e^{i2\pi/3}e^{i\alpha/3} - e^{i\alpha/3}| = \rho^{1/3}\cdot|\,(e^{i2\pi/3}-1)e^{i\alpha/3}|$$

$$= \rho^{1/3}\cdot|\,e^{i2\pi/3}-1|\cdot|e^{i\alpha/3}| = \rho^{1/3}\cdot\left|\cos\Big(\frac{2\pi}{3}\Big) + i\sin\Big(\frac{2\pi}{3}\Big) - 1\right|\cdot 1 = \rho^{1/3}\cdot\left|-\frac{1}{2} - 1 + i\frac{\sqrt{3}}{2}\right|$$

$$= \rho^{1/3}\cdot\sqrt{\Big(-\frac{3}{2}\Big)^2 + \Big(\frac{\sqrt{3}}{2}\Big)^2} = \sqrt{3}\,\rho^{1/3}.$$

15.1.4.13. (a) $r^2e^{i2\theta} = z^2 = i4 = 4e^{i\pi/2} \iff 4 = r^2$ and $e^{i\pi/2} = e^{i2\theta} \iff r = 2$ and $\frac{\pi}{2} + 2\pi k = 2\theta$ for some integer $k \iff r = 2$ and $\theta = \frac{\pi}{4} + \pi k$ for some integer k. So, there are exactly two solutions for z: $2e^{i\pi/4} = \sqrt{2} + i\sqrt{2}$ and $2e^{i5\pi/4} = -\sqrt{2} - i\sqrt{2}$.

(b) $r^2e^{i2\theta} = z^2 = -2 - i2\sqrt{3} = 4e^{-i2\pi/3} \iff 4 = r^2$ and $e^{-i2\pi/3} = e^{i2\theta} \iff r = 2$ and $-\frac{2\pi}{3} + 2\pi k = 2\theta$ for some integer $k \iff r = 2$ and $\theta = -\frac{\pi}{3} + \pi k$ for some integer k. So, there are exactly two solutions for z: $2e^{-i\pi/3} = 1 - i\sqrt{3}$ and $2e^{i2\pi/3} = -1 + i\sqrt{3}$.

(c) $r^3e^{i3\theta} = z^3 = 27 = 27e^{i0} \iff 27 = r^3$ and $e^{i0} = e^{i3\theta} \iff r = 3$ and $0 + 2\pi k = 3\theta$ for some integer $k \iff r = 3$ and $\theta = \frac{2\pi}{3}\,k$ for some integer k. So, there are exactly three solutions for z: $e^{i0} = 3 + i0$, $3e^{i2\pi/3} = -\frac{3}{2} + i\,\frac{3\sqrt{3}}{2}$, and $3e^{i4\pi/3} = -\frac{3}{2} - i\,\frac{3\sqrt{3}}{2}$.

(d) $r^3e^{i3\theta} = z^3 = -2 - i2 = 2\sqrt{2}e^{-i3\pi/4} \iff 2\sqrt{2} = r^3$ and $e^{-i3\pi/4} = e^{i3\theta} \iff r = \sqrt{2}$ and $-\frac{3\pi}{4} + 2\pi k = 3\theta$ for some integer $k \iff r = \sqrt{2}$ and $\theta = -\frac{\pi}{4} + \frac{2\pi}{3}\,k$ for some integer k. So, there are exactly three solutions for

z: $\sqrt{2}e^{-i\pi/4} = 1 - i$, $\sqrt{2}e^{i5\pi/12} = \sqrt{2}\cos\left(\frac{5\pi}{12}\right) + i\sqrt{2}\sin\left(\frac{5\pi}{12}\right)$, and $\sqrt{2}e^{-i11\pi/12} = \sqrt{2}\cos\left(\frac{11\pi}{12}\right) - i\sqrt{2}\sin\left(\frac{11\pi}{12}\right)$.

(e) $r^3e^{i3\theta} = z^3 = -4 + i4\sqrt{3} = 8e^{i2\pi/3} \iff 8 = r^3$ and $e^{i2\pi/3} = e^{i3\theta} \iff r = 2$ and $\frac{2\pi}{3} + 2\pi k = 3\theta$ for some integer $k \iff r = 2$ and $\theta = \frac{2\pi}{9} + \frac{2\pi}{3}k$ for some integer k. So, there are exactly three solutions for z: $2e^{i2\pi/9} = 2\cos\left(\frac{2\pi}{9}\right) + i2\sin\left(\frac{2\pi}{9}\right)$, $2e^{i8\pi/9} = 2\cos\left(\frac{8\pi}{9}\right) + 2\sin\left(\frac{8\pi}{9}\right)$, and $2e^{-i4\pi/9} = 2\cos\left(\frac{4\pi}{9}\right) - 2\sin\left(\frac{4\pi}{9}\right)$.

(f) $r^4e^{i4\theta} = z^4 = -81 = 81e^{i\pi} \iff 81 = r^4$ and $e^{i\pi} = e^{i4\theta} \iff r = 3$ and $\theta = \frac{\pi}{4} + \frac{\pi}{2}k$ for some integer k. So, there are exactly four solutions for z: $3e^{i\pi/4} = \dfrac{3}{\sqrt{2}} + i\,\dfrac{3}{\sqrt{2}}$, $3e^{i3\pi/4} = -\dfrac{3}{\sqrt{2}} + i\,\dfrac{3}{\sqrt{2}}$, $3e^{-i3\pi/4} = -\dfrac{3}{\sqrt{2}} - i\,\dfrac{3}{\sqrt{2}}$, and $3e^{-i\pi/4} = \dfrac{3}{\sqrt{2}} - i\,\dfrac{3}{\sqrt{2}}$.

15.1.4.15. $r^3e^{i3\theta} = -1 - i\sqrt{3} = 2\,e^{-i2\pi/3} \iff 2 = r^3$ and $e^{-i2\pi/3} = e^{i3\theta} \iff r = 2^{1/3}$ and $\theta = -\frac{2\pi}{9} + \frac{2\pi}{3}k$ for some integer k. So, there are exactly three solutions for z: $2^{1/3}\,e^{-i2\pi/9}$, $2^{1/3}\,e^{i4\pi/9}$, and $2^{1/3}\,e^{-i8\pi/9}$.

15.1.4.17. (a) $\dfrac{1-z}{1+z} = z \iff z \neq -1$ and $1 - z = z(1+z)$; the latter can be rewritten as $0 = z^2 + 2z - 1 = (z+1)^2 - 2 \iff (z+1)^2 = 2 = 2e^{i\cdot 0} \iff z+1$ is $\sqrt{2}e^{i\cdot 0}$ or $\sqrt{2}e^{i\pi}$, that is, $z+1$ is $\sqrt{2}$ or $-\sqrt{2}$. The solutions are $z = -1 \pm \sqrt{2}$, both of which satisfy $z \neq -1$. [As in Example 15.3(b) in Section 15.1, there are only two solutions of $(z+1) = 2$.]

(b) $(z-i)^3 = z^3 \iff z^3 - 3iz^2 - 3z + i = z^3 \iff 0 = -3iz^2 - 3z + i$. The quadratic formula gives

$$(\star) \quad z = \frac{3 \pm \sqrt{(-3)^2 - 4(-i3)(i)}}{-6i} = \frac{3 \pm \sqrt{-3}}{-6i} = i \cdot \frac{3 \pm i\sqrt{3}}{6}.$$

The solutions given in $(\star)$ are

$$z = \mp\frac{\sqrt{3}}{6} + i\,\frac{1}{2}.$$

15.1.4.19. $w = f(z) = \dfrac{1}{z}$ and $z \neq 0 \iff z = f(w) = \dfrac{1}{w}$ and $w \neq 0$.

(a) Define $A = \{z : |z| = 3\}$. We have $w \in f(A) \iff 3 = |z| = \left|\dfrac{1}{w}\right| = \dfrac{1}{|w|}$ and $w \neq 0 \iff |w| = \dfrac{1}{3}$ and $w \neq 0$, so the image of A under the inversion mapping is $f(A) = \left\{w : |w| = \dfrac{1}{3}\right\}$. Also, see the picture.

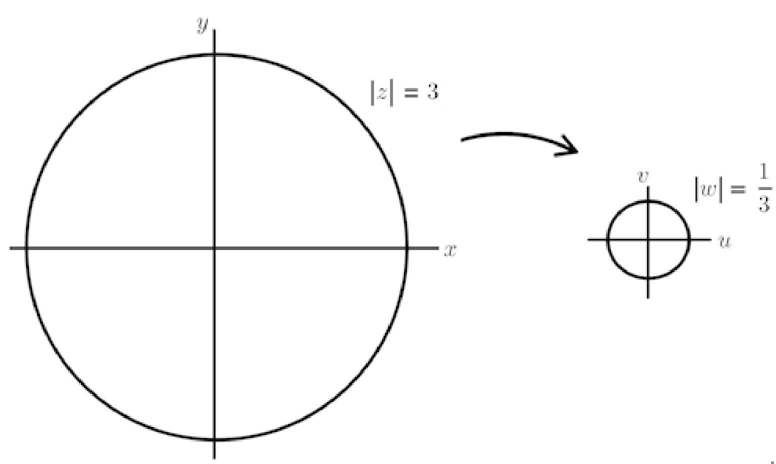


Figure 3: Transformation in 15.1.4.19(a)

(b) Define $A = \{z : |z-1| = 1\}$. Note that $0 \in A$. We have

$w \in f(A) \iff 1 = |z-1| = \left|\frac{1}{w} - 1\right| = \left|\frac{1-w}{w}\right| = \frac{|1-w|}{|w|}$ and $w \neq 0 \iff |w| = |1-w|$ and $w \neq 0$.

Let $w = u + iv$, where u, v are real.

Then, $|w| = |1-w| \implies u^2 + v^2 = |w|^2 = |1-w|^2 = |(1-u) + iv|^2 = (1-u)^2 + v^2 = 1 - 2u + u^2 + v^2$ $\iff 0 = 1 - 2u \iff u = \dfrac{1}{2}$, so the image of $\widehat{A} \triangleq \{z : |z-1| = 1,\ z \neq 0\}$ under the inversion mapping is *contained in* the line $L \triangleq \left\{w = \frac{1}{2} + iv : -\infty < v < \infty\right\}$.

Can we change *contained in* to *equals*? We use analysis similar to that used in Example 15.4 in Section 15.1: Does the image of $\widehat{A}$ include *all* of the points in the line L in the $w-$plane?

Every z in $A = \{z : |z-1| = 1\}$ is of the form

$$z = 1 + e^{it}$$

for some real t. We define

$$\widehat{A} = \{z \neq 0 : z \text{ is in } A\} = \left\{1 + e^{it} : -\pi < t < \pi\right\}.$$

It follows that for every z in $\widehat{A}$,

$$u + iv = w = f(z) = \frac{1}{1+e^{it}} = \frac{1}{(1+\cos t) + i\sin t}.$$

Rationalizing the denominator gives

$$u + iv = \frac{(1+\cos t) - i\sin t}{(1+\cos t)^2 + \sin^2 t} = \frac{(1+\cos t) - i\sin t}{1 + 2\cos t + \cos^2 t + \sin^2 t} = \frac{(1+\cos t) - i\sin t}{2 + 2\cos t}$$

$$= \frac{1}{2}\left(\frac{1+\cos t}{1+\cos t} + i\,\frac{-\sin t}{1+\cos t}\right) = \frac{1}{2} - i\,\frac{\sin t}{2(1+\cos t)}.$$

For any $w = u + iv$ satisfying $\mathcal{R}e(w) = \dfrac{1}{2}$, we need to show that there is at least one value of t in the interval $-\pi < t < \pi$ for which $g(t) \triangleq -\dfrac{\sin t}{2(1+\cos t)} = v$. [By the way, we chose the interval to be $-\pi < t < \pi$ in order to avoid t for which the denominator, $1 + \cos t$, is zero.] We calculate

$$(\star)\ \lim_{t\to-\pi^+} -\frac{\sin t}{2(1+\cos t)} = \lim_{t\to-\pi^+} -\frac{(1-\cos t)\sin t}{2(1+\cos t)(1-\cos t)} = \lim_{t\to-\pi^+} -\frac{(1-\cos t)}{2\sin t} = \lim_{t\to-\pi^+} -\frac{\approx 2}{\approx 0^+} = -\infty,$$

and similarly

$$(\star\star)\ \lim_{t\to\pi^-} g(t) = \lim_{t\to\pi^-} -\frac{\sin t}{2(1+\cos t)} = \lim_{t\to\pi^-} -\frac{(1-\cos t)}{2\sin t} = \infty.$$

Because $g(t)$ is continuous for $-\pi < t < \pi$, $(\star)$ and $(\star\star)$ imply $g(t)$ takes on all values in the interval $(-\infty, \infty)$. This concludes the explanation why $f(\widehat{A}) = L = \left\{w = \frac{1}{2} + iv : -\infty < v < \infty\right\}$, a vertical line in the $w-$plane. Also, see the picture.

(c) Define $A = \{z : |z+2| = 2\}$. Note that $0 \in A$. We have

$w \in f(A) \iff 2 = |z+2| = \left|\frac{1}{w} + 2\right| = \left|\frac{1+2w}{w}\right| = \frac{|1+2w|}{|w|}$ and $w \neq 0 \iff 2|w| = |1+2w|$ and $w \neq 0$.

Let $w = u + iv$, where u, v are real.

Then, $2|w| = |1+2w| \implies 4u^2 + 4v^2 = (2|w|)^2 = |1+2w|^2 = |(1+2u) + i2v|^2 = (1+2u)^2 + 4v^2 =$ $1 + 4u + 4u^2 + 4v^2 \iff 0 = 1 + 4u \iff u = -\dfrac{1}{4}$, so the image of $\widehat{A} \triangleq \{z : |z+2| = 2,\ z \neq 0\}$ under the inversion mapping is *contained in* the line $L \triangleq \left\{w = -\frac{1}{4} + iv : -\infty < v < \infty\right\}$.

Can we change *contained in* to *equals*? We use analysis similar to that used in Example 15.4 in Section 15.1: Does the image of $\widehat{A}$ include *all* of the points in the line L in the $w-$plane?

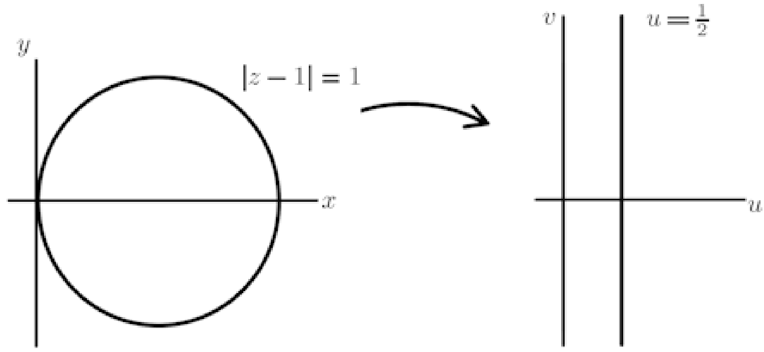


Figure 4: Transformation in 15.1.4.19(b)

Every z in $A = \{z : |z+2| = 2\}$ is of the form

$$z = -2 + 2e^{it}$$

for some real t. We define

$$\widehat{A} = \{z \neq 0 : z \text{ is in } A\} = \left\{-2 + 2e^{it} : 0 < t < 2\pi\right\}.$$

It follows that for every z in $\widehat{A}$,

$$u + iv = w = f(z) = \frac{1}{-2 + 2e^{it}} = \frac{1}{(-2 + 2\cos t) + i2\sin t} = \frac{1}{2\big((-1+\cos t) + i\sin t\big)}.$$

Rationalizing the denominator gives

$$u + iv = \frac{(-1+\cos t) - i\sin t}{2\big((-1+\cos t)^2 + \sin^2 t\big)} = \frac{(-1+\cos t) - i\sin t}{2\big(1 - 2\cos t + \cos^2 t + \sin^2 t\big)} = \frac{(-1+\cos t) - i\sin t}{4(1-\cos t)}$$

$$= \frac{1}{4}\left(\frac{-1+\cos t}{1-\cos t} + i\,\frac{-\sin t}{1-\cos t}\right) = -\frac{1}{4} - i\,\frac{\sin t}{4(1-\cos t)}\,.$$

For any $w = u + iv$ satisfying $\mathcal{R}e(w) = -\dfrac{1}{4}$, we need to show that there is at least one value of t in the interval $0 < t < 2\pi$ for which $g(t) \triangleq -\dfrac{\sin t}{4(1-\cos t)} = v$. [By the way, we chose the interval to be $0 < t < 2\pi$ in order to avoid t for which the denominator, $1 - \cos t$, is zero.] We calculate

$$(\star)\ \lim_{t\to 0^+} -\frac{\sin t}{4(1-\cos t)} = \lim_{t\to 0^+} -\frac{(1+\cos t)\sin t}{4(1+\cos t)(1-\cos t)} = \lim_{t\to 0^+} -\frac{(1+\cos t)}{4\sin t} = \lim_{t\to 0^+} -\frac{\approx 2}{\approx 0^+} = -\infty,$$

and similarly

$$(\star\star)\ \lim_{t\to 2\pi^-} g(t) = \lim_{t\to 2\pi^-} -\frac{\sin t}{2(1+\cos t)} = \lim_{t\to 2\pi^-} -\frac{(1+\cos t)}{4\sin t} = \infty.$$

Because $g(t)$ is continuous for $0 < t < 2\pi$, $(\star)$ and $(\star\star)$ imply $g(t)$ takes on all values in the interval $(-\infty, \infty)$. This concludes the explanation why $f(\widehat{A}) = L = \left\{w = -\frac{1}{4} + iv : -\infty < v < \infty\right\}$, a vertical line in the $w-$plane. Also, see the picture.

(d) Define $A = \{z : |2z + i| = 1\}$. Note that $0 \in A$. We have

$w \in f(A) \iff 1 = |2z+i| = \left|\dfrac{2}{w} + i\right| = \left|\dfrac{2+iw}{w}\right| = \dfrac{|2+iw|}{|w|}$ and $w \neq 0 \iff |w| = |2+iw|$ and $w \neq 0$.

Let $w = u + iv$, where u, v are real.

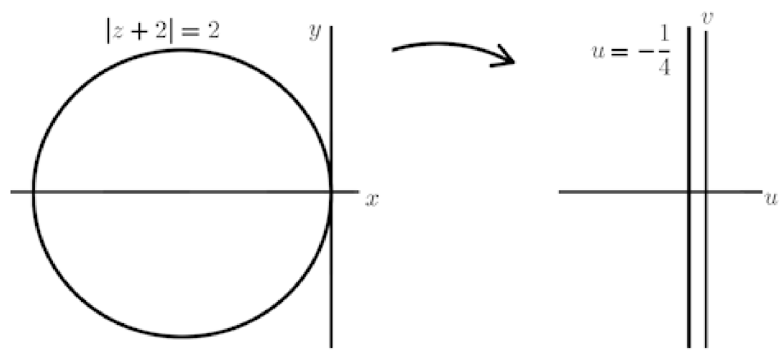


Figure 5: Transformation in 15.1.4.19(c)

Then, $|w|=|2+iw| \Longrightarrow u^2+v^2=|w|^2=|2+iw|^2=|(2-v)+iu|^2=(2-v)^2+u^2=4-4v+v^2+u^2$ $\Longleftrightarrow 0=4(1-v) \Longleftrightarrow v=1$, so the image of $\widehat{A} \triangleq \{z:\ |2z+i|=1,\ z\neq 0\}$ under the inversion mapping is *contained in* the line $L \triangleq \{w=u+i:\ -\infty<v<\infty\}$.

Can we change *contained in* to *equals*? We use analysis similar to that used in Example 15.4 in Section 15.1: Does the image of $\widehat{A}$ include *all* of the points in the line L in the $w-$plane?

Every z in $A=\{z:\ |2z+i|=1\}=\left\{z:\ \left|z+\frac{i}{2}\right|=\frac{1}{2}\right\}$ is of the form

$$z=-\frac{i}{2}+\frac{1}{2}\,e^{it}$$

for some real t. We define

$$\widehat{A}=\{z\neq 0:\ z \text{ is in } A\}=\left\{\frac{i}{2}\Big(-i+e^{it}\Big):\ -\pi<t<\pi\right\}.$$

It follows that for every z in $\widehat{A}$,

$$u+iv=w=f(z)=\frac{1}{\frac{1}{2}\big(-i+e^{it}\big)}=\frac{2}{\cos t+i(-1+\sin t)}.$$

Rationalizing the denominator gives

$$u+iv=\frac{2\big(\cos t-i(-1+\sin t)\big)}{\cos^2 t+(-1+\sin t)^2}=\frac{2\big(\cos t-i(-1+\sin t)\big)}{\cos^2 t+1-2\sin t+\sin^2 t}=\frac{2\big(\cos t-i(-1+\sin t)\big)}{2(1-\sin t)}$$

$$=\frac{\cos t}{1-\sin t}+i\,\frac{1-\sin t}{1-\sin t}=\frac{\cos t}{1-\sin t}+i\,.$$

For any $w=u+iv$ satisfying $\mathcal{I}m(w)=1$, we need to show that there is at least one value of t in the interval $\frac{\pi}{2}<t<\frac{5\pi}{2}$ for which $g(t)\triangleq\frac{\cos t}{1-\sin t}=u$. [By the way, we chose the interval to be $\frac{\pi}{2}<t<\frac{5\pi}{2}$ in order to avoid t for which the denominator, $1-\sin t$, is zero.] We calculate

$$(\star)\ \lim_{t\to\frac{\pi}{2}^+}\frac{\cos t}{1-\sin t}=\lim_{t\to\frac{\pi}{2}^+}\frac{\cos t(1+\sin t)}{(1-\sin t)(1+\sin t)}=\lim_{t\to\frac{\pi}{2}^+}\frac{1+\sin t}{\cos t}=\lim_{t\to\frac{\pi}{2}^+}\frac{\approx 2}{\approx 0^-}=-\infty,$$

and similarly

$$(\star\star)\ \lim_{t\to\frac{5\pi}{2}^-}g(t)=\lim_{t\to\frac{5\pi}{2}^-}\frac{\cos t}{1-\sin t}=\lim_{t\to\frac{5\pi}{2}^-}\frac{1+\sin t}{\cos t}=\infty.$$

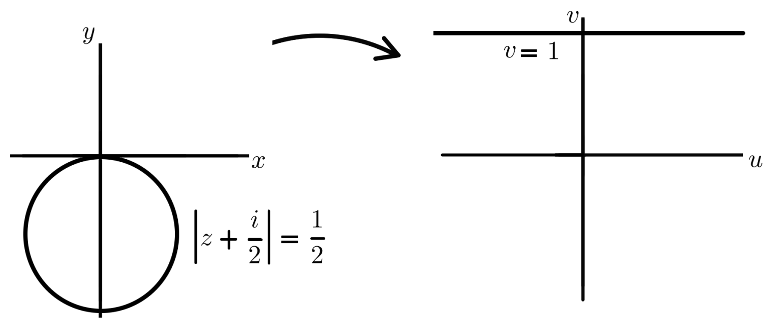


Figure 6: Transformation in 15.1.4.19(d)

Because $g(t)$ is continuous for $\frac{\pi}{2} < t < \frac{5\pi}{2}$, $(\star)$ and $(\star\star)$ imply $g(t)$ takes on all values in the interval $(-\infty, \infty)$. This concludes the explanation why $f(\widehat{A}) = L = \{w = u + i : -\infty < u < \infty\}$, a vertical line in the $w-$plane. Also, see the picture.

15.1.4.21. $w = f(z) = iz - 3 \iff w + 3 = iz \iff z = \frac{1}{i}(w+3) = -i(w+3) \triangleq f^{-1}(w)$

(a) $A = \{z : |z| = 1\}$, so w is in $f(A) \iff 1 = |z| = |-i(w+3)| = |-i|\ |w+3| = 1 \cdot |w+3| \iff |w+3| = 1$. So, the image of A under the mapping $f(z)$ is $f(A) = \{w : |w+3| = 1\}$, the circle of radius 1 and center at -3 in the $w-$plane.

(b) The line $\mathcal{R}e(z) = 2$ is A, so $w = u + iv$ is in $f(A) \iff z = -i(w+3) = -i\big((u+iv)+3\big) = v - i(u+3)$ has $v = \mathcal{R}e(z) = 2 \iff v = 2$. So, the image of A under the mapping $f(z)$ is the horizontal line $f(A) = \{w : \mathcal{I}m(w) = 2\} = \{u + i2 : -\infty < u < \infty\}$.

(c) The line $\mathcal{I}m(z) = -1$ is A, so $w = u + iv$ is in $f(A) \iff z = -i(w+3) = -i\big((u+iv)+3\big) = v - i(u+3)$ has $-(u+3) = \mathcal{I}m(z) = -1 \iff u = -2$. So, the image of A under the mapping $f(z)$ is the horizontal line $f(A) = \{w : \mathcal{R}e(w) = -2\} = \{-2 + iv : -\infty < v < \infty\}$.

(d) The line $\mathcal{R}e(z) = -\mathcal{I}m(z)$ is A, so $w = u+iv$ is in $f(A) \iff z = -i(w+3) = -i\big((u+iv)+3\big) = v-i(u+3)$ has $v = \mathcal{R}e(z) = -\mathcal{I}m(z) = (u+3) \iff v = u+3$. So, the image of A under the mapping $f(z)$ is the line $f(A) = \{w = u+iv : v = u+3\} = \{u + i(u+3) : -\infty < u < \infty\}$.

15.1.4.23. (a) $LHS = |a + Re^{i\theta}| = |a + R\cos\theta + iR\sin\theta| = \sqrt{(a+R\cos\theta)^2 + (R\sin\theta)^2}$ and
$RHS = |a+Re^{-i\theta}| = |a+R\cos\theta - iR\sin\theta| = \sqrt{(a+R\cos\theta)^2 + (-R\sin\theta)^2} = \sqrt{(a+R\cos\theta)^2 + (R\sin\theta)^2}$,

so $|a + Re^{i\theta}| = |a + Re^{-i\theta}|$.

(b) $RHS = |ae^{i\theta} + R| = |e^{i\theta}(a + Re^{-i\theta})| = |e^{i\theta}|\ |a + Re^{-i\theta}| = 1 \cdot |a + Re^{-i\theta}| = |a + Re^{i\theta}| = LHS$, by part (a).

Section 15.2.5

15.2.5.1. (a) $\lim_{z\to -1} \dfrac{3(z^2-1)}{z+1} = \lim\limits_{z\to -1} \dfrac{3(z+1)(z-1)}{(z+1)} = \lim\limits_{z\to -1} \dfrac{3(z-1)}{1} = \dfrac{3(-1-1)}{1} = -6$

(b) $\lim_{z\to -1} \dfrac{2(z^3+1)}{z+1} = \lim\limits_{z\to -1} \dfrac{2(z+1)(z^2-z+1)}{(z+1)} = \lim\limits_{z\to -1} \dfrac{2(z^2-z+1)}{1} = \dfrac{2(1-(-1)+1)}{1} = 6$

(c) *Method 1*: $\lim_{z\to -1} \dfrac{2(z^3+1)}{3(z^2-1)} = \lim\limits_{z\to -1} \dfrac{2\cancel{(z+1)}(z^2-z+1)}{3\cancel{(z+1)}(z-1)} = \lim\limits_{z\to -1} \dfrac{2(z^2-z+1)}{3(z-1)} = \dfrac{2(1-(-1)+1)}{3(-1-1)} = -1$

15.2.5.3. (a) From the result of problem 15.1.5.1(a), $\lim_{z\to -1} f(z) = .. = -6$. Also, only at $z = -1$ is $(z+1)$, the denominator of the rational function $f(z)$, equal to zero, so $f(z)$ is continuous at all $z \neq -1$. So, an extension of $f(z)$ that is continuous everywhere is given by

$$\widetilde{f}(z) \triangleq \left\{ \begin{array}{cc} \dfrac{3(z^2-1)}{z+1}, & z \neq -1 \\ \\ -6, & z = -1 \end{array} \right\}.$$

(b) From the result of problem 15.1.5.1(b), $\lim_{z\to -1} f(z) = .. = 6$. Also, only at $z = -1$ is $(z+1)$, the denominator of the rational function $f(z)$, equal to zero, so $f(z)$ is continuous at all $z \neq -1$. So, an extension of $f(z)$ that is continuous everywhere is given by

$$\widetilde{g}(z) \triangleq \left\{ \begin{array}{cc} \dfrac{2(z^3+1)}{z+1}, & z \neq -1 \\ \\ 6, & z = -1 \end{array} \right\}.$$

15.2.5.5. (a) $f'(z) = i2z + 2$ exists everywhere because $f(z)$ is a polynomial. [see Theorem 15.4(b) in Section 15.2]

(b) $g'(z) = 4\pi(2z - i)$ exists everywhere because $p(z) \triangleq (2z - i)$ is a polynomial, $q(z) \triangleq \pi z^2$ is a polynomial, and $g(z) = q\big(p(z)\big)$. [see the chain rule, Theorem 15.6(e) in Section 15.2]

(c) $h(z) = (z^2+1)(z^3+2z+4)$ has, by the product rule,

$$h'(z) = (z^2+1)'(z^3+2z+4) + (z^2+1)(z^3+2z+4)' = (2z)(z^3+2z+4) + (z^2+1)(3z^2+2)$$
$$= 2z^4 + 4z^2 + 8z + 3z^4 + 2z^2 + 3z^2 + 2 = 5z^4 + 9z^2 + 8z + 2$$

existing everywhere. [see Theorem 15.6(c) in Section 15.2]

(d) $k(z) = \dfrac{z+i2}{z-i}$ has, by the quotient rule,

$$k'(z) = \frac{(z+i2)'(z-i) - (z+i2)(z-i)'}{(z-i)^2} = \frac{(z-i) - (z+i2)}{(z-i)^2} = \frac{-i3}{(z-i)^2}$$

existing everywhere except at $z = i$. [see Theorem 15.6(d) in Section 15.2]

(e) $\ell(z) = \dfrac{1}{z^2 - i2z - 4} = (z^2 - i2z - 4)^{-1}$ has, by the chain rule,

$$\ell'(z) = (-1)(z^2 - i2z - 4)^{-2} \cdot (z^2 - i2z - 4)' = (-1)(z^2 - i2z - 4)^{-2} \cdot (2z - i2) = \frac{-2(z-i)}{(z^2 - i2z - 4)^2}$$

existing everywhere except where $0 = (z^2 - i2z - 4) = (z-i)^2 - 3$, that is, existing everywhere except at $z = i \pm \sqrt{3}$.

(f) $m(z) = \dfrac{\pi^2}{z^2(z-1)}$ has, by the chain rule,

$$m'(z) = \pi^2(-1)\big((z^2(z-1)\big)^{-2} \cdot \big(z^2(z-1)\big)' = (-1)\pi^2\big((z^2(z-1)\big)^{-2} \cdot (3z^2-2z) = \frac{-\pi^2 z(3z-2)}{z^4(z-1)^2}$$

existing everywhere except where $0 = z^2(z-1)$, that is, existing everywhere except at $z=0$ and except at $z=1$.

15.2.5.7. When x, y are real we have $f(z) = (-1+2x)y + i(x - x^2 + y^2) = u(x,y) + iv(x,y)$, so the CR (Cauchy-Riemann) equations are

$$\left\{ \begin{array}{l} 2y = \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 2y \\ \\ -1+2x = \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = -1+2x \end{array} \right\}.$$

In the $z-$plane, the CR equations satisfied everywhere. The functions $u(x,y) = (-1+2x)y$ and $v(x,y) = x - x^2 + y^2$ are polynomials in x, y, hence are continuous everywhere.

(a) By Theorem 15.8 in Section 15.2, $f(z)$ is differentiable everywhere in the $z-$plane.

(b) By Theorem 15.8 in Section 15.2, nowhere in the $z-$plane is $f(z)$ not differentiable.

(c) By Theorem 15.8 in Section 15.2, everywhere there exists

$$f'(z) = f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\,\frac{\partial v}{\partial x}(x,y) = 2y + i(1-2x) = i - 2i(x+iy) = i - i2z.$$

15.2.5.9. When x, y are real we have $f(z) = (-1-2x)y + i(x - x^2 - y^2) = u(x,y) + iv(x,y)$, so the CR (Cauchy-Riemann) equations are

$$\left\{ \begin{array}{l} -2y = \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = -2y \\ \\ -1-2x = \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = -1+2x \end{array} \right\}.$$

In the $z-$plane, the CR equations satisfied if, and only if, $-1-2x = -1+2x$, if and only if, $x = 0$. The functions $u(x,y) = (-1-2x)y$ and $v(x,y) = x - x^2 - y^2$ are polynomials in x, y, hence are continuous everywhere.

(a) By Theorem 15.8 in Section 15.2, $f(z)$ is differentiable only on the vertical line $x = 0$ in the $z-$plane, that is, the imaginary axis in the $z-$plane.

(b) By Theorem 15.8 in Section 15.2, $f(z)$ is not differentiable at all points in the half planes $\mathcal{R}e(z) < 0$ and $\mathcal{R}e(z) > 0$ in the $z-$plane.

(c) By Theorem 15.8 in Section 15.2, where it exists, namely at points $z = 0 + iy$, we have

$$f'(z) = f'(0+iy) = \frac{\partial u}{\partial x}(0,y) + i\,\frac{\partial v}{\partial x}(0,y) = -2y + i(1-0) = i + 2i(0+iy) = i(1 + z - \bar{z}).$$

15.2.5.11. When x, y are real we have $f(z) = z\,\mathcal{R}e(z) = (x+iy)x = x^2 + ixy = u(x,y) + iv(x,y)$, so the CR (Cauchy-Riemann) equations are

$$\left\{ \begin{array}{l} 2x = \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = x \\ \\ 0 = \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = -y \end{array} \right\}.$$

In the $z-$plane, the CR equations satisfied if, and only if, $2x = x$ and $0 = -y$, if and only if, $x = 0$ and $y = 0$, if and only if, $z = 0$. The functions $u(x, y) = x^2$ and $v(x, y) = xy$ are polynomials in x, y, hence are continuous everywhere.

(a) By Theorem 15.8 in Section 15.2, $f(z)$ is differentiable only at the point $z = 0$ in the $z-$plane.

(b) By Theorem 15.8 in Section 15.2, $f(z)$ is not differentiable at all points $z \neq 0$ in the $z-$plane.

(c) By Theorem 15.8 in Section 15.2, where it exists, namely only at $z = 0 + i0$, we have

$$f'(0) = f'(0 + i0) = \frac{\partial u}{\partial x}(0, 0) \ + i\, \frac{\partial v}{\partial x}(0, 0) = 0 + i(-0) = 0.$$

15.2.5.13. When x, y are real we have $f(z) = (x - iy)(2 - x^2 - y^2) = x(2 - x^2 - y^2) - iy(2 - x^2 - y^2) = u(x, y) + iv(x, y)$, so the CR (Cauchy-Riemann) equations are

$$\left\{ \begin{array}{l} 2 - y^2 - 3x^2 = \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = -2 + x^2 + 3y^2 \\ \\ -2xy = \dfrac{\partial u}{\partial y} = -\ \dfrac{\partial v}{\partial x} = -2xy \end{array} \right\}.$$

In the $z-$plane, the CR equations satisfied if, and only if, $2 - y^2 - 3x^2 = -2 + x^2 + 3y^2$, if and only if, $4 = 4x^2 + 4y^2$, that is, if and only if, $x^2 + y^2 = 1$. The functions $u(x, y) = x(2 - x^2 - y^2)$ and $v(x, y) = -y(2 - x^2 - y^2)$ are polynomials in x, y, hence are continuous everywhere.

By Theorem 15.8 in Section 15.2, f is differentiable only at the points on the circle $x^2 + y^2 = 1$.

15.2.5.15. $\widehat{\mathbf{T}}$ be a unit tangent vector at a point on the streamline $\Psi = k_2$. Because $\boldsymbol{\nabla}\Psi$ is normal to the streamline at that point, $\boldsymbol{\nabla}\Psi \neq \mathbf{0}$ and $0 = \boldsymbol{\nabla}\Psi \bullet \widehat{\mathbf{T}}$. Define

$$\mathbf{q}_1 \triangleq \frac{1}{||\boldsymbol{\nabla}\Psi||}\, \boldsymbol{\nabla}\Psi.$$

It follows that

$$\left\{ \mathbf{q}_1, \widehat{\mathbf{T}} \right\}$$

is an o.n. basis for $\mathbb{R}^2$. Express $\mathbf{v}$ in terms of this basis to get

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{q}_1 \rangle\, \mathbf{q}_1 + \langle \boldsymbol{\nabla}\Psi, \widehat{\mathbf{T}} \rangle \widehat{\mathbf{T}} = 0 \cdot \mathbf{q}_1 \ + \ \mathit{constant} \cdot \widehat{\mathbf{T}}.$$

This says that $\mathbf{v}$ is parallel to the tangent vector to the streamline, that is, the streamline gives the flow of fluid particles.

Section 15.3.3

15.3.3.1. When x, y are real we have $f(z) = x-y+i(x+y) = u(x,y)+iv(x,y)$, so the CR (Cauchy-Riemann) equations are

$$\left\{\begin{array}{l} 1 = \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 1 \\[2ex] -1 = \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = -1 \end{array}\right\}.$$

In the $z-$plane, the CR equations satisfied everywhere. The functions $u(x,y) = x - y$ and $v(x,y) = x + y$ are polynomials in x, y, hence are continuous everywhere.

By Theorem 15.8 in Section 15.2, $f(z)$ is differentiable everywhere in the $z-$plane, hence $f(z)$ is analytic everywhere in the $z-$plane. Be definition, that means that $f(z)$ is an entire function. Also, everywhere

$$f'(z) = f'(x+iy) = \frac{\partial u}{\partial x}(x,y) \, + i\,\frac{\partial v}{\partial x}(x,y) = 1 + i.$$

15.3.3.3. Using Definition 15.4 in Section 15.2, for any $z_0 \neq 0$,

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{\dfrac{1}{z-i} - \dfrac{1}{z_0 - i}}{z - z_0} = \lim_{z\to z_0} \frac{\dfrac{z_0 - i}{(z_0-i)(z-i)} - \dfrac{z-i}{(z_0-i)(z-i)}}{z - z_0}$$

$$= \lim_{z\to z_0} \frac{\dfrac{\cancel{(z_0 - z)}}{(z_0-i)(z-i)}}{\cancel{(z - z_0)}} = \lim_{z\to z_0} \frac{-1}{(z_0-i)(z-i)} = -\frac{-1}{(z_0-i)^2}\,.$$

Since every point z_0 in the set $S = \{z : z \neq i\}$ has a disk $\mathcal{D}_r(i)$ contained in S, $f(z)$ is analytic in the set S. There, $f'(z) = -\dfrac{1}{(z-i)^2}\,.$

15.3.3.5. Assume that the unspecified domain $\mathcal{D}$ is the whole complex plane or, perhaps, the largest set in the complex plane on which u and its first and second partial derivatives are continuous. Later, we may have to specify the domain $\mathcal{D}$.

The given function $u(x,y) = x - 2xy$ is a real-valued function, as should be the desired harmonic conjugate function v.

We want v to satisfy the Cauchy-Riemann equations, hence

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\big[\, x - 2xy \,\big] = 1 - 2y \quad \Longrightarrow \quad v = \int (1-2y)\partial y = y - y^2 + g(x),$$

where $g(x)$ is an arbitrary function of x alone. Substitute v into the other Cauchy-Riemann equation to get

$$-2x = \frac{\partial}{\partial y}\big[\, x - 2xy \,\big] = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\big[\, y - y^2 + g(x) \,\big] = -g'(x),$$

hence $g'(x) \equiv x^2$. So, $v(x,y) = y - y^2 + x^2 + c$ is a harmonic conjugate of u, for any real constant c.

15.3.3.7. First, we will find a harmonic conjugate function v for the given function $u(x,y) = x - 2xy$.

We want v to satisfy the Cauchy-Riemann equations, hence

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\big[\, x - 2xy \,\big] = 1 - 2y \quad \Longrightarrow \quad v = \int (1-2y)\partial y = y - y^2 + g(x),$$

where $g(x)$ is an arbitrary function of x alone. Substitute v into the other Cauchy-Riemann equation to get

$$-2x = \frac{\partial}{\partial y}\big[\, x - 2xy \,\big] = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\big[\, y - y^2 + g(x) \,\big] = -g'(x),$$

hence $g'(x) \equiv x^2$. So, $v(x,y) = y - y^2 + x^2 + c$ is a harmonic conjugate of u, for any real constant c.

It follows that the simplest such desired function $f(z)$ is given by

$$f(z) = u(x,y) + iv(x,y) = (x - 2xy) + i(y - y^2 + x^2) = x + iy + \big(-2xy + i(x^2 - y^2)\big) = z + iz^2.$$

Further, because $f(z)$ is a polynomial we have $f'(z) = 1 + 2iz$.

15.3.3.9. First, we will find a harmonic conjugate function v for the given function
$u(x,y) = y^2 + x^3 - x^2 - 3xy^2 + 2$.

We want v to satisfy the Cauchy-Riemann equations, hence

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\big[\, y^2+x^3-x^2-3xy^2+2 \,\big] = 3x^2-2x-3y^2 \implies v = \int (3x^2-2x-3y^2)\partial y = 3x^2y-2xy-y^3+g(x),$$

where $g(x)$ is an arbitrary function of x alone. Substitute v into the other Cauchy-Riemann equation to get

$$2y-6xy = \frac{\partial}{\partial y}\big[\, y^2+x^3-x^2-3xy^2+2 \,\big] = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\big[\, 3x^2y-2xy-y^3+g(x) \,\big] = -6xy+2y-g'(x),$$

hence $g'(x) \equiv 0$. So, $v(x,y) = 3x^2y - 2xy - y^3 + c$ is a harmonic conjugate of u, for any real constant c.

It follows that the simplest such desired function $f(z)$ is given by

$$\begin{aligned} f(z) = u(x,y) + iv(x,y) &= (y^2 + x^3 - x^2 - 3xy^2 + 2) + i(3x^2y - 2xy - y^3) \\ &= 2 - (x^2 + i2xy - y^2) + \big(x^3 + i3x^2y - 3xy^2 - iy^3\big) \\ &= 2 - z^2 + z^3. \end{aligned}$$

Further, because $f(z)$ is a polynomial we have $f'(z) = -2z + 3z^2$.

15.3.3.11. For $z = x + iy \neq (0,0)$, we want v to satisfy the Cauchy-Riemann equations, hence,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\Big[\, \frac{2y}{x^2+y^2} \,\Big] = -\frac{2xy}{(x^2+y^2)^2}$$

hence, using the substitution $w = x^2 + y^2$, with $\frac{\partial w}{\partial y} = 2y$,

$$v = \int \Big(-\frac{xy}{(x^2+y^2)^2} \Big)\, \partial y = \int \Big(-\frac{x}{w^2} \Big)\, \partial w = \frac{x}{w} + g(x) = \frac{x}{x^2+y^2} + g(x),$$

where $g(x)$ is an arbitrary function of x alone. Substitute v into the other Cauchy-Riemann equation to get

$$\begin{aligned} \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{\partial}{\partial y}\Big[\, \frac{y}{x^2+y^2} \,\Big] = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} &= -\frac{\partial}{\partial x}\Big[\, \frac{x}{x^2+y^2} + g(x) \Big] \\ &= -\frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} - g'(x), \end{aligned}$$

hence

$$g'(x) \equiv -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} = -\frac{2}{x^2+y^2} + \frac{2(x^2+y^2)}{(x^2+y^2)^2} = 0,$$

hence $g(x) = c$, for an arbitrary constant c. So,

$$v(x,y) = \frac{x}{x^2+y^2} + c$$

is a harmonic conjugate of u, for any real constant c, on the domain $\mathcal{D} = \{z = x + iy : \, x^2 + y^2 > 0\}$.

15.3.3.13. The given function $U(r,\theta)=v_\infty\Big(r+\dfrac{a^2}{r}\Big)\cos\theta$ is a real-valued function, defined on the domain $r\neq 0$ in the $z-$plane, as should be the desired harmonic conjugate function v.

We want v to satisfy the Cauchy-Riemann equations, hence

$$\frac{1}{r}\,\frac{\partial V}{\partial\theta}=\frac{\partial U}{\partial r}=\frac{\partial}{\partial r}\Big[v_\infty\Big(r+\frac{a^2}{r}\Big)\cos\theta\Big]=v_\infty\Big(1-\frac{a^2}{r^2}\Big)\cos(\theta)$$

so

$$V=\int v_\infty\Big(r-\frac{a^2}{r}\Big)\cos(\theta)\,\partial\theta=v_\infty\Big(r-\frac{a^2}{r}\Big)\sin(\theta)+g(r),$$

where $g(r)$ is an arbitrary function of r alone. Substitute V into the other Cauchy-Riemann equation to get

$$-v_\infty\Big(1+\frac{a^2}{r^2}\Big)\sin\theta=\frac{1}{r}\,\frac{\partial}{\partial\theta}\Big[\,v_\infty\Big(r+\frac{a^2}{r}\Big)\cos\theta\Big]=\frac{1}{r}\,\frac{\partial U}{\partial\theta}=-\,\frac{\partial V}{\partial r}=-\,\frac{\partial}{\partial r}\Big[v_\infty\Big(r-\frac{a^2}{r}\Big)\sin(\theta)+g(r)\Big]$$

$$=-v_\infty\Big(1+\frac{a^2}{r^2}\Big)\sin(\theta)-g'(r),$$

hence $g'(r)\equiv 0$. So, $V=v_\infty\Big(r-\dfrac{a^2}{r}\Big)\sin(\theta)+c$ is a harmonic conjugate of U, for any real constant c.

15.3.3.15. The given function

$$U(r,\theta)=u(x,y)=\frac{2xy}{(x^2+y^2)^2}=\frac{2r^2\cos(\theta)\sin(\theta)}{(r^2)^2}=\frac{2\cos(\theta)\sin(\theta)}{r^2}=\frac{\sin(2\theta)}{r^2}$$

is a real-valued function, defined on the domain $r\neq 0$ in the $z-$plane, as should be the desired harmonic conjugate function v.

We want v to satisfy the Cauchy-Riemann equations, hence

$$\frac{1}{r}\,\frac{\partial V}{\partial\theta}=\frac{\partial U}{\partial r}=\frac{\partial}{\partial r}\Big[\frac{\sin(2\theta)}{r^2}\Big]=-\,\frac{2\sin(2\theta)}{r^3}\quad\Longrightarrow\quad V=\int-\,\frac{2\sin(2\theta)}{r^2}\,\partial\theta=\frac{\cos(2\theta)}{r^2}+g(r)$$

where $g(r)$ is an arbitrary function of r alone. Substitute V into the other Cauchy-Riemann equation to get

$$\frac{2\cos(2\theta)}{r^3}=\frac{1}{r}\,\frac{\partial}{\partial\theta}\Big[\frac{\sin(2\theta)}{r^2}\Big]=\frac{1}{r}\,\frac{\partial U}{\partial\theta}=-\,\frac{\partial V}{\partial r}=-\,\frac{\partial}{\partial r}\Big[\frac{\cos(2\theta)}{r^2}+g(r)\Big]=\frac{2\cos(2\theta)}{r^3}-g'(r),$$

hence $g'(r)\equiv 0$. So, $V=\dfrac{\cos(2\theta)}{r^2}+c$ is a harmonic conjugate of U, for any real constant c. We must convert back to the original independent variables to get that a harmonic conjugate of $u(x,y)=\dfrac{2xy}{(x^2+y^2)^2}$ is given by

$$v(x,y)=V(r,\theta)=\frac{\cos(2\theta)}{r^2}+c=\frac{r^2\big(\cos^2(\theta)-\sin^2(\theta)\big)}{r^4}+c=\frac{x^2-y^2}{(x^2+y^2)^2}+c,$$

for any real constant c.

15.3.3.17. *Method 1*: (Tricky) $r^n\sin n\theta=-(-r^n\sin n\theta)=\mathcal{R}e(-iz^n)$, so

$$-i\big(z^n\big)=-i\Big(\mathcal{R}e(-iz^n)+i\,\mathcal{I}m(-iz^n)\Big)=-i\Big(r^n\sin(n\theta)+i\big(-r^n\cos(n\theta)\big)\Big)\triangleq U(r,\theta)+i\,V(r,\theta),$$

hence $V(r,\theta)\triangleq -r^n\cos(n\theta)$ is a harmonic conjugate for $U(r,\theta)\triangleq r^n\sin n\theta$.

Method 2: Use the Cauchy-Riemann equations in polar coordinates, namely, the result of problem 15.3.3.12.

15.3.3.19. (a) For all $z=x+iy$, where x and y are real,

$$f(z)=e^{\alpha z}=e^{(a+ib)(x+iy)}=e^{(ax-by)+i(bx+ay)}=e^{(ax-by)}e^{i(bx+ay)}=e^{(ax-by)}\big(\cos(bx+ay)+i\sin(bx+ay)\big)$$

$$= e^{(ax-by)}\cos(bx+ay) + i\,e^{(ax-by)}\sin(bx+ay) \triangleq u(x,y) + v(x,y).$$

(b) First, check the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\big[\, e^{(ax-by)}\cos(bx+ay)\,\big] = e^{(ax-by)}\left(a\cos(bx+ay) - b\sin(bx+ay)\right),$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\big[\, e^{(ax-by)}\sin(bx+ay)\,\big] = e^{(ax-by)}\left(-b\sin(bx+ay) + a\cos(bx+ay)\right),$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\big[\, e^{(ax-by)}\cos(bx+ay)\,\big] = e^{(ax-by)}\left(-b\cos(bx+ay) - a\sin(bx+ay)\right),$$

and

$$-\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\big[\, e^{(ax-by)}\sin(bx+ay)\,\big] = e^{(ax-by)}\left(-a\sin(bx+ay) - b\cos(bx+ay)\right).$$

So, the Cauchy-Riemann equations are satisfied everywhere in the $z-$plane.

Because (1) the functions $\cos(\theta)$, $\sin(\theta)$ are everywhere continuous in θ, and (2) the functions defined by $p(x,y) \triangleq ax - by$ and $q(x,y) \triangleq bx + ay$ are polynomials in x and y and thus are continuous everywhere in the $z = (x+iy)-$plane, Theorem 15.8 in Section 15.2 implies that $f(z) = e^{\alpha z}$ is entire, that is, analytic everywhere.

15.3.3.21. *Method 1*: If $u(x,y)$ has harmonic conjugate $v(x,y)$ then $\frac{\partial u}{\partial y} \equiv -\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} \equiv \frac{\partial v}{\partial y}$ on some domain. Define $w(x,y) = -u(x,y)$. It follows that, $\frac{\partial v}{\partial x} \equiv \frac{\partial w}{\partial y}$ and $\frac{\partial v}{\partial y} \equiv -\frac{\partial w}{\partial x}$ on that domain, that is, $w = -u$ is a harmonic conjugate of v on that domain.

Method 2: (Tricky): If $u(x,y)$ has harmonic conjugate $v(x,y)$ then $f(z) \triangleq u(x,y) + iv(x,y)$ is analytic on some domain. It follows that $g(z) \triangleq -if(z) = v(x,y) - iu(x,y)$ is analytic on that domain, hence $v(x,y)$ has harmonic conjugate $-u(x,y)$ on that domain.

15.3.3.23. This is potential flow for potential function $\Phi(x,y)$ if the velocity is

$$\mathbf{v}(x,y) = 2y\,\hat{\boldsymbol{\imath}} + 2x\,\hat{\boldsymbol{\jmath}} = \frac{\partial \Phi}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial \Phi}{\partial y}\,\hat{\boldsymbol{\jmath}}.$$

As discussed in Example 15.23 in Section 15.3 and Example 15.14 in Section 15.2, the streamline function $\Psi(x,y)$ is to be chosen so that

$$f(z) = \Phi(x,y) + i\Psi(x,y)$$

is analytic on some domain. As we saw in the present section, Ψ is a harmonic conjugate of Φ. So, by the Cauchy-Riemann equations, we want Ψ to satisfy

$$(\star) \quad \frac{\partial \Psi}{\partial y} \equiv \frac{\partial \Phi}{\partial x} = 2y$$

and

$$(\star\star) \quad \frac{\partial \Psi}{\partial x} \equiv -\frac{\partial \Phi}{\partial y} = -2x$$

on some domain $\mathcal{D}$. From $(\star\star)$, we integrate to get

$$\Psi = \int (-2x)\,\partial x = -x^2 + g(y),$$

where $g(y)$ is an arbitrary function of y alone.

Substitute this into $(\star)$, the first Cauchy-Riemann equation, to get

$$2y = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left[\,-x^2 + g(y)\,\right] = g'(y)$$

This gives us $g'(y) = 2y$, so $g(y) = y^2 + \widetilde{c}$, where $\widetilde{c}$ is an arbitrary real constant. For convenience, choose $\widetilde{c} = 0$ to get the streamline function

$$\Psi(x,y) = -x^2 + y^2.$$

So, the solution curves, that is, the streamlines, are the hyperbolas $-x^2y + y^2 = c =$constant.

Section 15.4.6

15.4.6.1. $Arg(z)=\frac{3\pi}{4} \Longleftrightarrow z=\sqrt{2}\,\gamma e^{i3\pi/4}=-\gamma+i\gamma$, for some (real) $\gamma>0$. The stated problem requires that

$$2^2=|z-i|^2=|-\gamma+i\gamma-i|^2=|-\gamma+i(\gamma-1)|^2=(-\gamma)^2+(\gamma-1)^2=2\gamma^2-2\gamma+1,$$

hence

$$2\gamma^2-2\gamma-3=0.$$

This implies that

$$\gamma=\frac{2\pm\sqrt{2^2-4\cdot 2\cdot(-3)}}{2\cdot 2}=\frac{2\pm 2\sqrt{7}}{4}.$$

The solutions of the original problem are

$$z=-\gamma+i\gamma=(-1+i)\gamma=\frac{(-1+i)(1\pm\sqrt{7})}{2}.$$

15.4.6.3. (a) $\tan\theta=\dfrac{1}{-1}$ and $-1+i$ is in the second quadrant together imply that

$$arg(-1+i)=\left\{\frac{3\pi}{4}+2\pi k:\ k \text{ is any integer}\right\} \quad\text{and}\quad Arg(-1+i)=\frac{3\pi}{4}$$

(b) $\tan\theta=\dfrac{-1}{-\sqrt{3}}$ and $-\sqrt{3}-i$ is in the third quadrant together imply that

$$arg(-\sqrt{3}-i)=\left\{-\frac{5\pi}{6}+2\pi k: k \text{ is any integer}\right\} \quad\text{and}\quad Arg(-\sqrt{3}-i)=-\frac{5\pi}{6}$$

(c) $arg(2e^{i\,\frac{5\pi}{3}})=\left\{\frac{5\pi}{3}+2\pi k: k \text{ is any integer}\right\}$ and $Arg\big(2e^{i\,\frac{5\pi}{3}}\big)=-\frac{\pi}{3}$

(d) Using results of parts (a) and (b) of this problem,

$$arg\Big(\frac{-\sqrt{3}-i}{-1+i}\Big)=arg\Big(\frac{2e^{i7\pi/6}}{\sqrt{2}e^{i3\pi/4}}\Big)=arg\Big(\sqrt{2}\,e^{i5\pi/12}\Big)=\left\{\frac{5\pi}{12}+2\pi k: k \text{ is any integer}\right\}$$

and $Arg\Big(\dfrac{-\sqrt{3}-i}{-1+i}\Big)=\dfrac{5\pi}{12}$.

15.4.6.5. (a) False; a clue is that $Arg\big(2Log(z)\big)$ is in the interval $(-2\pi,2\pi]$ while $Arg\big(Log(z^2)\big)$ is in the interval $(-\pi,\pi]$. An explicit counterexample is given by $z=2e^{i3\pi/4}$, for which $Log(z^2)=Log(4e^{i3\pi/2})=\ln 4-i\,\frac{\pi}{2}$ but $2Log(z)=2\Big(\ln 2+i\,\frac{3\pi}{4}\Big)=\ln 4+i\,\frac{3\pi}{2}$.

(b) True, because for all $z\neq 0$,

$$Log(\sqrt{z})=Log(z^{\backslash 1/2\backslash})=Log\Big(|z|^{1/2}e^{iArg(z)/2}\Big)=\ln(|z|^{1/2})+i\frac{Arg(z)}{2}=\frac{1}{2}\ln|z|+i\frac{Arg(z)}{2}$$
$$=\frac{1}{2}\Big(\ln|z|+iArg(z)\Big)=\frac{1}{2}Log(z).$$

15.4.6.7. (a) A clue is that $\mathcal{I}m\big(Log(e^z)\big)$ lies in the interval $(-\pi,\pi]$. So, only if $\mathcal{I}m(z)$ lies in the interval $(-\pi,\pi]$ can it be true that $Log\,(e^z)=z$.

Recall that, by definition, $w=Log(\hat{z})$ is the unique w for which both $e^w=\hat{z}$ and $-\pi<\mathcal{I}m(z)\le\pi$.

Assume that z satisfies $-\pi<\mathcal{I}m(z)\le\pi$. Then z is an example of a complex number w_0 that satisfies both $e^{w_0}=e^z$ and $-\pi<\mathcal{I}m(w_0)\le\pi$. It follows that $w_0=z$ is the unique w for which $e^w=e^z$ and $-\pi<\mathcal{I}m(w)\le\pi$, so we have that $z=w$. So, $Log\,(e^z)=z$ for all z for which $-\pi<\mathcal{I}m(z)\le\pi$.

(b) When it exists, $Log(z)$ is an element of the set $\{w : e^w = z\}$, so by definition of that set, $e^{Log(z)} = z$. So, $e^{Log(z)} = z$ is true for all $z \neq 0$, the set of which is the punctured plane. [This explains why Theorem 15.16 in Section 15.4 is true.]

15.4.6.9. $\dfrac{e^z}{e^z+1} = i \iff e^z = i(e^z+1) \iff (1-i)e^z = i \iff$

$$e^z = \frac{i}{1-i} = \frac{e^{i\pi/2}}{\sqrt{2}\,e^{-i\pi/4}} = 2^{-1/2}\,e^{i3\pi/4} = e^{-\frac{1}{2}\ln 2}\,e^{i3\pi/4} = e^{-\frac{1}{2}\ln 2 + i\,\frac{3\pi}{4}}.$$

So, the solutions of the original equation $\dfrac{e^z}{e^z+1} = i$ are given by

$$z = -\frac{1}{2}\ln 2 + i\left(\frac{3\pi}{4} + 2n\pi\right), \quad \text{where } n \text{ is any integer.}$$

15.4.6.11. Define the principal square root function by $\sqrt{z}$, that is $z^{\backslash 1/2\backslash} \triangleq \sqrt{|z|}\,e^{i\,Arg(z)/2}$.

A clue is that $Arg\left(\sqrt{z^n}\right)$ lies in the interval $(-\pi,\pi]$, while $n\,Arg\left(\sqrt{z}\right)$ lies in the interval $(-n\pi, n\pi]$.

Ex. For $z = e^{i5\pi/6}$ and $n = 2$, we have

$$Arg\left(\sqrt{z^n}\right) = Arg\left(\sqrt{e^{i5\pi/3}}\right) = Arg\left(\sqrt{e^{-i\pi/3}}\right) = Arg\left(e^{-i\pi/6}\right) = -\frac{\pi}{6}$$

versus

$$n\,Arg\left(\sqrt{z}\right) = 2Arg\left(\sqrt{e^{i5\pi/6}}\right) = 2Arg\left(e^{i5\pi/12}\right) = \frac{5\pi}{6}.$$

15.4.6.13. $(2i)^i \triangleq e^{ilog(2i)} = e^{i\cdot\{\ln|2i| + iArg(2i) + i2\pi k:\, k=\text{integer}\}} =$

$$= e^{i\cdot\left\{\ln|2| + i\frac{\pi}{2} + i2\pi k:\, k=\text{integer}\right\}} = \left\{e^{i\cdot\left(\ln|2| + i\frac{\pi}{2} + i2\pi k\right)} : k = \text{integer}\right\} = \left\{e^{-\frac{\pi}{2} - 2\pi k}\cdot e^{i\ln 2} : k \text{ is any integer}\right\}.$$

15.4.6.15. Recall that the polar exponential form is $re^{i\theta}$, where $-\pi < \theta \leq \pi$. From Example 15.28(b) in Section 15.4, $Log(-\sqrt{3}+i) = \ln 2 + i\,\frac{5\pi}{6}$, so

$$\sqrt[3]{-\sqrt{3}+i} = (-\sqrt{3}+i)^{\backslash\frac{1}{3}\backslash} \triangleq e^{\frac{1}{3}\,Log(-\sqrt{3}+i)} = e^{\frac{1}{3}\,(\ln 2 + i\,\frac{5\pi}{6})} = e^{\frac{1}{3}\,\ln 2}\cdot e^{i5\pi/18} = \sqrt[3]{2}\,e^{i5\pi/18}.$$

15.4.6.17. $\sqrt[3]{z} \triangleq e^{\frac{1}{3}\,Log(z)}$. We note that the exponential function is entire and $Log(z)$ is differentiable everywhere except the ray $Arg(z) = \pi$, so $\sqrt[3]{z}$ is also differentiable everywhere except the ray $Arg(z) = \pi$, also known as the non-positive real axis.

15.4.6.19. There no solution of the equation $e^w = 0$ because for all $w = u + iv$ with u, v being real,

$$|e^w| = |e^u\,(\cos v + i\sin v)| = \sqrt{\left(e^u\,\cos v\right)^2 + \left(e^u\,\sin v\right)^2} = \sqrt{e^{2u}\left(\cos^2 v + \sin^2 v\right)} = \sqrt{e^{2u}} = e^u > 0.$$

15.4.6.21. If α is an integer then $z^\alpha \triangleq e^{\alpha log(z)} = \left\{e^{\alpha\left(\ln|z| + i\left(Arg(z) + 2\pi k\right)\right)} : k \text{ is any integer}\right\}$

$$= \left\{e^{\alpha\left(\ln|z| + iArg(z)\right)}\cdot e^{i2\pi\alpha k} : k \text{ is any integer}\right\} = \left\{e^{\alpha\left(\ln|z| + iArg(z)\right)}\cdot 1 : k \text{ is any integer}\right\}$$

$$= \left\{ e^{\alpha\left(\,\ln 2 + iArg(z)\right)} \right\}$$

is a single value.

15.4.6.23. If x is a real, positive number, then

$$Log(x) = \ln|x| + iArg(x+i0) = \ln|x| + i\cdot 0 = \ln x.$$

15.4.6.25. Because $x>0$, for all h sufficiently small we have that $x+h>0$, so

$$\frac{\partial v}{\partial x}\,(x,0) = \lim_{h\to 0}\frac{v(x+h,0)-v(x,0)}{h} = \lim_{h\to 0}\frac{\cos^{-1}\left(\frac{x+h}{\sqrt{(x+h)^2+0^2}}\right)-0}{h} = \lim_{h\to 0}\frac{\cos^{-1}\left(\frac{x+h}{(x+h)}\right)-0}{h} = \lim_{h\to 0}\frac{\cos^{-1}(1)-0}{h}$$

$$= \lim_{h\to 0}\frac{0-0}{h} = 0 = -\frac{\partial u}{\partial y}(x,0).$$

Regarding the second of the Cauchy-Riemann equations, (15.33) in Section 15.4, because $x>0$ and $y=0$, we calculate

$$\frac{\partial v}{\partial y}(x,0^+) = \lim_{y\to 0^+}\frac{v(x,y)-v(x,0)}{y} \triangleq g'(0+)$$

where we keep x fixed and $g(y) \triangleq \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$. But we know from Calculus I that $g'(0+)$ exists by the chain rule and equals

$$-\lim_{y\to 0^+}\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}}\cdot\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\lim_{y\to 0^+}\frac{1}{\sqrt{\frac{y^2}{x^2+y^2}}}\cdot\frac{-xy}{(x^2+y^2)^{3/2}}$$

$$= -\lim_{y\to 0^+}\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{-xy}{(x^2+y^2)^{3/2}} = -\lim_{y\to 0^+}\frac{\sqrt{x^2+y^2}}{y}\cdot\frac{-xy}{(x^2+y^2)^{3/2}} = \lim_{y\to 0^+}\frac{x}{x^2+y^2}\Big|_{as\ y\to 0^+} = \frac{1}{x},$$

and, similarly,

$$\frac{\partial v}{\partial y}(x,0^-) = \lim_{y\to 0^-}\frac{v(x,y)-v(x,0)}{y} \triangleq k'(0-)$$

where we keep x fixed and $k(y) \triangleq -\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$. But we know from Calculus I that $k'(0-)$ exists by the chain rule and equals

$$\lim_{y\to 0^-}\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}}\cdot\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \ldots = \lim_{y\to 0^-}\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{-xy}{(x^2+y^2)^{3/2}}$$

$$= \lim_{y\to 0^-}\frac{\sqrt{x^2+y^2}}{-y}\cdot\frac{-xy}{(x^2+y^2)^{3/2}} = \ldots = \lim_{y\to 0^-}\frac{x}{x^2+y^2} = \frac{1}{x}.$$

Because the left- and right-hand limits are equal, there exists $\dfrac{\partial v}{\partial y}(0,y)$ and it equals $-\dfrac{\partial u}{\partial x}(0,y)$.

Section 15.5.1

15.5.1.1. Using Theorem 15.18 in Section 15.5 with $z_1 = \frac{\pi}{4}$ and $z_2 = i$, along with (15.40) and (15.41) in Section 15.5, we have

$$\sin\left(\frac{\pi + i4}{4}\right) = \sin\frac{\pi}{4}\,\cos(i) + \cos\frac{\pi}{4}\,\sin(i) = \frac{1}{\sqrt{2}}\,\cosh(1) + i\,\frac{1}{\sqrt{2}}\,\sinh(1) = \frac{1}{\sqrt{2}}\big(\cosh(1) + i\,\sinh(1)\big).$$

15.5.1.3. For $z = x + iy$, where x, y are real, the equation is

$$2 = \cos(x + iy) = \cos x\,\cos(iy) - \sin x\,\sin(iy) = \cos x\,\cosh y - i\sin x\,\sinh y.$$

Separating the real and imaginary parts gives the system of equations

$$\left\{\begin{array}{lll}(1) & 2 = & \cos x\,\cosh y \\ & & \\ (2) & 0 = & -\sin x\,\sinh y\end{array}\right\}.$$

Equation (2) is easier to solve than (1): $0 = \sin x\,\sinh y \iff$ (i) $0 = \sin x$ or (ii) $0 = \sinh y$.
(2)(i) is true only for $x = n\pi$ for any integer n. (2)(ii) is true only for $y = 0$.

Substitute $x = n\pi$ into the first equation, (1):

$$2 = \cos x\,\cosh y = \cos(n\pi)\,\cosh y = (-1)^n\cosh y.$$

For $n = \mathit{even} = 2k$, (1) becomes $\cosh y = 2$, whose solutions would be $y = \mathit{arccosh}(2)$ if we had defined such a function. Instead, there is a solution technique of independent interest:

$$2 = \cosh y \triangleq \frac{e^y + e^{-y}}{2} \qquad\iff\qquad 4 = e^y - e^{-y} \qquad\iff\qquad 4e^y = e^y\cdot(e^y - e^{-y}) = (e^y)^2 - 1,$$

so substituting $w = e^y$ turns that equation into $4w = w^2 - 1$, that is, $w^2 - 4w - 1 = 0$, whose solutions are

$$e^y = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}.$$

This gives us only $y = \ln(2 + \sqrt{5})$, because $0 < e^y$ cannot equal $2 - \sqrt{5} < 0$.

So far, our only solutions are

$$z = x + iy = 2k\pi + i\ln(2 + \sqrt{5}).$$

For $n = \mathit{odd} = 2\ell - 1$, (1) becomes $0 < \cosh y = -2$, which has no solution.

Substitute $y = 0$ into the first equation, (1) to get $2 = \cos x\cosh y = \cos x\cosh 0 = \cos x\cdot 1 = \cos x \le 1$, which has no solution.

Putting everything together, the set of solutions is

$$\left\{2k\pi + i\ln(2 + \sqrt{5}) : \text{ any integer } k\right\}.$$

15.5.1.5. For $z = x + iy$, where x, y are real, the equation is

$$i3 = \sin(x + iy) = \sin x\,\cos(iy) + \cos x\,\sin(iy) = \sin x\,\cosh y + i\,\cos x\,\sinh y.$$

Separating the real and imaginary parts gives the system of equations

$$\left\{\begin{array}{lll}(1) & 0 = & \sin x\,\cosh y \\ & & \\ (2) & 3 = & \cos x\,\sinh y\end{array}\right\}.$$

Equation (1) is easier to solve than (2): $0 = \sin x\,\cosh y \iff$ (i) $0 = \sin x$ or (ii) $0 = \cosh y$.

(1)(ii) is never true. (1)(i) is true only for $x = n\pi$, for any integer n.

Substitute $x = n\pi$ into the second equation, (2):

$$3 = \cos x \,\sinh y = \cos(n\pi)\sinh y = (-1)^n \sinh y.$$

For $n = even = 2k$, (2) becomes $\sinh y = 3$, whose solutions would be $y = arcsinh(-3)$ if we had defined such a function. Instead, there is a solution technique of independent interest:

$$3 = \sinh y \triangleq \frac{e^y - e^{-y}}{2} \quad \iff \quad 6 = e^y - e^{-y} \quad \iff \quad 6e^y = e^y \cdot (e^y - e^{-y}) = (e^y)^2 - 1,$$

so substituting $w = e^y$ turns that equation into $6w = w^2 - 1$, that is, $w^2 - 6w - 1 = 0$, whose solutions are

$$e^y = \frac{6 \pm \sqrt{40}}{2} = 3 \pm \sqrt{10}.$$

This gives us only $y = \ln(3 + \sqrt{10})$, because $0 < e^y$ cannot equal $3 - \sqrt{10} < 0$.

So far, our only solutions are

$$z = x + iy = n\pi + i\,\ln(3 + \sqrt{10}).$$

For $n = odd = 2\ell - 1$, (2) becomes $\sinh y = -3$, whose solutions are found using

$$-3 = \sinh y \triangleq \frac{e^y - e^{-y}}{2} \quad \iff \quad -6 = e^y - e^{-y} \quad \iff \quad -6e^y = e^y \cdot (e^y - e^{-y}) = (e^y)^2 - 1,$$

so substituting $w = e^y$ turns that equation into $-6w = w^2 - 1$, that is, $w^2 + 6w - 1 = 0$, whose solutions are

$$e^y = \frac{-6 \pm \sqrt{40}}{2}.$$

Again, this gives us only $y = \ln(-3 + \sqrt{10})$.

Putting everything together, the set of solutions is

$$\Big\{2k\pi + i\,\ln\Big(3 + \sqrt{10}\Big) : \text{ any integer } k\Big\} \;\cup\; \Big\{(2\ell - 1)\pi + i\,\ln\Big(-3 + \sqrt{10}\Big) : \text{ any integer } \ell\Big\}.$$

15.5.1.7. For $z = x + iy$, where x, y are real, the equation is

$$-i\sinh\frac{\pi}{2} = \sin(x + iy) = \sin x\,\cos(iy) + \cos x\,\sin(iy) = \sin x\,\cosh y + i\,\cos x\,\sinh y.$$

Separating the real and imaginary parts gives the system of equations

$$\left\{\begin{array}{lrl} (1) & 0 = & \sin x\,\cosh y \\ & & \\ (2) & -\sinh\frac{\pi}{2} = & \cos x\,\sinh y \end{array}\right\}.$$

Equation (1) is easier to solve than (2): $0 = \sin x\,\cosh y \iff$ (i) $0 = \sin x$ or (ii) $0 = \cosh y$.

(1)(ii) is never true. (1)(i) is true only for $x = n\pi$, for any integer n.

Substitute $x = n\pi$ into the second equation, (2): $-\sinh\frac{\pi}{2} = \cos x\,\sinh y = \cos(n\pi)\sinh y = (-1)^n \sinh y$, that is,

$$\sinh y = (-1)^{n+1}\sinh\frac{\pi}{2}$$

For $n = even = 2k$, (2) becomes $\sinh y = -\sinh\frac{\pi}{2}$, hence $y = -\frac{\pi}{2}$.

For $n = odd = 2\ell - 1$, (2) becomes $\sinh y = \sinh\frac{\pi}{2}$, hence $y = \frac{\pi}{2}$.

Putting everything together, the set of solutions is

$$\Big\{n\pi + i\,(-1)^{n+1}\frac{\pi}{2} : \text{ any integer } n\Big\}.$$

15.5.1.9. For $z = x + iy$, where x, y are real, the equation is

$$i \sinh 3 = \cos(x + iy) = \cos x \, \cos(iy) - \sin x \, \sin(iy) = \cos x \, \cosh y - i \sin x \, \sinh y.$$

Separating the real and imaginary parts gives the system of equations

$$\left\{ \begin{array}{lrl} (1) & 0 = & \cos x \, \cosh y \\ \\ (2) & \sinh 3 = & -\sin x \, \sinh y \end{array} \right\}.$$

Equation (1) is easier to solve than (2): $0 = \cos x \, \cosh y \iff$ (i) $0 = \cos x$ or (ii) $0 = \cosh y$.
(1)(i) is true only for $x = (n - \frac{1}{2})\pi$ for any integer n. (1)(ii) is never true.

Substitute $x = \left(n - \frac{1}{2}\right)\pi$ into the first equation, (2): $\sinh 3 = -\sin x \, \sinh y = -\sin\left(\left(n - \frac{1}{2}\right)\pi\right) \sinh y = (-1)^n \sinh y$, that is,

$$\sinh y = (-1)^n \sinh 3$$

For $n = \textit{even} = 2k$, (2) becomes $\sinh y = \sinh 3$, hence $y = 3$.

For $n = \textit{odd} = 2\ell - 1$, (2) becomes $\sinh y = -\sinh 3$, hence $y = -3$.

Putting everything together, the set of solutions is

$$\left\{ \left(n - \frac{1}{2}\right)\pi + i\,3\,(-1)^n : \text{ any integer } n \right\}.$$

15.5.1.11. For $z = x + iy$, where x, y are real, the equation is

$$\sin(x + iy) = \sin x \, \cos(iy) + \cos x \, \sin(iy) = \sin x \, \cosh y + i \, \cos x \, \sinh y$$

$$= -\cos(x + iy) = -\big(\cos x \, \cos(iy) - \sin x \, \sin(iy)\big) = -\cos x \, \cosh y + i \sin x \, \sinh y.$$

Separating the real and imaginary parts gives the system of equations

$$\left\{ \begin{array}{rr} \sin x \, \cosh y = & -\cos x \, \cosh y \\ \\ \cos x \, \sinh y = & \sin x \, \sinh y \end{array} \right\},$$

that is

$$\left\{ \begin{array}{lrr} (1) & \cosh y \cdot (\cos x + \sin x) = & 0 \\ \\ (2) & \sinh y \cdot (\cos x - \sin x) = & 0 \end{array} \right\}.$$

The first equation is easier to analyze than the second: (1) requires that $\cosh y = 0$ or $\cos x + \sin x = 0$. But, $\cosh y = 0$ is never true, because $\cosh y \ge 1$ for all real y. So, (1) requires that $\cos x + \sin x = 0$, that is, $\sin x = -\cos x$, that is,

$$\tan x = -1.$$

The latter has solutions $x = -\frac{\pi}{4} + n\pi$, where n is any integer.

Substitute those value of x into (2) to get

$$0 = \sinh y \cdot (\cos x - \sin x) = \sinh y \cdot \left(\cos\left(-\frac{\pi}{4} + n\pi\right) - \sin\left(-\frac{\pi}{4} + n\pi\right) \right) = \sinh y \cdot \left((-1)^n \frac{1}{\sqrt{2}} - (-1)^{n+1} \frac{1}{\sqrt{2}} \right)$$

$$= \sqrt{2} \, \sinh y,$$

which implies that $y = 0$.

Putting everything together, the set of solutions is

$$\left\{ -\frac{\pi}{4} + n\pi + i\,0 : \text{ any integer } n \right\}.$$

15.5.1.13. Start on the more complicated side and use basic facts to get to the other side:

$$\sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \frac{e^{iz_1} - e^{-iz_1}}{i2} \cdot \frac{e^{iz_2} + e^{-iz_2}}{2} + \frac{e^{iz_1} + e^{-iz_1}}{2} \cdot \frac{e^{iz_2} - e^{-iz_2}}{i2}$$

$$= \frac{1}{i4}\Big(e^{i(z_1+z_2)} - \cancel{e^{i(-z_1+z_2)}} + \cancel{e^{i(z_1-z_2)}} - e^{-i(z_1+z_2)}\Big) + \frac{1}{i4}\Big(e^{i(z_1+z_2)} + \cancel{e^{i(-z_1+z_2)}} - \cancel{e^{i(z_1-z_2)}} - e^{-i(z_1+z_2)}\Big)$$

$$= \frac{1}{i4} \cdot 2 \cdot \Big(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\Big) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{i2} \triangleq \sin(z_1 + z_2),$$

thus establishing (15.39) in Section 15.5.

15.5.1.15. For $z = x + iy$, where x, y are real, the equation is

$$b = \cos(x + iy) = \cos x \, \cos(iy) - \sin x \, \sin(iy) = \cos x \, \cosh y - i \sin x \, \sinh y.$$

Separating the real and imaginary parts gives the system of equations

$$\left\{ \begin{array}{lll} (1) & b = & \cos x \, \cosh y \\ & & \\ (2) & 0 = & -\sin x \, \sinh y \end{array} \right\}.$$

Equation (2) is easier to solve than (1): $0 = \sin x \, \sinh y \iff$ (i) $0 = \sin x$ or (ii) $0 = \sinh y$.
(2)(i) is true only for $x = n\pi$ for any integer n. (2)(ii) is true only for $y = 0$.

Substitute $x = n\pi$ into the first equation, (1):

$$b = \cos x \, \cosh y = \cos(n\pi) \, \cosh y = (-1)^n \cosh y.$$

For $n = \mathit{even} = 2k$, (1) becomes $\cosh y = b$, which has no solution unless $b = 1$, because $-1 \le b < 1$ would imply that $\cosh y < 1$. If $b = 1$, then the only solution for y would be $y = 0$.

For $n = \mathit{odd} = 2\ell - 1$, (1) becomes $\cosh y = -b$, which has no solution unless $b = -1$, because $-1 < b \le 1$ would imply that $\cosh y < 1$. If $b = -1$, then the only solution for y would be $y = 0$.

To summarize what we have so far, for $|b| = 1$, $x = n\pi$ for some integer(s) n and $y = 0$ gives a solution(s) for $z = x + iy$. For $|b| < 1$ case (2)(i) gives no solution for z.

So, we see that $|b| \le 1$ implies that $y = \mathcal{I}m(z) = 0$, that is, the only solutions for z are real.

15.5.1.17. *Method 1*: Let $z = x + iy$, where x, y are real. Using the result of problem 15.5.1.12, we want to solve

$$b = \cosh(z) = \cosh x \, \cos y + i \sinh x \, \sin y.$$

Separating the real and imaginary parts gives the system of equations

$$\left\{ \begin{array}{lll} (1) & b = & \cosh x \, \cos y \\ & & \\ (2) & 0 = & \sinh x \, \sin y \end{array} \right\}.$$

Equation (2) is easier to solve than (1): $0 = \sinh x \, \sin y \iff$ (i) $0 = \sinh x$ or (ii) $0 = \sin y$.
(2)(i) is true only for $x = 0$. (2)(ii) is true only for $y = n\pi$, where n is any integer.

Substitute $y = n\pi$ into the first equation, (1):

$$b = \cosh x \, \cos y = \cos(n\pi) \, \cosh x = (-1)^n \cosh x.$$

For $n = \mathit{even} = 2k$, (1) becomes $\cosh x = b$, which has no solution unless $b = 1$, because $-1 \le b < 1$ would imply that $\cosh x < 1$. If $b = 1$, then the only solution for x would be $x = 0$.

For $n = \mathit{odd} = 2\ell - 1$, (1) becomes $\cosh x = -b$, which has no solution unless $b = -1$, because $-1 < b \le 1$ would imply that $\cosh x < 1$. If $b = -1$, then the only solution for x would be $x = 0$.

To summarize what we have so far, for $|b| = 1$, $y = n\pi$ for some integer(s) n and $x = 0$ gives a solution(s) for $z = x + iy$. For $|b| < 1$ case (2)(i) gives no solution for z.

So, we see that $|b| \le 1$ implies that $x = \mathcal{R}e(z) = 0$, that is, the only solutions for z are imaginary.

Method 2: A generalization of (15.40) in Section 15.5 to all complex numbers gives us that $\cosh(z) = \cos(iz)$. So, $b = \cosh(z)$ if, and only if, $b = \cos(iz)$. But we know from the result of problem 15.5.1.15 that $\cos(iz) = b$ only if $\mathcal{I}m(iz) = 0$, that is, only if $\mathcal{R}e(z) = 0$. So, if $|b| \le 1$ then $x = \mathcal{R}e(z) = 0$, that is, the only solutions for z are imaginary.

Section 15.6.4

15.6.4.1. (a) $\dfrac{1}{z-2}=\dfrac{-1}{2-z}=\dfrac{-1}{2\left(1-\frac{z}{2}\right)}=\dfrac{-1}{2}\cdot\dfrac{-1}{\left(1-\frac{z}{2}\right)}=\dfrac{-1}{2}\sum_{j=0}^{\infty}\left(\dfrac{z}{2}\right)^j=\sum_{j=0}^{\infty}-(2^{-j-1})z^j$

converges for $|z|<2$, so

$$f(z)=3z^{-1}+\sum_{j=0}^{\infty}-(2^{-j-1})\cdot z^j$$

is a Laurent series that converges for $0<|z|<2$.

(b) $\dfrac{1}{z-2}=\dfrac{1}{z\left(1-\frac{2}{z}\right)}=\dfrac{1}{z}\cdot\dfrac{1}{\left(1-\frac{2}{z}\right)}=\dfrac{1}{z}\sum_{j=0}^{\infty}\left(\dfrac{2}{z}\right)^j=\sum_{j=0}^{\infty}2^j\cdot z^{-j-1}$ converges for $|z|>2$, so

$$f(z)=3z^{-1}+\sum_{j=1}^{\infty}2^{j-1}\cdot z^{-j}=5z^{-1}+\sum_{j=2}^{\infty}2^{j-1}\cdot z^{-j}$$

is a Laurent series that converges for $|z|>2$.

15.6.4.3. For the first term we have

$$(1)\quad \frac{1}{z-1}=\frac{-1}{1-z}=\sum_{j=0}^{\infty}(-1)z^j \quad \text{converges for } |z|<1$$

$$(2)\quad \frac{1}{z-1}=\frac{1}{z\left(1-\frac{1}{z}\right)}=\sum_{j=0}^{\infty}z^{-j-1}=\sum_{k=1}^{\infty}z^{-k} \quad \text{converges for } |z|>1.$$

For the second term we have

$$(3)\quad \frac{1}{z+2}=\frac{1}{2+z}=\frac{1}{2}\cdot\frac{1}{1-\left(-\frac{z}{2}\right)}=\sum_{j=0}^{\infty}\frac{(-1)^j}{2^{j+1}}\cdot z^j \quad \text{converges for } |z|<2$$

$$(4)\quad \frac{1}{z+2}=\frac{1}{z\left(1-\left(-\frac{2}{z}\right)\right)}=\sum_{j=0}^{\infty}(-2)^j z^{-j-1}=\sum_{k=1}^{\infty}(-2)^{k-1}z^{-k} \quad \text{converges for } |z|>2.$$

(a) For $1<|z|<2$, combining series (2) and (3), we have

$$f(z)=\frac{1}{z-1}+\frac{1}{z+2}=\sum_{k=1}^{\infty}z^{-k}+\sum_{j=0}^{\infty}\frac{(-1)^j}{2^{j+1}}\cdot z^j.$$

(b) For $0<|z|<1$, combining series (1) and (3), we have

$$f(z)=\frac{1}{z-1}+\frac{1}{z+2}=\sum_{j=0}^{\infty}(-1)z^j+\sum_{j=0}^{\infty}\frac{(-1)^j}{2^{j+1}}\cdot z^j=\sum_{j=0}^{\infty}\left(-1+\frac{(-1)^j}{2^{j+1}}\right)z^j.$$

15.6.4.5. Ex. 1: Partial fractions gives $f(z)=\dfrac{A}{z}+\dfrac{B}{z-1}$ for some constants A,B to be determined. Multiply through by $z(z-1)$, the denominator of the LHS, to get

$$1=A(z-1)+Bz.$$

Substitute in $z=0$ to get $1=-A$, hence $A=-1$; substitute in $z=1$ to get $1=B$. Geometric series implies that

$$f(z)=-\frac{1}{z}+\frac{1}{z-1}=-\frac{1}{z}+\frac{1}{1-z}=-\frac{1}{z}+\sum_{j=0}^{\infty}z^j$$

is a Laurent series that converges in the domain $0<|z|<1$.

Ex. 2: Without use of partial fractions,

$$f(z)=\frac{1}{z(z-1)}=\frac{1}{z}\cdot\frac{1}{z-1}=\frac{1}{z}\cdot\frac{1}{z(1-z^{-1})}=\frac{1}{z}\cdot\frac{1}{z}\sum_{\ell=0}^{\infty}z^{-\ell}=\sum_{\ell=2}^{\infty}z^{-\ell},$$

which is a Laurent series that converges in the domain $0<\dfrac{1}{z}<\infty$. that is, the domain $1<|z|<\infty$.

15.6.4.7. Partial fractions gives $f(z)=\dfrac{A}{z+2}+\dfrac{B}{2z+1}$ for some constants A,B to be determined. Multiply through by $(z+2)(2z+1)$, the denominator of the LHS, to get

$$-(z+5)=A(2z+1)+B(z+2).$$

Substitute in $z=-2$ to get $-3=-3A$, hence $A=1$; substitute in $z=-\frac{1}{2}$ to get $-\frac{9}{2}=\frac{3}{2}B$, hence $B=-3$. Geometric series implies that

$$f(z)=\frac{1}{z+2}-\frac{3}{2z+1}=\frac{1}{2\big(1-(-z/2)\big)}-\frac{3}{1-(-2z)}=\frac{1}{2}\sum_{j=0}^{\infty}\left(-\frac{z}{2}\right)^j-3\sum_{j=0}^{\infty}(-2z)^j.$$

That is,

$$(\star)\quad f(z)=\sum_{j=0}^{\infty}\left(\frac{1}{2}\left(-\frac{1}{2}\right)^j-3(-2)^j\right)z^j.$$

Both series converge as long as both $\left|-\frac{z}{2}\right|<1$ and $|-2z|<1$. So, $(\star)$ gives a Laurent series in the domain for $0<|z|<\frac{1}{2}$.

Using again the partial fractions of part (a),

$$f(z)=\frac{1}{z+2}-\frac{3}{2z+1}=\frac{1}{2\big(1-(-z/2)\big)}-\frac{3}{2z\big(1-(-2z)^{-1}\big)}$$

hence

$$(\star\star)\quad f(z)=\frac{1}{2}\sum_{j=0}^{\infty}\left(-\frac{z}{2}\right)^j-\frac{3}{2z}\sum_{j=0}^{\infty}\left(-\frac{1}{2z}\right)^j$$

Both series converge as long as both $\left|-\frac{z}{2}\right|<1$ and $\left|-\frac{1}{2z}\right|<1$, that is, as long as $|z|<2$ and $|z|>\frac{1}{2}$. So, $(\star\star)$ gives a Laurent series in the domain for $\frac{1}{2}<|z|<2$.

15.6.4.9. $f(z)=z\,e^{1/z}=z\cdot\displaystyle\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{1}{z}\right)^j=z\cdot\left(1+\frac{1}{z}+\sum_{k=2}^{\infty}\frac{1}{k!}\cdot z^{-k}\right)=\sum_{k=1}^{\infty}\frac{1}{k!}\cdot z^{-k+1}+(1+z)$

$$=\sum_{j=0}^{\infty}\frac{1}{(j+1)!}\cdot z^{-j}+(1+z).$$

This Laurent series converges for $0<|z|<\infty$, by using the ratio test:

$$\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right|=\lim_{j\to\infty}\left|\frac{\frac{1}{(j+2)!}\cdot z^{-j-1}}{\frac{1}{(j+1)!}\cdot z^{-j}}\right|=\lim_{j\to\infty}\left|\frac{1}{(j+2)}\cdot z^{-1}\right|=0$$

for all $z \neq 0$.

15.6.4.11. By the Binomial Theorem, $(x+1)^j = \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!}\, x^\ell 1^{j-\ell}$. With $x = 3$, we have

$$4^j = (3+1)^j = \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!}\, 3^\ell 1^{j-\ell} = j! \cdot \sum_{\ell=0}^{j} \frac{1}{\ell!(j-\ell)!}\, 3^\ell,$$

which implies $(\star)$.

Next, generalize this to give an explanation of why the law of exponents (15.27)(i) in Section 15.4, that is, $e^{z_1+z_2} = e^{z_1}e^{z_2}$, is true: The Binomial Theorem states that

$$(z_1+z_2)^j = \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!}\, z_1^\ell z_2^{j-\ell},$$

hence

$$\frac{1}{j!}\,(z_1+z_2)^j = \sum_{\ell=0}^{j} \frac{1}{\ell!(j-\ell)!}\, z_1^\ell z_2^{j-\ell}.$$

Apply this and the convolution formula for the product of two series of complex numbers, namely,

$$\left(\sum_{\ell=0}^{\infty} a_\ell\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{j=0}^{\infty}\left(\sum_{\ell=0}^{j} a_\ell b_{j-\ell}\right)$$

[Note that this follows from the convolution formula in Section 15.6 by taking $z = 1$.]

$$e^{z_1}e^{z_2} = \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!}\, z_1^\ell\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!}\, z_2^n\right) = \sum_{j=0}^{\infty}\left(\sum_{\ell=0}^{j} \frac{1}{\ell!}\, z_1^\ell \frac{1}{(k-\ell)!}\, z_2^{k-\ell}\right) = \sum_{j=0}^{\infty} \frac{1}{j!}\,(z_1+z_2)^j = e^{z_1+z_2}.$$

Section 15.7.2

15.7.2.1. Ex: $f(z) = \dfrac{1}{(z+i)(z-i)(z+1)^3} = \dfrac{1}{(z^2+1)(z+1)^3}$

15.7.2.3. First, define $h(z) = (e^z - 1)^2$, and we will find the order of the zero of $h(z)$at $z_0 = 0$:

$$\begin{aligned} & h(0) = (e^0 - 1)^2 = 0 \\ h'(z) = 2(e^z - 1), \quad & h'(0) = 2(e^0 - 1) = 0 \\ h''(z) = 2e^z, \quad & h''(0) = 2 \neq 0, \end{aligned}$$

so $h(z)$ has a zero of order two at $z_0 = 0$. By Theorem 15.26(b) in Section 15.7, $f(z) \triangleq \dfrac{1}{h(z)} = \dfrac{1}{(e^z-1)^2}$ has a pole of order two at $z_0 = 0$.

15.7.2.5. $f(z) = \dfrac{1}{e^z+1} = \dfrac{1}{h(z)}$, so we will study the zeros of $h(z) \triangleq e^z + 1$. First, we need to find the zero(s) of $h(z)$:

$$0 = e^z + 1 \quad \Longleftrightarrow \quad e^z = -1 = e^{i\pi} \quad \Longleftrightarrow \quad z = i\pi + i2\pi k, \quad \text{where } k \text{ in any integer.}$$

Fix any integer k. We have

$$\begin{aligned} & h\big(i(2k+1)\pi\big) = 0 \\ h'(z) = e^z, \quad & h'\big(i(2k+1)\pi\big) = -1 \neq 0, \end{aligned}$$

so $h(z)$ has a zero of order one at $z_k \triangleq i(2k+1)\pi$. By Theorem 15.26(b) in Section 15.7, $f(z) \triangleq \dfrac{1}{h(z)}$ has a simple pole at $z_k \triangleq i(2k+1)\pi$ for every integer k, and nowhere else.

15.7.2.7. Ex: $f(z) = \dfrac{(z+i)(z-i)}{(z+1)^3} = \dfrac{z^2+1}{(z+1)^3}$

15.7.2.9. Because $f(z)$ has a pole of order m at z_0, $f(z) = \dfrac{h(z)}{(z-z_0)^m}$ for some function h that is analytic at z_0 and satisfies $h(z_0) \neq 0$. We have

$$f'(z) = \frac{-mh(z) + (z-z_0)h'(z)}{(z-z_0)^{m+1}} = \frac{\widetilde{h}(z)}{(z-z_0)^{m+1}},$$

where $\tilde{h}(z) \triangleq -mh(z) + (z-z_0)h'(z)$ is also analytic at z_0, by the anticipated Theorem 15.4 in Section 15.9, and $\widetilde{h}(z_0) = -mh(z_0) \neq 0$. It follows that $f'(z)$ has a pole of order $m+1$ at z_0.

15.7.2.11. By Theorem 15.24 in Section 15.7, $h(z)$ having a zero of order m at z_0 implies there is a function $k(z)$ such that $h(z) = (z-z_0)^m k(z)$, where $k(z_0) \neq 0$ and $k(z)$ is analytic at z_0. So,

$$(\star)\ f(z) = \frac{g(z)}{h(z)} = \frac{g(z)}{(z-z_0)^m k(z)} = \frac{g(z)/k(z)}{(z-z_0)^m}.$$

Because $k(z_0) \neq 0$, and we were given that $g(z_0) \neq 0$, and both $g(z)$ and $k(z)$ are analytic at z_0 it follows that the function $\dfrac{g(z)}{k(z)}$ is analytic at z_0 and non-zero at z_0. By this and $(\star)$, Theorem 15.26(a) in Section 15.7 applies to justify that $f(z)$ has a pole of order m at z_0.

15.7.2.13. By Theorem 15.24 in Section 15.7, because $g(z)$ has a zero of order m at z_0, and $h(z)$ has a zero of order n at z_0, there are functions $k(z)$ and $\ell(z)$ that are analytic and non-zero at z_0 and such that

$$g(z) = k(z)(z - z_0)^m \quad \text{and} \quad h(z) = \ell(z)(z - z_0)^n,$$

So,

$$f(z) \triangleq g(z)h(z) = k(z)\ell(z)(z - z_0)^m (z - z_0)^n$$

that is,

$$(\star)\ f(z) = \big(k(z)\ell(z)\big)(z - z_0)^{m+n}.$$

Because $k(z)\ell(z)$ is analytic and non-zero at z_0, it follows that $f(z)$ has a zero of order $(m + n)$ at z_0.

15.7.2.15. By Theorem 15.24 in Section 15.7, there is a function $\ell(z)$ that is analytic and non-zero at z_0 for which $h(z) = \ell(z)(z - z_0)^n$. By Theorem 15.27 in Section 15.7, there is a function $k(z)$ that is analytic and non-zero at z_0 for which $g(z) = (z - z_0)^{-m}k(z)$.

So,

$$(\star) \quad f(z) \triangleq g(z)h(z) = (z - z_0)^{n-m}\big(k(z)\ell(z)\big).$$

Because $\big(k(z)\ell(z)\big)$ is analytic and non-zero at z_0, $(\star)$ implies that $f(z)$ has a removable singularity at z_0. The extension of $f(z)$ defined in Theorem 15.25 in Section 15.7 has a zero of order $n - m$, by Theorem 15.24 in Section 15.7.

Section 15.8.3

15.8.3.1. Parametrizing the curve as $z(t)=3e^{it}$, $0\le t\le 2\pi$,

$$\oint_{|z|=3}\overline{z}\,dz=\int_0^{2\pi}3e^{-it}\left(i3e^{it}\,dt\right)=\int_0^{2\pi}9\,i\,dt=i18\pi.$$

15.8.3.3. First, partial fractions gives

$$\frac{z}{(z-i)(z-2i)}=\frac{A}{z-2i}+\frac{B}{z-i}\qquad\Longleftrightarrow\qquad z=A(z-i)+B(z-2i).$$

Substituting in $z=i$ implies $i=B(-i)$, hence $B=-1$, and substituting in $z=2i$ implies $2i=A(i)$, hence $A=2$. So,

$$f(z)\triangleq\frac{z}{(z-i)(z-2i)}=\frac{2}{z-2i}-\frac{1}{z-i}.$$

In the domain $\mathcal{D}\triangleq\mathcal{D}_0\left(\frac{7}{4}\right)$, for example, the function $\dfrac{2}{z-2i}$ is analytic, so the Cauchy-Goursat Theorem (Theorem 15.36 in Theorem 15.8) implies

$$\oint_{|z|=\frac{3}{2}}\frac{2}{z-2i}\,dz=0.$$

The positively oriented circle $|z|=\frac{3}{2}$ can be deformed in $\mathcal{D}$ to the positively oriented circle $|z-i|=\frac{1}{2}$, so the Deformation Theorem (Theorem 15.35 in Theorem 15.8) and Theorem 15.32 in Section 15.8 together imply

$$\oint_{|z|=\frac{3}{2}}\frac{1}{z-i}\,dz=\oint_{|z-i|=\frac{1}{2}}\frac{1}{z-i}\,dz=2\pi i.$$

Putting everything together yields

$$\oint_{|z|=\frac{3}{2}}\frac{z}{(z-i)(z-2i)}\,dz=\oint_{|z|=\frac{3}{2}}\frac{2}{z-2i}\,dz-\oint_{|z|=\frac{3}{2}}\frac{1}{z-i}\,dz=0-2\pi i=-2\pi i.$$

15.8.3.5. Use two things from Section 15.4.4 in Section 15.4:

$$(1)\quad Log_{\frac{3\pi}{4}}(z)=w \text{ is the unique } w \text{ in } log(z) \text{ satisfying } -\pi+\frac{3\pi}{4}<\mathcal{I}m(w)\le\pi+\frac{3\pi}{4}$$

and

$$(2)\quad \frac{d}{dz}\left[Log_{\frac{3\pi}{4}}(z)\right]=\frac{1}{z},\ \text{for all } z\neq 0 \text{ for which } \frac{3\pi}{4}+\pi \text{ is not in } arg(z).$$

An example of such a domain $\mathcal{D}$ is shown in the figure. So, we can calculate

$$\int_{\mathcal{C}}\frac{1}{z}\,dz=Log_{\frac{3\pi}{4}}(2)-Log_{\frac{3\pi}{4}}(-2)=\big(\ln|2|+i\cdot 0\big)-\big(\ln|2|+i\pi\big)=-i\,\pi.$$

15.8.3.7. (a) Given a function $f(z)=u(x,y)+iv(x,y)$ and writing $z=x+iy$, where x and y are real,

$$\int_{\mathcal{C}}f(z)\,dz=\int_{\mathcal{C}}\big(u(x,y)+iv(x,y)\big)\,d(x+iy)=\int_{\mathcal{C}}\big(u(x,y)+iv(x,y)\big)\,(dx+idy)$$

$$=\int_{\mathcal{C}}\big(u(x,y)dx-v(x,y)dy+i\big(u(x,y)dy+v(x,y)dx\big)\big)\,,$$

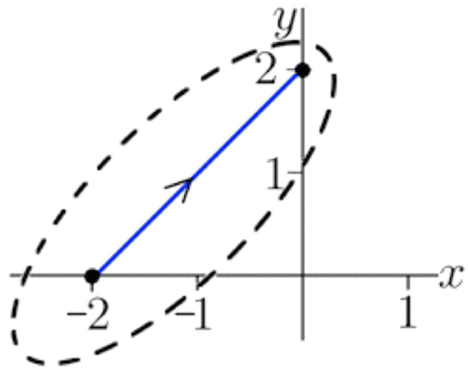


Figure 7: Answer key for problem 15.8.3.5

hence

$$(\star) \qquad \int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} (u\,dx - v\,dy) + i\int_{\mathcal{C}} (v\,dx + u\,dy).$$

(b) A complex number is zero if, and only if, both its real and imaginary parts are zero. In order to have path independence of the contour integral $\int_{\mathcal{C}} f(z)\,dz$, we must have path independence of both the real and imaginary parts of the right hand side of ($\star$), that is, $\displaystyle\int_{\mathcal{C}} (u\,dx - v\,dy)$ and $\displaystyle\int_{\mathcal{C}} (v\,dx + u\,dy)$.

(c) Assume that the functions $u(x, y)$ and $v(x, y)$ are nice enough that Green's Theorem, that is, Theorem 7.13 in Section 7.3, applies to the vector field $\mathbf{F} = u\,\hat{\boldsymbol{\imath}} - v\,\hat{\boldsymbol{\jmath}}$. Then

$$\int_{\mathcal{C}} (u\,dx - v\,dy) = \int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}.$$

So, in order to have path independence of $\displaystyle\int_{\mathcal{C}} (u\,dx - v\,dy)$, for any two simple, smooth paths $\mathcal{C}_1$ and $\mathcal{C}_2$ that have the same initial and terminal points and that do not intersect, the contour $\mathcal{C} \triangleq \mathcal{C}_1 \cup (-\mathcal{C}_2)$ is closed, simple, and smooth, hence

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \bullet d\mathbf{r} + \int_{-\mathcal{C}_2} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \bullet d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F} \bullet d\mathbf{r} = 0.$$

But, Green's Theorem implies then that

$$0 = \int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathcal{C}} \left(u\,\hat{\boldsymbol{\imath}} - v\,\hat{\boldsymbol{\jmath}}\right) \bullet d\mathbf{r} = \oiint_{\mathcal{D}} \left(\frac{\partial(-v)}{\partial x} - \frac{\partial(u)}{\partial y}\right) dA = \oiint_{\mathcal{D}} \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dA.$$

This being true for all such simple, smooth paths $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{D}$, we must have

$$0 \equiv \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

which can be rewritten as

$$(\star\star) \qquad \frac{\partial u}{\partial y} \equiv -\frac{\partial v}{\partial x},$$

which states that the vector field $\mathbf{F}$ satisfies the exactness criterion (6.43) in Section 6.4.

Similarly, define the vector field $\mathbf{G} = v\,\hat{\boldsymbol{\imath}} + u\,\hat{\boldsymbol{\jmath}}$. Then

$$\int_{\mathcal{C}} (v\,dx + u\,dy) = \int_{\mathcal{C}} \mathbf{G} \bullet d\mathbf{r}.$$

So, in order to have path independence of $\int_{\mathcal{C}} (v\,dx + u\,dy)$, for any two simple, smooth paths $\mathcal{C}_1$ and $\mathcal{C}_2$ that have the same initial and terminal points and that do not intersect, the contour $\mathcal{C} \triangleq \mathcal{C}_1 \cup (-\mathcal{C}_2)$ is closed, simple, and smooth, hence

$$\int_{\mathcal{C}} \mathbf{G} \bullet d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{G} \bullet d\mathbf{r} + \int_{-\mathcal{C}_2} \mathbf{G} \bullet d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{G} \bullet d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{G} \bullet d\mathbf{r} = 0.$$

But, Green's Theorem implies then that

$$0 = \int_{\mathcal{C}} \mathbf{G} \bullet d\mathbf{r} = \int_{\mathcal{C}} \left(v\,\hat{\boldsymbol{\imath}} + u\,\hat{\boldsymbol{\jmath}}\right) \bullet d\mathbf{r} = \oiint_{\mathcal{D}} \left(\frac{\partial(u)}{\partial x} - \frac{\partial(v)}{\partial y} \right) dA$$

This being true for all such simple, smooth paths $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{D}$, we must have

$$0 \equiv \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y},$$

which can be rewritten as

$$(\star\star\star) \quad \frac{\partial u}{\partial x} \equiv \frac{\partial v}{\partial y},$$

which states that the vector field $\mathbf{G}$ satisfies the exactness criterion (6.43) in Section 6.4.

(d) $(\star\star)$ implies that the second Cauchy-Riemann equation, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, must be satisfied. $(\star\star\star)$ implies that the first Cauchy-Riemann equation, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, must be satisfied.

(e) In Example 15.54 in Section 15.8 we were doing a contour integral of

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} - \frac{y)}{x^2+y^2}.$$

This is similar to problems 7.2.5.19's integrand, $\left(\frac{x}{x^2+y^2}\,\hat{\boldsymbol{\imath}} + \frac{y}{x^2+y^2}\,\hat{\boldsymbol{\jmath}} \right)$. Moreover,

$$\frac{i}{z} = \frac{i}{x+iy} = \frac{i(x-iy)}{(x+iy)(x-iy)} = \frac{y}{x^2+y^2} + i\,\frac{x)}{x^2+y^2}.$$

is similar to problems 7.2.5.20's integrand, $\left(-\frac{y}{x^2+y^2}\,\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\,\hat{\boldsymbol{\jmath}} \right)$.

In Section 7.2, it takes a lot of work to find a potential function in order to use the fundamental theorem of line integrals. In Example 15.54 in Section 15.8, there was little effort needed to find an anti-derivative because we know that $\frac{d}{dz}\left[\, Log_\sigma(z) \,\right] = \frac{1}{z}$.

Section 15.9.4

15.9.4.1. Partial fractions gives

$$f(z) = \frac{1}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2} \quad \Longleftrightarrow \quad 1 = A(z+1)(z-2) + Bz(z-2) + Cz(z+1).$$

where A, B, C are constants to be determined.

Substitute in $z=0$ to get $1=-2A$; substitute in $z=-1$ to get $1=3B$; and substitute in $z=2$ to get $1=6C$. So,

$$f(z) = -\frac{1}{2}\cdot\frac{1}{z} \;+\; \frac{1}{3}\cdot\frac{1}{z+1} \;+\; \frac{1}{6}\cdot\frac{1}{z-2}.$$

So, the singularities of f are at $z=0,-1,2$, and the corresponding residues are

$$\mathsf{Res}\Big[\,f;\ 0\,\Big] = \mathsf{Res}\Big[-\frac{1}{2}\cdot\frac{1}{z};\ 0\Big] = -\frac{1}{2}, \quad \mathsf{Res}\Big[\,f;\ -1\,\Big] = \mathsf{Res}\Big[\frac{1}{3}\cdot\frac{1}{z+1};\ -1\Big] = \frac{1}{3},$$

and

$$\mathsf{Res}\Big[\,f;\ 2\,\Big] = \mathsf{Res}\Big[\frac{1}{6}\cdot\frac{1}{z-2};\ 2\Big] = \frac{1}{6}.$$

15.9.4.3. First, let us find where the denominator, $z\sin z$, is zero: For $z = x+iy$, where x, y are real, $0 = \sin z$ gives

$$0 + i0 = \sin(x+iy) = \sin x\,\cosh(y) + i\cos x\,\sinh(y) \Longleftrightarrow \left\{\begin{array}{lll} (1) & 0 = & \sin x\,\cosh y \\ & & \\ (2) & 0 = & \cos x\,\sinh y \end{array}\right\}.$$

Because $\cosh y \ge 1$ for all y, equation (1) is true only for (1) $x = n\pi$, where n is an integer. Substitute $x = n\pi$ into equation (2) to get $0 = \cos(n\pi)\sinh y = (-1)^n \sinh y$, hence $y = 0$.

So, the only zeros of $z\sin z$ are at $z = n\pi$, where n is an integer. The only singularities of $f(z) = \dfrac{1}{z\sin z}$ in $D_2(\frac{\pi}{2})$ are at $z=0$ and at $z=\pi$.

$h(z) \triangleq z\sin z$ has a zero of order two at $z=0$, because

$$\begin{aligned} & & h(0) &= 0 \\ h'(z) &= \sin z + z\cos z, & h'(0) &= 0 \\ h''(z) &= 2\cos z - z\sin z, & h''(0) &= 2 \ne 0. \end{aligned}$$

By Theorem 15.26 in Section 15.7, $f(z) \triangleq \dfrac{1}{h(z)}$ has a pole of order two at $z=0$.

$h(z) \triangleq z\sin z$ has a zero of order one at $z=\pi$, because

$$\begin{aligned} & & h(\pi) &= 0 \\ h'(z) &= \sin z + z\cos z, & h'(\pi) &= -\pi \ne 0. \end{aligned}$$

By Theorem 15.26 in Section 15.7, $f(z) \triangleq \dfrac{1}{h(z)}$ has a simple pole at $z=\pi$.

We can use Theorem 15.41 in Section 15.9 to find the residues:

$$\mathsf{Res}[\,f;\ 0\,] = \frac{1}{2!}\lim_{z\to 0}\left(\frac{d}{dz}\Big[\,(z-0)^2\cdot f(z)\Big]\right) = \frac{1}{2!}\lim_{z\to 0}\left(\frac{d}{dz}\Big[\,z^2\cdot\frac{1}{z\sin z}\,\Big]\right) = \frac{1}{2!}\lim_{z\to 0}\left(\frac{d}{dz}\Big[\,\frac{z}{\sin z}\,\Big]\right)$$

$$= \frac{1}{2!}\lim_{z\to 0}\frac{(z)'\sin z - z(\sin z)'}{\sin^2 z} = \frac{1}{2!}\lim_{z\to 0}\frac{\sin z - z\cos z}{\sin^2 z}$$

which equals, using L'Hôpital's Rule

$$= \frac{1}{2!}\lim_{z\to 0}\frac{(\sin z - z\cos z)'}{(\sin^2 z)'} = \frac{1}{2!}\lim_{z\to 0}\frac{\cos z - 1\cdot\cos z - z\cdot(-\sin z)}{2\sin z\cos z} = \frac{1}{2!}\lim_{z\to 0}\frac{z\,\sin z}{2\sin z\cos z} = \frac{1}{2!}\lim_{z\to 0}\frac{z}{2\cos z} = 0.$$

For the other residue, eventually using L'Hôpital's Rule we have

$$\mathsf{Res}[\,f;\ \pi\,] = \lim_{z\to\pi}(z-\pi)\,f(z) = \lim_{z\to\pi}\ (z-\pi)\cdot\frac{1}{z\sin z} = \left(\lim_{z\to\pi}\ \frac{1}{z}\right)\left(\lim_{z\to\pi}\ \frac{(z-\pi)}{\sin z}\right) = \frac{1}{\pi}\left(\lim_{z\to\pi}\ \frac{(z-\pi)'}{(\sin z)'}\right)$$

$$= \frac{1}{\pi}\left(\lim_{z\to\pi}\ \frac{1}{\cos z}\right) = -\frac{1}{\pi}\,.$$

In summary, in $D_2(\frac{\pi}{2})$, the only singularities of f are at $z = 0, \pi$ and the corresponding residues are $\mathsf{Res}[\,f;\ 0\,] = 0, \mathsf{Res}[\,f;\ \pi\,] = -\frac{1}{\pi}$.

15.9.4.5. $\displaystyle\oint_{|z|=\frac{3}{2}}\left(\frac{z\cos z}{(z-1)^2}+\frac{z\cos z}{z+2}\right)dz = \oint_{|z|=\frac{3}{2}}\frac{z\cos z}{(z-1)^2}\,dz+\oint_{|z|=\frac{3}{2}}\frac{z\cos z}{z+2}\,dz = \oint_{|z|=\frac{3}{2}}\frac{z\cos z}{(z-1)^2}\,dz+0,$

because $\dfrac{z\cos z}{z+2}$ is analytic on and inside the curve $\mathcal{C}:\ |z| = \dfrac{3}{2}$.

For the first term, whose integrand has a pole of order $2 = m+1$ at $z = 1$, use Theorem 15.39, that is, Cauchy's Integral Formula in general, to get

$$\oint_{|z|=\frac{3}{2}}\left(\frac{z\cos z}{(z-1)^2}+\frac{z\cos z}{z+2}\right)dz = \oint_{|z|=\frac{3}{2}}\frac{z\cos z}{(z-1)^2}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\Big[\,z\cos z\,\Big]\Big|_{z=1} = \frac{2\pi i}{1!}\Big(\,\cos z - z\sin z\,\Big)\Big|_{z=1}$$

$$= 2\pi i\big(\cos(1)-\sin(1)\big).$$

15.9.4.7. $\dfrac{e^z}{z-i\pi}$ is analytic on and inside the curve $\mathcal{C}:\ |z| = 3$, so $\displaystyle\oint_{|z|=3}\frac{e^z}{z-i\pi}\,dz = 0.$

15.9.4.9. Define $f(z) \triangleq \dfrac{1}{(z+3)(z-3)}$. The only singularity of $f(z)$ on or inside the curve $\mathcal{C}:\ |z-1| = 3$ is at $z = 3$. Using Theorem 15.42 in Section 15.9, as in Example 15.64 in Section 15.9, we have

$$\oint_{|z-1|=3}\frac{1}{z^2-9}\,dz = \oint_{\mathcal{C}} f(z)\,dz = 2\pi i\cdot\mathsf{Res}[\,f;\ 3\,] = 2\pi i\cdot\left(\cancel{(z-3)}\cdot\frac{1}{\cancel{(z-3)}(z+3)}\right)\Big|_{at\ \ z=3} = 2\pi i\Big(\frac{1}{6}\Big) = \frac{\pi i}{3}\,.$$

15.9.4.11. The only singularities of $f(z) \triangleq \dfrac{z}{\sin z}$ on or inside the curve $\mathcal{C}:\ |z-\frac{\pi}{2}| = 2$ are at $z = 0$ and $z = \pi$.

The singularity at $z = 0$ is a removable singularity because L'Hôpital's Rule implies that there exists

$$\lim_{z\to 0}\ \frac{z}{\sin z} = \lim_{z\to 0}\ \frac{(z)'}{(\sin z)'} = \lim_{z\to 0}\ \frac{1}{\cos z} = 1.$$

By Theorem 15.41(a) in Section 15.9, $\mathsf{Res}[\,f;\ 0\,] = 0$.

For the residue at $z = \pi$, first we find the order of the pole there. First, we note that $h(z) \triangleq \sin z$ has a simple zero at $z = \pi$, because

$$h(\pi) = 0$$

$$h'(z) = \cos z, \quad h'(\pi) = -1 \neq 0,$$

By Theorem 15.24 in Section 15.7, $\sin z = (z-\pi)^1 k(z)$, where $k(z)$ is analytic at $z=\pi$ and $k(\pi)\neq 0$. So,

$$f(z) = \frac{z}{\sin z} = \frac{z}{(z-\pi)^1 k(z)} = \frac{\frac{z}{k(z)}}{(z-\pi)^1},$$

hence $f(z)$ has a simple pole at $z=\pi$.

It follows from Theorem 15.41(c) in Section 15.9, along with use of L'Hôpital's Rule, that

$$\mathsf{Res}[\,f;\ \pi\,] = \lim_{z\to\pi}(z-\pi)\cdot\frac{z}{\sin z} = \left(\lim_{z\to\pi} z\right)\left(\lim_{z\to\pi}\frac{(z-\pi)}{\sin z}\right) = (\pi)\left(\lim_{z\to\pi}\frac{(z-\pi)'}{(\sin z)'}\right) = (\pi)\left(\lim_{z\to\pi}\frac{1}{\cos z}\right) = -\pi.$$

So, by Theorem 15.42 in Section 15.9,

$$\oint_{|z-\frac{\pi}{2}|=2}\frac{z}{\sin z}\,dz = \oint_{\mathcal{C}} f(z)\,dz = 2\pi i\cdot(\mathsf{Res}[\,f;\ 0\,] + \mathsf{Res}[\,f;\ \pi\,]) = 2\pi i\cdot(0+(-\pi)) = -2\pi^2 i.$$

15.9.4.13. The denominator, $z^2+\omega^2 = (z+i\omega)(z-i\omega)$, is zero at $z=-i\omega$ and $z=i\omega$, both of which are inside the curve $\mathcal{C}:\ |z|=R$. Define $f(z)\triangleq \dfrac{ze^{i\xi z}}{z^2+\omega^2}$. Cauchy's residue theorem, as stated in Theorem 15.42 in Section 15.9, implies that

$$\oint_{|z-1|=3}\frac{ze^{i\xi z}}{z^2+\omega^2}\,dz = \oint_{\mathcal{C}} f(z)\,dz = 2\pi i\cdot(\mathsf{Res}[\,f;\ -i\omega\,] + \mathsf{Res}[\,f;\ i\omega\,])$$

$$= 2\pi i\cdot\left(\left(\cancel{(z+i\omega)}\cdot\frac{z\,e^{i\xi z}}{\cancel{(z+i\omega)}(z-i\omega)}\right)\Big|_{at\ z=-i\omega} + \left(\cancel{(z-i\omega)}\cdot\frac{z\,e^{i\xi z}}{\cancel{(z-i\omega)}(z+i\omega)}\right)\Big|_{at\ z=i\omega}\right)$$

$$= 2\pi i\left(\frac{-i\omega\,e^{i\xi(-i\omega)}}{-i2\omega} + \frac{i\omega\,e^{i\xi(i\omega)}}{i2\omega}\right) = 2\pi i\cdot\frac{e^{\xi\omega}+e^{-\xi\omega}}{2} = 2\pi i\cosh(\xi\omega).$$

Section 15.10.5

15.10.5.1. Parametrize the unit circle $|z| = 1$ by $\mathcal{C} : z = z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$, which has positive orientation. For z on $\mathcal{C}$,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = (2iz)^{-1}(z^2 - 1).$$

On $\mathcal{C}$, we have $\dfrac{dz}{d\theta}(\theta) = ie^{i\theta} = iz$, hence $d\theta = \dfrac{dz}{iz}$. By Theorem 15.29 in Section 15.8,

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \oint_{|z|=1} \frac{1}{1 + \big((2iz)^{-1}(z^2-1)\big)^2}\,\frac{dz}{iz} = \oint_{|z|=1} \frac{-(2z)^2}{-(2z)^2 + (z^2-1)^2}\,\frac{dz}{iz}$$

$$= 4i \oint_{|z|=1} \frac{z}{-4z^2 + (z^2-1)^2}\,dz = 4i \oint_{|z|=1} \frac{z}{z^4 - 6z^2 + 1}\,dz.$$

The four singularities are where the denominator is zero, that is, where $0 = z^4 - 6z^2 + 1 = \big(z^2\big)^2 - 6z^2 + 1$, that is, where

$$z^2 = \frac{6 \pm \sqrt{6^2 - 4}}{2} = 3 \pm 2\sqrt{2},$$

that is,

$$z = \pm\sqrt{3 + 2\sqrt{2}}, \quad \pm\sqrt{3 - 2\sqrt{2}}.$$

Of these singularities, only $z_3 = \sqrt{3 - 2\sqrt{2}}$ and $z_4 = -\sqrt{3 - 2\sqrt{2}}$ are inside $\mathcal{C} : |z| = 1$. Because

$$f(z) \triangleq \frac{z}{z^4 - 6z^2 + 1} = \frac{z}{(z^2 - 3 - 2\sqrt{2})\left(z - \sqrt{3 - 2\sqrt{2}}\right)\left(z + \sqrt{3 - 2\sqrt{2}}\right)},$$

both z_3 and z_4 are simple poles of $f(z)$. It is relatively straightforward to use Theorem 15.41(c) in Section 15.9 to calculate that

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta}\,d\theta = 4i \oint_{|z|=1} f(z)\,dz = 4i \cdot 2\pi i\ \big(\,\mathsf{Res}[\,f(z);\ z_3\,] + \mathsf{Res}[\,f(z);\ z_4\,]\,\big)$$

$$= -8\pi\cancel{\left(z - \sqrt{3 - 2\sqrt{2}}\right)} \frac{z}{(z^2 - 3 - 2\sqrt{2})\cancel{\left(z - \sqrt{3 - 2\sqrt{2}}\right)} \cdot \left(z + \sqrt{3 - 2\sqrt{2}}\right)}\Bigg|_{at\ z=\sqrt{3-2\sqrt{2}}}$$

$$-8\pi\cancel{\left(z + \sqrt{3 - 2\sqrt{2}}\right)} \frac{z}{(z^2 - 3 - 2\sqrt{2})\cancel{\left(z + \sqrt{3 - 2\sqrt{2}}\right)} \cdot \left(z - \sqrt{3 - 2\sqrt{2}}\right)}\Bigg|_{at\ z=-\sqrt{3-2\sqrt{2}}}$$

$$= -8\pi \cdot \frac{1}{(3 - 2\sqrt{2}) - 3 - 2\sqrt{2}} \cdot \left(\frac{\sqrt{3 - 2\sqrt{2}}}{\left(\sqrt{3 - 2\sqrt{2}} + \sqrt{3 - 2\sqrt{2}}\right)} + \frac{-\sqrt{3 - 2\sqrt{2}}}{\left(-\sqrt{3 - 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}\right)} \right)$$

$$= -8\pi \cdot \frac{1}{-4\sqrt{2}} \cdot \left(\frac{1}{2} + \frac{1}{2}\right) = \pi\sqrt{2}$$

15.10.5.3. Using the change of variables $\phi = \theta - \pi$,

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta + \int_\pi^{2\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta + \int_0^{\pi} \frac{1}{1+\sin^2(\phi+\pi)}\,d\phi$$

$$= 2\int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta,$$

because $\big(\sin(\phi+\pi)\big)^2 = \big(-\sin(\phi)\big)^2 = \sin^2\phi$. So,

$$\int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \frac{1}{2}\cdot\pi\sqrt{2} = \frac{\pi}{\sqrt{2}},$$

by using the result of problem 15.10.5.1.

15.10.5.5. Similar to work in Example 15.66 in Section 15.10, define $g(x) = \dfrac{x^2}{(x^2+4)^2}$ and $f(z) = \dfrac{z^2}{(z^2+4)^2}$. We will explain why the improper integral $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx$ exists and equals

$$2\pi i\,\mathsf{Res}[\,f(z);\ i2\,] = \ldots = \frac{\pi}{4}.$$

First, the improper integral $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx$ exists because both

$$\lim_{b\to\infty}\int_0^b \frac{dx}{(x^2+4)^2} \quad\text{and}\quad \lim_{a\to-\infty}\int_a^0 \frac{dx}{(x^2+4)^2}$$

exist, by a Comparison Theorem for definite integrals.

Because both of the improper integrals $\displaystyle\int_0^{\infty} g(x)dx$ and $\displaystyle\int_{-\infty}^{0} g(x)dx$ are convergent, we have

$$\int_{-\infty}^{\infty} g(x)\,dx = \lim_{R\to\infty}\int_{-R}^{0} g(x)\,dx + \lim_{R\to\infty}\int_0^R g(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} g(x)\,dx.$$

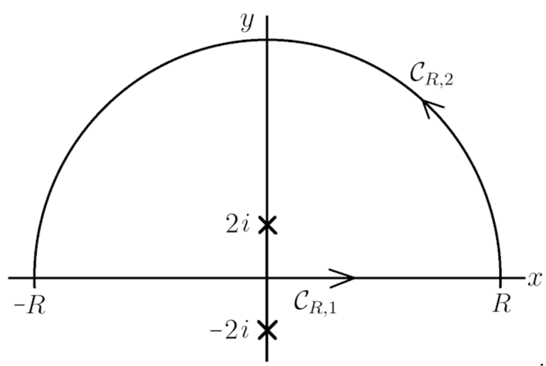


Figure 8: Complex integration to evaluate a real integral

Consider the contour $\mathcal{C}_R = \mathcal{C}_{R,1} + \mathcal{C}_{R,2}$ shown in the figure, which also indicates by $\times$ the singularities of $f(z)$ at $z = \pm i2$. We calculate

$$\int_{\mathcal{C}_{R,2}} f(z)\,dz = \int_{\mathcal{C}_{R,2}} \frac{z^2}{(z^2+4)^2}\,dz = \int_0^{\pi} \frac{R^2 e^{i2\theta}}{\big((Re^{i\theta})^2+4\big)^2}\, iRe^{i\theta}\,d\theta = \int_0^{\pi} \frac{R^2 e^{i2\theta}}{(R^2e^{i2\theta}+4)^2}\, iRe^{i\theta}\,d\theta.$$

Using work similar to that which explained Lemma 15.1 in Section 15.10, as $R\to\infty$,

$$\left|\int_{\mathcal{C}_{R,2}} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{R^2 e^{i2\theta}}{(R^2e^{i2\theta}+4)^2}\, iRe^{i\theta}\,d\theta\right| \le \int_0^{\pi} \left|\frac{R^2 e^{i2\theta}}{(R^2e^{i2\theta}+4)^2}\, iRe^{i\theta}\right| d\theta \le \int_0^{\pi} \frac{R^3}{(R^2-4)^2}\,d\theta \to 0.$$

Also, $\mathcal{C}_{R,1} : z = x + i0$, $-R \le x \le R$, so

$$\int_{\mathcal{C}_{R,1}} f(z)\,dz = \int_{-R}^{R} g(x)\,dx.$$

Putting things together, we have

$$\lim_{R\to\infty}\int_{\mathcal{C}_{R,1}+\mathcal{C}_{R,2}} f(z)\,dz = \lim_{R\to\infty}\left(\int_{\mathcal{C}_{R,2}} f(z)\,dz + \int_{-R}^{R} g(x)\,dx\right) = 0 + \int_{-\infty}^{\infty} g(x)\,dx.$$

For $R > 2$,

$$\int_{\mathcal{C}_{R,1}+\mathcal{C}_{R,2}} f(z)\,dz = 2\pi i\,\mathsf{Res}[\,f(z);\ i2\,].$$

Finally,

$$\int_{-\infty}^{\infty}\frac{x^2dx}{\left(x^2+4\right)^2} = \int_{\mathcal{C}_{R,1}+\mathcal{C}_{R,2}} f(z)\,dz = 2\pi i\,\mathsf{Res}[\,f(z);\ 2i\,] = \frac{2\pi i}{1!}\frac{d}{dz}\Big[\cancel{(z-2i)^2}\frac{z^2}{\cancel{(z-2i)^2}(z+2i)^2}\Big]\Big|_{z=2i}$$

$$= \frac{2\pi i}{1!}\frac{d}{dz}\Big[\frac{z^2}{(z+2i)^2}\Big]\Big|_{z=2i} = \frac{2\pi i}{1!}\Big(\frac{(z^2)'(z+2i)^2 - z^2\big((z+2i)^2\big)'}{(z+2i)^4}\Big)\Big|_{z=2i}$$

$$= \frac{2\pi i}{1!}\Big(\frac{(2z)(z+2i)^2 - z^2 2(z+2i)}{(z+2i)^4}\Big)\Big|_{z=2i} = \frac{2\pi i}{1!}\Big(\frac{(2z)(z+2i) - 2z^2}{(z+2i)^3}\Big)\Big|_{z=2i}$$

$$= \frac{2\pi i}{1!}\Big(\frac{4iz}{(z+2i)^3}\Big)\Big|_{z=2i} = \frac{2\pi i}{1!}\Big(-\frac{8}{(4i)^3}\Big) = \frac{\pi}{4}.$$

In the last steps of the calculations, we used the fact that $f(z) = \dfrac{z^2}{(z-2i)^2\,(z+2i)^2}$ has a pole of order two at $z = 2i$, along with Theorem 15.41(b) in Section 15.9.

15.10.5.7. Parametrize the unit circle $|z| = 1$ by $\mathcal{C} : z = z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$, which has positive orientation. For z on $\mathcal{C}$,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i} = \frac{z+z^{-1}}{2} = (2z)^{-1}(z^2+1).$$

On $\mathcal{C}$, we have $\dfrac{dz}{d\theta}(\theta) = ie^{i\theta} = iz$, hence $d\theta = \dfrac{dz}{iz}$. By Theorem 15.29 in Section 15.8,

$$\int_0^{2\pi}\frac{1}{2+\cos^2\theta}\,d\theta = \oint_{|z|=1}\frac{1}{2+\big((2z)^{-1}(z^2+1)\big)^2}\,\frac{dz}{iz} = \oint_{|z|=1}\frac{(2z)^2}{2(2z)^2+\left(z^2+1\right)^2}\,\frac{dz}{iz}$$

$$= -4i\oint_{|z|=1}\frac{z}{8z^2+\left(z^2+1\right)^2}\,dz = -4i\oint_{|z|=1}\frac{z}{z^4+10z^2+1}\,dz.$$

The four singularities are where the denominator is zero, that is, where $0 = z^4+10z^2+1 = \left(z^2\right)^2+10z^2+1$, that is, where

$$z^2 = \frac{-10\pm\sqrt{10^2-4}}{2} = -5\pm 2\sqrt{6},$$

that is,

$$z = \pm i\sqrt{5+2\sqrt{6}},\quad \pm i\sqrt{5-2\sqrt{6}}.$$

Of these singularities, only $z_3 = i\sqrt{5-2\sqrt{6}}$ and $z_4 = -i\sqrt{5-2\sqrt{6}}$ are inside $\mathcal{C} : |z| = 1$. Because

$$f(z) \triangleq \frac{z}{z^4+10z^2+1} = \frac{z}{(z^2+5+2\sqrt{6})\left(z-i\sqrt{5-2\sqrt{6}}\right)\left(z+i\sqrt{5-2\sqrt{6}}\right)},$$

both z_3 and z_4 are simple poles of $f(z)$. It is relatively straightforward to use Theorem 15.41(c) in Section 15.9 to calculate that

$$\int_0^{2\pi} \frac{1}{2+\cos^2\theta}\,d\theta = -4i \oint_{|z|=1} f(z)\,dz = -4i\cdot 2\pi i\ \left(\ \mathsf{Res}[\,f(z);\ z_3\,] + \mathsf{Res}[\,f(z);\ z_4\,]\ \right)$$

$$= 8\pi\cancel{\left(z-i\sqrt{5-2\sqrt{6}}\right)}\frac{z}{(z^2+5+2\sqrt{6})\cancel{\left(z-i\sqrt{5-2\sqrt{6}}\right)}\cdot\left(z+i\sqrt{5-2\sqrt{6}}\right)}\Big|_{at\ z=i\sqrt{5-2\sqrt{6}}}$$

$$+8\pi\cancel{\left(z+i\sqrt{5-2\sqrt{6}}\right)}\frac{z}{(z^2+5+2\sqrt{6})\cancel{\left(z+i\sqrt{5-2\sqrt{6}}\right)}\cdot\left(z-i\sqrt{5-2\sqrt{6}}\right)}\Big|_{at\ z=-i\sqrt{5-2\sqrt{6}}}$$

$$= 8\pi\cdot\frac{1}{-(5-2\sqrt{6})+5+2\sqrt{6}}\cdot\left(\frac{\sqrt{5-2\sqrt{6}}}{\left(\sqrt{5-2\sqrt{6}}+\sqrt{5-2\sqrt{6}}\right)} + \frac{-\sqrt{5-2\sqrt{6}}}{\left(-\sqrt{5-2\sqrt{6}}-\sqrt{5-2\sqrt{6}}\right)}\right)$$

$$= 8\pi\cdot\frac{1}{4\sqrt{6}}\cdot\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{2\pi}{\sqrt{6}} = \sqrt{\frac{2}{3}}\cdot\pi.$$

15.10.5.9. Because $\cos\omega x = \mathcal{R}e\left(e^{i\omega x}\right)$, we have

$$P.\ v.\ \int_{-\infty}^{\infty} \frac{x^2\cos\omega x}{x^4+1}\,dx = \mathcal{R}e\left(P.\ v.\int_{-\infty}^{\infty}\frac{x^2}{x^4+1}\,e^{i\omega x}\,dx\right).$$

Let $f(z) \triangleq \dfrac{z^2}{z^4+1}\,e^{i\omega z}$ and use the same contour $\mathcal{C}_R = \mathcal{C}_{R,1} + \mathcal{C}_{R,2}$ found in Example 15.66 in Section 15.10 and shown in Figure 15.32. At any point on $\mathcal{C}_{R,2}$ given by $z = Re^{i\theta} = x+iy$ that lies in the *upper* half-plane, that is, has $y > 0$, we have

$$\left|e^{i\omega z}\right| = \left|e^{i\omega(x+iy)}\right| = \left|e^{i\omega x}\right|\ \left|e^{-\omega y}\right| = 1\cdot e^{-\omega y} < 1.$$

So,

$$\left|\int_{\mathcal{C}_{R,2}} \frac{z^2}{z^4+1}\,e^{i\omega z}\,dz\right| = \left|\int_0^{\pi}\frac{\left(Re^{i\theta}\right)^2}{\left(Re^{i\theta}\right)^4+1}\,e^{i\omega Re^{i\theta}}\,iRe^{i\theta}\,d\theta\right| \le \int_0^{\pi}\frac{R^2\left|e^{i\omega(x+iy)}\right|}{|R^4e^{i4\theta}+1|}\cdot\left|iRe^{i\theta}\right|\,d\theta$$

$$< \int_0^{\pi}\frac{R^3}{R^4-1}\,d\theta \to 0,\quad \text{as}\quad R\to\infty.$$

So, similar to the work of Example 15.66 in Section 15.10, the convergent improper integral is given by

$$\int_{-\infty}^{\infty}\frac{x^2\cos\omega x}{x^2+1}\,dx = P.\ v.\int_{-\infty}^{\infty}\frac{x^2\cos\omega x}{x^2+1}\,dx \triangleq \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{x^2\cos\omega x}{x^2+1}\,dx\right) = \lim_{R\to\infty}\mathcal{R}e\left(\int_{\mathcal{C}_{R,1}}\frac{z^2}{z^4+1}\,e^{i\omega z}\,dz\right).$$

We need to find the singularities of $f(z)$ that are inside $\mathcal{C}_R$:

$$0 = z^4+1 \quad\Longleftrightarrow\quad z^4 = -1 = e^{i\pi},$$

which gives four zeros: $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, and $e^{i7\pi/4}$, of which only $z_1 \triangleq e^{i\pi/4}$ and $z_2 \triangleq e^{i3\pi/4}$ are inside $\mathcal{C}_R$. So, using L'Hôpital's Rule

$$\int_{-\infty}^{\infty} \frac{x^2\cos\omega x}{x^2+1}\,dx = \mathcal{R}e\Big(2\pi i\,\mathsf{Res}\Big[\frac{z^2e^{i\omega z}}{z^4+1};\ e^{i\pi/4}\Big] + \mathsf{Res}\Big[\frac{z^2e^{i\omega z}}{z^4+1};\ e^{i3\pi/4}\Big]\Big)$$

$$= \mathcal{R}e\left(2\pi i\left(\lim_{z\to e^{i\pi/4}} \frac{(z-e^{i\pi/4})z^2e^{i\omega z}}{z^4+1} + \lim_{z\to e^{i3\pi/4}} \frac{(z-e^{i3\pi/4})z^2e^{i\omega z}}{z^4+1}\right)\right)$$

$$= \mathcal{R}e\left(2\pi i\left(\lim_{z\to e^{i\pi/4}} \frac{\big((z-e^{i\pi/4})z^2e^{i\omega z}\big)'}{(z^4+1)'} + \lim_{z\to e^{i3\pi/4}} \frac{\big((z-e^{i3\pi/4})z^2e^{i\omega z}\big)'}{(z^4+1)'}\right)\right)$$

$$= \mathcal{R}e\Big(2\pi i\Big(\lim_{z\to e^{i\pi/4}} \frac{z^2e^{i\omega z}+2z(z-e^{i\pi/4})e^{i\omega z}+i\omega(z-e^{i\pi/4})z^2e^{i\omega z}}{4z^3}$$

$$+ \lim_{z\to e^{i3\pi/4}} \frac{z^2e^{i\omega z}+2z(z-e^{i\pi/4})e^{i\omega z}+i\omega(z-e^{i\pi/4})z^2e^{i\omega z}}{4z^3}\Big)\Big)$$

$$= \mathcal{R}e\Big(2\pi i\Big(\frac{e^{i\pi/2}e^{i\omega\left(\frac{1}{\sqrt2}+i\,\frac{1}{\sqrt2}\right)}}{4e^{i3\pi/4}} + \frac{e^{i3\pi/2}e^{i\omega\left(-\frac{1}{\sqrt2}+i\,\frac{1}{\sqrt2}\right)}}{4e^{i\pi/4}}\Big)\Big)$$

$$= \mathcal{R}e\left(\frac{\pi i}{2}\left(ie^{i\omega/\sqrt2}e^{-\omega/\sqrt2}e^{-i3\pi/4} - ie^{-i\omega/\sqrt2}e^{-\omega/\sqrt2}e^{-i\pi/4}\right)\right)$$

$$= -\frac{\pi}{2}\,e^{-\omega/\sqrt2}\,\mathcal{R}e\left(\left(\cos\left(\frac{\omega}{\sqrt2}\right)+i\sin\left(\frac{\omega}{\sqrt2}\right)\right)\left(-\frac{1}{\sqrt2}-i\,\frac{1}{\sqrt2}\right) - \left(\cos\left(\frac{\omega}{\sqrt2}\right)-i\sin\left(\frac{\omega}{\sqrt2}\right)\right)\left(\frac{1}{\sqrt2}-i\,\frac{1}{\sqrt2}\right)\right)\Big)$$

$$= -\frac{\pi}{2}\,e^{-\omega/\sqrt2}\left(\sqrt2\left(-\cos\left(\frac{\omega}{\sqrt2}\right)+\sin\left(\frac{\omega}{\sqrt2}\right)\right)\right) = \frac{\pi}{\sqrt2}\,e^{-\omega/\sqrt2}\left(\cos\left(\frac{\omega}{\sqrt2}\right)-\sin\left(\frac{\omega}{\sqrt2}\right)\right)$$

$$= \pi\,e^{-\omega/\sqrt2}\,\cos\left(\frac{\omega}{\sqrt2}-\frac{\pi}{4}\right).$$

Chapter Sixteen

Section 16.1.5

16.1.5.1. When $c = 0$ and $d \neq 0$, the Möbius transformation is $M(z) = \dfrac{az+b}{d}$. So, $M(z)$ is not constant if, and only if, $a \neq 0$, which is true if and only if $ad - bc = ad - 0 = ad \neq 0$.

16.1.5.3. Define $\zeta = g(z) \triangleq \dfrac{3+z}{3-z}$. By Theorem 16.2 in Section 16.1, $h(z) \triangleq Log\big(g(z)\big)$ is conformal wherever $g(z)$ is both conformal and not equal to a non-positive real number. By Theorem 16.5 in Section 16.1, $g(z)$ is conformal, including having $g'(z) \neq 0$, everywhere except at $z = \pm 3$.

For which z is $\zeta = \dfrac{3+z}{3-z}$ a non-positive real number? To answer this it helps to invert the dependence to get z as a function of ζ:

$$\zeta = \frac{3+z}{3-z} \iff (3-z)\zeta = 3+z \iff 3\zeta - 3 = (1+\zeta)z \iff z = k(\zeta) \triangleq \frac{3(\zeta-1)}{\zeta+1} = 3 - \frac{6}{\zeta+1}.$$

The set of all z for which ζ is real and ≤ 0 is the set $k\big(\,(-\infty, 0]\,\big)$. Basic Calculus graphical techniques give that real $\zeta \leq 0 \iff z \leq -3$ or $z > 3$.

So, the set of z for which ζ is *not* a non-positive real number is everywhere except the intervals $(-\infty, -3]$ and $(3, \infty)$ on the real axis.

Finally, we conclude that $g(z)$ is conformal on

$$\{z : \mathcal{I}m(z) \neq 0\} \cup \{z : z \text{ is real and } -3 < z < 3\}.$$

16.1.5.5. $\cosh z$ is an entire function whose derivative is $\sinh z$. The latter is zero, if, and only if, $0 = e^z - e^{-z}$, if, and only if, $e^z = e^{-z}$, if, and only if, $1 = e^{2z} = e^{2x}(\cos 2y + i \sin 2y)$, if, and only if,

$$\left\{ \begin{array}{ll} (1) & e^{2x}\cos 2y = 1 \\ (2) & e^{2x}\sin 2y = 0 \end{array} \right\}.$$

Equation (2) is satisfied if, and only if, $y = \dfrac{n\pi}{2}$ for some integer n, in which case equation (1) requires

$$1 = e^{2x}\cos n\pi = e^{2x} \cdot (-1)^n,$$

which implies n must be even and $x = 0$. So, $\cosh z$ is conformal at all z except $z = 0 + ik\pi$, where k is any integer.

Section 16.2.5

16.2.5.1. With $\delta = -1$ and $\gamma = \dfrac{5}{2}$, Theorem 16.6 in Section 16.2 says to first choose α which satisfies the quadratic equation

$$0 = \delta\alpha^2 + (\gamma^2 - \delta^2 - 1)\alpha + \delta = -\alpha^2 + \left(\left(\frac{5}{2}\right)^2 - (-1)^2 - 1\right)\alpha + (-1)$$

that is

$$0 = \alpha^2 - \frac{17}{4}\alpha + 1,$$

so

$$\alpha_\pm = \frac{\frac{17}{4} \pm \sqrt{\left(\frac{17}{4}\right)^2 - 4}}{2} = \frac{\frac{17}{4} \pm \frac{15}{4}}{2} = \left\{ \begin{array}{ll} 4, & + \\ \\ \frac{1}{4}, & - \end{array} \right\}.$$

After that, we choose $\beta_\pm = \dfrac{1}{\alpha_\pm}$ to find Möbius transformations that both map the given circles to concentric circles centered at the origin:

$$M_+(z) = \frac{z-\alpha_+}{z-\beta_+} = \frac{z-4}{z-\frac{1}{4}} \cdot \frac{4}{4} = \frac{4(z-4)}{4z-1}$$

or

$$M_-(z) = \frac{z-\alpha_-}{z-\beta_-} = \frac{z-\frac{1}{4}}{z-4} \cdot \frac{4}{4} = \frac{4z-1}{4(z-4)}\,.$$

16.2.5.3. First, rewrite $|z| = 3$ as $1 = \left|\dfrac{z}{3}\right| = |\widetilde{z}|$, where $\widetilde{z} = \dfrac{z}{3}$, that is, $3\widetilde{z} = z$. After that, rewrite $|z-1| = 1$ as $|3\widetilde{z} - 1| = 1$, that is, as $\left|\widetilde{z} - \dfrac{1}{3}\right| = \dfrac{1}{3}$. So, in terms of the variable $\widetilde{z}$, we have $\delta = \frac{1}{3}$ and $\gamma = \frac{1}{3}$, so the Möbius transformation should have α satisfy the quadratic equation

$$0 = \delta\alpha^2 + (\gamma^2 - \delta^2 - 1)\alpha + \delta = \frac{1}{3}\,\alpha^2 + \left(\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 - 1\right)\alpha + \frac{1}{3}$$

that is

$$0 = \alpha^2 - 3\alpha + 1,$$

so

$$\alpha_\pm = \frac{3 \pm \sqrt{3^2-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = \left\{\begin{array}{cc} \dfrac{3+\sqrt{5}}{2}, & + \\[2ex] \dfrac{3-\sqrt{5}}{2}, & - \end{array}\right\}.$$

After that, we choose

$$\beta_\pm = \frac{1}{\alpha_\pm} = \frac{2}{3 \pm \sqrt{5}} = \frac{2(3 \mp \sqrt{5})}{3^2 - 5} = \frac{3 \mp \sqrt{5}}{2}$$

to find Möbius transformations that both map the given circles to concentric circles centered at the origin:

$$M_+(z) = \frac{\widetilde{z} - \alpha_+}{\widetilde{z} - \beta_+} = \frac{\frac{z}{3} - \frac{3+\sqrt{5}}{2}}{\frac{z}{3} - \frac{3-\sqrt{5}}{2}} \cdot \frac{6}{6} = \frac{2z - 9 - 3\sqrt{5}}{2z - 9 + 3\sqrt{5}}$$

or

$$M_-(z) = \frac{\widetilde{z} - \alpha_-}{\widetilde{z} - \beta_-} = \frac{\frac{z}{3} - \frac{3-\sqrt{5}}{2}}{\frac{z}{3} - \frac{3+\sqrt{5}}{2}} \cdot \frac{6}{6} = \frac{2z - 9 + 3\sqrt{5}}{2z - 9 - 3\sqrt{5}}\,.$$

16.2.5.5. First, translate the center of one of the given circles to the origin. For example, probably the easiest choice is to let $z = \widetilde{z} + 2$, that is, $\widetilde{z} = z - 2$, so that in terms of $\widetilde{z}$ the given circles are $|\widetilde{z}| = |z-2| = 1$ and $|\widetilde{z} + 1| = |z-1| = 3$.

With $\delta = -1$ and $\gamma = 3$, Theorem 16.6 in Section 16.2 says to choose α which satisfies the quadratic equation

$$0 = \delta\alpha^2 + (\gamma^2 - \delta^2 - 1)\alpha + \delta = -\alpha^2 + \left(3^2 - (-1)^2 - 1\right)\alpha + (-1)$$

that is

$$0 = \alpha^2 - 7\alpha + 1,$$

so

$$\alpha_\pm = \frac{7 \pm \sqrt{7^2-4}}{2} = \frac{7 \pm \sqrt{45}}{2} = \left\{\begin{array}{cc} \dfrac{7+3\sqrt{5}}{2}, & + \\[2ex] \dfrac{7-3\sqrt{5}}{2}, & - \end{array}\right\}.$$

After that, we choose
$$\beta_\pm = \frac{1}{\alpha_\pm} = \frac{2}{7 \pm \sqrt{45}} = \frac{2(7 \mp \sqrt{45})}{7^2 - 45} = \frac{7 \mp \sqrt{45}}{2}$$
to find Möbius transformations that both map the given circles to concentric circles centered at the origin:
$$M_+(z) = \frac{\widetilde{z} - \alpha_+}{\widetilde{z} - \beta_+} = \frac{(z-2) - \frac{7+3\sqrt{5}}{2}}{(z-2) - \frac{7-3\sqrt{5}}{2}} \cdot \frac{2}{2} = \frac{2z - 11 - 3\sqrt{5}}{2z - 11 + 3\sqrt{5}}$$
or
$$M_-(z) = \frac{\widetilde{z} - \alpha_-}{\widetilde{z} - \beta_-} = \frac{(z-2) - \frac{7-3\sqrt{5}}{2}}{(z-2) - \frac{7+3\sqrt{5}}{2}} \cdot \frac{2}{2} = \frac{2z - 11 + 3\sqrt{5}}{2z - 11 - 3\sqrt{5}}\,.$$

16.2.5.7. First, use a preliminary rotation to get the second given circle to have its center on the real axis: Write $i = e^{i\pi/2}$ and let $\widetilde{z} = ze^{-i\pi/2}$, that is, $\widetilde{z}e^{i\pi/2} = z$, so the second circle can be rewritten as
$$\frac{5}{2} = |z - i| = |\widetilde{z}e^{i\pi/2} - i\,| = |\widetilde{z}e^{i\pi/2} - e^{i\pi/2}| = |(\widetilde{z} - 1)e^{i\pi/2}| = |\widetilde{z} - 1|\;|e^{i\pi/2}| = |\widetilde{z} - 1| \cdot 1 = |\widetilde{z} - 1|.$$
The first given circle can be rewritten as
$$1 = |z| = |\widetilde{z}e^{i\pi/2}| = |\widetilde{z}|\;|e^{i\pi/2}| = |\widetilde{z}| \cdot 1 = |\widetilde{z}|.$$

So, with $\delta = 1$ and $\gamma = \dfrac{5}{2}$, Theorem 16.6 in Section 16.2 says to choose α which satisfies the quadratic equation
$$0 = \delta\alpha^2 + (\gamma^2 - \delta^2 - 1)\alpha + \delta = \alpha^2 + \left(\left(\frac{5}{2}\right)^2 - 1^2 - 1\right)\alpha + 1$$
that is
$$0 = \alpha^2 + \frac{17}{4}\,\alpha + 1,$$
so
$$\alpha_\pm = \frac{-\frac{17}{4} \pm \sqrt{\left(\frac{17}{4}\right)^2 - 4}}{2} = \frac{-\frac{17}{4} \pm \frac{15}{4}}{2} = \left\{ \begin{array}{ll} -\frac{1}{4}, & + \\ -4, & - \end{array} \right\}.$$
After that, we choose $\beta_\pm = \dfrac{1}{\alpha_\pm}$ to find Möbius transformations that both map the given circles to concentric circles centered at the origin:
$$M_+(z) = \frac{\widetilde{z} - \alpha_+}{\widetilde{z} - \beta_+} = \frac{ze^{-i\pi/2} - \left(-\frac{1}{4}\right)}{ze^{-i\pi/2} - (-4)} \cdot \frac{4i}{4i} = \frac{4z + i}{4z + 16i}$$
or
$$M_-(z) = \frac{z - \alpha_-}{z - \beta_-} = \frac{ze^{-i\pi/2} - (-4)}{ze^{-i\pi/2} - \left(-\frac{1}{4}\right)} \cdot \frac{4i}{4i} = \frac{4z + 16i}{4z + i}\,.$$

16.2.5.9. We can rewrite the circle in the $w-$plane as $1 = \left|\frac{1}{2}(w - i)\right|$, so if $z = \frac{1}{2}(w - i) = M^{-1}(w)$ then the Möbius transformation $w = M(z)$ would have $|z| = 1 \mapsto |w - i| = 2$. We have
$$z = \frac{1}{2}(w - i) \quad \Longleftrightarrow \quad 2z = w - i \quad \Longleftrightarrow \quad w = 2z + i = M(z).$$
So, the desired Möbius transformation is $w = M(z) = 2z + i$.

16.2.5.11. For example, choose on the circle $|z| = 1$ the points z_1, z_2, z_3 to be $1, i, -1$; we chose this order because the order z_1, z_2, z_3 corresponds to travel counterclockwise on the circle $|z| = 1$. Choose on the line $\mathcal{I}m(w) = 0$ the points w_1, w_2 to be $-1, 0$. As in Example 16.8 in Section 16.2, we want to have

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = (z, z_1, z_2, z_3) = (w, w_1, w_2, \infty) = \frac{(w - w_1)}{(w_2 - w_1)}\,,$$

that is,

$$\frac{\big(z - (1)\big)\big((i) - (-1)\big)}{\big(z - (-1)\big)\big((i) - (1)\big)} = \frac{\big(w - (-1)\big)}{\big(0 - (-1)\big)}\,,$$

that is

$$(1)(1 + i)(z - 1) = (w + 1)(-1 + i)(z + 1),$$

so

$$w + 1 = \frac{(1 + i)(z - 1)}{(-1 + i)(z + 1)}$$

hence

$$w = -1 + \frac{(1 + i)(z - 1)}{(-1 + i)(z + 1)} = \frac{-(-1 + i)(z + 1) + (1 + i)(z - 1)}{(-1 + i)(z + 1)} = \frac{2z - i2}{(-1 + i)(z + 1)} \triangleq M(z)\,.$$

So, $M(z) \triangleq \dfrac{2z - i2}{(-1 + i)(z + 1)}$ is a Möbius transformation that does what we want it to do in this problem.

16.2.5.13. For example, choose on the line $\mathcal{R}e(z) = 1$ the points z_1, z_2, to be $1, 1 + i$. Choose on the circle $|w| = 2$ the points w_1, w_2, w_3 to be $2, i2, -2$; we chose this order because the order w_1, w_2, w_3 corresponds to travel counterclockwise on the circle $|w| = 2$. As in Example 16.8 in Section 16.2, we want to have

$$\frac{(z - z_1)}{(z_2 - z_1)} = (z, z_1, z_2, \infty) = (w, w_1, w_2, w_3) = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}\,,$$

that is,

$$\frac{\big(z - (1)\big)}{\big((1 + i) - (1)\big)} = \frac{\big(w - (2)\big)\big((i2) - (-2)\big)}{\big(w - (-2)\big)\big((i2) - (2)\big)}\,,$$

that is

$$(z - 1)\cancel{2}(-1 + i)(w + 2) = (i)\cancel{2}(1 + i)(w - 2).$$

So

$$\Big((-1 + i)(z - 1) - i(1 + i)\Big)w = -2i(1 + i) - 2(-1 + i)(z - 1),$$

that is,

$$\big((-1 + i)z + 2(1 - i)\big)w = 2(1 - i)z$$

hence

$$w = \frac{2(1 - i)z}{(-1 + i)z + 2(1 - i)} = \frac{-2z}{z - 2} \triangleq M(z)\,.$$

So, $M(z) \triangleq \dfrac{-2z}{z - 2}$ is a Möbius transformation that does what we want it to do in this problem.

16.2.5.15. First, we will find a Möbius transformation that maps the circle $|z| = 2$ to the line $\mathcal{R}e(w) = 0$. For example, choose on the circle $|z| = 2$ the points z_1, z_2, z_3 to be $2, i2, -2$; we chose this order because the order z_1, z_2, z_3 corresponds to travel counterclockwise on the circle $|z| = 2$. Choose on the line $\mathcal{R}e(w) = 0$ the points w_1, w_2 to be $0, i$. [These choices of orientations of the circle and the line may, or may not, produce the desired map from the circle's inside, $|z| < 2$, to the line's left, $\mathcal{R}e(w) < 0$. If not, we will go back and change one of the orientations.]

As in Example 16.8 in Section 16.2, we want to have

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = (z,z_1,z_2,z_3) = (w,w_1,w_2,\infty) = \frac{(w-w_1)}{(w_2-w_1)}\,,$$

that is,

$$\frac{\big(z-(2)\big)\big((i2)-(-2)\big)}{\big(z-(-2)\big)\big((i2)-(2)\big)} = \frac{(w-0)}{(i-0)}\,,$$

that is

$$(i)\cancel{2}(1+i)(z-2) = w\cancel{2}(-1+i)(z+2),$$

so

$$w = \frac{i(1+i)(z-2)}{(-1+i)(z+2)} = \frac{(-1+i)(z-2)}{(-1+i)(z+2)} = \frac{z-2}{z+2} \triangleq M(z)\,.$$

To check the orientation, continuity of the map $M(z)$ implies that it suffices to check the map at a single point inside the circle: We have $M(0)=-1$, so, yes, $M(z)$ maps $|z|<2$ to $\mathcal{R}e(w)<0$.

16.2.5.17. First, we will find a Möbius transformation that maps the circle $|z|=2$ to the line $\mathcal{R}e(w)=1$. For example, choose on the circle $|z|=2$ the points z_1,z_2,z_3 to be $-2, i2, 2$; we chose this order because the order z_1,z_2,z_3 corresponds to travel clockwise on the circle $|z|=2$. Choose on the line $\mathcal{R}e(w)=1$ the points w_1,w_2 to be $1, 1+i$. [These choices of orientations of the circle and the line may, or may not, produce the desired map from the circle's outside, $|z|>2$, to the line's left, $\mathcal{R}e(w)<1$. If not, we will go back and change one of the orientations.]

As in Example 16.8 in Section 16.2, we want to have

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = (z,z_1,z_2,z_3) = (w,w_1,w_2,\infty) = \frac{(w-w_1)}{(w_2-w_1)}\,,$$

that is,

$$\frac{\big(z-(-2)\big)\big((i2)-(2)\big)}{\big(z-(2)\big)\big((i2)-(-2)\big)} = \frac{(w-1)}{\big((1+i)-(1)\big)}\,,$$

that is

$$(i)\cancel{2}(-1+i)(z+2) = (w-1)\cancel{2}(1+i)(z-2),$$

so

$$w = 1+\frac{i(-1+i)(z+2)}{(1+i)(z-2)} = 1+\frac{-(1+i)(z+2)}{(1+i)(z-2)} = 1+\frac{-(z+2)}{z-2} = \frac{-4}{z-2} \triangleq M(z)\,.$$

To check the orientation, continuity of the map $M(z)$ implies that it suffices to check the map at a single point inside the circle: We have $M(3)=-4$, so, yes, $M(z)$ maps $|z|>2$ to $\mathcal{R}e(w)<1$.

16.2.5.19. Assume $0\neq\alpha$ is real. For $0<\theta\le 2\pi$, $\big|M(e^{i\theta})\big|^2 = \left|\dfrac{e^{i\theta}-\alpha}{e^{i\theta}-\frac{1}{\alpha}}\right|^2 = \left|\dfrac{(\cos\theta-\alpha)+i\sin\theta}{(\cos\theta-\frac{1}{\alpha})+i\sin\theta}\right|^2$

$$= \frac{(\cos\theta-\alpha)^2+\sin^2\theta}{(\cos\theta-\frac{1}{\alpha})^2+\sin^2\theta} = \frac{1+\alpha^2-2\alpha\cos\theta}{1+\alpha^{-2}-2\alpha^{-1}\cos\theta} = \frac{\alpha^2\left(1+\alpha^2-2\alpha\cos\theta\right)}{\alpha^2\left(1+\alpha^{-2}-2\alpha^{-1}\cos\theta\right)} = \frac{\alpha^2\left(1+\alpha^2-2\alpha\cos\theta\right)}{\alpha^2+1-2\alpha\cos\theta} = \alpha^2.$$

So, yes, $M(\mathcal{C}_1) = \{w : |w|=|\alpha|\}$.

16.2.5.21. (a) $M(\delta+i\gamma)-M(\delta-i\gamma) =$

$$= \frac{\delta+i\gamma-\alpha}{\delta+i\gamma-\beta} - \frac{\delta-i\gamma-\alpha}{\delta-i\gamma-\beta} = \frac{(\delta+i\gamma-\alpha)(\delta-i\gamma-\beta)-(\delta-i\gamma-\alpha)(\delta+i\gamma-\beta)}{(\delta-\beta)^2+\gamma^2},$$

whose numerator is

$$(\delta+i\gamma)(\delta-i\gamma)-\alpha(\delta-i\gamma)-\beta(\delta+i\gamma)+\alpha\beta-(\delta+i\gamma)(\delta-i\gamma)+\alpha(\delta+i\gamma)+\beta(\delta-i\gamma)-\alpha\beta = i2\gamma(\alpha-\beta).$$

So, $M(\delta + i\gamma) = M(\delta - i\gamma) \iff \alpha = \beta$.

(b) $\alpha\beta = 1$, so $\alpha = \beta \iff \alpha^2 = 1 \iff \alpha = \pm 1$.

(c) $M(\delta + i\gamma) = M(\delta - i\gamma) \iff \alpha = \pm 1$ follows directly from parts (a) and (b).

16.2.5.23. [Note the change to the problem, specifically to the definition of $w_0$in the Errata Webpage.]
First, note that z is in $\mathcal{C}$ if, and only if, $z = z_0 + Re^{i\theta}$ for some real θ, so

$$f(\mathcal{C}) = \left\{ w = \frac{1}{z_0 + Re^{i\theta}} : 0 \le \theta \le 2\pi \right\}.$$

Assume $|z_0| \ne R$, and define $w_0 = \dfrac{\overline{z_0}}{|z_0|^2 - R^2}$.

So, w is in $f(\mathcal{C})$ if, and only if,

$$|w - w_0| = \left| \frac{1}{z_0 + Re^{i\theta}} - \frac{\overline{z_0}}{|z_0|^2 - R^2} \right| = \frac{\left| |z_0|^2 - R^2 - \overline{z_0}\left(z_0 + Re^{i\theta}\right) \right|}{\left|(z_0 + Re^{i\theta})\left(|z_0|^2 - R^2\right)\right|} = \frac{\left| \cancel{|z_0|^2} - R^2 - \cancel{|z_0|^2} - \overline{z_0}Re^{i\theta} \right|}{\left|z_0 + Re^{i\theta}\right| \, \left||z_0|^2 - R^2\right|}$$

$$= \frac{|-R| \, \left|R + \overline{z_0}e^{i\theta}\right|}{\left|z_0 + Re^{i\theta}\right| \, \left||z_0|^2 - R^2\right|} = \frac{R \, \left|\overline{R + \overline{z_0}e^{i\theta}}\right|}{\left|z_0 + Re^{i\theta}\right| \, \left||z_0|^2 - R^2\right|} = \frac{R \cdot 1 \cdot \left|R + z_0 e^{-i\theta}\right|}{\left|z_0 + Re^{i\theta}\right| \, \left||z_0|^2 - R^2\right|}$$

$$= \frac{R \cdot |e^{i\theta}| \cdot \left|R + z_0 e^{-i\theta}\right|}{\left|z_0 + Re^{i\theta}\right| \, \left||z_0|^2 - R^2\right|} = \frac{R \cdot \left|e^{i\theta}\left(R + z_0 e^{-i\theta}\right)\right|}{\left|z_0 + Re^{i\theta}\right| \, \left||z_0|^2 - R^2\right|} = \frac{R \cdot \cancel{\left|Re^{i\theta} + z_0\right|}}{\cancel{\left|Re^{i\theta} + z_0\right|} \, \left||z_0|^2 - R^2\right|}$$

$$= \frac{R}{\left||z_0|^2 - R^2\right|},$$

as we desired.

16.2.5.25. Define $g(\varphi) \triangleq \dfrac{\sin\varphi}{1+\cos\varphi}$. Using L'Hôpital's Rule in the "$\frac{0}{0}$ " case, we calculate that

$$\lim_{\varphi\to\pi^-} g(\varphi) = \lim_{\varphi\to\pi^-} \frac{(\sin\varphi)'}{(1+\cos\varphi)'} = \lim_{\varphi\to\pi^-} \frac{\cos\varphi}{-\sin\varphi} = \lim_{\varphi\to\pi^-} \frac{\approx(-1)}{\approx 0-} = +\infty$$

and $\lim_{\varphi\to 0^+} g(\varphi) = 0$, so

$$(1) \quad \text{the image } g\big((0,\pi)\big) \text{ is the interval } (0,\infty).$$

Similarly,

$$\lim_{\varphi\to\pi^+} g(\varphi) = \lim_{\varphi\to\pi^+} \frac{(\sin\varphi)'}{(1+\cos\varphi)'} = \lim_{\varphi\to\pi^+} \frac{\cos\varphi}{-\sin\varphi} = \lim_{\varphi\to\pi^+} \frac{\approx(-1)}{\approx 0+} = -\infty\,.$$

and $g(2\pi) = 0$, so

$$(2) \quad \text{the image } g\big((\pi,2\pi]\big) \text{ is the interval } (-\infty,0].$$

Together, (1) and (2) imply that, as we desired to explain,

$$\left\{\frac{\sin\varphi}{1+\cos\varphi} : 0 < \varphi \le 2\pi \text{ with } \varphi \neq \pi\right\} = (-\infty,\infty).$$

Section 16.3.5

16.3.5.1. From the Students' Solutions Manual for problem 16.2.5.1, with $\delta = -1$ and $\gamma = \dfrac{5}{2}$ we get that $M_+(z) = \dfrac{4(z-4)}{4z-1}$ and $M_-(z) = \dfrac{4z-1}{4(z-4)}$ map the circles $|z| = 1$ and $|z+1| = \frac{5}{2}$ to circles in the $\zeta-$plane. Does either of these map $\mathcal{D} \triangleq \{z : |z| > 1 \text{ and } |z+1| < \frac{5}{2}\}$, the region between the two circles, to $\mathcal{E}$, an annulus in the $\zeta-$plane?

By continuity, it suffices to test a map on a single point in $\mathcal{D}$. For example, $z = -2$ is in $\mathcal{D}$.

According to the derivation in the solution of problem 16.2.5.1, $\alpha = 4$ produces the map $M_+(z) = \dfrac{4(z-4)}{4z-1}$ which should map $|z| = 1$ to the circle $|\zeta| = |\alpha| = 4$ and should map the circle $|z+1| = \frac{5}{2}$ to the circle $|\zeta| = \dfrac{|\delta - \alpha|}{\gamma} = \dfrac{|-1-4|}{5/2} = 2$. So, $\mathcal{E} \triangleq \{\zeta : 2 < |\zeta| < 4\}$. We calculate that $M_+(-2) = \dfrac{4(-2-4)}{4\cdot(-2)-1} = \dfrac{24}{9}$, hence $M_+(-2)$ is in $\mathcal{E}$.

Define $M(z) = \dfrac{4(z-4)}{4z-1}$. Then $\zeta = M(z)$ maps the BC $\phi(z) = 0$ on $|z| = 1$ to the BC $\Phi(\zeta) = 0$ on $|\zeta| = 4$ and maps the BC $\phi(z) = 1$ on $|z+1| = \frac{5}{2}$ to the BC $\Phi(\zeta) = 1$ on $|\zeta| = 2$.

The next step is to solve the PDE-BVP consisting of the Laplace equation in the annulus $\mathcal{E} \triangleq \{\zeta : 2 < |\zeta| < 4\}$ and BCs $\Phi(\zeta) = 0$ on $|\zeta| = 4$ and $\Phi(\zeta) = 1$ on $|\zeta| = 2$. Because the BCs do not involve angular dependence, the problem reduces to solving

$$\frac{1}{R}\frac{d}{dR}\left[R\,\frac{d\Phi}{dR}\right] = 0.$$

This gives

$$\Phi = c_1 + c_2 \ln R,$$

where c_1, c_2 are arbitrary constants. The boundary conditions are

$$0 = \Phi(4) = c_1 + c_2 \ln 4 \quad \text{and} \quad 1 = \Phi(2) = c_1 + c_2 \ln 2,$$

and it's easy to see that the solution is $c_1 = 2, c_2 = -\dfrac{1}{\ln 2}$. The solution in the $\zeta-$plane is $\Phi = 2 - \dfrac{1}{\ln 2}\ \ln R$.

Noting that $R = \sqrt{\xi^2 + \eta^2}$, we have $\ln R = \frac{1}{2}\ \ln(\xi^2 + \eta^2)$, so

$$\Phi = \Phi(\xi, \eta) = 2 - \frac{1}{2\ln 2}\ \ln(\xi^2 + \eta^2)\,.$$

To transform back to the original problem in the $x + iy = z$ plane, we note that

$$\xi + i\eta = \zeta = M(z) = \frac{4(z-4)}{4z-1} = 4\cdot\frac{(x+iy)-4}{4(x+iy)-1} = 4\cdot\frac{(x-4)+iy}{(4x-1)+i4y}$$

$$= 4\cdot\frac{\big((x-4)+iy\big)\big((4x-1)-i4y\big)}{(4x-1)^2+16y^2} = 4\cdot\frac{\big(4x^2+4y^2-17x+4\big)+i15y}{(4x-1)^2+16y^2}\,.$$

So,

$$\xi = \frac{4(4x^2+4y^2-17x+4)}{(4x-1)^2+i4y^2}\,,\quad \eta = \frac{60y}{(4x-1)^2+16y^2}\,.$$

The solution of the original problem is

$$\phi = \phi(x,y) = 2 - \frac{1}{2\ln 2}\ \ln\left(\frac{16(4x^2+4y^2-17x+4)^2+(60y)^2}{\big((4x-1)^2+16y^2\big)^2}\right)\,.$$

16.3.5.3. From the Students' Solutions Manual for problem 16.2.5.3, the original circles are rewritten as with $1 = \left|\frac{z}{3}\right| = |\widetilde{z}|$, where $\widetilde{z} = \frac{z}{3}$. After that, rewrite $|z-1| = 1$ as $|3\widetilde{z} - 1| = 1$, that is, as $\left|\widetilde{z} - \frac{1}{3}\right| = \frac{1}{3}$.

So, with $\delta = \frac{1}{3}$ and $\gamma = \frac{1}{3}$ we get that $M_+(z) = \frac{2z-9-3\sqrt{5}}{2z-9+3\sqrt{5}}$ and $M_-(z) = \frac{2z-9+3\sqrt{5}}{2z-9-3\sqrt{5}}$ map the circles $|z| = 3$ and $|z-1| = 1$ to circles in the $\zeta-$plane. Does either of these map $\mathcal{D} \triangleq \{z : |z| < 3 \text{ and } |z-1| > 1\}$, the region between the two circles, to $\mathcal{E}$, an annulus in the $\zeta-$plane?

By continuity, it suffices to test a map on a single point in $\mathcal{D}$. For example, $z = -1$ is in $\mathcal{D}$.

According to the derivation in the solution of problem 16.2.5.3, $\alpha = \frac{3+\sqrt{5}}{2}$ produces the map $M_+(z) = \frac{\widetilde{z} - \frac{3+\sqrt{5}}{2}}{\widetilde{z} - \frac{3-\sqrt{5}}{2}}$, which should map $|\widetilde{z}| = 1$ to the circle $|\zeta| = |\alpha| = \frac{3+\sqrt{5}}{2}$ and should map the circle $\left|\widetilde{z} - \frac{1}{3}\right| = \frac{1}{3}$ to the circle $|\zeta| = \frac{|\delta - \alpha|}{\gamma} = \frac{\left|\frac{1}{3} - \frac{3+\sqrt{5}}{2}\right|}{\frac{1}{3}} \cdot \frac{6}{6} = \frac{7+3\sqrt{5}}{2}$. So, $\mathcal{E} \triangleq \left\{\zeta : \frac{3+\sqrt{5}}{2} < |\zeta| < \frac{7+3\sqrt{5}}{2}\right\}$. We calculate that $M_+(-1) = \frac{2(-1)-9-3\sqrt{5}}{2(-1)-9+3\sqrt{5}} = \frac{-11-3\sqrt{5}}{-11+3\sqrt{5}}$, hence $M_+(-1)$ is in $\mathcal{E}$.

Define $M(z) = \frac{2z-9-3\sqrt{5}}{2z-9+3\sqrt{5}}$. Then $\zeta = M(z)$ maps the BC $\phi(z) = 1$ on $|z| = 3$ to the BC $\Phi(\zeta) = 1$ on $|\zeta| = \frac{3+\sqrt{5}}{2}$, and $\zeta = M(z)$ maps the BC $\phi(z) = -1$ on $|z-1| = 1$ to the BC $\Phi(\zeta) = -1$ on $|\zeta| = \frac{7+3\sqrt{5}}{2}$.

The next step is to solve the PDE-BVP consisting of the Laplace equation in the annulus

$$\mathcal{E} \triangleq \left\{\zeta : \frac{3+\sqrt{5}}{2} < |\zeta| < \frac{7+3\sqrt{5}}{2}\right\}$$

and BCs $\Phi(\zeta) = 1$ on $|\zeta| = \frac{3+\sqrt{5}}{2}$ and $\Phi(\zeta) = -1$ on $|\zeta| = \frac{7+3\sqrt{5}}{2}$. Because the BCs do not involve angular dependence, the problem reduces to solving

$$\frac{1}{R}\frac{d}{dR}\left[R\frac{d\Phi}{dR}\right] = 0.$$

This gives

$$\Phi = c_1 + c_2 \ln R,$$

where c_1, c_2 are arbitrary constants. The boundary conditions are

$$1 = \Phi\Big(\frac{3+\sqrt{5}}{2}\Big) = c_1 + c_2 \ln\Big(\frac{3+\sqrt{5}}{2}\Big) \quad \text{and} \quad -1 = \Phi\Big(\frac{7+3\sqrt{5}}{2}\Big) = c_1 + c_2 \ln\Big(\frac{7+3\sqrt{5}}{2}\Big),$$

and it's easy to see that the solution is

$$c_2 = \frac{2}{\ln\left(\frac{3+\sqrt{5}}{2}\right) - \ln\left(\frac{7+3\sqrt{5}}{2}\right)} = \frac{2}{\ln\left(\frac{3+\sqrt{5}}{7+3\sqrt{5}}\right)} = \frac{2}{\ln\left(\frac{(3+\sqrt{5})(7-3\sqrt{5})}{7^2-(3\sqrt{5})^2}\right)} = \frac{2}{\ln\left(\frac{3-\sqrt{5}}{2}\right)}$$

and

$$c_1 = 1 - c_2 \ln\Big(\frac{3+\sqrt{5}}{2}\Big) = 1 - \frac{2}{\ln\left(\frac{3-\sqrt{5}}{2}\right)} \cdot \ln\Big(\frac{3+\sqrt{5}}{2}\Big)$$

$$= \left(\ln\Big(\frac{3-\sqrt{5}}{2}\Big)\right)^{-1} \left(\ln\Big(\frac{3-\sqrt{5}}{2}\Big) - \ln\Big(\Big(\frac{3+\sqrt{5}}{2}\Big)^2\Big)\right)$$

$$= \left(\ln\Big(\frac{3-\sqrt{5}}{2}\Big)\right)^{-1} \left(\ln\Big(\frac{3-\sqrt{5}}{2}\Big) - \ln\Big(\frac{7+3\sqrt{5}}{2}\Big)\right) = \left(\ln\Big(\frac{3-\sqrt{5}}{2}\Big)\right)^{-1} \cdot \ln\Big(\frac{3-\sqrt{5}}{7+3\sqrt{5}}\Big)$$

$$= \left(\ln\left(\frac{3-\sqrt{5}}{2}\right)\right)^{-1} \cdot \ln\left(\frac{(3-\sqrt{5})(7-3\sqrt{5})}{7^2-(3\sqrt{5})^2}\right) = \left(\ln\left(\frac{3-\sqrt{5}}{2}\right)\right)^{-1} \cdot \ln(9-4\sqrt{5})\,.$$

The solution in the $\zeta-$plane is

$$\Phi = \left(\ln\left(\frac{3-\sqrt{5}}{2}\right)\right)^{-1} \cdot \left(\ln(9-4\sqrt{5}) - 2\ln R\right).$$

Noting that $R = \sqrt{\xi^2+\eta^2}$, we have $\ln R = \frac{1}{2}\ln(\xi^2+\eta^2)$, so

$$\Phi = \Phi(\xi,\eta) = \left(\ln\left(\frac{3-\sqrt{5}}{2}\right)\right)^{-1} \cdot \left(\ln(9-4\sqrt{5}) - \ln(\xi^2+\eta^2)\right)\,.$$

To transform back to the original problem in the $x+iy=z$ plane, we note that

$$\xi + i\eta = \zeta = M(z) = \frac{2z-9-3\sqrt{5}}{2z-9+3\sqrt{5}} = \frac{2(x+iy)-9-3\sqrt{5}}{2(x+iy)-9+3\sqrt{5}}$$

$$= \frac{\big((2x-9-3\sqrt{5})+i2y\big)\big((2x-9+3\sqrt{5})-i2y\big)}{(2x-9+3\sqrt{5})^2+4y^2} = \frac{\big((2x-9)^2-(3\sqrt{5})^2+4y^2\big)+i12\sqrt{5}\,y}{(2x-9+3\sqrt{5})^2+4y^2}\,.$$

So,

$$\xi = \frac{4\big(x^2+y^2-9x+9\big)}{(2x-9+3\sqrt{5})^2+4y^2}\,,\quad \eta = \frac{12\sqrt{5}\,y}{(2x-9+3\sqrt{5})^2+4y^2}\,.$$

The solution of the original problem is

$$\phi = \phi(x,y) = \left(\ln\left(\frac{3-\sqrt{5}}{2}\right)\right)^{-1} \cdot \left(\ln(9-4\sqrt{5}) - \ln\left(\frac{16\big(x^2+y^2-9x+9\big)^2+720y^2}{\big((2x-9+3\sqrt{5})^2+4y^2\big)^2}\right)\right)\,.$$

16.3.5.5. *Method 1*: Because the BCs $\phi = 1$ on $|z|=1$ and $\frac{\partial\phi}{\partial n} = 0$ on $|z-i| = \frac{5}{2}$, the obvious solution to the whole, original problem is $\phi(x,y) \equiv 1$.

Method 2: From the Students' Solutions Manual for problem 16.2.5.7, first, translate the center of one of the given circles to the origin. For example, let $z = \widetilde{z} + i$, that is, $\widetilde{z} = z - i$, so ...

16.3.5.7. This problem is an example of (16.22) in Section 16.3, with, specifically, $N=3$, $\phi_1 = 1$, $\alpha_1 = -\dfrac{\pi}{3}$, $\phi_2 = -1$, $\alpha_1 = \dfrac{\pi}{3}$, and $\phi_3 = 1$. In the $\xi + i\eta = \zeta-$plane, the problem becomes that of (16.25), with

$$\eta_1 = \tan\frac{\alpha_1}{2} = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}},$$

and

$$\eta_2 = \tan\frac{\alpha_2}{2} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}.$$

The solution is given in (16.28) using the information specific to this problem:

$$\phi(x,y) = 1 + \frac{2}{\pi}\cos^{-1}\left(\frac{R^2-(x^2+y^2)}{\sqrt{\left(2Ry + \frac{1}{\sqrt{3}}\left((x+R)^2+y^2\right)\right)^2 + \left(R^2-(x^2+y^2)\right)^2}}\right)$$

$$-\frac{2}{\pi}\cos^{-1}\left(\frac{R^2-(x^2+y^2)}{\sqrt{\left(2Ry-\frac{1}{\sqrt{3}}\left((x+R)^2+y^2\right)\right)^2+\left(R^2-(x^2+y^2)\right)^2}}\right).$$

16.3.5.9. (a) $g'(\rho)=\left(\rho+a^2\rho^{-1}\right)'=1-a^2\rho^{-2}=\rho^{-2}\left(\rho^2-a^2\right)$, so $g(\rho)$ is decreasing for $0<\rho<a$ and increasing for $a<\rho$.

(b) $f(\rho)\triangleq\left|\rho-a^2\rho^{-1}\right|=\left|\rho^{-1}(\rho^2-a^2)\right|=|\rho|^{-1}(\rho^2-a^2)\cdot sgn(\rho^2-a^2)$, so $f(\rho)=-\rho+a^2\rho^{-1}$ for $0<\rho<a$ and $f(\rho)=\rho-a^2\rho^{-1}$ for $a<\rho$.

For $0<\rho<a$, $f'(\rho)=\left(-\rho+a^2\rho^{-1}\right)'=-1-a^2\rho^{-2}<0$, so $f(\rho)$ is decreasing for $0<\rho<a$.

For $a<\rho<\infty$, $f'(\rho)=\left(\rho-a^2\rho^{-1}\right)'=1+a^2\rho^{-2}>0$, so $f(\rho)$ is increasing for $\rho>a$.

Recall that we defined a circle by $\mathcal{C}_\rho : |\zeta|=\rho,\ -\pi<\varphi\le\pi$ and an ellipse, the image of $\mathcal{C}_\rho$ under the Zhukovskii map, $x+iy=z=J(\zeta)\triangleq\frac{1}{2}\left(\zeta+\frac{a^2}{\zeta}\right)$. Note that by convention, a is a positive real number.

Further, we calculated in (16.31) that

$$\mathcal{E}_\rho:\ 1=\frac{x^2}{\frac{1}{4}\left(\rho+\frac{a^2}{\rho}\right)^2}+\frac{y^2}{\frac{1}{4}\left(\rho-\frac{a^2}{\rho}\right)^2},$$

that is,

$$\mathcal{E}_\rho:\ 1=\frac{x^2}{\frac{1}{4}\left(g(\rho)\right)^2}+\frac{y^2}{\frac{1}{4}\left(f(\rho)\right)^2},$$

On the $x-$axis, the major axis of the ellipse has length $2g(\rho)$, and on the $y-$axis, the major axis of the ellipse has length $2f(\rho)$.

(c) Fix any ρ satisfying $\rho>a$. If ζ_0 is in the exterior of $\mathcal{C}_\rho$, that is, if $\rho_0\triangleq|\zeta_0|>\rho$, then $g(\rho_0)>g(\rho)$ and $f(\rho_0)>f(\rho)$, because both $g(\rho)$ and $f(\rho)$ are increasing for $\rho>a$. It follows that $J(\zeta_0)$, being on the ellipse $\mathcal{E}_{\rho_0}$, is in the exterior of $\mathcal{E}_\rho$.

(d) The interior of $\mathcal{C}_a$ is the open disk $\mathcal{D}_a$. The punctured disk $\mathcal{A}_{0,a}(0)=\{\zeta : 0<|\zeta|<a\}$ omits the origin, which is not in the domain of the map $J(\zeta)$.

We want to explain why the image of $\mathcal{A}_{0,a}(0)$ under the Zhukovskii map, $J(\zeta)$, is the whole $z-$plane, except for the finite interval $\{x+i0 : -a\le x\le a\}$ on the real axis. [Note change found on the Errata Webpage.]

Choose any $z=x+iy$, *where* $y\neq 0$. We will solve for $\zeta\neq 0$ the equation

$$z=J(\zeta)\triangleq\frac{1}{2}\left(\zeta+\frac{a^2}{\zeta}\right).$$

that is,

$$2\zeta z=\zeta^2+a^2$$

that is,

$$\zeta^2-2z\zeta+a^2=0.$$

The solutions are

$$\zeta=\frac{2z\pm\sqrt{4z^2-4a^2}}{2},$$

that is,

$$(\star)\quad \zeta=\zeta_\pm\triangleq z\pm\sqrt{z^2-a^2}.$$

As long as z is not in the finite interval $\{x+i0:\,-a\le x\le a\}$ on the real axis, the function $h(z)\triangleq\sqrt{z^2-a^2}$ is well defined and analytic.

We want to explain why

$$J\big(\mathcal{A}_{0,a}(0)\big)=\{z\text{ in }\mathbb{C}:\,z\ne x+i0\text{ for any real }x\text{ with }-a\le x\le a\}.$$

First, we note that for every z *not* in $\{x+i0:\,-a\le x\le a\}$, the formula for $\zeta_\pm$ given in $(\star)$ produces at least one solution ζ for the equation $J(\zeta)=z$ *and*, most importantly, $\zeta_\pm$ is (are) not in the circle $\mathcal{C}_\rho:|\zeta|=\rho$.

Why is the latter true? Because, for example, if $\zeta_+=ae^{i\theta}$ for some real θ, then $(\star)$ implies that

$$\pm\sqrt{z^2-a^2}=(\zeta_+-z)=(ae^{i\theta}-z),$$

hence

$$\cancel{z^2}-a^2=(ae^{i\theta}-z)=a^2e^{i2\theta}-2ae^{i\theta}z+\cancel{z^2},$$

hence

$$z=\frac{a^2(e^{i2\theta}+1)}{2ae^{i\theta}}=a\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}=a\cos(\theta),$$

hence z is real and z is in $\{x+i0:\,-a\le x\le a\}$, contradicting what we assumed.

In addition, $\zeta_\pm\ne 0$, because if $0=z\pm\sqrt{z^2-a^2}$, then $z=\mp\sqrt{z^2-a^2}$, hence $z^2=(\mp\sqrt{z^2-a^2})^2=z^2-a^2$, hence $0=-a^2$, which would give a contradiction.

So, $\zeta_\pm$ given in $(\star)$ produces at least one solution $\zeta\ne 0$ for the equation $J(\zeta)=z$ *and* $\zeta_\pm$ is (are) not in the circle $\mathcal{C}_\rho:|\zeta|=\rho$.

Now, $J(\zeta)$ has an interesting property: $J(\zeta)=J\Big(\dfrac{a}{\zeta}\Big)$, for all $\zeta\ne 0$. Note that if ζ satisfies $|\zeta|<a$, then $\widetilde{\zeta}\triangleq\dfrac{a}{\zeta}$ satisfies $|\widetilde{\zeta}|>a$.

It follows that the equation $z=J(\zeta)$ has at least one solution satisfying $|\zeta|<a$. Whether that solution is ζ_+ or ζ_-, we don't care, because we have managed to explain why the image of $\mathcal{A}_{0,a}(0)$ under the Zhukovskii map, $J(\zeta)$, is the whole $z-$plane, except for the finite interval $\{x+i0:\,-a\le x\le a\}$ on the real axis.

(e) Fix any ρ satisfying $0<\rho<a$. If ζ_0 is in the interior of $\mathcal{C}_\rho$, that is, if $0\le\rho_0\triangleq|\zeta_0|<\rho$, then $g(\rho_0)>g(\rho)$ and $f(\rho_0)>f(\rho)$, because both $g(\rho)$ and $f(\rho)$ are decreasing for $0<\rho<a$. It follows that $J(\zeta_0)$, being on the ellipse $\mathcal{E}_{\rho_0}$, is in the exterior of $\mathcal{E}_\rho$.

(f) Fix any ρ satisfying $0<\rho<a$. We know from part (d) that the image of $\mathcal{A}_{0,a}(0)$ under the Zhukovskii map, $J(\zeta)$, is the whole $z-$plane, except for the finite interval $\{x+i0:\,-a\le x\le a\}$ on the real axis. In other words, if z is not in the finite interval $\{x+i0:\,-a\le x\le a\}$ then there is at least one ζ with $|\zeta|<a$ and $J(\zeta)=z$. It follows that if z is not in the finite interval $\{x+i0:\,-a\le x\le a\}$ then there is at least one $\widetilde{\zeta}=\dfrac{a}{\zeta}$ with $|\widetilde{\zeta}|>a$ and $J(\widetilde{\zeta})=z$. Note that such $\widetilde{\zeta}$ is in the exterior of $\mathcal{C}_\rho$, because $|\widetilde{\zeta}|>a>\rho$.

Further, if z is in the finite interval $\{x+i0:\,-a\le x\le a\}$ on the real axis, then we can find a $\zeta=ae^{i\varphi}$ with $J(\zeta)=x+iy$, where

$$x=\frac{1}{2}\left(\rho+\frac{a^2}{\rho}\right)\cos\varphi\Big|_{\rho=a}=\frac{1}{2}\,g(a)\cos\varphi=a\,\cos\varphi\quad\text{and}\quad y=\frac{1}{2}\left(\rho-\frac{a^2}{\rho}\right)\sin\varphi\Big|_{\rho=a}=0.$$

So, $\{x+i0:\,-a\le x\le a\}$ is in the image of $J(\mathcal{C}_a)$. So, every z in the complex plane is in the image under $J(\zeta)$ of the exterior of $\mathcal{C}_\rho$.

16.3.5.11. We are asked to start from

$$(16.32)\qquad 0=|\zeta|^2-(2|\zeta_0|\cos\nu)|\zeta|-(a^2-2a|\zeta_0|\cos\delta)$$

and define $\rho\triangleq\dfrac{|\zeta|}{a}>0$ and $\varepsilon\triangleq\dfrac{|\zeta_0|}{a}>0$. This gives us

$$0=(a\rho)^2-(2|a\varepsilon|\cos\nu)a\rho-(a^2-2a|a\varepsilon|\cos\delta).$$

Divide all terms by a^2 to get

$$0 = \rho^2 - (2\varepsilon \cos \nu)\rho - (1 - 2\varepsilon \cos \delta).$$

Solve this using the quadratic formula to get

$$\rho = \frac{2\varepsilon \cos \nu \pm \sqrt{(2\varepsilon \cos \nu)^2 + 4(1 - 2\varepsilon \cos \delta)}}{2} = \varepsilon \cos \nu \pm \sqrt{\varepsilon^2 \cos^2 \nu + 1 - 2\varepsilon \cos \delta}.$$

For small ε, Taylor's series gives the approximation $\sqrt{1 + 2\varepsilon\alpha + \varepsilon^2\beta} \approx 1 + \alpha$, so

$$\rho = \varepsilon \cos \nu \pm \sqrt{\varepsilon^2 \cos^2 \nu + 1 - 2\varepsilon \cos \delta} \varepsilon \cos \nu \pm (\approx\ 1 - \varepsilon \cos \delta) \approx \varepsilon \cos \nu + (1 - \varepsilon \cos \delta),$$

where we chose the + sign because $\rho = |\zeta|/a > 0$.

Note that if $\nu = \delta$, then for small ε,

$$\rho = \varepsilon \cos \nu \pm \sqrt{\varepsilon^2 \cos^2 \nu + 1 - 2\varepsilon \cos \delta} = \varepsilon \cos \nu \pm \sqrt{(1 - \varepsilon \cos \nu)^2}$$

$$= \varepsilon \cos \nu \pm (1 - \varepsilon \cos \nu) = \varepsilon \cos \nu + (1 - \varepsilon \cos \nu) = 1,$$

again, because $\rho = |\zeta|/a > 0$.

We arrive at

$$(16.33) \qquad \rho \approx 1 + \varepsilon \cdot \left\{ \begin{array}{ll} \cos \nu - \cos \delta, & \nu \neq \delta \\ 0, & \nu = \delta \end{array} \right\}.$$

16.3.5.13. The center is at $\zeta_0 = -0.07 + i0.07 = |\zeta_0| \cos \delta + i|\zeta_0| \sin \delta = |\zeta_0| e^{i\delta}$ and $a = 1$. So, the camber ratio is

$$H \approx \frac{|\zeta_0| \sin \delta}{a} = \frac{0.07}{1} = 0.07$$

and the thickness ratio is

$$T \approx \frac{3\sqrt{3}\, |\zeta_0|\ |\cos \delta|}{4a} = \frac{3\sqrt{3}\, |0.07|}{4 \cdot 1} \approx 0.0909327.$$

The airfoil is shown in the figure, which was drawn using (6.30) in Section 6.3, that is, $x(\xi, \eta) \triangleq \xi \cdot \dfrac{1 + \xi^2 + \eta^2}{2(\xi^2 + \eta^2)}$, $y(\xi, \eta) = \eta \cdot \dfrac{-1 + \xi^2 + \eta^2}{2(\xi^2 + \eta^2)}$. The lack of a cusp at the tail is an artifact of drawing thick curves in order to be visible in the printed copy.

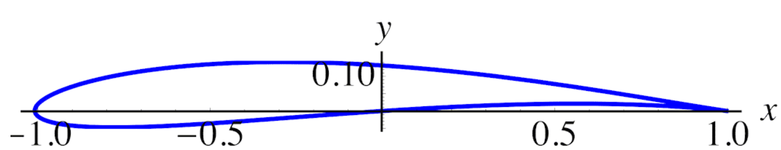


Figure 1: Answer key for problem 16.3.5.13

The lift is

$$F_y \approx 4\pi a \left(\frac{1}{2} \varrho_0\, v_\infty^2\right)\left(1 + \frac{4T}{3\sqrt{3}}\right) \sin(\alpha + 2H) = 2\pi \cdot 1 \cdot \varrho_0\, v_\infty^2 (1 + 0.07) \sin(\alpha + 0.14) = 2.14\pi \varrho_0\, v_\infty^2 \sin(\alpha + 0.14).$$

Chapter Seventeen

Section 17.1.2

17.1.2.1. By definition, the Fourier transform is the improper integral

$$F(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x)\, e^{-i\omega x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\, e^{-i\omega x}\, dx\,,$$

as long as the latter two improper integrals exist.

Separately we calculate that there exists

$$\int_{-\infty}^{0} f(x)\, e^{-i\omega x}\, dx = \lim_{a\to-\infty} \left(\int_{a}^{0} e^{-|x|}\, e^{-i\omega x}\, dx \right) = \lim_{a\to\infty} \left(\int_{a}^{0} e^{-(-x)}\, e^{-i\omega x}\, dx \right) = \lim_{a\to-\infty} \left(\int_{a}^{0} e^{(1-i\omega)x}\, dx \right)$$

$$= \lim_{a\to-\infty} \left[\frac{e^{(1-i\omega)x}}{1-i\omega} \right]_a^0 = \lim_{a\to-\infty} \left(\frac{1-e^{(1-i\omega)a}}{1-i\omega} \right) = \frac{1-0}{1-i\omega} = \frac{1}{1-i\omega}\,,$$

because $|e^{(1-i\omega)a}| = |e^a| \cdot |e^{-i\omega a}| = |e^a| \cdot 1 = e^a \to 0$ as $a \to -\infty$.

Similarly, there exists

$$\int_{0}^{\infty} f(x)\, e^{-i\omega x}\, dx = \lim_{b\to\infty} \left(\int_{0}^{b} e^{-|x|}\, e^{-i\omega x}\, dx \right) = \lim_{b\to\infty} \left(\int_{0}^{b} e^{-(x)}\, e^{-i\omega x}\, dx \right) = \lim_{b\to\infty} \left(\int_{0}^{b} e^{(-1-i\omega)x}\, dx \right)$$

$$= \lim_{b\to\infty} \left[\frac{e^{(-1-i\omega)x}}{-1-i\omega} \right]_0^b = \lim_{b\to\infty} \left(\frac{e^{(-1-i\omega)b}-1}{-1-i\omega} \right) = \frac{0-1}{-1-i\omega} = -\frac{1}{-1-i\omega}\,,$$

because $|e^{(-1-i\omega)b}| = |e^{-b}| \cdot |e^{-i\omega b}| = |e^{-b}| \cdot 1 = e^{-b} \to 0$ as $b \to \infty$.

Putting the two improper integrals together, we have that the original improper integral exists and is the Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x)\, e^{-i\omega x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\omega} - \frac{1}{-1-i\omega} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\omega} \cdot \frac{-1-i\omega}{-1-i\omega} - \frac{1}{-1-i\omega} \cdot \frac{1-i\omega}{1-i\omega} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{-2}{(1-i\omega)(-1-i\omega)} \right) = \frac{-2}{(-1-\omega^2)\sqrt{2\pi}}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\omega^2}\,.$$

By the way, this agrees with entry F.6 of Table 17.1 at the end of Section 17.1.

17.1.2.3. First, assume $\omega \ge 0$. We will calculate the Fourier transform using residues and the "real integral" method as used in Example 17.1 in Section 17.1. Let $\mathcal{C}_R = \mathcal{C}_{R,1} + \mathcal{C}_{R,2}$, as shown in part (a) of the figure.

Let $z = x + iy$ and extend $f(z) = \dfrac{z}{1+z^2}$ to be a function of the complex variable z. We chose $\mathcal{C}_{R,2}$ to be the bottom half circle because the factor

$$e^{-i\omega z} = e^{-i\omega(x+iy)} = e^{\omega y} e^{-i\omega x} \to 0, \text{ as } y \to -\infty.$$

Noting that $\mathcal{C}_R$ is *negatively* oriented, we calculate

$$\int_{\mathcal{C}_R} \frac{z\, e^{-i\omega z}}{1+z^2}\, dz = -2\pi i\, \mathsf{Res}\left[\frac{z\, e^{-i\omega z}}{1+z^2};\ -i \right] = -2\pi i \lim_{z\to -i} (z+i) \cdot \frac{z\, e^{-i\omega z}}{z^2+1} = -2\pi i \lim_{z\to -i} \frac{z e^{-i\omega z}}{z-i}$$

$$= -2\pi i \cdot \frac{-ie^{-\omega}}{-i2} = -i\pi e^{-\omega}\,.$$

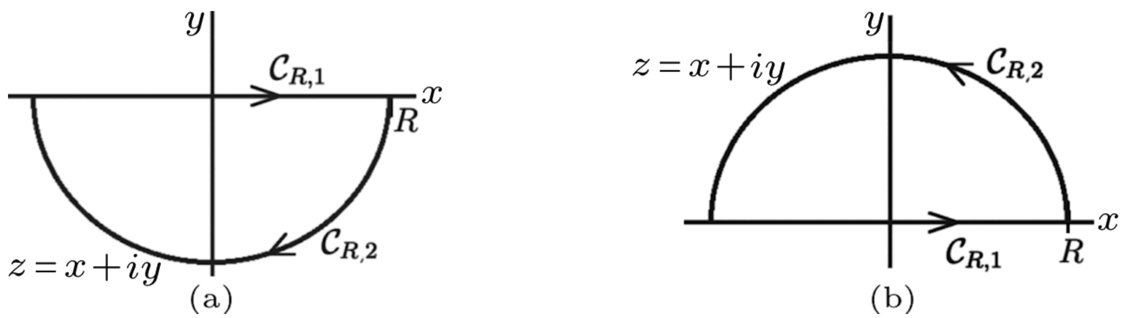


Figure 1: Large semicircular contours in (a) the lower half plane and (b) the upper half plane

Next, we will use the line integral over $\mathcal{C}_R$ explain why the Fourier transform of $f(x)$ is given by

$$\mathcal{F}\left[\frac{x}{1+x^2}\right](\omega) = \frac{1}{\sqrt{2\pi}} \cdot \left(-i\pi e^{-\omega}\right), \text{ for } \omega \geq 0.$$

Suppose $R > 1$. We want that, as $R \to \infty$,

$$\frac{1}{\sqrt{2\pi}} \cdot \pi e^{-\omega} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{C}_{R,1}} f(z)\, e^{-i\omega z}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(x)\, e^{-i\omega x} dx \to \mathcal{F}[\, f(x)\,](\omega).$$

To get this we will use that (1) the improper integrals $\displaystyle\int_0^R \frac{x}{1+x^2}\, e^{-i\omega x} dx$ and $\displaystyle\int_{-R}^0 \frac{x}{1+x^2}\, e^{-i\omega x} dx$ converge and (2) $\displaystyle\int_{\mathcal{C}_{R,2}} \frac{z}{1+z^2}\, e^{-i\omega z}\, dz \to 0$, as $R \to \infty$.

Result (1) follows from consideration of the real and imaginary parts being rewritten as alternating series of terms such as

$$(-1)^{2k-1} a_{2k-1} \triangleq \int_{(2k-1)\pi/\omega}^{2k\pi/\omega} \frac{x}{1+x^2}\ \sin(\omega x) dx < 0 \quad \text{and} \quad (-1)^{2k} a_{2k} \triangleq \int_{2k\pi/\omega}^{(2k+1)\pi/\omega} \frac{x}{1+x^2}\ \sin(\omega x) dx > 0.$$

I'm sorry that I've included this problem because of the need to use this tricky idea.

Result (2) follows from calculating on $\mathcal{C}_{R,2} :\ z = Re^{-i\theta}$, $0 \leq \theta \leq \pi$, that

$$\left|\frac{z\, e^{-i\omega z}}{1+z^2}\right| = \left|\frac{Re^{-i\theta} e^{-i(\omega Re^{-i\theta})}}{1+R^2 e^{-i2\theta}}\right| = \frac{R|e^{-i\theta}|\,|e^{-i\omega R\cos\theta - \omega R \sin\theta}|}{|1+R^2 e^{-i2\theta}|} = \frac{R|e^{-i\theta}|}{|1+R^2 e^{-i2\theta}|} \cdot |e^{-i(\omega R \cos\theta)}| \cdot |e^{-\omega R \sin\theta}|$$

$$= \frac{R \cdot 1}{|R^2 e^{-i2\theta} + 1|} \cdot 1 \cdot e^{-\omega R \sin\theta} \leq \frac{R}{R^2 - 1} \cdot 1 \cdot 1 = \frac{R}{R^2-1}\,,$$

because $0 \leq \theta \leq \pi$ and we assumed $\omega \geq 0$. So,

$$\left|\int_{\mathcal{C}_{R,2}} \frac{z\, e^{-i\omega z}}{1+z^2}\, dz\right| \leq \int_0^\pi \frac{R}{R^2-1}\, d\theta = \frac{\pi R}{R^2-1} \to 0, \text{ as } R \to \infty.$$

We conclude that for $\omega \geq 0$,

$$\mathcal{F}[\, f(x)\,] = \frac{1}{\sqrt{2\pi}} \cdot \left(-i\pi e^{-\omega}\right).$$

Second, if $\omega \leq 0$ then we can use the same method but applied to the *positively* oriented contour $\mathcal{C}_R$ in the upper half plane shown in Figure 17.1(b), to get, for $\omega \leq 0$,

$$\mathcal{F}[\, f(x)\,](\omega) = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i\, \mathsf{Res}\left[\frac{z\, e^{-i\omega z}}{1+z^2};\ i\right] = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \lim_{z\to i}\, (z-i) \cdot \frac{z\, e^{-i\omega z}}{z^2+1} = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \lim_{z\to i} \frac{ze^{-i\omega z}}{z+i}$$

$$= \sqrt{\frac{\pi}{2}} \cdot 2i \cdot \frac{ie^{\omega}}{i2} = \sqrt{\frac{\pi}{2}} \cdot (ie^{\omega})\,.$$

Putting the two cases for ω together, we have that

$$\mathcal{F}\left[\frac{x}{1+x^2}\right](\omega) = \sqrt{\frac{\pi}{2}} \cdot \left(-i\,sgn(\omega)e^{-|\omega|}\right).$$

By the way, Mathematica writes the final conclusion as

$$\mathcal{F}\left[\frac{x}{1+x^2}\right](\omega) = i\sqrt{\frac{\pi}{2}}\ \left(e^{\omega}Step(-\omega) - e^{-\omega}Step(\omega)\right).$$

17.1.2.5. The conclusion of Example 17.1 in Section 17.1 but with $a = 1$, gives $\mathcal{F}\left[\frac{1}{1+t^2}\right](\omega) = \sqrt{\frac{\pi}{2}} \cdot e^{-|\omega|}$. This implies

$$\mathcal{F}^{-1}\left[e^{-|\omega|}\right](\omega) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+t^2}.$$

17.1.2.7. Assume β is a constant. Using the method of Theorem 9.9(b) in Section 9.4 and the result of Example 17.3 in Section 17.1, we have

$$\mathcal{F}^{-1}\left[\,e^{-\beta\omega^2}\,\right] = \mathcal{F}\left[\,e^{-\beta\omega^2}\,\right](-x) = \left(\frac{1}{\sqrt{2\beta}} \cdot e^{-x^2/(4\beta)}\right)\Big|_{x\mapsto -x} = \frac{1}{\sqrt{2\beta}} \cdot e^{-(-x)^2/(4\beta)} = \frac{1}{\sqrt{2\beta}} \cdot e^{-x^2/(4\beta)}\,.$$

17.1.2.9. Assume that $f(x)$ is an odd, real valued function on $(-\infty, \infty)$. Without paying attention to issues of convergence of improper integrals, we calculate that

$$\mathcal{F}[\,f(x)\,] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \left(\cos(\omega x) - i\sin(\omega x)\right)dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \cos(\omega x)dx\ -\ i\,\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \sin(\omega x)dx.$$

Now, on the interval $-\infty < x < \infty$, $\cos(\omega x)$ is an even function of x and $\sin(\omega x)$ is an odd function of x. Because $f(x)$ is an odd, real valued function on $(-\infty, \infty)$, on that interval $f(x)\cos(\omega x)$ is an odd function of x and $f(x)\sin(\omega x)$ is an even function of x. Using Theorem 9.2 in Section 9.1, we have that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \cos(\omega x)dx = 0 \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \sin(\omega x)dx = 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\ \sin(\omega x)dx.$$

So, it is plausible that

$$\mathcal{F}[\,f(x)\,] = -i\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\ \sin(\omega x)\,dx.$$

17.1.2.11. Assume that $f(x)$ is an even, real valued function on $(-\infty, \infty)$. Without paying attention to issues of convergence of improper integrals, we calculate that

$$\mathcal{F}[\,f(x)\,] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \left(\cos(\omega x) - i\sin(\omega x)\right)dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \cos(\omega x)dx\ -\ i\,\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \sin(\omega x)dx.$$

(b) Because $f(x)$ is real, both $\int_{-\infty}^{\infty} f(x)\, \cos(\omega x) dx$ and $\int_{-\infty}^{\infty} f(x)\, \sin(\omega x) dx$ are real. It follows that $\mathcal{F}_c[\, f(x)\,](\omega) = \mathcal{R}e\big(\mathcal{F}[\, f(x)\,](\omega)\big)$.

(a) Now, on the interval $-\infty < x < \infty$, $\cos(\omega x)$ is an even function of x and $\sin(\omega x)$ is an odd function of x. Because $f(x)$ is an even, real valued function on $(-\infty, \infty)$, on that interval $f(x)\cos(\omega x)$ is an even function of x and $f(x)\sin(\omega x)$ is an odd function of x. Using Theorem 9.2 in Section 9.1, we have that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \cos(\omega x) dx = 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\, \cos(\omega x) dx \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \sin(\omega x) dx = 0.$$

So, ignoring issues of convergence of improper integrals, the real valued function of ω given by the Fourier transform of $f(x)$ satisfies

$$\mathcal{R}e\big(\mathcal{F}[\, f(x)\,](\omega)\big) = \mathcal{F}[\, f(x)\,] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\, \cos(\omega x)\, dx \triangleq \mathcal{F}_c[\, f(x)\,](\omega),$$

which is is called the Fourier cosine transform of $f(x)$.

Section 17.2.2

17.2.2.1. Because $u(0,t)$ is specified, it makes sense to use the Fourier sine transform,

$$U(\omega, t) \triangleq \mathcal{F}_{s,x}[\, u(x,t)\,](\omega) \triangleq \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} u(x,t)\, \sin(\omega x)\, dx.$$

Take the Fourier sine transform of both sides of the PDE to get

$$\mathcal{F}_{s,x}\Big[\, \frac{\partial u}{\partial t}(x,t)\,\Big](\omega) = \alpha \mathcal{F}_{s,x}\Big[\, \frac{\partial^2 u}{\partial x^2}(x,t)\,\Big](\omega).$$

We assume that we can interchange the two operations of (i) partial differentiation with respect to t and (ii) taking the Fourier sine transform with respect to x. So, Theorem 17.2(b) in Section 17.2 applied to this PDE, using the assumptions that both $u(x,t) \to 0$ and $\frac{\partial u}{\partial x}(x,t) \to 0$ as $|x| \to \infty$, implies

$$\frac{\partial U}{\partial t}(\omega, t) = \alpha\, \omega \cdot \sqrt{\frac{2}{\pi}} \cdot u(0,t) - \alpha\, \omega^2 U.$$

Using the BC $u(0,t) = 0$, $0 < t < \infty$, this becomes

$$(\star) \quad \frac{\partial U}{\partial t} = -\alpha\, \omega^2 U.$$

We can solve $(\star)$ as if it were the first order ODE of exponential decay, so $U(\omega, t) = C(\omega) e^{-\alpha \omega^2 t}$, where $C(\omega)$ is an arbitrary function of ω.

The initial condition $u(x,0) = f(x)$ implies that

$$F_s(\omega) \triangleq \mathcal{F}_{s,x}\,[\, u(x,0)\,]\,(\omega) = U(\omega, 0) = 0 + C(\omega),$$

so

$$U(\omega, t) = F_s(\omega) e^{-\alpha \omega^2 t}.$$

To find the solution $u(x,t)$, use the inverse Fourier sine transform

$$u(x,t) = \mathcal{F}^{-1}_{s,x}\Big[\, F_s(\omega) e^{-\alpha \omega^2 t}\,\Big](x)\,.$$

This is relatively easy to calculate by using the convolution theorem result (17.21) in Theorem 17.5 in Section 17.2. To prepare for that we recall from Example 17.6 in Section 17.2, or implicitly from Example 17.3 in Section 17.1, that $g(x) \triangleq \frac{1}{\sqrt{2\alpha t}} e^{-x^2/(4\alpha t)}$ satisfies

$$G_c(\omega) = \mathcal{F}_c[\, g(x)\,](\omega) = \mathcal{F}\big[\, g(x)\,\big] = e^{-\alpha\omega^2 t}.$$

Using (17.21)'s convolution result and our definition that $F_s(\omega) = \mathcal{F}_{s,x}\,[\, f(x)\,]$, we have that

$$u(x,t) = \mathcal{F}^{-1}_{s,x}\Big[\, F_s(\omega)e^{-\alpha\omega^2 t}\Big](x) = \frac{1}{2\sqrt{2\alpha t}} \int_0^\infty \left(e^{-|x-\xi|^2/(4\alpha t)} - e^{-(x+\xi)^2/(4\alpha t)}\right) f(\xi)\, d\xi.$$

Using the IC $u(x,0) = f(x) = \left\{ \begin{array}{ll} 2, & 0 < x < \pi \\ 0, & \pi < x < \infty \end{array} \right\}$, the solution of the problem is

$$u(x,t) = \frac{1}{\sqrt{2\alpha t}} \int_0^\pi \left(e^{-(x-\xi)^2/(4\alpha t)} \ - \ e^{-(x+\xi)^2/(4\alpha t)}\right) d\xi\,.$$

By the way, this can be rewritten in terms of the function $\operatorname{erf}(\theta) \triangleq \frac{2}{\sqrt{\pi}} \int_0^\theta e^{-y^2}\, dy$: Using the substitutions $y = x - \xi$ and $Y = x + \xi$, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\alpha t}} \left(\int_0^\pi e^{-(x-\xi)^2/(4\alpha t)} d\xi \ - \ \int_0^\pi e^{-(x+\xi)^2/(4\alpha t)} d\xi \right) \\
&= \frac{1}{\sqrt{2\alpha t}} \cdot \frac{\sqrt{\pi}}{2} \left(\frac{2}{\sqrt{\pi}} \int_x^{x-\pi} e^{-y^2/(4\alpha t)}(-dy) \ - \ \frac{2}{\sqrt{\pi}} \int_x^{x+\pi} e^{-Y^2/(4\alpha t)} dY \right) \\
&= \frac{\sqrt{\pi}}{2\sqrt{2\alpha t}} \left(\frac{2}{\sqrt{\pi}} \int_{x-\pi}^{x} e^{-y^2/(4\alpha t)} dy \ - \ \frac{2}{\sqrt{\pi}} \int_x^{x+\pi} e^{-Y^2/(4\alpha t)} dY \right) \\
&= \frac{\sqrt{\pi}}{2\sqrt{2\alpha t}} \cdot \frac{2}{\sqrt{\pi}} \left(\int_0^x e^{-y^2/(4\alpha t)} dy \ - \ \int_0^{x-\pi} e^{-y^2/(4\alpha t)} dy \ - \ \int_0^{x+\pi} e^{-Y^2/(4\alpha t)} dY \ + \int_0^x e^{-Y^2/(4\alpha t)} dY \right) \\
&= \frac{\sqrt{\pi}}{2\sqrt{2\alpha t}} \left(2\operatorname{erf}\Big(\frac{x}{2\sqrt{\alpha t}}\Big) - \operatorname{erf}\Big(\frac{x+\pi}{2\sqrt{\alpha t}}\Big) - \operatorname{erf}\Big(\frac{x-\pi}{2\sqrt{\alpha t}}\Big) \right).
\end{aligned}$$

17.2.2.3. Follow the hint by taking the Fourier transforms with respect to both x and y, that is, use

$$U(\omega,\nu,t) \triangleq \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty u(x,y,t) e^{-i\omega x} e^{-i\nu y} dx dy.$$

Take the Fourier transforms with respect to x and y of both sides of the PDE, and assume also that $u(x,y,t) \to 0$, $\frac{\partial u}{\partial y}(x,y,t) \to 0$, and $\frac{\partial u}{\partial x}(x,y,t) \to 0$, as $|x| \to \infty$ or $|y| \to \infty$, to get

$$\frac{\partial U}{\partial t}(\omega,\nu,t) = -\alpha(\omega^2 + \nu^2) U(\omega,\nu,t).$$

We can solve this as if it were the first order ODE of exponential decay, so $U(\omega,\nu,t) = C(\omega,\nu)e^{-\alpha\omega^2 t}$, where $C(\omega,\nu)$ is an arbitrary function of ω.

The initial condition $u(x,y,0) = f(x,y)$ implies that

$$F(\omega,\nu) \triangleq \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(x,y) e^{-i\omega x} e^{-i\nu y} dx dy = U(\omega,\nu,0) = 0 + C(\omega,\nu),$$

so

$$U(\omega,\nu,t) = F(\omega,\nu) e^{-\alpha(\omega^2+\nu^2)t} = F(\omega,\nu) e^{-\alpha\omega^2 t} e^{-\alpha\nu^2 t}.$$

To find the solution $u(x,y,t)$, use the inverse Fourier transforms and the convolution theorem result (9.47) in Theorem 9.10 in Section 9.4 to get

$$u(x,y,t) = \mathcal{F}_y^{-1}\Big[\mathcal{F}_x^{-1}\Big[F(\omega,\nu)e^{-\alpha\omega^2 t}e^{-\alpha\nu^2 t}\Big]\Big](x,y,t) = \mathcal{F}_y^{-1}\Big[e^{-\alpha\nu^2 t}\cdot\mathcal{F}_x^{-1}\Big[F(\omega,\nu)e^{-\alpha\omega^2 t}\Big]\Big](x,y,t)\,,$$

hence

$$(\star)\quad u(x,y,t) = \Big(\mathcal{F}_y^{-1}\Big[e^{-\alpha\nu^2 t}\Big]\Big) \underset{y}{*} \Big(\mathcal{F}_y^{-1}\Big[\mathcal{F}_x^{-1}\Big[e^{-\alpha\omega^2 t}F(\omega,\nu)\Big]\Big]\Big)(x,y,t).$$

From Example 17.6 in Section 17.2 we have that

$$\mathcal{F}_x^{-1}\Big[e^{-\alpha\omega^2 t}\Big] = \frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)} \quad \text{and} \quad \mathcal{F}_y^{-1}\Big[e^{-\alpha\nu^2 t}\Big] = \frac{1}{\sqrt{2\alpha t}}\,e^{-y^2/(4\alpha t)}\,.$$

Using (9.47)'s convolution result in Section 9.4 and our definition that $F(\omega,\nu) = \mathcal{F}_x\Big[\mathcal{F}_x\Big[f(x,y)\Big]\Big]$, we have from $(\star)$ that

$$(\star\star)\quad u(x,y,t) = \left(\frac{1}{\sqrt{2\alpha t}}\,e^{-y^2/(4\alpha t)}\right) \underset{y}{*} \left(\mathcal{F}_y^{-1}\left[\Big(\frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)}\Big) \underset{x}{*} \Big(\mathcal{F}_x^{-1}\Big[F(\omega,\nu)\Big]\Big)\right]\right)(x,y,t).$$

Note that

$$\mathcal{F}_y^{-1}\left[\Big(\frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)}\Big) \underset{x}{*} \Big(\mathcal{F}_x^{-1}\Big[F(\omega,\nu)\Big]\Big)\right] = \mathcal{F}_y^{-1}\left[\Big(\frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)}\Big) \underset{x}{*} \Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega,\nu)e^{i\omega x}d\omega\Big)\right]$$

$$= \mathcal{F}_y^{-1}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\alpha t}}\,e^{-(x-\xi)^2/(4\alpha t)}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega,\nu)e^{i\omega\xi}d\omega\right)\right]$$

$$= \mathcal{F}_y^{-1}\left[\frac{1}{2\pi}\cdot\frac{1}{\sqrt{2\alpha t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x-\xi)^2/(4\alpha t)}\,F(\omega,\nu)e^{i\omega\xi}d\omega\right]$$

$$= \frac{1}{(\sqrt{2\pi})^3}\cdot\frac{1}{\sqrt{2\alpha t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x-\xi)^2/(4\alpha t)}\,F(\omega,\nu)e^{i\omega\xi}\,e^{i\nu y}d\omega\,d\nu$$

$$= \frac{1}{(\sqrt{2\pi})}\cdot\frac{1}{\sqrt{2\alpha t}}\int_{-\infty}^{\infty}e^{-(x-\xi)^2/(4\alpha t)}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\omega,\nu)e^{i\omega\xi}\,e^{i\nu y}d\omega\,d\nu\right)$$

$$= \frac{1}{(\sqrt{2\pi})}\cdot\frac{1}{\sqrt{2\alpha t}}\int_{-\infty}^{\infty}e^{-(x-\xi)^2/(4\alpha t)}\,f(\xi,y)\,d\xi.$$

So, $(\star\star)$ gives that the solution of the whole problem can be rewritten as

$$u(x,y,t) = \left(\frac{1}{\sqrt{2\alpha t}}\,e^{-y^2/(4\alpha t)}\right) \underset{y}{*} \left(\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\alpha t}}\int_{-\infty}^{\infty}e^{-(x-\xi)^2/(4\alpha t)}\,f(\xi,y)\,d\xi\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\alpha t}}e^{-(y-\eta)^2/(4\alpha t)}\left(\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\alpha t}}\int_{-\infty}^{\infty}e^{-(x-\xi)^2/(4\alpha t)}\,f(\xi,y)\,d\xi\right)d\eta$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2\alpha t}\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(y-\eta)^2/(4\alpha t)}\,e^{-(x-\xi)^2/(4\alpha t)}\,f(\xi,y)\,d\xi\,d\eta$$

$$= \frac{1}{4\pi\alpha t}\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\left((x-\xi)^2+(y-\eta)^2\right)/(4\alpha t)}\,u(\xi,\eta,0)\,d\xi\,d\eta\,.$$

17.2.2.5. Because $\frac{\partial u}{\partial x}(0,t)$ is specified, it makes sense to use the Fourier cosine transform,

$$U(\omega,t) \triangleq \mathcal{F}_{c,x}[\,u(x,y)\,](\omega) \triangleq \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,y)\,\cos(\omega x)\,dx.$$

We assume that we can interchange the two operations of (i) partial differentiation with respect to y and (ii) taking the Fourier cosine transform with respect to x. Take the Fourier cosine transform of both sides of the PDE to get, using the assumptions that both $u(x,y)\to 0$ and $\frac{\partial u}{\partial x}(x,y)\to 0$ as $|x|\to\infty$,

$$0 = \mathcal{F}_{c,x}\Big[\,\frac{\partial^2 u}{\partial x^2}\,\Big] + \mathcal{F}_{c,x}\Big[\,\frac{\partial^2 u}{\partial y^2}\,\Big] = -\sqrt{\frac{2}{\pi}}\cdot\frac{\partial u}{\partial x}(0,y) - \omega^2 U(\omega,y) + \frac{\partial^2 U}{\partial x^2}(\omega,y).$$

Using the BC $\dfrac{\partial u}{\partial x}(0,t)=0$, $0<t<\infty$, the PDE becomes

$$(\star)\quad \frac{\partial^2 U}{\partial x^2}(\omega,y) - \omega^2 U(\omega,y) = 0.$$

We can solve $(\star)$ as if it were a second order linear, homogeneous ODE, so, using a bit of "clairvoyance,"

$$U(\omega,y) = A(\omega)\sinh\big(\omega(H-y)\big) + B(\omega)\sinh(\omega y),$$

where $A(\omega)$ and $B(\omega)$ are arbitrary functions of ω.

The remaining two BCs imply that

$$F_c(\omega) \triangleq \mathcal{F}_{c,x}\,[\,u(x,0)\,]\,(\omega) = U(\omega,0) = A(\omega)\sinh(\omega H) + B(\omega)\cdot 0,$$

hence $A(\omega) = \dfrac{F_c(\omega)}{\sinh(\omega H)}$, and

$$0 = \mathcal{F}_{c,x}\,[\,u(x,H)\,]\,(\omega) = U(\omega,0) = A(\omega)\cdot 0 + B(\omega)\cdot\sinh(\omega H),$$

hence $B(\omega)=0$.

So,

$$U(\omega,y) = \frac{F_c(\omega)}{\sinh(\omega H)}\,\sinh\big(\omega(H-y)\big) = \frac{\sinh\big(\omega(H-y)\big)}{\sinh(\omega H)}\cdot F_c(\omega).$$

To find the solution $u(x,t)$, use the inverse Fourier cosine transform

$$u(x,y) = \mathcal{F}^{-1}_{c,x}\left[\,\frac{\sinh\big(\omega(H-y)\big)}{\sinh(\omega H)}\cdot F_c(\omega)\,\right](x)\,.$$

We will use the convolution theorem result (17.22) in Theorem 17.5 in Section 17.2. First, recall the BC $u(x,0)=f(x)$, and note that in the Errata webpage we have the hint that "You may assume that $\mathcal{F}^{-1}_{c,x}\left[\,\dfrac{\sinh(a\omega)}{\sinh(b\omega)}\,\right] = \sqrt{\frac{\pi}{2}}\cdot\dfrac{\sin\big(\pi a/b\big)}{b\big(\cosh(\pi x/b)+\cos(\pi a/b)\big)}$, for $x>0$, as long as $|\mathcal{R}e(a)|<|\mathcal{R}e(b)|$."

It follows from the latter that

$$\mathcal{F}^{-1}_{c,x}\left[\,\frac{\sinh\big((H-y)\omega\big)}{\sinh(H\omega)}\,\right] = \sqrt{\frac{\pi}{2}}\cdot\frac{\sin\big(\frac{\pi}{H}\,(H-y)\big)}{H}\cdot\frac{1}{\cosh\big(\frac{\pi x}{H}\big)+\cos\big(\frac{\pi}{H}\,(H-y)\big)},$$

for $x>0$ and $|H-y|<|H|$, that is, for $x>0$ and $-H<y<H$. In the original problem, we had $0<y<H$ in the strip.

Using this, along with (17.22)'s convolution result and our definition that $F_c(\omega)=\mathcal{F}_{c,x}\,[\,f(x)\,]$, we have that

$$u(x,y) = \mathcal{F}^{-1}_{c,x}\left[\,\frac{\sinh\big((H-y)\omega\big)}{\sinh(H\omega)}\cdot F_c(\omega)\,\right](x) = \left(\mathcal{F}^{-1}_{c,x}\left[\,\frac{\sinh\big((H-y)\omega\big)}{\sinh(H\omega)}\,\right]\right) * \Big(\mathcal{F}^{-1}_{c,x}\,[\,F_c(\omega)\,]\Big)$$

that is,

$$u(x,y)=\frac{\sqrt{\pi/2}}{2H}\cdot\sin\Big(\frac{\pi}{H}\,(H-y)\Big)\cdot$$
$$\cdot\int_0^\infty\left(\frac{1}{\cosh\left(\frac{\pi(x-\xi)}{H}\right)+\cos\left(\frac{\pi}{H}\,(H-y)\right)}+\frac{1}{\cosh\left(\frac{\pi(x+\xi)}{H}\right)+\cos\left(\frac{\pi}{H}\,(H-y)\right)}\right)f(\xi)\,d\xi,$$

for $x>0$ and $0<y<H$.

17.2.2.7. $\mathcal{F}_c^{-1}\big[\,e^{-\alpha\omega^2 t}\,\big](x)\triangleq\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\alpha\omega^2 t}\,\cos(\omega x)\,d\omega=\frac{1}{2}\cdot\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty e^{-\alpha\omega^2 t}\,\cos(\omega x)\,d\omega$, because $e^{-\alpha\omega^2 t}\,\cos(\omega x)$ is an even function of ω and the improper integrals $\int_0^\infty ...d\omega$ and $\int_{-\infty}^\infty ...d\omega$ are easily shown to converge, as long as $t>0$.

Continuing, we have

$$\mathcal{F}_c^{-1}\big[\,e^{-\alpha\omega^2 t}\,\big](x)=\mathcal{R}e\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\alpha\omega^2 t}\,e^{i\omega x}\,d\omega\right)=\mathcal{R}e\left(\mathcal{F}^{-1}\big[\,e^{-\alpha\omega^2 t}\,\big](x)\right)=\frac{1}{\sqrt{2\alpha t}}\,e^{-(-x)^2/(4\alpha t)}$$
$$=\frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)},$$

by using Table 17.1's entry F.9 with $\beta=\alpha t$ and Theorem 17.3(a) in Section 17.1.

17.2.2.9. Let A be a constant.
(a) Because $u(0,t)$ is specified, it makes sense to use the Fourier sine transform,

$$U(\omega,t)\triangleq\mathcal{F}_{s,x}[\,u(x,t)\,](\omega)\triangleq\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\,\sin(\omega x)\,dx.$$

Take the Fourier sine transform of both sides of the PDE to get

$$\mathcal{F}_{s,x}\Big[\,\frac{\partial u}{\partial t}(x,t)\,\Big](\omega)=\alpha\mathcal{F}_{s,x}\Big[\,\frac{\partial^2 u}{\partial x^2}(x,t)\,\Big](\omega).$$

We assume that we can interchange the two operations of (i) partial differentiation with respect to t and (ii) taking the Fourier sine transform with respect to x. So, Theorem 17.2(b) in Section 17.2 applied to this PDE, using the assumptions that both $u(x,t)\to 0$ and $\frac{\partial u}{\partial x}(x,t)\to 0$ as $|x|\to\infty$, implies

$$\frac{\partial U}{\partial t}(\omega,t)=\alpha\,\omega\cdot\sqrt{\frac{2}{\pi}}\cdot u(0,t)\;-\;\alpha\,\omega^2 U.$$

Using the BC $u(0,t)=A$, $0<t<\infty$, this becomes

$$(\star)\quad\frac{\partial U}{\partial t}=A\alpha\,\omega\cdot\sqrt{\frac{2}{\pi}}\;-\;\alpha\,\omega^2 U.$$

We can solve $(\star)$ as if it were a first order linear ODE:

$$\frac{\partial U}{\partial t}(\omega,t)+\alpha\,\omega^2 U=A\alpha\,\omega\cdot\sqrt{\frac{2}{\pi}}$$

has integrating factor $\mu(t)=e^{\alpha\omega^2 t}$, so the ODE can be rewritten as

$$\frac{d}{dt}\big[\,e^{\alpha\omega^2 t}U\,\big]=A\alpha\,\omega\cdot\sqrt{\frac{2}{\pi}}\cdot e^{\alpha\omega^2 t},$$

whose solutions are, implicitly,

$$e^{\alpha\omega^2 t}U(\omega,t) = A\,\alpha\,\omega\sqrt{\frac{2}{\pi}}\int e^{\alpha\omega^2 t}dt + C(\omega) = A\,\alpha\,\omega\sqrt{\frac{2}{\pi}}\cdot\frac{1}{\alpha\,\omega^2}\,e^{\alpha\omega^2 t} + C(\omega),$$

by using the substitution $v = \alpha\omega^2 t$, with $dv = \alpha\omega^2 dt$.

Explicitly,

$$U(\omega,t) = \frac{A\sqrt{2}}{\omega\sqrt{\pi}} + C(\omega)e^{-\alpha\omega^2 t}\,,$$

where $C(\omega)$ is an arbitrary function of ω.

The initial condition $u(x,0) = 0$ implies that

$$0 = \mathcal{F}_{s,x}\left[\,u(x,0)\,\right](\omega) = U(\omega,0) = \sqrt{\frac{2}{\pi}}\cdot\frac{A}{\omega} + C(\omega),$$

so $C(\omega) = -\sqrt{\frac{2}{\pi}}\cdot\frac{A}{\omega}$ and

$$U(\omega,t) = \sqrt{\frac{2}{\pi}}\cdot\frac{A}{\omega}\Big(1 - e^{-\alpha\omega^2 t}\Big),$$

as was desired.

(b) To find the solution $u(x,t)$, use the inverse Fourier sine transform

$$u(x,t) = \mathcal{F}^{-1}_{s,x}\Big[\sqrt{\frac{2}{\pi}}\cdot\frac{A}{\omega}\Big(1 - e^{-\alpha\omega^2 t}\Big)\Big](x) = \sqrt{\frac{2}{\pi}}\cdot A\,\mathcal{F}^{-1}_{s,x}\Big[\,\frac{1}{\omega}\,\Big](x)\ -\ \sqrt{\frac{2}{\pi}}\cdot A\,\mathcal{F}^{-1}_{s,x}\Big[\,\frac{1}{\omega}e^{-\alpha\omega^2 t}\,\Big](x)\,.$$

The first involves

$$(17.36)\qquad \frac{2}{\pi}\int_0^\infty \frac{\sin\omega x}{\omega}\,d\omega = 1,\ \text{for } x > 0,$$

which the instructions say we may *assume* is true.

(c) The second of the inverse Fourier sine transforms involves calculating that

$$(17.37)\qquad \frac{2}{\pi}\int_0^\infty \frac{\sin\omega x}{\omega}\,e^{-\alpha\omega^2 t}\,d\omega = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right).$$

We follow the problem's hints to establish (17.37) by using the following method: Problem 17.2.2.8 gives the fact that

$$\frac{1}{2}\cdot\sqrt{\frac{\pi}{a}}\,e^{-x^2/(4a)} = I(x) \triangleq \int_0^\infty e^{-a\omega^2}\,\cos(\omega x)\,d\omega.$$

Note that $\{(\omega,v)\,:\,0\le v\le x,\,0\le\omega<\infty\} = \{(\omega,v)\,:\,0\le\omega<\infty,\,0\le v\le x\}$, hence interchange of the order of integrations gives us

$$(17.38)\ \int_0^x I(v)\,dv = \int_0^x\left(\int_0^\infty e^{-a\omega^2}\cos(\omega v)d\omega\right)dv = \int_0^\infty e^{-a\omega^2}\left(\int_0^x\cos(\omega v)dv\right)d\omega = \int_0^\infty\frac{\sin\omega x}{\omega}\,e^{-a\omega^2}d\omega.$$

After that, use the substitution $a = \alpha t$ and the left hand side of (17.38) to get

$$\int_0^x \frac{1}{2}\cdot\sqrt{\frac{\pi}{\alpha t}}\,e^{-v^2/(4\alpha t)}\,dv = \int_0^x I(v)\,dv = \int_0^\infty \frac{\sin\omega x}{\omega}\,e^{-\alpha t\omega^2}\,d\omega,$$

hence

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\omega x}{\omega}\,e^{-\alpha\omega^2 t}\,d\omega = \frac{2}{\pi}\cdot\frac{1}{2}\cdot\sqrt{\frac{\pi}{\alpha t}}\int_{v=0}^{v=x} e^{-v^2/(4\alpha t)}\,dv.$$

Substitute into the latter $y = \dfrac{v}{\sqrt{4\alpha t}}$ and $dy = \dfrac{1}{\sqrt{4\alpha t}}\,dv$, to get

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\omega x}{\omega}\,e^{-\alpha\omega^2 t}\,d\omega = \frac{2}{\pi}\cdot\frac{1}{2}\cdot\sqrt{\frac{\pi}{\alpha t}}\int_0^{x/(\sqrt{4\alpha t})} e^{-y^2}\,\sqrt{4\alpha t}\,dy = \frac{2}{\sqrt{\pi}}\int_0^{x/(2\sqrt{\alpha t})} e^{-y^2}\,dy = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right),$$

using the definition of $\operatorname{erf}(\theta)$ in Definition 7.21 in Section 7.7, thus establishing (17.37).
(d) Finish up by finding that the solution of the original problem is

$$\begin{aligned}
u(x,t) &= \sqrt{\frac{2}{\pi}}\cdot A\,\mathcal{F}^{-1}_{s,x}\Big[\,\frac{1}{\omega}\,\Big](x) \;-\; \sqrt{\frac{2}{\pi}}\cdot A\,\mathcal{F}^{-1}_{s,x}\Big[\,\frac{1}{\omega}e^{-\alpha\omega^2 t}\,\Big](x)\\
&= \sqrt{\frac{2}{\pi}}\cdot A\cdot\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\omega x}{\omega}\,d\omega \;-\; \sqrt{\frac{2}{\pi}}\cdot A\cdot\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\omega x}{\omega}\,e^{-\alpha\omega^2 t}\,d\omega\\
&= A\cdot 1 \;-\; A\cdot\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \triangleq A\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right),
\end{aligned}$$

as was desired.

17.2.2.11. Using the result of Example 17.3 in Section 17.1 with $\beta = \dfrac{1}{4\alpha t}$,

$$\mathcal{F}\big[\,e^{-x^2/(4\alpha t)}\,\big](\omega) = \mathcal{F}\big[\,e^{-\beta x^2}\,\big](\omega) = \frac{1}{\sqrt{2\beta}}\cdot e^{-\omega^2/(4\beta)} = \frac{1}{\sqrt{2\cdot\dfrac{1}{4\alpha t}}}\cdot e^{-\omega^2/\left(4\cdot\frac{1}{4\alpha t}\right)} = \sqrt{2\alpha t}\,e^{-\alpha\omega^2 t},$$

hence

$$\mathcal{F}\left[\,\frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)}\,\right](\omega) = e^{-\alpha\omega^2 t}.$$

Now, $\dfrac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)}$ is an even function of x, so the result of problem 17.1.2.11 implies that

$$\mathcal{F}_{c,x}\left[\,\frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)}\,\right](\omega) = \mathcal{F}\left[\,\frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)}\,\right](\omega) = e^{-\alpha\omega^2 t}.$$

It follows that

$$\mathcal{F}_c^{-1}\big[\,e^{-\alpha\omega^2 t}\,\big](x) = \frac{1}{\sqrt{2\alpha t}}\,e^{-x^2/(4\alpha t)},$$

which is (17.27)(b) in Section 17.2, as was desired.

Section 17.3.2

17.3.2.1. The only singularities of $F(s) \triangleq \dfrac{s+1}{(s+1)^2+4}$ are at $s = -1 \pm i2$.

Ex: Use as a Bromwich contour the positively oriented curve $\mathcal{C}:\ s = 0 + iy,\ -\infty < y < \infty$. We use the residue theorem and the positively oriented curve $\mathcal{C}_R = \mathcal{C}_{R,1} + \mathcal{C}_{R,2}$ shown in the figure:

$$\mathcal{C}_{R,1}:\ z = 0 + iy,\ -R \le y \le R \qquad\text{and}\qquad \mathcal{C}_{R,2}:\ z = 0 + Re^{i\theta},\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}.$$

We calculate, for any $R > 1 + \sqrt{6}$, that

$$\frac{1}{2\pi i}\left(\int_{\mathcal{C}_{R,1}} F(s)e^{st}\,ds + \int_{\mathcal{C}_{R,2}} F(s)e^{st}\,ds\right) = \frac{1}{2\pi i}\int_{\mathcal{C}_R} F(s)e^{st}\,ds = \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{s+1}{(s+1)^2+4}\,e^{st}\,ds$$

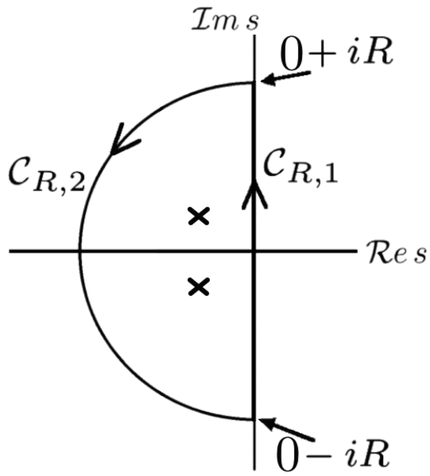


Figure 2: Answer key for problem 17.3.2.1

$$= \mathsf{Res}\Big[\frac{(s+1)e^{st}}{(s+1)^2+4}; -1+i2\Big] + \mathsf{Res}\Big[\frac{(s+1)e^{st}}{(s+1)^2+4}; -1-i2\Big] = \left(\frac{(s+1)e^{st}}{(s+1)+i2}\Big|_{s=-1+i2}\right) + \left(\frac{(s+1)e^{st}}{(s+1)-i2}\Big|_{s=-1-i2}\right)$$

$$= \left(\frac{i2e^{(-1+i2)t}}{i4}\right) + \left(\frac{-i2e^{(-1-i2)t}}{-i4}\right) = e^{-t}\left(\frac{e^{i2t}}{2} + \frac{e^{-i2t}}{2}\right) = e^{-t}\cos(2t).$$

For s on the left half circle $\mathcal{C}_{R,2}:\ s = 0 + Re^{i\theta}$,

$$|\,F(s)\,| = \left|\,\frac{s+1}{(s+1)^2+4}\,\right| = \frac{|1+Re^{i\theta}|}{|(1+Re^{i\theta})^2+4|} = \frac{|1+Re^{i\theta}|}{|R^2e^{i2\theta}+2Re^{i\theta}+5|} \le \frac{R+1}{R^2-2R-5} \to 0, \quad \text{as } R \to \infty,$$

and

$$\mathcal{R}e(s) = \mathcal{R}e\left(0 + Re^{i\theta}\right) \le 0,\ \text{hence}\ |\,e^{st}\,| \le e^{0\cdot t} = 1, \quad \text{for } t \ge 0.$$

Choose left half circles with $R = R_k$. Similarly to an argument in Example 17.1 in Section 17.1, we see that $\dfrac{|1+R_ke^{i\theta}|}{|R_k^2e^{i2\theta}+2R_ke^{i\theta}+5|} \le \dfrac{R_k+1}{R_k^2-2R_k-5}$, hence

$$\Big|\int_{\mathcal{C}_{R,2}} F(s)\,e^{st}\,ds\Big| \le \int_{\mathcal{C}_{R,2}} |\,F(s)\,|\,|e^{st}|\,ds \le \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{|1+R_ke^{i\theta}|}{|R_k^2e^{i2\theta}+2R_ke^{i\theta}+5|}\ e^{0\cdot t}\ d\theta \le \pi\cdot 1\cdot \frac{R_k+1}{R_k^2-2R_k-5} \to 0,$$

as $k \to \infty$. So,

$$\mathcal{L}^{-1}\Big[\frac{s+1}{(s+1)^2+4}\Big](t) = \frac{1}{2\pi i}\ P.V.\ \int_{0-i\infty}^{0+i\infty} \frac{s+1}{(s+1)^2+4}\,e^{st}\,ds = \frac{1}{2\pi i}\ \lim_{R\to\infty}\int_{\mathcal{C}_{R,1}} \frac{s+1}{(s+1)^2+4}\,e^{st}\,ds$$

$$= \lim_{R\to\infty}\ \frac{1}{2\pi i}\int_{\mathcal{C}_R} F(s)e^{st}\,ds = \lim_{R\to\infty}\left(\mathsf{Res}\Big[\frac{(s+1)e^{st}}{(s+1)^2+4}; -1+i2\Big] + \mathsf{Res}\Big[\frac{(s+1)e^{st}}{(s+1)^2+4}; -1-i2\Big]\right)$$

$$= \lim_{R\to\infty} e^{-t}\cos(2t) = e^{-t}\cos(2t).$$

17.3.2.3. Let $\mathcal{L}_t$ denote the Laplace transformation operation with respect to t and define $U(x,s) \triangleq \mathcal{L}_t[\,u(x,t)\,](s) = \displaystyle\int_0^\infty u(x,t)\,e^{-st}\,dt$. By a property of Laplace transforms, the PDE and initial conditions in this problem imply

$$\mathcal{L}_t\Big[\frac{\partial^2 u}{\partial t^2}(x,t)\Big](s) = s^2U(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0) = s^2U(x,s) - s\cdot 0 - 0,$$

hence the PDE implies

$$(\star) \quad s^2 U(x,s) = c^2 \frac{\partial^2 U}{\partial x^2},\ 0 < x < L,$$

along with boundary conditions

$$U(0,s) = \mathcal{L}_t[\,u(0,t)\,](s) = \mathcal{L}_t[\,\sin 2t\,](s) = \frac{2}{s^2+4}, \quad U(L,s) = \mathcal{L}_t[\,u(L,t)\,](s) = \mathcal{L}_t[\,0\,](s) = 0.$$

The solutions of the "ODE" $(\star)$, that is, $\dfrac{\partial^2 U}{\partial x^2} - \dfrac{s^2}{c^2} U(x,s) = 0$, are, by using "clairvoyance" as in Example 9.32 in Section 9.6 and Section 11.3,

$$(\star\star) \quad U(x,s) = A(s) \sinh\left(\frac{s}{c}(L-x)\right) + B(s) \sinh\left(\frac{sx}{c}\right),$$

for arbitrary functions $A(s), B(s)$. The boundary conditions give

$$0 = U(L,s) = A(s)\cdot 0 + B(s)\sinh\left(\frac{sL}{c}\right) \quad \text{and} \quad \frac{2}{s^2+4} = U(0,s) = A(s)\sinh\left(\frac{sL}{c}\right),$$

hence $B(s) = 0$. The solution of the entire problem has Laplace transform

$$(3\star) \quad U(x,s) = A(s)\sinh\left(\frac{s}{c}(L-x)\right) = \frac{2}{(s^2+4)\sinh\left(\frac{sL}{c}\right)} \cdot \sinh\left(\frac{s}{c}(L-x)\right).$$

We will find

$$(4\star) \quad u(x,t) = \mathcal{L}_t^{-1}\left[2\sinh\left(\frac{s}{c}(L-x)\right) \Big/ \left((s^2+4)\sinh\left(\frac{sL}{c}\right)\right)\right](t).$$

Recall that for real u, v

$$\sinh(u+iv) = \sinh u \cos v + i \cosh u \sin v.$$

The poles of $U(x,s)$, as a function of the complex variable s, are where $s^2+4=0$ or $\sinh\left(\frac{sL}{c}\right) = 0$, that is, are at

$$s_{0,\pm} = \pm\, i2 \quad \text{and} \quad s_{n,\pm} = \pm\, i\,\frac{n\pi c}{L},\ n = 1, 2, \ldots .$$

Note that $s = 0$ is a removable singularity of $U(x,s)$.

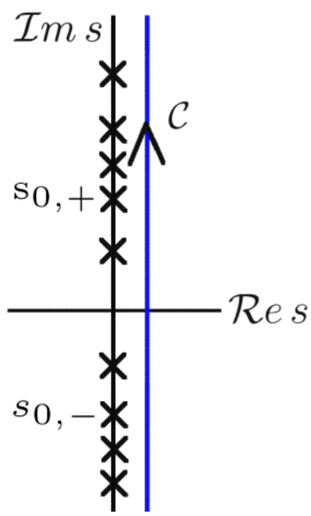


Figure 3: Answer key for problem17.3.2.3

The assumption that "$\frac{2L}{\pi c}$ is not an integer" guarantees that all of the poles are simple. In addition, for any small but positive real number ε, for $0 < x < L$ there are left half circles $S_{\varepsilon,k}$ on which

$$\left|\sinh\left(\frac{s}{c}(L-x)\right)\Big/\sinh\left(\frac{sL}{c}\right)\right|$$

is bounded, so the factor of $\left|\frac{2}{s^2+4}\right|$ in $|U(x,s)|$ shows that $U(x,s)$ is of class $\mathbf{L}$ in s.

So, we can apply Theorem 17.8 in Section 17.3 to conclude that

$$u(x,t) = \sum_{j=0}^{\infty}\sum_{\epsilon=\pm 1}\mathsf{Res}\big[\,U(x,s)e^{st};\ s_{j,\epsilon}\,\big].$$

Let us calculate those residues: First, for $j=0$ and $\epsilon = \pm 1$, $s_{0,\epsilon} = i2\epsilon$, so

$$\mathsf{Res}\big[\,U(x,s)e^{st};\ s_{0,\epsilon}\,\big] = 2e^{st}\sinh\left(\frac{s}{c}(L-x)\right)\Big/\left((s+2i\epsilon)\sinh\left(\frac{sL}{c}\right)\right)\Big|_{s=2i\epsilon}$$

$$= \frac{2e^{i2\epsilon t}}{i4\epsilon}\cdot\frac{\sinh\left(\frac{i2\epsilon}{c}(L-x)\right)}{\sinh\left(\frac{i2\epsilon L}{c}\right)} = \frac{e^{i2\epsilon t}}{i2\epsilon}\cdot\frac{i\,\sin\left(\frac{2\epsilon}{c}(L-x)\right)}{i\,\sin\left(\frac{2\epsilon L}{c}\right)} = \frac{e^{\pm\, i2t}}{\pm\, i2}\cdot\frac{\pm\,\sin\left(\frac{2(L-x)}{c}\right)}{\pm\,\sin\left(\frac{2L}{c}\right)}$$

$$= \pm\,\frac{e^{\pm\, i2t}}{i2}\cdot\frac{\sin\left(\frac{2(L-x)}{c}\right)}{\sin\left(\frac{2L}{c}\right)}.$$

Add the two residues from the poles $s_{0,\pm}$ to get

$$\frac{e^{i2t}-e^{-i2t}}{2i}\cdot\frac{\sin\left(\frac{2(L-x)}{c}\right)}{\sin\left(\frac{2L}{c}\right)},$$

that is,

$$\sin 2t\cdot\sin\left(\frac{2(L-x)}{c}\right)\Big/\sin\left(\frac{2L}{c}\right).$$

Similarly, for $n \geq 1$ and $\epsilon = \pm 1$, $s_{n,\epsilon} = i\epsilon\cdot\frac{n\pi c}{L}$, so

$$\mathsf{Res}\big[\,U(x,s)e^{st};\ s_{n,\epsilon}\,\big] = \lim_{s\to i\epsilon n\pi c/L} 2\left(s-\frac{i\epsilon n\pi c}{L}\right)\cdot e^{st}\cdot\frac{\sinh\left(\frac{s}{c}(L-x)\right)}{(4+s^2)}\cdot\frac{1}{\sinh\left(\frac{sL}{c}\right)}$$

$$= 2e^{i\epsilon n\pi ct/L}\cdot\frac{\sinh\left(\frac{i\epsilon n\pi(L-x)}{L}\right)}{4-\left(\frac{n\pi c}{L}\right)^2}\cdot\lim_{s\to i\epsilon n\pi c/L}\frac{-\frac{i\epsilon n\pi c}{L}}{\sinh\left(\frac{sL}{c}\right)}.$$

L'Hôpital's Rule in the "$\frac{0}{0}$" case gives that this residue equals

$$= 2e^{i\epsilon n\pi ct/L}\cdot\frac{\sinh\left(\frac{i\epsilon n\pi(L-x)}{L}\right)}{4-\left(\frac{n\pi c}{L}\right)^2}\cdot\frac{1}{\frac{L}{c}\,\cosh(i\epsilon n\pi)} = 2e^{i\epsilon n\pi ct/L}\cdot\frac{i\epsilon\sin\left(\frac{n\pi(L-x)}{L}\right)}{4-\left(\frac{n\pi c}{L}\right)^2}\cdot\frac{1}{\frac{L}{c}\,(-1)^n}.$$

Noting that $i = \frac{-1}{i}$, *adding* the two residues from the poles $s_{n,\pm}$ gives

$$\frac{4c}{L}\,(-1)^{n+1}\cdot\frac{e^{in\pi ct/L}-e^{-in\pi ct/L}}{2i}\cdot\sin\left(\frac{n\pi(L-x)}{L}\right)\Big/\left(4-\left(\frac{n\pi c}{L}\right)^2\right),$$

that is,

$$\frac{4c}{L}\,(-1)^{n+1}\cdot\sin\left(\frac{n\pi ct}{L}\right)\cdot\sin\left(\frac{n\pi(L-x)}{L}\right)\Big/\left(4-\left(\frac{n\pi c}{L}\right)^2\right).$$

Putting everything together, we get that the solution of the whole problem is

$$u(x,t)=\frac{\sin\left(\frac{2(L-x)}{c}\right)}{\sin\left(\frac{2L}{c}\right)}\cdot\sin 2t\;+\;\frac{4c}{L}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4-\left(\frac{n\pi c}{L}\right)^2}\cdot\sin\left(\frac{n\pi(L-x)}{L}\right)\cdot\sin\left(\frac{n\pi ct}{L}\right).$$

17.3.2.5. Let $\mathcal{L}_t$ denote the Laplace transformation operation with respect to t and define $U(x,s)\triangleq \mathcal{L}_t[\,u(x,t)\,](s)=\int_0^\infty u(x,t)\,e^{-st}\,dt$. By a property of Laplace transforms, the PDE and initial conditions in this problem together imply that

$$\mathcal{L}_t\Big[\,\frac{\partial^2 u}{\partial t^2}(x,t)\,\Big](s)=s^2U(x,s)-su(x,0)-\,\frac{\partial u}{\partial t}(x,0)=s^2U(x,s)-s\cdot 0-0,$$

hence the PDE implies

$$(\star)\quad s^2U(x,s)=c^2\frac{\partial^2 U}{\partial x^2},\ 0<x<L,$$

along with boundary conditions

$$U(0,s)=\mathcal{L}_t[\,u(0,t)\,](s)=\mathcal{L}_t[\,0\,](s)=0,\quad \frac{\partial U}{\partial x}(L,s)=\mathcal{L}_t[\,\frac{\partial u}{\partial x}(L,t)\,](s)=\mathcal{L}_t[\,\cos t\,](s)=\frac{s}{s^2+1}.$$

The solutions of the "ODE" $(\star)$, that is, $\dfrac{\partial^2 U}{\partial x^2}-\dfrac{s^2}{c^2}U(x,s)=0$, are, by using "clairvoyance" as in Example 9.32 in Section 9.6 and Section 11.3,

$$(\star\star)\quad U(x,s)=A(s)\cosh\left(\frac{s}{c}(L-x)\right)+B(s)\sinh\left(\frac{sx}{c}\right),$$

for arbitrary functions $A(s),B(s)$. The boundary conditions give

$$\frac{s}{s^2+1}=\frac{\partial U}{\partial x}(L,s)=A(s)\cdot 0+B(s)\cdot\frac{s}{c}\,\cosh\left(\frac{sL}{c}\right)\quad\text{and}\quad 0=U(0,s)=A(s)\cosh\left(\frac{sL}{c}\right)+B(s)\cdot 0,$$

hence $A(s)=0$. The solution of the entire problem has Laplace transform

$$(3\star)\quad U(x,s)=B(s)\sinh\left(\frac{sx}{c}\right)=\frac{c}{(s^2+1)\cosh\left(\frac{sL}{c}\right)}\cdot\sinh\left(\frac{sx}{c}\right).$$

We will find

$$(4\star)\quad u(x,t)=\mathcal{L}_t^{-1}\left[\,c\,\sinh\left(\frac{sx}{c}\right)\Big/\left((s^2+1)\cosh\left(\frac{sL}{c}\right)\right)\right](t).$$

Recall from the result of problem 15.5.1.12(a) that for real u,v

$$\cosh(u+iv)=\cosh u\cos v+i\sinh u\sin v.$$

The poles of $U(x,s)$, as a function of the complex variable s, are where $s^2+1=0$ or $\cosh\left(\frac{sL}{c}\right)=0$, that is, are at

$$s_{0,\pm}=\pm\,i\quad\text{and}\quad s_{n,\pm}=\pm\,i\,\frac{(n-\frac{1}{2})\pi c}{L},\ n=1,2,...\,.$$

Note that $s=0$ is a removable singularity of $U(x,s)$.

The assumption that "$\frac{2L}{\pi c}$ is not an integer" guarantees that all of the poles are simple. In addition, for any small but positive real number ε, for $0<x<L$ there are left half circles $S_{\varepsilon,k}$ on which

$$\left|\sinh\left(\frac{sx}{c}\right)\Big/\cosh\left(\frac{sL}{c}\right)\right|$$

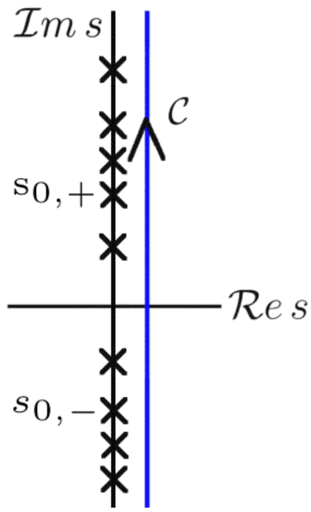


Figure 4: Answer key for problem17.3.2.5

is bounded, so the factor of $\left|\frac{1}{s^2+1}\right|$ in $|U(x,s)|$ shows that $U(x,s)$ is of class **L** in s.

So, we can apply Theorem 17.8 in Section 17.3 to conclude that

$$u(x,t) = \sum_{j=0}^{\infty} \sum_{\epsilon=\pm 1} \mathsf{Res}\big[\, U(x,s)e^{st};\ s_{j,\epsilon}\,\big].$$

Let us calculate those residues: First, for $j=0$ and $\epsilon=\pm 1$, $s_{0,\epsilon} = i\epsilon$, so

$$\mathsf{Res}\big[\, U(x,s)e^{st};\ s_{0,\epsilon}\,\big] = c\, e^{st} \sinh\left(\frac{sx}{c}\right) \Big/ \left((s+i\epsilon)\cosh\left(\frac{sL}{c}\right)\right)\Big|_{s=i\epsilon}$$

$$= \frac{c\, e^{2\epsilon t}}{i2\epsilon} \cdot \sinh\left(\frac{i\epsilon x}{c}\right) \Big/ \cosh\left(\frac{i\epsilon L}{c}\right) = \frac{c\, e^{i\epsilon t}}{i2\epsilon} \cdot i \sin\left(\frac{\epsilon x}{c}\right) \Big/ \cos\left(\frac{\epsilon L}{c}\right). = \frac{c\, e^{\pm\, it}}{\pm i2} \cdot \frac{\pm\, i \sin\left(\frac{x}{c}\right)}{\cos\left(\frac{L}{c}\right)}$$

$$= \frac{c\, e^{\pm\, it}}{2} \cdot \frac{\sin\left(\frac{x}{c}\right)}{\cos\left(\frac{L}{c}\right)}.$$

Add the two residues from the poles $s_{0,\pm}$ to get

$$c\, \frac{e^{it}+e^{-it}}{2} \cdot \frac{\sin\left(\frac{x}{c}\right)}{\sin\left(\frac{L}{c}\right)},$$

that is,

$$c\, \cos t \cdot \sin\left(\frac{x}{c}\right) \Big/ \sin\left(\frac{L}{c}\right).$$

Similarly, for $n \geq 1$ and $\epsilon = \pm 1$, $s_{n,\epsilon} = i\epsilon\, \frac{(n-\frac{1}{2})\pi c}{L}$, so

$$\mathsf{Res}\big[\, U(x,s)e^{st};\ s_{n,\epsilon}\,\big] = \lim_{s \to i\epsilon(n-\frac{1}{2})\pi c/L} c \cdot e^{st} \cdot \frac{\sinh\left(\frac{sx}{c}\right)}{1+s^2} \cdot \frac{1}{\cosh\left(\frac{sL}{c}\right)}$$

$$= c \cdot e^{i\epsilon(n-\frac{1}{2})\pi ct/L} \cdot \frac{\sinh\left(\dfrac{i\epsilon\big(n-\frac{1}{2}\big)\pi x}{L}\right)}{1-\left(\dfrac{(n-\frac{1}{2})\pi c}{L}\right)^2} \quad \cdot \lim_{s\to i\epsilon(n-\frac{1}{2})\pi c/L} \frac{\left(s-\dfrac{i\epsilon(n-\frac{1}{2})\pi c}{L}\right)}{\cosh\left(\dfrac{sL}{c}\right)}.$$

L'Hôpital's Rule in the "$\frac{0}{0}$ " case gives that this residue equals

$$= c \cdot e^{i\epsilon\left(n-\frac{1}{2}\right)\pi ct/L} \cdot \frac{i\sin\left(\dfrac{\epsilon\big(n-\frac{1}{2}\big)\pi x}{L}\right)}{1-\left(\dfrac{(n-\frac{1}{2})\pi c}{L}\right)^2} \cdot \frac{1}{\dfrac{L}{c}\sinh\left(i\epsilon(n-\dfrac{1}{2})\pi\right)}$$

$$= c \cdot e^{i\epsilon(n-\frac{1}{2})\pi ct/L} \cdot \frac{i\epsilon\sin\left(\dfrac{(n-\frac{1}{2})\pi x}{L}\right)}{1-\left(\dfrac{(n-\frac{1}{2})\pi c}{L}\right)^2)} \cdot \frac{1}{\frac{i\epsilon L}{c}(-1)^{n+1}}.$$

Noting that $i = \frac{-1}{i}$, *adding* the two residues from the poles $s_{n,\pm}$ gives

$$\frac{2c^2}{L}(-1)^{n+1} \cdot \frac{e^{i(n-\frac{1}{2})\pi ct/L} + e^{-i(n-\frac{1}{2})\pi ct/L}}{2} \cdot \frac{\sin\left(\dfrac{(n-\frac{1}{2})\pi x}{L}\right)}{1-\left(\dfrac{(n-\frac{1}{2})\pi c}{L}\right)^2},$$

that is,

$$\frac{2c^2}{L}(-1)^{n+1} \cdot \cos\left(\frac{(n-\frac{1}{2})\pi ct}{L}\right) \cdot \frac{\sin\left(\dfrac{(n-\frac{1}{2})\pi x}{L}\right)}{1-\left(\dfrac{(n-\frac{1}{2})\pi c}{L}\right)^2}.$$

Putting everything together, we get that the solution of the whole problem is

$$u(x,t) = \frac{\sin\left(\frac{x}{c}\right)}{\sin\left(\frac{L}{c}\right)} \cdot \cos t \; + \; \frac{2c^2}{L}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1-\left(\dfrac{(n-\frac{1}{2})\pi c}{L}\right)^2} \cdot \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) \cdot \cos\left(\frac{(n-\frac{1}{2})\pi ct}{L}\right).$$

17.3.2.7. Let $\mathcal{L}_t$ denote the Laplace transformation operation with respect to t and define

$$U(x,s) \triangleq \mathcal{L}_t[\,u(x,t)\,](s) = \int_0^{\infty} u(x,t)\, e^{-st}\, dt.$$

Note that $c = 1$ was assumed. By a property of Laplace transforms, the PDE, that is, $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$, and initial conditions imply that

$$\mathcal{L}_t\Big[\,\frac{\partial^2 u}{\partial t^2}(x,t)\,\Big](s) = s^2 U(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0) = s^2 U(x,s) - s\cdot 0 - 0,$$

hence the PDE implies

$$(\star) \quad s^2\, U(x,s) = \frac{\partial^2 U}{\partial x^2},\; 0 < x < \pi,$$

along with boundary conditions

$$U(0,s) = \mathcal{L}_t[\,u(0,t)\,](s) = \mathcal{L}_t[\,\sin t\,](s) = \frac{1}{s^2+1}, \quad U(\pi,s) = \mathcal{L}_t[\,u(\pi,t)\,](s) = \mathcal{L}_t[\,0\,](s) = 0.$$

The solutions of the “ODE" $(\star)$, that is, $\dfrac{\partial^2 U}{\partial x^2} - s^2\, U(x,s) = 0$, are, by using “clairvoyance" as in Example 9.32 in Section 9.6 and Section 11.3,

$$(\star\star)\quad U(x,s) = A(s)\sinh\big(s(\pi - x)\big) + B(s)\sinh(sx),$$

for arbitrary functions $A(s), B(s)$. The boundary conditions give

$$\frac{1}{s^2+1} = U(0,s) = A(s)\sinh(s\pi) \quad \text{and} \quad 0 = U(\pi,s) = A(s)\cdot 0 + B(s)\sinh(s\pi),$$

hence $B(s) = 0$. The solution of the entire problem has Laplace transform

$$(3\star)\quad U(x,s) = A(s)\sinh\big(s(\pi - x)\big) = \frac{1}{(s^2+1)\sinh(s\pi)}\cdot\sinh\big(s(\pi - x)\big)\,.$$

We will find

$$(4\star)\quad u(x,t) = \mathcal{L}_t^{-1}\left[\frac{\sinh\big(s(\pi - x)\big)}{(s^2+1)\sinh(s\pi)}\right](t)\,.$$

Recall that for real u, v

$$\sinh(u + iv) = \sinh u \cos v + i\cosh u \sin v.$$

The poles of $U(x,s)$, as a function of the complex variable s, are where $s^2 + 1 = 0$ or $\sinh(s\pi) = 0$, that is, the poles are at

$$s_{0,\pm} = \pm\, i \quad \text{and} \quad s_{n,\pm} = \pm\, in,\ n = 1, 2, ...\ .$$

We see that $s_{0,\pm} = s_{1,\pm}$, so we have poles of order two at $s_{0,\pm}$ and simple poles at $s_{n,\pm} = \pm\, in,\ n = 2, 3, , ...$

Note that $s = 0$ is a removable singularity of $U(x,s)$, because there exists

$$\lim_{s\to 0}\ \frac{s}{\sinh(s\pi)} = \lim_{s\to 0}\ \frac{(s)'}{\big(\sinh(s\pi)\big)'} = \lim_{s\to 0}\ \frac{1}{\pi\cosh(s\pi)} = \frac{1}{\pi}\,.$$

For any small but positive real number ε, for $0 < x < \pi$ there are left half circles $S_{\varepsilon,k}$ on which

$$\Big|\sinh\big(s(\pi - x)\big)\Big/\sinh(s\pi)\Big|$$

is bounded, so the factor of $\left|\frac{1}{s^2+1}\right|$ in $|U(x,s)|$ shows that $U(x,s)$ is of class $\mathbf{L}$ in s.

We can apply Theorem 17.7 in Section 17.3 to conclude that

$$u(x,t) = \lim_{s\to i}\left(\frac{d}{ds}\Big[\,(s-i)^2 U(x,s)\, e^{st}\,\Big]\right) + \lim_{s\to -i}\left(\frac{d}{ds}\Big[\,(s+i)^2 U(x,s)\, e^{st}\,\Big]\right) + \sum_{j=2}^{\infty}\ \sum_{\epsilon=\pm 1}\mathsf{Res}\big[\,U(x,s)e^{st};\ s_{j,\epsilon}\,\big].$$

Let us calculate those residues: First, denoting $' = \dfrac{d}{ds}$, we have

$$\lim_{s\to i}\frac{d}{ds}\Big[\,(s-i)^2 U(x,s)\, e^{st}\,\Big] = \lim_{s\to i}\frac{d}{ds}\Big[\,(s-i)^2\cdot\frac{\sinh\big(s(\pi - x)\big)e^{st}}{(s^2+1)\sinh(s\pi)}\,\Big] = \lim_{s\to i}\frac{d}{ds}\Big[\,\frac{(s-i)\cdot\sinh\big(s(\pi - x)\big)e^{st}}{(s+i)\sinh(s\pi)}\,\Big]$$

$$= \lim_{s\to i}\left(\frac{\big((s-i)\cdot\sinh\big(s(\pi - x)\big)e^{st}\big)'\cdot(s+i)\sinh(s\pi) - \big((s-i)\cdot\sinh\big(s(\pi - x)\big)e^{st}\cdot\big((s+i)\sinh(s\pi)\big)'}{(s+i)^2\sinh^2(s\pi)}\right).$$

In the latter, the numerator is

$$\Big(\sinh\big(s(\pi - x)\big)e^{st} + (s-i)(\pi - x)\cosh(s(\pi - x)\big)e^{st} + (s-i)\sinh\big(s(\pi - x)\big)te^{st}\Big)(s+i)\sinh(s\pi)$$

$$-(s-i)\sinh\big(s(\pi - x)\big)e^{st}\cdot\big(\sinh(s\pi) + (s+i)\pi\cosh(s\pi)\big)$$

$$= e^{st}\Big((s+i)\sinh(s\pi)\sinh\big(s(\pi-x)\big) + (s-i)(s+i)(\pi-x)\sinh(s\pi)\cosh(s(\pi-x))$$

$$+(s-i)(s+i)t\sinh(s\pi)\sinh\big(s(\pi-x)\big) - (s-i)\sinh(s\pi)\sinh\big(s(\pi-x)\big) - (s-i)(s+i)\pi\cosh(s\pi)\sinh\big(s(\pi-x)\big)\Big)$$

So, separating terms in the numerator that have factors of *both* $(s-i)$ and $\sinh(s\pi)$, in which case we cancel a factor of $\sinh(s\pi)$ in the denominator, from terms that don't have factors of *both* $(s-i)$ and $\sinh(s\pi)$, we have

$$(5\star)\quad \lim_{s\to i}\left(\frac{d}{ds}\Big[(s-i)^2U(x,s)\,e^{st}\Big]\right)$$

$$= e^{it}\lim_{s\to i}\left(\frac{(s-i)(s+i)(\pi-x)\cosh(s(\pi-x)) + (s-i)(s+i)t\sinh\big(s(\pi-x)\big) - (s-i)\sinh\big(s(\pi-x)\big)}{(s+i)^2\sinh(s\pi)}\right)$$

$$+e^{it}\lim_{s\to i}\left(\frac{(s+i)\sinh(s\pi)\sinh\big(s(\pi-x)\big) - (s-i)(s+i)\pi\cosh(s\pi)\sinh\big(s(\pi-x)\big)}{(s+i)^2\sinh^2(s\pi)}\right).$$

Note that for real θ, $\sinh(i\theta) = i\sin\theta$ and $\cosh(i\theta) = \cos\theta$. In particular, $\sinh(i\pi) = 0$ and $\cosh(i\pi) = -1$.

Concerning the first of the two limits in $(5\star)$, we have

$$\lim_{s\to i}\left(\frac{(s-i)(s+i)(\pi-x)\cosh\big(s(\pi-x)\big) + (s-i)(s+i)\,t\sinh\big(s(\pi-x)\big) - (s-i)\sinh\big(s(\pi-x)\big)}{(s+i)^2\sinh(s\pi)}\right)$$

$$= \lim_{s\to i}\left(\frac{(\pi-x)\cosh(s(\pi-x))}{(s+i)} + \frac{t\,\sinh\big(s(\pi-x)\big)}{(s+i)} - \frac{\sinh\big(s(\pi-x)\big)}{(s+i)^2}\right)\cdot\left(\frac{(s-i)}{\sinh(s\pi)}\right)$$

$$= \left(\frac{(\pi-x)\cos(\pi-x)}{2i} + \frac{t\,i\sin(\pi-x)}{2i} - \frac{i\sin(\pi-x)}{-4}\right)\cdot\left(-\frac{1}{\pi}\right)$$

$$= -\frac{1}{\pi}\left(\frac{1}{2i}(\pi-x)\cos(\pi-x) + \frac{1}{2}\,t\sin(\pi-x) - \frac{1}{4i}\,\sin(\pi-x)\right),$$

after using L'Hôpital's Rule in the "$\frac{0}{0}$" case to get $\displaystyle\lim_{s\to i}\frac{(s-i)}{\sinh(s\pi)} = \lim_{s\to i}\frac{(s-i)'}{\big(\sinh(s\pi)\big)'} = \lim_{s\to i}\frac{1}{\pi\cosh(s\pi)} = -\frac{1}{\pi}$.

Concerning the second of the two limits in $(5\star)$, we have

$$\lim_{s\to i}\left(\frac{(s+i)\sinh(s\pi)\sinh\big(s(\pi-x)\big) - (s-i)(s+i)\pi\cosh(s\pi)\sinh\big(s(\pi-x)\big)}{(s+i)^2\sinh^2(s\pi)}\right)$$

$$= \lim_{s\to i}\left(\frac{\sinh\big(s(\pi-x)\big)}{(s+i)}\cdot\frac{\sinh(s\pi) - (s-i)\pi\cosh(s\pi)}{\sinh^2(s\pi)}\right).$$

L'Hôpital's Rule in the "$\frac{0}{0}$" case gives

$$\lim_{s\to i}\frac{\sinh(s\pi) - (s-i)\pi\cosh(s\pi)}{\sinh^2(s\pi)} = \lim_{s\to i}\frac{\big(\sinh(s\pi) - (s-i)\pi\cosh(s\pi)\big)'}{\big(\sinh^2(s\pi)\big)'}$$

$$= \lim_{s\to i}\frac{\cancel{\pi\cosh(s\pi)} - \cancel{\pi\cosh(s\pi)} - (s-i)\pi^2\sinh(s\pi)}{2\pi\sinh(s\pi)\cosh(s\pi)} = \lim_{s\to i}\frac{-(s-i)\pi}{2\cosh(s\pi)} = 0,$$

so the second of the two limits in $(5\star)$ is

$$\lim_{s\to i}\left(\frac{\sinh\big(s(\pi-x)\big)}{(s+i)}\cdot\frac{\sinh(s\pi) - (s-i)\pi\cosh(s\pi)}{\sinh^2(s\pi)}\right)$$

$$= \lim_{s\to i}\left(\frac{\sinh\big(s(\pi-x)\big)}{(s+i)}\right)\cdot \lim_{s\to i}\left(\frac{\sinh(s\pi)-(s-i)\pi\cosh(s\pi)}{\sinh^2(s\pi)}\right) = \frac{i\sin(\pi-x)}{2i}\cdot 0 = 0.$$

Putting things together, the limit in $(5\star)$ is

$$\lim_{s\to i}\left(\frac{d}{ds}\Big[\,(s+i)^2U(x,s)\,e^{st}\,\Big]\right) = -\,e^{it}\frac{1}{\pi}\left(\frac{1}{2i}\,(\pi-x)\cos(\pi-x)+\frac{1}{2}\,t\sin(\pi-x)-\frac{1}{4i}\,\sin(\pi-x)\right).$$

Second, denoting $' = \dfrac{d}{ds}$, we have

$$\lim_{s\to -i}\frac{d}{ds}\Big[\,(s+i)^2U(x,s)\,e^{st}\,\Big] = \lim_{s\to -i}\frac{d}{ds}\Big[\,(s+i)^2\cdot\frac{\sinh\big(s(\pi-x)\big)e^{st}}{(s^2+1)\sinh(s\pi)}\,\Big] = \lim_{s\to -i}\frac{d}{ds}\Big[\,\frac{(s+i)\cdot\sinh\big(s(\pi-x)\big)e^{st}}{(s-i)\sinh(s\pi)}\,\Big]$$

$$= \lim_{s\to -i}\left(\frac{\big((s+i)\cdot\sinh\big(s(\pi-x)\big)e^{st}\big)'(s-i)\sinh(s\pi)-\big((s+i)\cdot\sinh\big(s(\pi-x)\big)e^{st}\big((s-i)\sinh(s\pi)\big)'}{(s-i)^2\sinh^2(s\pi)}\right).$$

In the latter, the numerator is

$$\big(\sinh\big(s(\pi-x)\big)e^{st}+(s+i)(\pi-x)\cosh(s(\pi-x))e^{st}+(s+i)\sinh\big(s(\pi-x)\big)te^{st}\big)(s-i)\sinh(s\pi)$$

$$-(s+i)\sinh\big(s(\pi-x)\big)e^{st}\cdot\big(\sinh(s\pi)+(s-i)\pi\cosh(s\pi)\big)$$

$$= e^{st}\Big((s-i)\sinh(s\pi)\sinh\big(s(\pi-x)\big)+(s+i)(s-i)(\pi-x)\sinh(s\pi)\cosh(s(\pi-x))$$

$$+(s+i)(s-i)t\sinh(s\pi)\sinh\big(s(\pi-x)\big)-(s+i)\sinh(s\pi)\sinh\big(s(\pi-x)\big)-(s+i)(s-i)\pi\cosh(s\pi)\sinh\big(s(\pi-x)\big)\Big)$$

So, separating terms in the numerator that have factors of *both* $(s+i)$ and $\sinh(s\pi)$ from terms that don't, in which case we cancel a factor of $\sinh(s\pi)$ in the denominator, from terms that don't have factors of *both* $(s+i)$ and $\sinh(s\pi)$, we have

$$(6\star)\quad \lim_{s\to -i}\left(\frac{d}{ds}\Big[\,(s+i)^2U(x,s)\,e^{st}\,\Big]\right)$$

$$= e^{-it}\lim_{s\to -i}\left(\frac{(s+i)(s-i)(\pi-x)\cosh(s(\pi-x))+(s+i)(s-i)t\sinh\big(s(\pi-x)\big)-(s+i)\sinh\big(s(\pi-x)\big)}{(s-i)^2\sinh(s\pi)}\right)$$

$$+e^{-it}\lim_{s\to -i}\left(\frac{(s-i)\sinh(s\pi)\sinh\big(s(\pi-x)\big)-(s+i)(s-i)\pi\cosh(s\pi)\sinh\big(s(\pi-x)\big)}{(s-i)^2\sinh^2(s\pi)}\right).$$

Note that for real θ, $\sinh(i\theta)=i\sin\theta$ and $\cosh(i\theta)=\cos\theta$. In particular, $\sinh(i\pi)=0$ and $\cosh(i\pi)=-1$.

Concerning the first of the two limits in $(6\star)$, we have

$$\lim_{s\to -i}\left(\frac{(s+i)(s-i)(\pi-x)\cosh(s(\pi-x))+(s+i)(s-i)\,t\sinh\big(s(\pi-x)\big)-(s+i)\sinh\big(s(\pi-x)\big)}{(s-i)^2\sinh(s\pi)}\right)$$

$$= \lim_{s\to -i}\left(\frac{(\pi-x)\cosh(s(\pi-x))}{(s-i)}+\frac{t\sinh\big(s(\pi-x)\big)}{(s-i)}-\frac{\sinh\big(s(\pi-x)\big)}{(s-i)^2}\right)\cdot\left(\frac{(s+i)}{\sinh(s\pi)}\right)$$

$$= \left(\frac{(\pi-x)\cos(\pi-x)}{-2i}+\frac{t(-i)\sin(\pi-x)}{-2i}-\frac{(-i)\sin(x)}{-4}\right)\cdot\left(-\frac{1}{\pi}\right)$$

$$= -\,\frac{1}{\pi}\left(-\frac{1}{2i}\,(\pi-x)\cos(\pi-x)+\frac{1}{2}\,t\,\sin(\pi-x)+\frac{1}{4i}\,\sin(\pi-x)\right),$$

after using L'Hôpital's Rule in the "$\frac{0}{0}$" case to get $\lim_{s\to -i} \dfrac{(s+i)}{\sinh(s\pi)} = \lim_{s\to -i} \dfrac{(s+i)'}{\big(\sinh(s\pi)\big)'} = \lim_{s\to -i} \dfrac{1}{\pi\cosh(s\pi)} = -\dfrac{1}{\pi}$.

Concerning the second of the two limits in (6⋆), we have

$$\lim_{s\to -i}\left(\frac{(s-i)\sinh(s\pi)\sinh\big(s(\pi-x)\big)-(s+i)(s-i)\pi\cosh(s\pi)\sinh\big(s(\pi-x)\big)}{(s-i)^2\sinh^2(s\pi)}\right)$$

$$=\lim_{s\to -i}\left(\frac{\sinh\big(s(\pi-x)\big)}{(s-i)}\cdot\frac{\sinh(s\pi)-(s+i)\pi\cosh(s\pi)}{\sinh^2(s\pi)}\right).$$

L'Hôpital's Rule in the "$\frac{0}{0}$" case gives

$$\lim_{s\to -i}\frac{\sinh(s\pi)-(s+i)\pi\cosh(s\pi)}{\sinh^2(s\pi)}=\lim_{s\to -i}\frac{\big(\sinh(s\pi)-(s+i)\pi\cosh(s\pi)\big)'}{\big(\sinh^2(s\pi)\big)'}$$

$$=\lim_{s\to -i}\frac{\cancel{\pi\cosh(s\pi)}-\cancel{\pi\cosh(s\pi)}-(s+i)\pi^2\sinh(s\pi)}{2\pi\sinh(s\pi)\cosh(s\pi)}=\lim_{s\to -i}\frac{-(s+i)\pi}{2\cosh(s\pi)}=0,$$

so the second of the two limits in (6⋆) is

$$\lim_{s\to -i}\left(\frac{\sinh\big(s(\pi-x)\big)}{(s-i)}\cdot\frac{\sinh(s\pi)-(s+i)\pi\cosh(s\pi)}{\sinh^2(s\pi)}\right)$$

$$=\lim_{s\to -i}\left(\frac{\sinh\big(s(\pi-x)\big)}{(s-i)}\right)\cdot\lim_{s\to -i}\left(\frac{\sinh(s\pi)-(s+i)\pi\cosh(s\pi)}{\sinh^2(s\pi)}\right)=\frac{(-i)\sin(\pi-x)}{-2i}\cdot 0=0.$$

Putting things together, the limit in (6⋆) is

$$\lim_{s\to -i}\left(\frac{d}{ds}\Big[(s+i)^2U(x,s)\,e^{st}\Big]\right)=e^{-it}\cdot\frac{-1}{\pi}\left(-\frac{1}{2i}(\pi-x)\cos(\pi-x)+\frac{1}{2}t\,\sin(\pi-x)+\frac{1}{4i}\,\sin(\pi-x)\right)$$

Combining the residues for $s=\pm i$ gives

$$\lim_{s\to i}\left(\frac{d}{ds}\Big[(s-i)^2U(x,s)\,e^{st}\Big]\right)+\lim_{s\to -i}\left(\frac{d}{ds}\Big[(s+i)^2U(x,s)\,e^{st}\Big]\right)$$

$$=-\,e^{it}\frac{1}{\pi}\left(\frac{1}{2i}(\pi-x)\cos(\pi-x)+\frac{1}{2}t\sin(\pi-x)-\frac{1}{4i}\,\sin(\pi-x)\right)$$

$$-e^{-it}\cdot\frac{1}{\pi}\left(-\frac{1}{2i}(\pi-x)\cos(\pi-x)+\frac{1}{2}t\,\sin(\pi-x)+\frac{1}{4i}\,\sin(\pi-x)\right)$$

$$=-\frac{1}{\pi}\left((\pi-x)\sin(t)\cos(\pi-x)+\frac{1}{2}t\cos(t)\sin(\pi-x)-\frac{1}{2}\,\sin(t)\sin(\pi-x)\right).$$

Similarly, for $n\ge 2$ and $\epsilon=\pm 1$, $s_{n,\epsilon}=i\epsilon\cdot n$, so

$$\mathsf{Res}\big[\,U(x,s)e^{st};\ s_{n,\epsilon}\,\big]=\lim_{s\to i\epsilon n}(s-i\epsilon n)\cdot e^{st}\cdot\frac{\sinh\big(s(\pi-x)\big)}{1+s^2}\cdot\frac{1}{\sinh(s\pi)}=e^{i\epsilon nt}\cdot\frac{\sinh\big(i\epsilon n(\pi-x)\big)}{1-n^2}\cdot\lim_{s\to i\epsilon n}\frac{(s-i\epsilon n)}{\sinh(s\pi)}.$$

L'Hôpital's Rule in the "$\frac{0}{0}$" case gives that this residue equals

$$=e^{i\epsilon nt}\cdot\frac{\sinh\big(i\epsilon n(\pi-x)\big)}{1-n^2}\cdot\frac{1}{\pi\cosh(i\epsilon n\pi)}=e^{i\epsilon nt}\cdot\frac{i\epsilon\sin\big(n(\pi-x)\big)}{1-n^2}\cdot\frac{1}{\pi(-1)^n}\ .$$

Noting that $i = \frac{-1}{i}$, *adding* the two residues from the poles $s_{n,\pm}$ gives

$$\frac{2}{\pi}\,(-1)^{n+1} \cdot \frac{e^{int} - e^{-int}}{2i} \cdot \frac{\sin\big(n(\pi - x)\big)}{1 - n^2},$$

that is,

$$\frac{2}{\pi}\,(-1)^{n+1} \cdot \sin(nt) \cdot \sin\big(n(\pi - x)\big)\Big/(1 - n^2).$$

Putting everything together, we get that the solution of the whole problem is

$$u(x,t) = -\frac{1}{\pi}\left((\pi - x)\sin(t)\cos(\pi - x) + \frac{1}{2}\,t\cos(t)\sin(\pi - x) - \frac{1}{2}\,\sin(t)\sin(\pi - x)\right)$$

$$+\ \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{1 - n^2} \cdot \sin\big(n(\pi - x)\big) \cdot \sin(nt).$$

17.3.2.9. Substituting $q = \sqrt{s}$,

$$\frac{1}{s-1} = \frac{1}{q^2 - 1} = \frac{A}{q-1} + \frac{B}{q+1} \quad \Longleftrightarrow \quad 1 = A(q+1) + B(q-1).$$

Substituting in $q = 1$ implies $1 = 2A$; substituting in $q = -1$ implies $1 = -2B$, so

$$F(s) = \frac{e^{-x\sqrt{s}}}{s-1} = \left(e^{-x\sqrt{s}}\right) \cdot \frac{1}{s-1} = e^{-x\sqrt{s}}\left(\frac{1/2}{q-1} - \frac{1/2}{q+1}\right) = \frac{e^{-x\sqrt{s}}}{2}\,\left(\frac{1}{\sqrt{s}\,-1} - \frac{1}{\sqrt{s}\,+1}\right).$$

Using entry L2.9 of Table 17.2 with $a = 1$ and $k = x$ gives

$$\mathcal{L}^{-1}\left[\frac{e^{-x\sqrt{s}}}{1+\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}e^{-x^2/(4t)} - e^{x}e^{t}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}} + \sqrt{t}\right)$$

and with $a = -1$ and $k = x$ gives

$$\mathcal{L}^{-1}\left[\frac{e^{-x\sqrt{s}}}{-1+\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}e^{-x^2/(4t)} + e^{-x}e^{t}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}} - \sqrt{t}\right).$$

So,

$$\begin{aligned}\mathcal{L}^{-1}\Big[F(s)\Big] &= \mathcal{L}^{-1}\left[\frac{e^{-x\sqrt{s}}}{s-1}\right] = -\frac{1}{2}\mathcal{L}^{-1}\left[\frac{e^{-x\sqrt{s}}}{1+\sqrt{s}}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{e^{-x\sqrt{s}}}{-1+\sqrt{s}}\right]\\ &= -\frac{1}{2\sqrt{\pi t}}e^{-x^2/(4t)} + \frac{1}{2}\,e^{x}e^{t}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}} + \sqrt{t}\right) + \frac{1}{2\sqrt{\pi t}}e^{-x^2/(4t)} + \frac{1}{2}e^{-x}e^{t}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}} - \sqrt{t}\right)\\ &= \frac{e^{t}}{2}\left(e^{-x}\mathrm{erfc}\Big(\frac{x}{2\sqrt{t}} - \sqrt{t}\Big) + e^{x}\mathrm{erfc}\Big(\frac{x}{2\sqrt{t}} + \sqrt{t}\Big)\right).\end{aligned}$$

17.3.2.11. The Errata webpage adds that we should assume ICs $u(x,0) = v(x,0) = 0$ for $0 < x < \infty$.

Let $\mathcal{L}_t$ denote the Laplace transformation operation with respect to t and define

$$U(x,s) \triangleq \mathcal{L}_t[\,u(x,t)\,](s) = \int_0^{\infty} u(x,t)\,e^{-st}\,dt \quad \text{and} \quad V(x,s) \triangleq \mathcal{L}_t[\,v(x,t)\,](s) = \int_0^{\infty} v(x,t)\,e^{-st}\,dt.$$

By a property of Laplace transforms and using the ICs $u(x,0) = v(x,0) = 0$,

$$\mathcal{L}_t\Big[\,\frac{\partial u}{\partial t}(x,t)\,\Big](s) = s\,U(x,s) - u(x,0) = s\,U(x,s) - 0 = s\,U(x,s)$$

and

$$\mathcal{L}_t\Big[\,\frac{\partial v}{\partial t}(x,t)\,\Big](s) = s\,V(x,s) - v(x,0) = s\,V(x,s) - 0 = s\,V(x,s),$$

hence the PDE system becomes

$$\left\{\begin{array}{l} s\,U(x,s) = \alpha\,\dfrac{\partial^2 U}{\partial x^2} \\[2ex] s\,V(x,s) = \alpha\,\dfrac{\partial^2 V}{\partial x^2} + \beta\,s\,U(x,s) \end{array}\right\}$$

along with boundary conditions

$$U(0,s) = \mathcal{L}_t[\,u(0,t)\,](s) = \mathcal{L}_t[\,1\,](s) = \frac{1}{s} \quad \text{and} \quad \infty > \lim_{x\to\infty}|U(x,s)| = \lim_{x\to\infty}|\mathcal{L}_t[\,u(x,t)\,](s)|$$

and

$$\frac{\partial V}{\partial x}(0,s) = \mathcal{L}_t\Big[\,\frac{\partial v}{\partial x}(0,t)\,\Big](s) = \mathcal{L}_t[\,-1\,](s) = -\,\frac{1}{s} \quad \text{and} \quad \infty > \lim_{x\to\infty}|V(x,s)| = \lim_{x\to\infty}|\mathcal{L}_t[\,v(x,t)\,](s)|.$$

First, let's solve

$$(\star) \qquad \left\{\begin{array}{c} \dfrac{\partial^2 U}{\partial x^2} - \dfrac{s}{\alpha}\,U(x,s) = 0 \\[2ex] U(0,s) = \dfrac{1}{s} \\[2ex] \lim_{x\to\infty}|U(x,s)| < \infty \end{array}\right\}.$$

The first equation can be considered as a second order linear ODE for the x dependence, so the solutions are

$$U(x,s) = A(s)e^{(x/\sqrt{\alpha})\sqrt{s}} \;+\; B(s)e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$

The condition that $\lim_{x\to\infty}|U(x,s)| < \infty$ requires that $A(s) \equiv 0$, hence

$$U(x,s) = B(s)e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$

The BC $U(0,s) = \dfrac{1}{s}$ implies that $B(s) = \dfrac{1}{s}$, hence the solution of the first PDE has Laplace transform

$$U(x,s) = \frac{1}{s}\cdot e^{-(x/\sqrt{\alpha})\sqrt{s}}$$

With $k = (x/\sqrt{\alpha})$, entry L2.7 of Table 17.2 at the end of Section 17.3 gives

$$u(x,t) = \mathcal{L}_t^{-1}\left[\,\frac{1}{s}\cdot e^{(x/\sqrt{\alpha})\sqrt{s}}\,\right] = \operatorname{erfc}\left(\frac{x/\sqrt{\alpha}}{2\sqrt{t}}\right) = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right).$$

It follows that, using the definition of $\operatorname{erfc}(\theta)$ found after the problem 17.2.2.7, for $x > 0$ we have

$$\mathcal{L}_t\left[\,\frac{\partial u}{\partial t}\,\right] = s\,\mathcal{L}_t\,[\,u\,] - u(x,0) = s\cdot\frac{1}{s}\cdot e^{-(x/\sqrt{\alpha})\sqrt{s}} \;-\; \lim_{t\to 0^+}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$= e^{-(x/\sqrt{\alpha})\sqrt{s}} \;-\; \lim_{t\to 0^+}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{x/(2\sqrt{\alpha t})} e^{-y^2}\,dy\right)$$

$$= e^{-(x/\sqrt{\alpha})\sqrt{s}} \;-\; \lim_{t\to\infty}\big(1 - \operatorname{erf}(\theta)\big) = e^{-(x/\sqrt{\alpha})\sqrt{s}} \;-\; 0 = e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$

$\Big[$Or,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{x/(2\sqrt{\alpha t})} e^{-y^2}\,dy\right] = -\frac{2}{\sqrt{\pi}}e^{-(x/(2\sqrt{\alpha t}))^2}\cdot\frac{\partial}{\partial t}\left[\frac{x}{(2\sqrt{\alpha t})}\right] = -\frac{2}{\sqrt{\pi}}\,e^{-x^2/(4\alpha t)}\cdot\left(-\frac{x}{4\sqrt{\alpha t^3}}\right)$$

$$= \frac{x}{2\sqrt{\pi\alpha t^3}}\,e^{-x^2/(4\alpha t)}$$

has

$$\mathcal{L}_t\left[\frac{\partial u}{\partial t}\right] = \mathcal{L}_t\left[\frac{x}{2\sqrt{\pi\alpha t^3}}\,e^{-x^2/(4\alpha t)}\right] = e^{-(x/\sqrt{\alpha})\sqrt{s}}.\ \Big]$$

Substitute $\mathcal{L}_t\left[\dfrac{\partial u}{\partial t}\right] = e^{-(x/\sqrt{\alpha})\sqrt{s}}$ into the Laplace transform of the second PDE to get

$$s\,V(x,s) = \alpha\,\frac{\partial^2 V}{\partial x^2} + \beta\,e^{-(x/\sqrt{\alpha})\sqrt{s}},$$

that is

$$\frac{\partial^2 V}{\partial x^2} - \frac{s}{\alpha}\,V = -\,\frac{\beta}{\alpha}\,e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$

Considered as a nonhomogeneous, linear, second order ODE for the x dependence, the easiest way to solve this is by using the method of undetermined coefficients:

$$V(x,s) = C(s)e^{(x/\sqrt{\alpha})\sqrt{s}} + E(s)e^{-(x/\sqrt{\alpha})\sqrt{s}} + V_p(x,s),$$

where $V_p(x,s) = F(s)\,x\,e^{-(x/\sqrt{\alpha})\sqrt{s}}$ is the form for a particular solution. Substituting this, along with

$$\frac{\partial V_p}{\partial x} = F(s)\left(1 - \frac{x\sqrt{s}}{\sqrt{\alpha}}\right)e^{-(x/\sqrt{\alpha})\sqrt{s}}, \quad \text{hence} \quad \frac{\partial^2 V_p}{\partial x^2} = F(s)\left(-\frac{2\sqrt{s}}{\sqrt{\alpha}} + \frac{xs}{\alpha}\right)e^{-(x/\sqrt{\alpha})\sqrt{s}},$$

into the ODE gives

$$-\,\frac{\beta}{\alpha}\,e^{-(x/\sqrt{\alpha})\sqrt{s}} = \frac{\partial^2 V_p}{\partial x^2} - \frac{s}{\alpha}\,V_p = F(s)\left(-\frac{2\sqrt{s}}{\sqrt{\alpha}} + \frac{xs}{\alpha}\right)e^{-(x/\sqrt{\alpha})\sqrt{s}} \;-\; \frac{s}{\alpha}\,F(s)\,x\,e^{-(x/\sqrt{\alpha})\sqrt{s}}$$

$$= -F(s)\,\frac{2\sqrt{s}}{\sqrt{\alpha}}e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$

So

$$F(s) = \frac{\beta}{2\sqrt{\alpha s}},$$

and thus

$$V(x,s) = C(s)e^{(x/\sqrt{\alpha})\sqrt{s}} \;+\; E(s)e^{-(x/\sqrt{\alpha})\sqrt{s}} \;+\frac{\beta}{2\sqrt{\alpha s}}\,x\,e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$

The condition that $\lim_{x\to\infty}|V(x,s)| < \infty$ requires that $C(s)\equiv 0$, hence

$$V(x,s) = E(s)e^{-(x/\sqrt{\alpha})\sqrt{s}} \;+\frac{\beta}{2\sqrt{\alpha s}}\,x\,e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$

The BC for $\dfrac{\partial V}{\partial x}$ at $x=0$ implies that

$$-\frac{1}{s} = \frac{\partial V}{\partial x}(0,s) = -\,\frac{\sqrt{s}}{\sqrt{\alpha}}\,E(s) \;+\frac{\beta}{2\sqrt{\alpha s}}$$

hence

$$E(s) = \frac{\sqrt{\alpha}}{s\sqrt{s}} \;+\frac{\beta}{2s}.$$

The solution of the second PDE has Laplace transform

$$V(x,s)=\left(\frac{\sqrt{\alpha}}{s\sqrt{s}}\ +\frac{\beta}{2s}\right)\cdot e^{-(x/\sqrt{\alpha})\sqrt{s}}\ +\frac{\beta}{2\sqrt{\alpha s}}\,x\,e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$

With $k=(x/\sqrt{\alpha})$, entry L2.6 of Table 17.2 at the end of Section 17.3 gives

$$\mathcal{L}_t^{-1}\left[\frac{1}{\sqrt{s}}\cdot e^{-(x/\sqrt{\alpha})\sqrt{s}}\right]=\frac{1}{\sqrt{\pi t}}\,e^{-x^2/(4\alpha t)}\,.$$

With $k=(x/\sqrt{\alpha})$, entry L2.7 of Table 17.2 at the end of Section 17.3 gives

$$\mathcal{L}_t^{-1}\left[\frac{1}{s}\cdot e^{-(x/\sqrt{\alpha})\sqrt{s}}\right]=\operatorname{erfc}\left(\frac{x}{\sqrt{2\alpha t}}\right)\,.$$

With $k=(x/\sqrt{\alpha})$, entry L2.8 of Table 17.2 at the end of Section 17.3 gives

$$\mathcal{L}_t^{-1}\left[\frac{1}{s\sqrt{s}}\cdot e^{-(x/\sqrt{\alpha})\sqrt{s}}\right]=2\cdot\sqrt{\frac{t}{\pi}}\cdot e^{-x^2/(4\alpha t)}-\frac{x}{\sqrt{\alpha}}\cdot\operatorname{erfc}\left(\frac{x}{\sqrt{2\alpha t}}\right)\,.$$

The solution of the second PDE is

$$\begin{aligned}v(x,t)&=\mathcal{L}_t^{-1}\left[\,V(x,s)\,\right]\\ &=\sqrt{\alpha}\cdot\left(2\cdot\sqrt{\frac{t}{\pi}}\cdot e^{-x^2/(4\alpha t)}-\frac{x}{\sqrt{\alpha}}\cdot\operatorname{erfc}\left(\frac{x}{\sqrt{2\alpha t}}\right)\right)+\frac{\beta}{2}\cdot\operatorname{erfc}\left(\frac{x}{\sqrt{2\alpha t}}\right)+\frac{\beta}{2\sqrt{\alpha}}\cdot x\cdot\frac{1}{\sqrt{\pi t}}\,e^{-x^2/(4\alpha t)}\end{aligned}$$

that is,

$$v(x,t)=\left(\frac{\beta}{2}-x\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)+\left(2\sqrt{\frac{\alpha t}{\pi}}+\frac{\beta x}{2\sqrt{\pi\alpha t}}\right)e^{-x^2/(4\alpha t)}.$$

In summary, the solution of the original system of PDEs is

$$\left\{\begin{array}{l}u(x,t)=\operatorname{erfc}\left(\dfrac{x}{2\sqrt{\alpha t}}\right)\\[2ex] v(x,t)=\left(\dfrac{\beta}{2}-x\right)\operatorname{erfc}\left(\dfrac{x}{2\sqrt{\alpha t}}\right)+\left(2\sqrt{\dfrac{\alpha t}{\pi}}+\dfrac{\beta x}{2\sqrt{\pi\alpha t}}\right)e^{-x^2/(4\alpha t)}\end{array}\right\}.$$

Section 17.4.1

17.4.1.1. First, if $f(r)$ is a function of only r, then in spherical coordinates $\Delta f(r)=\frac{1}{r}\frac{d}{dr}\Big[r\frac{df}{dr}(r)\Big]$, so, indeed,

$$\Delta^2 f(r)=\frac{1}{r}\frac{d}{dr}\Big[r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\big[r\frac{df}{dr}(r)\big]\right]\Big].$$

Let $g(r)=\Delta f(r)=\frac{1}{r}\frac{d}{dr}\Big[r\frac{df}{dr}(r)\Big]$. We have, by definition of the Hankel transform, that

$$\mathcal{H}_0\left[\,\Delta^2 f(r)\,\right]=\mathcal{H}_0\left[\,\Delta g(r)\,\right]\triangleq\int_0^\infty\frac{1}{r}\frac{d}{dr}\Big[r\frac{dg}{dr}(r)\Big]\,J_0(kr)r\,dr=\int_0^\infty J_0(kr)\Big(\frac{d}{dr}\Big[r\frac{dg}{dr}(r)\Big]\,dr\Big).$$

The definition of improper integral and integration by parts with $u=J_0(kr)$, $du=\Big(\frac{d}{dr}\big[\,J_0(kr)\,\big]\Big)dr$ and $dv=\Big(\frac{d}{dr}\big[\,r\frac{dg}{dr}(r)\,\big]\Big)dr$, give

$$\mathcal{H}_0\left[\,\Delta^2 f(r)\,\right]=\lim_{b\to\infty}\int_0^b J_0(kr)\Big(\frac{d}{dr}\Big[r\frac{dg}{dr}(r)\Big]\,dr\Big)=\lim_{b\to\infty}\left(r\frac{dg}{dr}(r)\,J_0(kr)\Big|_0^b\ -\int_0^b r\frac{dg}{dr}(r)\,\Big(\frac{d}{dr}\big[\,J_0(kr)\,\big]\Big)dr\right)$$

$$= \lim_{b\to\infty}\left(b\,\frac{dg}{dr}(b)\,J_0(kb) - 0 \;+\; \int_0^b \Big(r\,\frac{d}{dr}\big[\,J_0(kr)\,\big]\Big)\Big(\frac{dg}{dr}(r)\,dr\Big)\right)$$

$$= \lim_{b\to\infty}\left(\frac{d}{dr}\Big[\,r\frac{dg}{dr}(r)\Big]\Big|_{r=b}\cdot J_0(kb) + k\int_0^b \Big(r\,\frac{d}{dr}\big[\,J_0(kr)\,\big]\Big)\Big(\frac{dg}{dr}(r)\,dr\Big)\right)$$

$$= 0 + \lim_{b\to\infty}\int_0^b \Big(r\,\frac{d}{dr}\big[\,J_0(kr)\,\big]\Big)\Big(\frac{dg}{dr}(r)\,dr\Big),$$

as long as $\lim_{r\to 0^+}|g'(r)|<\infty$ and $\sqrt{r}g'(r)$ is bounded as $r\to\infty$, after using the fact that $\sqrt{r}\dfrac{d}{dr}\big[\,J_0(kr)\,\big] = \sqrt{r}\big(-kJ_1(kr)\big)\to 0$, as $r\to\infty$.

Further, use integration by parts with $dv = \dfrac{dg}{dr}(r)\,dr$, $u = r\,\dfrac{d}{dr}\big[\,J_0(kr)\,\big]$, and $\dfrac{du}{dr} = \dfrac{d}{dr}\Big[\,r\,\dfrac{d}{dr}\big[\,J_0(kr)\,\big]\Big]$. This gives

$$\mathcal{H}_0\left[\,\Delta g(r)\,\right] = \lim_{b\to\infty}\int_0^b \Big(r\,\frac{d}{dr}\big[\,J_0(kr)\,\big]\Big)\Big(\frac{dg}{dr}(r)\,dr\Big)$$

$$= \lim_{b\to\infty}\left(g(r)\,\Big(r\,\frac{d}{dr}\big[\,J_0(kr)\,\big]\Big)\Big|_0^b - \int_0^b g(r)\Big(\frac{d}{dr}\Big[\,r\,\frac{d}{dr}\big[\,J_0(kr)\,\big]\Big]\Big)dr\Big)\right).$$

But, $J_0(kr)$ satisfies Bessel's equation of order zero, so $\dfrac{d}{dr}\Big[\,r\,\dfrac{d}{dr}\big[\,J_0(kr)\,\big]\Big] = -k^2\,r\,J_0(kr)$. It follows that

$$\mathcal{H}_0\left[\,\Delta g(r)\,\right] = \lim_{b\to\infty}\left(kg(b)\,k\,b\,J_1(kb) - 0 \;-k^2\int_0^b g(r)\,r\,J_0(kr)dr\right),$$

So,

$$\mathcal{H}_0\left[\,\Delta g(r)\,\right] = -k^2\mathcal{H}_0\left[\,g(r)\,\right],$$

as long as $\sqrt{r}g'(r)$ and $\sqrt{r}g(r)$ are bounded as $r\to\infty$, using the fact that $\sqrt{r}J_1(kr)\to 0$ as $r\to\infty$, as well as $\lim_{r\to 0^+}|g(r)|<\infty$.

We can use this result twice to solve problem 17.4.1.1:

$$\mathcal{H}_0\left[\,\Delta^2 f(r)\,\right] = \mathcal{H}_0\left[\,\Delta g(r)\,\right] = -k^2\mathcal{H}_0\left[\,g(r)\,\right] = -k^2\left(\mathcal{H}_0\left[\,\Delta f(r)\,\right]\right) = -k^2\left(-k^2\mathcal{H}_0\left[\,f(r)\,\right]\right) = k^4\mathcal{H}_0\left[\,f(r)\,\right],$$

as long as $\sqrt{r}g'(r)$, $\sqrt{r}g(r)$, $\sqrt{r}f'(r)$, and $\sqrt{r}f(r)$ are bounded as $r\to\infty$, where $g(r) = \Delta f(r) = \dfrac{1}{r}\dfrac{d}{dr}\Big[\,r\dfrac{df}{dr}(r)\Big]$.

In addition, we assumed that $\lim_{r\to 0^+}|g'(r)|<\infty$, $\lim_{r\to 0^+}|g(r)|<\infty$, $\lim_{r\to 0^+}|f'(r)|<\infty$, and $\lim_{r\to 0^+}|f(r)|<\infty$.

17.4.1.3. Define $C(k,z) = \mathcal{H}_0\left[\,c(r,z)\,\right]$. By Theorem 17.9 in Section 17.4, $\mathcal{H}_0\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\Big[r\dfrac{\partial}{\partial r}c(r,z)\Big]\right] = -k^2\mathcal{H}_0\left[\,c(r,z)\,\right]$. So, the PDE $0 = \nabla^2 c = \dfrac{\partial^2 c}{\partial z^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}\Big[\,r\,\dfrac{\partial}{\partial r}c(r,z)\Big]$ implies that

$$(\star)\quad 0 = -k^2 C + \frac{\partial^2 C}{\partial z^2}.$$

Considered as a second order ODE, the solutions are

$$C(k,z) = A(k)e^{-kz} + B(k)e^{kz}.$$

Physically, we expect that $\lim_{z\to\infty} C(r,z) = 0$, so we require that $B(k)\equiv 0$, so $C(k,z) = A(k)e^{-kz}$.

Denote

$$F(k) \triangleq \mathcal{H}_0\left[\,c(r,0)\,\right] = \mathcal{H}_0\left[\left\{\begin{array}{ll} c_0, & 0\le r<a \\ 0, & a\le r<\infty\end{array}\right\}\right].$$

The last BC, for $c(r,0)$, we need to satisfy by having

$$F(k) = \mathcal{H}_0\left[\, c(r,0)\,\right] = C(k,0) = A(k).$$

The Hankel transform of the solution of the whole problem is $C(k,z) = F(k)e^{-kz}$, so the solution of the whole problem is

$$c(r,z) = \mathcal{H}_0^{-1}\left[\, C(k,z)\,\right] = \mathcal{H}_0^{-1}\left[\, F(k)e^{-kz}\,\right] = \int_0^\infty F(k)e^{-kz}J_0(kr)k\,dk = \int_0^\infty \Big(\mathcal{H}_0\left[\, c(r,0)\,\right]\Big)e^{-kz}J_0(kr)k\,dk.$$

Using the same method as in Example 17.10 in Section 17.4, we have

$$c(r,z) = \int_0^\infty \left(\int_0^\infty c(\xi,0)J_0(k\xi)\xi\,d\xi\right)e^{-kz}J_0(rk)k\,dk = \int_0^\infty \left(\int_0^\infty e^{-kz}J_0(\xi k)J_0(rk)k\,dk\right)c(\xi,0)\,\xi\,d\xi.$$

Using problem 17.4.1.2's result (17.59), with $a = z$, $b = \xi$, and $c = r$, we have

$$c(r,z) = \int_0^\infty \left(-\frac{1}{\pi\sqrt{\xi r}}\cdot\frac{z}{\xi r}\,Q'_{-\frac{1}{2}}\left(\frac{z^2+\xi^2+r^2}{2\xi r}\right)\right)c(\xi,0)\xi\,d\xi$$

so the BC being $c(r,0) = \left\{\begin{array}{ll} c_0, & 0 \le r < a \\ 0, & a \le r < \infty \end{array}\right\}$ implies that the solution of the whole problem is

$$c(r,z) = -\frac{c_0 z}{\pi r^{3/2}}\int_0^a \frac{1}{\xi^{3/2}}\,Q'_{-\frac{1}{2}}\left(\frac{z^2+r^2+\xi^2}{2\xi r}\right)d\xi.$$

17.4.1.5. Take the Hankel transform of both sides of the PDE $\dfrac{\partial^2 w}{\partial t^2} = -\beta^2\nabla^4 w$ and denote $W(k,t) \triangleq \mathcal{H}_0\left[\, w(r,t)\,\right]$. Using the result of problem 17.4.1.1, the PDE implies

$$\frac{\partial^2 W}{\partial t^2}(k,t) = -\beta^2 k^4 w(k,t),$$

assuming $w(r,t)$ satisfies the conditions, as $r \to 0^+$ and as $r \to \infty$, which answers the question of problem 17.4.1.1.

Considered as the second order linear ODE of undamped harmonic oscillation, the solution of

$$\frac{\partial^2 W}{\partial t^2}(k,t) + (\beta^2 k^2)^2 w(k,t) = 0$$

is

$$W(k,t) = A(k)\cos(\beta\, k^2 t) + B(k)\sin(\beta\, k^2 t).$$

The appropriate initial conditions are

$$w(r,0) = f(r) \quad \text{and} \quad \frac{\partial w}{\partial t}(r,0) = g(r),\ 0 < r < \infty.$$

Defining the corresponding Hankel transforms by $F(k) \triangleq \mathcal{H}_0\left[\, f(r)\,\right]$ and $G(k) \triangleq \mathcal{H}_0\left[\, g(r)\,\right]$, we get

$$F(k) = \mathcal{H}_0\left[\, w(r,0)\,\right] = W(k,0) = A(k)$$

and

$$G(k) = \mathcal{H}_0\left[\,\frac{\partial w}{\partial t}(r,0)\,\right] = \frac{\partial W}{\partial t}(k,0) = \beta k^2 B(k).$$

So, the Hankel transform of the solution of the whole, original problem is

$$W(k,t) = F(k)\cos(\beta\, k^2 t) + \beta^{-1}k^{-2}G(k)\sin(\beta\, k^2 t).$$

So, the solution of the whole, original problem is

$$w(r,t)=\mathcal{H}_0^{-1}\left[F(k)\cos(\beta\,k^2t)+\frac{1}{\beta k^2}\,G(k)\sin(\beta\,k^2t)\right]$$

$$=\int_0^\infty \cos(\beta\,k^2t)\,F(k)\,J_0(rk)k\,dk+\int_0^\infty \sin(\beta\,k^2t)\,\frac{1}{\beta k^2}\,G(k)\,J_0(rk)k\,dk.$$

We can analyze this further using the method following Example 17.10 in Section 17.4: First,

$$\int_0^\infty \cos(\beta\,k^2t)\,F(k)\,J_0(rk)k\,dk=\int_0^\infty \cos(\beta\,k^2t)\left(\int_0^\infty f(\xi)\,J_0(k\xi)\xi\,d\xi\right)J_0(rk)k\,dk$$

$$=\int_0^\infty\left(\int_0^\infty \cos(\beta\,k^2t)\,J_0(k\xi)J_0(rk)k\,dk\right)f(\xi)\,\xi\,d\xi.$$

We can further rewrite this by noting that $\cos(\beta\,k^2t)=\frac{1}{2}\left(e^{i\beta\,k^2t}+e^{-i\beta\,k^2t}\right)$, so, using Weber's second exponential integral with $D=-i\beta$ and with $D=i\beta$

$$\int_0^\infty \cos(\beta\,k^2t)\,J_0(k\xi)J_0(rk)k\,dk=\int_0^\infty \frac{1}{2}\left(e^{-(-i\beta)k^2t}+e^{-(i\beta)k^2t}\right)J_0(k\xi)J_0(rk)k\,dk$$

$$=\frac{1}{-2i\beta t}\,\exp\left(-\frac{r^2+\xi^2}{-4i\beta t}\right)I_0\left(\frac{r\xi}{-2i\beta t}\right)+\frac{1}{2i\beta t}\,\exp\left(-\frac{r^2+\xi^2}{4i\beta t}\right)I_0\left(\frac{r\xi}{2i\beta t}\right).$$

It happens that $I_0(z)$ is an even function of z and $I_0(z)=J_0(iz)$, so

$$\int_0^\infty \cos(\beta\,k^2t)\,J_0(k\xi)J_0(rk)k\,dk=\frac{1}{\beta t}\cdot I_0\left(-i\,\frac{r\xi}{2\beta t}\right)\cdot\frac{1}{2i}\left(-\exp\left(-i\,\frac{r^2+\xi^2}{4\beta t}\right)+\exp\left(i\,\frac{r^2+\xi^2}{4\beta t}\right)\right)$$

$$=\frac{1}{\beta t}\cdot I_0\left(-i\,\frac{r\xi}{2\beta t}\right)\cdot\sin\left(\frac{r^2+\xi^2}{4\beta t}\right)$$

$$=\frac{1}{\beta t}\cdot J_0\left(\frac{r\xi}{2\beta t}\right)\cdot\sin\left(\frac{r^2+\xi^2}{4\beta t}\right).$$

So,

$$\int_0^\infty \cos(\beta\,k^2t)\,F(k)\,J_0(rk)k\,dk=\frac{1}{\beta t}\cdot\int_0^\infty J_0\left(\frac{r\xi}{2\beta t}\right)\cdot\sin\left(\frac{r^2+\xi^2}{4\beta t}\right)f(\xi)\,\xi\,d\xi.$$

Second, defining $h(r)\triangleq\mathcal{H}_0^{-1}\left[\frac{1}{\beta k^2}\,G(k)\right]$, we have

$$\int_0^\infty \sin(\beta\,k^2t)\left(\frac{1}{\beta k^2}\,G(k)\right)J_0(rk)k\,dk=\int_0^\infty \sin(\beta\,k^2t)\left(\int_0^\infty h(\xi)\,J_0(k\xi)\xi\,d\xi\right)J_0(rk)\,k\,dk$$

$$=\int_0^\infty\left(\int_0^\infty \sin(\beta\,k^2t)\,J_0(k\xi)J_0(rk)\,k\,dk\right)h(\xi)\,\xi\,d\xi.$$

We can further rewrite this by noting that $\sin(\beta\,k^2t)=\frac{1}{2i}\left(e^{i\beta\,k^2t}-e^{-i\beta\,k^2t}\right)$, so, using Weber's second exponential integral with $D=-i\beta$ and with $D=i\beta$

$$\int_0^\infty \sin(\beta\,k^2t)\,J_0(k\xi)J_0(rk)k\,dk=\int_0^\infty \frac{1}{2i}\left(e^{-(-i\beta)k^2t}-e^{-(i\beta)k^2t}\right)J_0(k\xi)J_0(rk)\,k\,dk$$

$$=\frac{1}{2\beta t}\,\exp\left(-\frac{r^2+\xi^2}{-4i\beta t}\right)I_0\left(\frac{r\xi}{-2i\beta t}\right)+\frac{1}{2\beta t}\,\exp\left(-\frac{r^2+\xi^2}{4i\beta t}\right)I_0\left(\frac{r\xi}{2i\beta t}\right).$$

It happens that $I_0(z)$ is an even function of z and $I_0(z) = J_0(iz)$, so

$$\int_0^\infty \cos(\beta\, k^2 t)\, J_0(k\xi) J_0(rk)\, k\, dk = \frac{1}{\beta t} \cdot I_0\left(-i\, \frac{r\xi}{2\beta t}\right) \cdot \frac{1}{2}\left(\exp\left(-i\, \frac{r^2+\xi^2}{4\beta t}\right) + \exp\left(i\, \frac{r^2+\xi^2}{4\beta t}\right)\right)$$

$$= \frac{1}{\beta t} \cdot I_0\left(-i\, \frac{r\xi}{2\beta t}\right) \cdot \cos\left(\frac{r^2+\xi^2}{4\beta t}\right)$$

$$= \frac{1}{\beta t} \cdot J_0\left(\frac{r\xi}{2\beta t}\right) \cdot \cos\left(\frac{r^2+\xi^2}{4\beta t}\right).$$

So,

$$\int_0^\infty \sin(\beta\, k^2 t)\left(\frac{1}{\beta k^2}\, G(k)\right) J_0(rk) k\, dk = \frac{1}{\beta t} \cdot \int_0^\infty J_0\left(\frac{r\xi}{2\beta t}\right) \cdot \cos\left(\frac{r^2+\xi^2}{4\beta t}\right) h(\xi)\, \xi\, d\xi.$$

To summarize, the solution of the original problem is

$$w(r,t) = \int_0^\infty \cos(\beta\, k^2 t)\, F(k)\, J_0(rk) k\, dk + \int_0^\infty \sin(\beta\, k^2 t)\, \frac{1}{\beta k^2}\, G(k)\, J_0(rk) k\, dk$$

that is,

$$w(r,t) = \frac{1}{\beta t} \cdot \int_0^\infty J_0\left(\frac{r\xi}{2\beta t}\right) \cdot \left(\sin\left(\frac{r^2+\xi^2}{4\beta t}\right) f(\xi) \; + \; \cos\left(\frac{r^2+\xi^2}{4\beta t}\right) h(\xi)\right) \xi\, d\xi,$$

where $f(r) = w(r,0)$ and $h(r) \triangleq \mathcal{H}_0^{-1}\left[\frac{1}{\beta k^2}\, \mathcal{H}_0\left[\frac{\partial w}{\partial t}(r,0)\right]\right]$.

Section 18.1.5

18.1.5.1. $0 = \begin{vmatrix} -1-\lambda & \sqrt{3} \\ \sqrt{3} & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) - 3 = \lambda^2 - 4 = (\lambda+2)(\lambda-2)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -2, \lambda_2 = 2$.

Immediately, this tells us that the equilibrium point at the origin is a saddle point, because there is one positive and one negative eigenvalue.

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 1 & \sqrt{3} & 0 \\ & & \\ \sqrt{3} & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} ① & \sqrt{3} & 0 \\ & & \\ 0 & 0 & 0 \end{array}\right], \text{ after } -\sqrt{3}\,R_1 + R_2 \to R_2$$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -2$.

$$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -3 & \sqrt{3} & 0 \\ & & \\ \sqrt{3} & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} ① & -\frac{1}{\sqrt{3}} & 0 \\ & & \\ 0 & 0 & 0 \end{array}\right], \text{ after } -\tfrac{1}{3}R_1 \to R_1, -\sqrt{3}\,R_1 + R_2 \to R_2$$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 2$

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

The straight line solutions are on the lines through the vectors $\mathbf{v}_1 = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$. The curved solutions approach the line through $\mathbf{v}_2$ as $t \to \infty$ and approach the line through $\mathbf{v}_1$ as $t \to -\infty$.

18.1.5.3. $0 = \begin{vmatrix} 1-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 8 = \lambda^2 + 2\lambda + 5 = (\lambda^2 + 2\lambda + 1) + 4 = (\lambda+1)^2 + 4$

$\Rightarrow$ eigenvalues are $\lambda_1 = -1 + 2i, \lambda_2 = -1 - 2i$.

Immediately, this tells us that the equilibrium point at the origin is a stable spiral point because the eigenvalues are $-1 \pm i2$.

18.1.5.5. I'm sorry, this is the same problem as 18.1.5.3.

18.1.5.7. $0 = \begin{vmatrix} -1-\lambda & \sqrt{3} \\ \sqrt{3} & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) - 3 = \lambda^2 - 4 = (\lambda+2)(\lambda-2)$

$\Rightarrow$ eigenvalues are $\lambda_1 = -2, \lambda_2 = 2$

Immediately, this tells us that the equilibrium point at the origin is a saddle point, because there is one positive and one negative eigenvalue.

$$[\,A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 1 & \sqrt{3} & 0 \\ & & \\ \sqrt{3} & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} ① & \sqrt{3} & 0 \\ & & \\ 0 & 0 & 0 \end{array}\right], \text{ after } -\sqrt{3}\,R_1 + R_2 \to R_2$$

$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_1 = -2$.

$$[\,A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -3 & \sqrt{3} & 0 \\ & & \\ \sqrt{3} & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} ① & -\frac{1}{\sqrt{3}} & 0 \\ & & \\ 0 & 0 & 0 \end{array}\right], \text{ after } -\tfrac{1}{3}R_1 \to R_1, -\sqrt{3}\,R_1 + R_2 \to R_2$$

$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, for any constant $c_1 \neq 0$, are the eigenvectors corresponding to eigenvalue $\lambda_2 = 2$

The general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \text{where } c_1, c_2 = \text{arbitrary constants.}$$

The straight line solutions are on the lines through the vectors $\mathbf{v}_1 = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$. The curved solutions approach the line through $\mathbf{v}_2$ as $t \to \infty$ and approach the line through $\mathbf{v}_1$ as $t \to -\infty$.

A phase plane picture is in the figure.

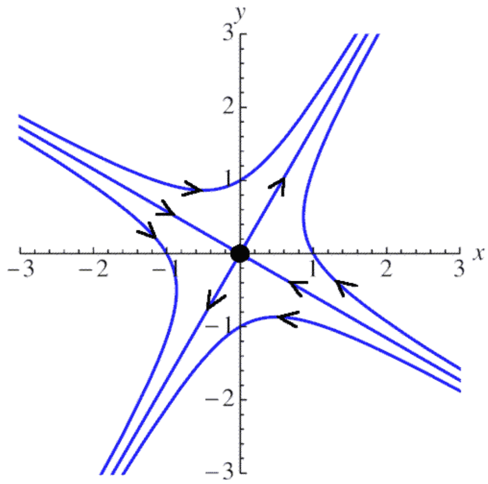


Figure 1: Answer key for problem 18.1.5.7

18.1.5.9. By uniqueness of solutions of an ODE system [unfortunately, not seen until Theorem 18.21 in Section 18.7 but analogous to Theorem 3.6 in Section 3.2], $f(t,x) = x - \frac{3}{2}x^2$ being continuously differentiable everywhere implies that the solution of the IVP $(\star\star)$ $\ddot{x} - x + \dfrac{3}{2}x^2 = 0$, $x(0) = 1$, $\dot{x}(0) = 0$, is unique. So, it will suffice to explain why $(\star)$ $\quad x(t) = 1 - \left(\dfrac{1-e^t}{1+e^t}\right)^2$ does solve IVP $(\star\star)$.

First, we note that

$$\dot{x}(t) = 0 - 2\left(\frac{1-e^t}{1+e^t}\right)\frac{d}{dt}\left[\frac{1-e^t}{1+e^t}\right] = -2\left(\frac{1-e^t}{1+e^t}\right)\cdot\frac{(1-e^t)'(1+e^t) - (1+e^t)'(1-e^t)}{(1+e^t)^2}$$

$$= -2\left(\frac{1-e^t}{1+e^t}\right)\cdot\frac{-2e^t}{(1+e^t)^2},$$

so

$$(\star\star\star) \quad \dot{x}(t) = \frac{4e^t(1-e^t)}{(1+e^t)^3} = \frac{4e^t - 4e^{2t}}{(1+e^t)^3}.$$

It follows that

$$\ddot{x}(t) = \frac{(4e^t - 8e^{2t})(1+e^t)^3 - 3e^t(1+e^t)^2(4e^t - 4e^{2t})}{(1+e^t)^6} = \frac{(4e^t - 8e^{2t})(1+e^t) - 3e^t(4e^t - 4e^{2t})}{(1+e^t)^4},$$

so

$$\ddot{x}(t) = \frac{4e^t - 16e^{2t} + 4e^{3t}}{(1+e^t)^4}.$$

Substitute this into the LHS of the ODE to get

$$\ddot{x} - x + \frac{3}{2}x^2 = \frac{4e^t - 16e^{2t} + 4e^{3t}}{(1+e^t)^4} - 1 + \left(\frac{1-e^t}{1+e^t}\right)^2 + \frac{3}{2}\cdot\left(1 - \left(\frac{1-e^t}{1+e^t}\right)^2\right)^2$$

$$= \frac{4e^t - 16e^{2t} + 4e^{3t}}{(1+e^t)^4} - \frac{(1+e^t)^4}{(1+e^t)^4} + \frac{(1-e^t)^2(1+e^t)^2}{(1+e^t)^4} + \frac{3}{2}\left(\frac{(1+e^t)^4}{(1+e^t)^4} - \frac{2(1-e^t)^2}{(1+e^t)^2} + \frac{(1-e^t)^4}{(1+e^t)^4}\right)$$

$$= \frac{1}{2(1+e^t)^4}\Big(2(4e^t - 16e^{2t} + 4e^{3t}) - \underline{2(1+e^t)^4} + \underline{2(1-e^t)^2(1+e^t)^2} + \underline{3(1+e^t)^4} - \underline{6(1-e^t)^2(1+e^t)^2} + 3(1-e^t)^4\Big)$$

$$= \frac{1}{2(1+e^t)^4}\Big(8e^t - 32e^{2t} + 8e^{3t} + (1 + 4e^t + 6e^{2t} + 4e^{3t} + e^{4t}) - 4(1 - 2e^{2t} + e^{4t}) + 3(1 - 4e^t + 6e^{2t} - 4e^{3t} + e^{4t})\Big)$$

$$= \frac{1}{2(1+e^t)^4}\Big(1 - 4 + 3 + (8 + 4 - 12)e^t + (-32 + 6 + 8 + 18)e^{2t} + (8 + 4 - 12)e^{3t} + (1 - 4 + 3)e^{4t}\Big) = 0,$$

as we desired. We also have

$$x(0) = 1 - \left(\frac{1-e^0}{1+e^0}\right)^2 = 1 - \left(\frac{1-1}{1+1}\right)^2 = 1 \quad \text{and} \quad \dot{x}(0) = \frac{4e^0(1-e^0)}{(1+e^0)^3} = 0,$$

as desired.

(b) Define $\mathbf{x}(t) = [\, x(t) \quad \dot{x}(t)\,]^T$ to be the solution of IVP $(\star\star)$ $\ddot{x} - x + \frac{3}{2}x^2 = 0$, $x(0) = 1$, $\dot{x}(0) = 0$, rewritten as a system. Let $\mathbf{x}(t;t_0) = [\, x(t;t_0) \quad \dot{x}(t;t_0)\,]^T$ to be the solution of IVP $(4\star)$ $\ddot{x} - x + \frac{3}{2}x^2 = 0$, $x(t_0) = 1$, $\dot{x}(t_0) = 0$, rewritten as a system.

Theorem 18.1 in Section 18.1 implies that $\mathbf{x}(t;t_0) = \mathbf{x}(t - t_0)$. Specifically with $t_0 = 2$, we have that

$$x(t-2) \triangleq 1 - \left(\frac{1-e^{t-2}}{1+e^{t-2}}\right)^2$$

solves the IVP $\ddot{x} - x + \frac{3}{2}x^2 = 0$, $x(2) = 1$, $\dot{x}(2) = 0$.

Section 18.2.3

18.2.3.1. Equilibrium points (x, y) for this LCCHS (linear constant coefficients homogeneous systems) $\dot{\mathbf{x}} = A\mathbf{x}$ satisfy

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Because $\det(A) = \begin{vmatrix} 1 & 4 \\ -1 & -3 \end{vmatrix} = 1 \neq 0$, the only equilibrium point is at $(x, y) = (0, 0)$.

(b) Because the system is linear and has constant coefficients, the Jacobian matrix is $\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0,0)\right] = A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$ and the linearization about the equilibrium point is also $\dot{\mathbf{x}} = A\mathbf{x}$.

The eigenvalues of A satisfy

$$0 = |\, A - \lambda I \,| = \begin{vmatrix} 1-\lambda & 4 \\ -1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2,$$

so the eigenvalues are $\lambda_1 = \lambda_2 = -1$. By Theorem 18.4 in Section 18.2, $(x, y) = (0, 0)$ is asymptotically stable.

18.2.3.3. (a) Equilibrium points (x_1, x_2) for this system satisfy

$$\left\{ \begin{array}{rl} 0 = & -x_1\big(1 - x_1^2 - x_2^2\big) \\ \\ 0 = & -x_2\big(1 - x_1^2 - x_2^2\big) \end{array} \right\}$$

The first equation is satisfied only when either (1) $x_1 = 0$ or (2) $x_1^2 + x_2^2 = 1$. The second equation is satisfied only when either (3) $x_2 = 0$ or (4) $x_1^2 + x_2^2 = 1$.

In principle, there are four possibilities:

$$(1) \quad \text{and} \quad (3): \quad x_1 = 0 \quad \text{and} \quad x_2 = 0: \ (x_1, x_2) = (0, 0)$$

$$(1) \quad \text{and} \quad (4): \quad x_1 = 0 \quad \text{and} \quad x_1^2 + x_2^2 = 1: \ (x_1, x_2) = (0, \pm 1)$$

$$(2) \quad \text{and} \quad (3): \quad x_1^2 + x_2^2 = 1 \quad \text{and} \quad x_2 = 0: \ (x_1, x_2) = (\pm 1, 0)$$

$$(2) \quad \text{and} \quad (4): \quad x_1^2 + x_2^2 = 1 \quad \text{and} \quad x_1^2 + x_2^2 = 1: \ \text{all points on the circle } x_1^2 + x_2^2 = 1.$$

To summarize, the equilibrium points consist of $(x_1, x_2) = (0, 0)$ and all points on the circle $x_1^2 + x_2^2 = 1$.

(b) The Jacobian matrix is

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(x_1, x_2)\right] = \begin{bmatrix} -1 + x_2^2 + 3x_1^2 & 2x_1x_2 \\ 2x_1x_2 & -1 + x_1^2 + 3x_2^2 \end{bmatrix}.$$

At the equilibrium point $(x_1, x_2) = (0, 0)$, the linearization is

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x},$$

so the eigenvalues are $\lambda_1 = \lambda_2 = -1$. By Theorem 18.4 in Section 18.2, $(x_1, x_2) = (0, 0)$ is asymptotically stable.

At an equilibrium point $(x_1, x_2) = (\alpha, \pm\sqrt{1 - \alpha^2})$, the Jacobian matrix is

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(x_1, x_2)\right] = \begin{bmatrix} -1 + (1 - \alpha^2) + 3\alpha^2 & \pm 2\alpha\sqrt{1 - \alpha^2} \\ \\ \pm 2\alpha\sqrt{1 - \alpha^2} & -1 + \alpha^2 + 3(1 - \alpha^2) \end{bmatrix},$$

whose eigenvalues λ satisfy

$$0 = \begin{vmatrix} 2\alpha^2 - \lambda & \pm 2\alpha\sqrt{1 - \alpha^2} \\ \\ \pm 2\alpha\sqrt{1 - \alpha^2} & 2(1 - \alpha^2) - \lambda \end{vmatrix} = (2\alpha^2 - \lambda)\big(2(1 - \alpha^2) - \lambda\big) - 4\alpha^2(1 - \alpha^2)$$

$$= \lambda^2 - 2\lambda + 0 = \lambda(\lambda - 2).$$

so the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2 > 0$. By Theorem 18.4 in Section 18.2, an eigenvalue being positive implies that all equilibria on the circle $x_1^2 + x_2^2 = 1$ are unstable.

18.2.3.5. (a) Equilibrium points (x, y) for this system satisfy

$$\left\{ \begin{array}{rc} 0 = & 8x - y^2 \\ \\ 0 = & x^2 - y \end{array} \right\}$$

The second equation is satisfied only when $y = x^2$. Substitute this into the first equation to get

$$0 = 8x - (x^2)^2 = 8x - x^4 = x(8 - x^3) = x(2 - x)(4 + 2x + x^2).$$

The solutions are $x = 0$, $x = 2$, and $x = -1 \pm i\sqrt{3}$, so the only real number solutions are $x = 0$ and $x = 2$. Correspondingly, because $y = x^2$, the only equilibria are at $(x, y) = (0, 0)$ and $(2, 4)$.

The Jacobian matrix is

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(x, y)\right] = \begin{bmatrix} 8 & -2y \\ 2x & -1 \end{bmatrix}.$$

At the equilibrium point $(x, y) = (0, 0)$, the linearization is

$$\dot{\mathbf{x}} = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x},$$

so the eigenvalues are $\lambda_1 = -1 < 0$, $\lambda_2 = 8 > 0$. By Theorem 18.4 in Section 18.2, $(x, y) = (0, 0)$ is an unstable equilibrium point.

At the equilibrium point $(x, y) = (2, 4)$, the linearization is

$$\dot{\mathbf{x}} = \begin{bmatrix} 8 & -8 \\ 4 & -1 \end{bmatrix} \mathbf{x},$$

so the eigenvalues satisfy

$$0 = \begin{vmatrix} 8-\lambda & -8 \\ 4 & -1-\lambda \end{vmatrix} = (8-\lambda)(-1-\lambda) + 32 = \lambda^2 - 7\lambda + 24 = \left(\lambda - \frac{7}{2}\right)^2 + \frac{47}{4},$$

so the eigenvalues are $\lambda_{1,2} = \frac{7}{2} \pm i\,\frac{\sqrt{47}}{2}$, whose real parts are positive. By Theorem 18.4 in Section 18.2, $(x, y) = (2, 4)$ is an unstable equilibrium point.

18.2.3.7. (a) Equilibrium points (x, y) for this system satisfy

$$\left\{ \begin{array}{rl} 0 = & 2\sin x + y \\ \\ 0 = & \sin x - 3y \end{array} \right\}$$

The first equation is satisfied only when $y = -2\sin x$, and the second equation is satisfied only when $y = \frac{1}{3}\sin x$, so x must satisfy

$$-2\sin x = \frac{1}{3}\sin x \quad \Longleftrightarrow \quad 0 = \frac{7}{3}\sin x \quad \Longleftrightarrow \quad \sin x = 0.$$

The solutions are $x = n\pi$, where n is any integer. Correspondingly, $y = -2\sin n\pi = 0$, so the only equilibria are at $(x, y) = (n\pi, 0)$, where n is any integer.

(b) The Jacobian matrix is

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(x, y)\right] = \begin{bmatrix} 2\cos x & 1 \\ \cos x & -3 \end{bmatrix}.$$

At an equilibrium point $(x, y) = (n\pi, 0)$, the linearization is

$$\dot{\mathbf{x}} = \begin{bmatrix} 2(-1)^n & 1 \\ (-1)^n & -3 \end{bmatrix} \mathbf{x},$$

so the eigenvalues satisfy

$$0 = \begin{vmatrix} 2(-1)^n - \lambda & 1 \\ (-1)^n & -3-\lambda \end{vmatrix} = \big(2(-1)^n - \lambda\big)(-3-\lambda) - (-1)^n = \lambda^2 + \big(3 - 2(-1)^n\big)\lambda - 7(-1)^n.$$

The eigenvalues are

$$\lambda = \frac{-3 + 2(-1)^n \pm \sqrt{\big(3 - 2(-1)^n\big)^2 + 28(-1)^n}}{2} = \frac{-3 + 2(-1)^n \pm \sqrt{13 + 16(-1)^n}}{2}.$$

If $n = even$, that is, if $n = 2k$ for some integer k, then the eigenvalues are

$$\lambda_1 = \frac{-1 + \sqrt{29}}{2} > 0 \quad \text{and} \quad \lambda_2 = \frac{-1 - \sqrt{29}}{2} < 0.$$

Because $\lambda_1 > 0$, Theorem 18.4 in Section 18.2 implies that the points $(x,y) = (2k\pi, 0)$ are unstable equilibria.

If $n = \mathit{odd}$, that is, if $n = 2\ell - 1$ for some integer ℓ, then the eigenvalues are

$$\lambda_1 = \frac{-5 + i\sqrt{3}}{2} \quad \text{and} \quad \lambda_2 = \frac{-5 - i\sqrt{3}}{2}.$$

Because all, that is, both, of the eigenvalues have negative real part, Theorem 18.4 in Section 18.2 implies that the points $(x,y) = \big((2\ell - 1)\pi, 0\big)$ are asymptotically stable equilibria.

18.2.3.9. (a) Equilibrium points (x,y) for this system satisfy

$$\left\{ \begin{array}{rrl} 0 = & -0.1x + 0.02xy = & -0.02x(5-y) \\ \\ 0 = & 0.2y - 0.4xy - 0.05y^2 = & 0.05y\big(4 - 8x - y\big) \end{array} \right\}$$

The first equation is satisfied only when either (1) $x = 0$ or (2) $y = 5$. The second equation is satisfied only when either (3) $y = 0$ or (4) $y = 4 - 8x$.

In principle, there are four possibilities:

$$\begin{array}{lllllll} (1) & \text{and} & (3): & x = 0 & \text{and} & y = 0: & (x,y) = (0,0) \\ \\ (1) & \text{and} & (4): & x = 0 & \text{and} & y = 4 - 8x: & (x,y) = (0,4) \\ \\ (2) & \text{and} & (3): & y = 5 & \text{and} & y = 0: & \mathit{impossible} \\ \\ (2) & \text{and} & (4): & y = 5 & \text{and} & y = 4 - 8x: & (x,y) = \left(-\frac{1}{8}, 5\right). \end{array}$$

To summarize, the equilibrium points consist of $(x,y) = (0,0)$, $(0,4)$, and $\left(-\frac{1}{8}, 5\right)$.

The Jacobian matrix is

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(x,y)\right] = \begin{bmatrix} -0.1 + 0.02y & 0.02x \\ -0.4y & 0.2 - 0.4x - 0.1y \end{bmatrix}.$$

(b) At the equilibrium point $(x,y) = (0,0)$, the linearization is

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \mathbf{x},$$

so the eigenvalues are $\lambda_1 = -0.1 < 0$ and $\lambda_2 = 0.2 > 0$. By Theorem 18.4 in Section 18.2, the equilibrium point $(x,y) = (0,0)$ is unstable.

At the equilibrium point $(x,y) = (0,4)$, the Jacobian matrix is the lower triangular matrix

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(x,y)\right] = \begin{bmatrix} -0.02 & 0 \\ -1.6 & -0.2 \end{bmatrix},$$

so the eigenvalues λ are $\lambda_1 = -0.2 < 0$, $\lambda_2 = -0.02 < 0$. By Theorem 18.4 in Section 18.2, the equilibrium point $(x,y) = (0,4)$ is asymptotically stable.

At the equilibrium point $(x,y) = \left(-\frac{1}{8}, 5\right)$, the Jacobian matrix is

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(x,y)\right] = \begin{bmatrix} 0 & -0.0025 \\ -2 & -0.25 \end{bmatrix},$$

so the eigenvalues λ satisfy

$$0 = \begin{vmatrix} -\lambda & -0.0025 \\ -2 & -0.25 - \lambda \end{vmatrix} = (-\lambda)(-0.25 - \lambda) - 0.005 = \lambda^2 + 0.25\lambda - 0.005 = (\lambda + 0.125)^2 - \frac{33}{1600}$$

whose eigenvalues are $\lambda_1 = -0.125 + \frac{\sqrt{33}}{40} \approx 0.018614 > 0$ and $\lambda_2 = -0.125 - \frac{\sqrt{33}}{40} < 0$. Because $\lambda_1 > 0$, Theorem 18.4 in Section 18.2 implies that the equilibrium point $(x, y) = \left(-\dfrac{1}{8}, 5\right)$ is unstable.

18.2.3.11. (a) Let $x_1 = \theta$ and $x_2 = \dot{\theta}$, so $\dot{x}_1 = x_2$ and $\dot{x}_2 = \ddot{\theta} = -2b\dot{\theta} - \omega^2\theta = -2bx_2 - \omega^2 x_1$. Let $\mathbf{x} = [\,x_1 \quad x_2\,]^T$. The equivalent system of two first order ODEs is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2b \end{bmatrix} \mathbf{x}.$$

(b) [This problem was corrected in the Errata webpage to include the assumption that $\omega > 0$ and $b \ge 0$ are constants.] Equilibrium points (x_1, x_2) for this LCCHS (linear constant coefficients homogeneous systems) $\dot{\mathbf{x}} = A\mathbf{x}$ satisfy

$$\begin{bmatrix} 0 & 1 \\ \omega^2 & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Because $\det(A) = \begin{vmatrix} 0 & 1 \\ -\omega^2 & -2b \end{vmatrix} = \omega^2 \neq 0$, the only equilibrium point is at $(x, y) = (0, 0)$. Because the system is linear and has constant coefficients, the Jacobian matrix is $\left[\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(0,0)\right] = A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2b \end{bmatrix}$ and the linearization about the equilibrium point is also $\dot{\mathbf{x}} = A\mathbf{x}$.

The eigenvalues of A satisfy

$$0 = |\,A - \lambda I\,| = \begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -2b - \lambda \end{vmatrix} = (-\lambda)(-2b - \lambda) + -\omega^2 = \lambda^2 + 2b\lambda + \omega^2 = (\lambda - b)^2 + (\omega^2 - b^2),$$

so the eigenvalues are either $\lambda = -b \pm i\sqrt{\omega^2 - b^2}$, when $\omega \ge b$, or $\lambda = -b \pm \sqrt{b^2 - \omega^2}$, when $b \ge \omega$.

By Theorem 18.4 in Section 18.2, $(\theta, \dot{\theta}) = (0, 0)$ is asymptotically stable if $b > 0$.

If $b = 0$, we cannot use linearization to decide if the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$ is stable or unstable. Why? Because Theorem 18.4 in Section 18.2 does not give conclusions when there is an eigenvalue whose real part is zero. [But, using results from Section 5.3 instead of using linearization, we could study stability of the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$.]

18.2.3.13. Suppose β is a constant. Study the stability of the origin $(x_1, x_2, x_3) = (0, 0, 0)$ for the system

$$\begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \\ \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 + \beta x_1\left(x_1^2 + x_2^2 + x_3^2\right) \\ \\ x_3 - x_1 + \beta x_2\left(x_1^2 + x_2^2 + x_3^2\right) \\ \\ x_1 - x_2 + \beta x_3\left(x_1^2 + x_2^2 + x_3^2\right) \end{bmatrix}.$$

Define $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \ge 0$.

If $\beta \neq 0$ then an equilibrium point (x_1, x_2, x_3) must satisfy $0 = \dot{x}_1 = x_2 - x_3 + \beta x_1\left(x_1^2 + x_2^2 + x_3^2\right)$, $0 = \dot{x}_2 = x_3 - x_1 + \beta x_2\left(x_1^2 + x_2^2 + x_3^2\right)$, and $0 = \dot{x}_3 = x_1 - x_2 + \beta x_3\left(x_1^2 + x_2^2 + x_3^2\right)$, hence

$$\begin{aligned} x_3 - x_2 &= \beta x_1\left(x_1^2 + x_2^2 + x_3^2\right) = \beta x_1 r^2 \\ x_1 - x_3 &= \beta x_2\left(x_1^2 + x_2^2 + x_3^2\right) = \beta x_2 r^2 \\ x_2 - x_1 &= \beta x_3\left(x_1^2 + x_2^2 + x_3^2\right) = \beta x_3 r^2. \end{aligned}$$

Multiply both sides of the first equation by x_1, multiply the second equation by x_2, and multiply the third equation by x_3, and then add, canceling all six terms on the LHS, to get

$$0 = (x_3 - x_2)x_1 + (x_1 - x_3)x_2 + (x_2 - x_1)x_3 = \beta(x_1^2 r^2 + x_2^2 r^2 + x_3^2 r^2) = \beta r^2(x_1^2 + x_2^2 + x_3^2) = \beta r^4.$$

Because $r^2 = r(t)^2 = x_1^2 + x_2^2 + x_3^2 \ge 0$, with equality only if $x_1 = x_2 = x_3 = 0$, it follows that $0 = \beta r^4$ and $\beta \neq 0$ imply that $x_1 = x_2 = x_3 = 0$. So, the only equilibrium point is at $(x_1, x_2, x_3) = (0, 0, 0)$, no matter what is the sign of $\beta \neq 0$.

Define

$$V(t) \triangleq \big(r(t)\big)^2 = \big(x_1(t)\big)^2 + \big(x_2(t)\big)^2 + \big(x_3(t)\big)^2.$$

Fix a value of $\beta \neq 0$. We calculate

$$\begin{aligned}\dot{V}(t) &= 2x_1(t)\dot{x}_1(t) + 2x_2\dot{x}_2(t) + 2x_3\dot{x}_3(t)\\ &= 2x_1\big(\cancel{x_2} - \cancel{x_3} + \beta x_1 r^2\big) + 2x_2\big(\cancel{x_3} - \cancel{x_1} + \beta x_2 r^2\big) + 2x_3\big(\cancel{x_1} - \cancel{x_2} + \beta x_3 r^2\big)\\ &= 2\beta r^2(x_1^2 + x_1^2 + x_1^2) = \beta r^4 = 2\beta V^2.\end{aligned}$$

The separable differential equation $\dot{V} = 2\beta V^2$ has solution

$$V(t) = \frac{1}{V_0 - 2\beta t},$$

where $V_0 \triangleq V(0)$.

If $\beta < 0$ then $\dot{V}(t) = 2\beta\big(V(t)\big)^2 < 0$ for all $\big(x_1(t), x_2(t), x_3(t)\big) \neq (0,0,0)$. It follows that $(0,0,0)$ is stable. In addition, $\big(x_1(t)\big)^2 + \big(x_2(t)\big)^2 + \big(x_3(t)\big)^2 = V(t) = \frac{1}{V_0 - 2\beta t} \to 0$ as $t \to \infty$, hence $(0,0,0)$ is an attractor. It follows that $(0,0,0)$ is asymptotically stable if $\beta < 0$.

If $\beta > 0$ then

$$\big(x_1(t)\big)^2 + \big(x_2(t)\big)^2 + \big(x_3(t)\big)^2 = V(t) = \frac{1}{V_0 - 2\beta t} \to \infty, \text{ as } t \to \frac{V_0}{2\beta}^{+}.$$

It follows that $(0,0,0)$ is unstable if $\beta > 0$.

Finally, if $\beta = 0$, then the system is the LCCHS $\dot{\mathbf{x}} = A\mathbf{x}$, where $\mathbf{x} = [\,x_1 \quad x_2 \quad x_3\,]^T$ and $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. Expanding the determinant along the first row, we see that A's eigenvalues λ satisfy

$$0 = |\,A - \lambda I\,| = \begin{vmatrix} -\lambda & 1 & -1 \\ -1 & -\lambda & 1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & 1 \\ 1 & -\lambda \end{vmatrix} + (-1) \cdot \begin{vmatrix} -1 & -\lambda \\ 1 & -1 \end{vmatrix}$$

$$= -\lambda(\lambda^2 + 1) - (\lambda - 1) - (1 + \lambda) = -\lambda^3 - 3\lambda = -\lambda(\lambda^2 + 3),$$

so the eigenvalues are $\lambda_1 = 0$, $\lambda_{2,3} = \pm i\sqrt{3}$. If $\beta = 0$, then $(0,0,0)$ is stable but not asymptotically stable, by Theorem 5.11 in Section 5.3.

Section 18.3.3

18.3.3.1. The first ODE in the system is $\dot{x} = x$, whose solutions are $x(t) = c_1 e^t$, where c_1 is an arbitrary constant. Substitute this into the second ODE to get the linear, first order ODE

$$\dot{y} = -c_1^2 e^{2t} - y, \quad \text{that is} \quad \dot{y} + y = -c_1^2 e^{2t}.$$

The integrating factor $\mu(t) = e^t$ turns this into $\frac{d}{dt}\big[\,e^t\,y\,\big] = -c_1^2 e^{3t}$, so

$$y(t) = e^{-t}\left(-\frac{1}{3}\,c_1^2 e^{3t} + c_2\right) = -\frac{1}{3}\,c_1^2 e^{2t} + c_2 e^{-t}.$$

The solution of the original system of ODEs is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ -\frac{1}{3}\,c_1^2 e^{2t} + c_2 e^{-t} \end{bmatrix},$$

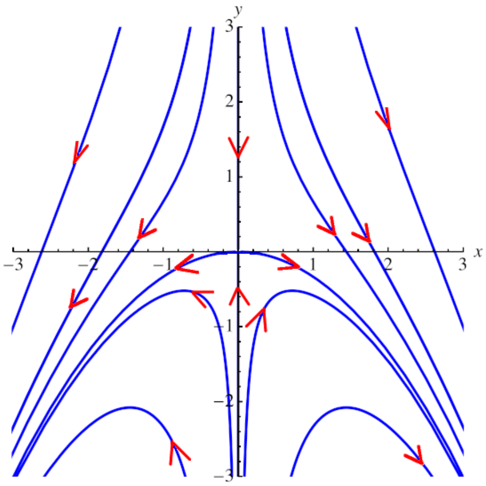


Figure 2: Answer key for problem18.3.3.1

where c_1 and c_2 are arbitrary constants. The phase plane picture is shown in the figure.

18.3.3.3. For the original, nonlinear system $(\star)$ $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} x_1^3 \\ \sin t \end{bmatrix}$, we will use a variation of parameters formula, as in work for Example 18.14 in Section 18.3.

Using the method of Section 5.2 we construct the principal fundamental matrix for the corresponding linear homogeneous system of ODEs $(\star\star)$ $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{x}$:

$$0 = \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 4 = (\lambda+2)(\lambda-2) \Rightarrow \text{ eigenvalues are } \lambda_1 = -2, \lambda_2 = 2$$

$$[\, A - \lambda_1 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ after } -R_1 + R_2 \to R_2,\ \tfrac{1}{3} R_1 \to R_1$$

$$\Rightarrow \mathbf{v}_1 = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_1 = -2$$

$$[\, A - \lambda_2 I \mid \mathbf{0}\,] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ after } 3R_1 + R_2 \to R_2,\ -R_1 \to R_1$$

$$\Rightarrow \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ for any constant } c_1 \neq 0, \text{ are the eigenvectors corresponding to eigenvalue } \lambda_2 = 2$$

The general solution of $(\star\star)$ is

$$\mathbf{x} = c_1 e^{-2t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which gives a fundamental matrix

$$Z(t) = \begin{bmatrix} e^{-2t} & e^{2t} \\ -3e^{-2t} & e^{2t} \end{bmatrix}$$

and thus principal fundamental matrix

$$X(t) = Z(t)\big(Z(0)\big)^{-1} = \begin{bmatrix} e^{-2t} & e^{2t} \\ -3e^{-2t} & e^{2t} \end{bmatrix} \left(\frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \right) = \frac{1}{4} \begin{bmatrix} e^{-2t} + 3e^{2t} & -e^{-2t} + e^{2t} \\ -3e^{-2t} + 3e^{2t} & 3e^{-2t} + e^{2t} \end{bmatrix}.$$

Because $(\star\star)$ has no $2\pi-$periodic solution, Theorem 18.6 in Section 18.3 guarantees that the original, nonlinear system $(\star)$ has a $2\pi-$periodic solution $\mathbf{x}(t)$ that satisfies

$$\mathbf{x}(t) = \frac{1}{4}\begin{bmatrix} e^{-2t}+3e^{2t} & -e^{-2t}+e^{2t} \\ -3e^{-2t}+3e^{2t} & 3e^{-2t}+e^{2t} \end{bmatrix}\cdot$$

$$\cdot\left(\mathbf{x}_0 + \int_0^t \left(\frac{1}{4}\begin{bmatrix} e^{-2s}+3e^{2s} & -e^{-2s}+e^{2s} \\ -3e^{-2s}+3e^{2s} & 3e^{-2s}+e^{2s} \end{bmatrix}\right)^{-1}\begin{bmatrix} x_1^3(s) \\ \sin s \end{bmatrix} ds\right),$$

that is,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{-2t}+3e^{2t} & -e^{-2t}+e^{2t} \\ -3e^{-2t}+3e^{2t} & 3e^{-2t}+e^{2t} \end{bmatrix}\cdot$$

$$\cdot\left(\mathbf{x}_0 + \int_0^t \left(\frac{1}{4}\begin{bmatrix} 3e^{-2s}+e^{2s} & e^{-2s}-e^{2s} \\ 3e^{-2s}-3e^{2s} & e^{-2s}+3e^{2s} \end{bmatrix}\right)\begin{bmatrix} x_1^3(s) \\ \sin s \end{bmatrix} ds\right),$$

that is,

$$\begin{bmatrix} x_1(t) \\ \\ x_2(t) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{-2t}+3e^{2t} & -e^{-2t}+e^{2t} \\ \\ -3e^{-2t}+3e^{2t} & 3e^{-2t}+e^{2t} \end{bmatrix}\left(\mathbf{x}_0 + \frac{1}{4}\int_0^t \begin{bmatrix} (3e^{-2s}+e^{2s})x_1^3(s)+(e^{-2s}-e^{2s})\sin s \\ \\ (3e^{-2s}-3e^{2s})x_1^3(s)+(e^{-2s}+3e^{2s})\sin s \end{bmatrix} ds\right).$$

This is a system of nonlinear Volterra integral equations which can be stated abstractly as $\mathbf{x} = N(\mathbf{x})$. The most straight forward way to find the solution function $\mathbf{x}(t)$ is to take a first guess for the solution as $\mathbf{x}_1(t) \equiv \mathbf{v}$ for some constant vector $\mathbf{v}$ and then calculate the sequence of vector valued functions $\mathbf{x}_2(t) \triangleq N\big(\mathbf{x}_1(t)\big)$, $\mathbf{x}_3(t) \triangleq N\big(\mathbf{x}_2(t)\big)$,... . Hopefully, the sequence of vector valued functions will converge to a solution $\mathbf{x}_\infty(t)$ which would be the $2\pi-$periodic solution of the original system $(\star)$.

Section 18.4.5

18.4.5.1. (a) $x^2 \ge 0$ for all x, and $y^2 \ge 0$ for all y, so $V(x,y) = x^2+3y^2 \ge 0$ for all (x,y). Also, $V(x,y) = x^2+3y^2 = 0$ only if $x = y = 0$. So, $V(x,y) = x^2+3y^2 > 0$ for all $(x,y) \ne (0,0)$. By definition, $V(x,y) = x^2+3y^2$ is positive definite with respect to $(0,0)$.

(b) $(\sin x)^2 \ge 0$ for all x, and $y^2 \ge 0$ for all y, so $V(x,y) = (\sin x)^2 + y^2 \ge 0$ for all (x,y). So, $V(x,y) = \sin^2 x + 3y^2$ is either positive semi-definite or positive definite with respect to $(\pi,0)$. But, $\sin^2 x \ne 0$ for all x with $0 < |x-\pi| < \pi$, so $V(x,y) > 0$ for all (x,y) in the punctured ball $\widehat{B}_\pi\big((\pi,0)\big)$. Thus, $V(x,y) = \sin^2 x + 3y^2$ is positive definite with respect to $(\pi,0)$.

(c) $V(x,y) = x^3 + (y-1)^6$ is indefinite with respect to $(0,1)$ because in every ball $B_R\big((0,1)\big)$ there exists points $\left(\pm\frac{R}{2},1\right)$ with $V\left(\pm\frac{R}{2},1\right) = \pm\frac{R^3}{8}$ whose sign is ±1, respectively.

18.4.5.3. $\dot{V}_{\mathbf{f}}(\mathbf{x}) = x\dot{x} + y\dot{y} = x\left(-y - x\sin^2 x\right) + y\left(x - y\sin^2 x\right) = -(x^2+y^2)\sin^2 x \le 0$ for all (x,y). So, $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is either negative semi-definite or negative definite with respect to the origin. But, $\dot{V}_{\mathbf{f}}(\mathbf{x})(0,y) = 0$ for all y, so $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is not negative definite. Because $V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{x}$ is positive definite, Theorem 18.7(a) in Section 18.4 implies that $(x,y) = (0,0)$ is stable. [We cannot decide whether or not $(x,y) = (0,0)$ is asymptotically stable without further information or theory.]

18.4.5.5. $\dot{V}_{\mathbf{f}}(\mathbf{x}) = x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 + x_4\dot{x}_4$

$$= x_1\left(-x_1 - 2x_2^2\right) + x_2\left(-x_2 + 2x_1x_2\right) + x_3\left(-3x_3 + x_4\right) + x_4\left(-x_3 - 2x_4\right)$$

$$= -x_1^2 - x_2^2 - 3x_3^2 - 2x_4^2 \le -x_1^2 - x_2^2 - x_3^2 - x_4^2 = -\mathbf{x}^T\mathbf{x}.$$

So, $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is negative definite. Because $V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{x}$ is positive definite, $\mathbf{x} = \mathbf{0}$ is asymptotically stable, by Theorem 18.7(b) in Section 18.4.

18.4.5.7. As long as the constants A and B are positive, $V(x,y) = Ax^2 + By^4$ is positive definite on $\mathbb{R}^2$. We calculate

$$\dot{V}_{\mathbf{f}}(\mathbf{x}) = 2Ax\dot{x}+4By^3\dot{y} = 2Ax(-x^3-3xy^4)+4By^3(x^2y-2y^3-y^5) = -2Ax^4-8By^6-4By^8+(-6A+4B)x^2y^4.$$

Choose, for example, $A = 2$ and $B = 3$. Then $V(x,y) = 2x^2 + 3y^4$ has $\dot{V}_{\mathbf{f}}(\mathbf{x}) = -4x^4 - 24y^6 - 12y^8 \le 0$ for all (x,y). Further, $\dot{V}_{\mathbf{f}}(\mathbf{x}) = 0$ only at (x,y) where $-4x^4$, $-24y^6$, and $-12y^8$ are all zero. So, $\dot{V}_{\mathbf{f}}(\mathbf{x}) = 0$ only at $(x,y) = (0,0)$. It follows that $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is negative definite on $\mathbb{R}^2$. Theorem 18.7(b) in Section 18.4 implies that $(x,y) = (0,0)$ is asymptotically stable.

18.4.5.9. As long as the constants A and B are positive, $V(x,y) = Ax^2 + By^2$ is positive definite on $\mathbb{R}^2$. We calculate

$$\dot{V}_{\mathbf{f}}(\mathbf{x}) = 2Ax\dot{x} + 2By\dot{y} = 2Ax(x^3y^2 - 3x^3) + 2By(-5x^4y - y) = -6Ax^4 - 2By^2 + (2A - 10B)x^4y^2.$$

Choose, for example, $A = 5$ and $B = 1$. Then $V(x,y) = 5x^2 + y^2$ has $\dot{V}_{\mathbf{f}}(\mathbf{x}) = -6x^4 - 2y^2 \le 0$ for all (x,y). Further, $\dot{V}_{\mathbf{f}}(\mathbf{x}) = 0$ only at (x,y) where $-6x^4$ and $-2y^2$ are both zero. So, $\dot{V}_{\mathbf{f}}(\mathbf{x}) = 0$ only at $(x,y) = (0,0)$. It follows that $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is negative definite on $\mathbb{R}^2$. Theorem 18.7(b) in Section 18.4 implies that $(x,y) = (0,0)$ is asymptotically stable.

18.4.5.11. $V(x,y) = \frac{9}{2}y^2 + \frac{1}{2}\,v^2$ is positive definite on $\mathbb{R}^2$. We calculate

$$\dot{V}_{\mathbf{f}}(\mathbf{x}) = 9y\dot{y} + v\dot{v} = 9y\big(v + y(y^2+v^2)\big) + v\big(-9y + v(y^2+v^2)\big) = 9(y^2+v^2)^2 \ge 0,$$

so $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is positive semi-definite on $\mathbb{R}^2$. Further, $\dot{V}_{\mathbf{f}}(\mathbf{x}) = 0$ only at (x,y) where $y^2 + v^2 = 0$. So, $\dot{V}_{\mathbf{f}}(\mathbf{x}) = 0$ only at $(x,y) = (0,0)$. It follows that $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is positive definite on $\mathbb{R}^2$. Theorem 18.10(b) in Section 18.4 implies that $(x,y) = (0,0)$ is unstable.

As to the linearization, the Jacobian matrix is

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(x,y)\right] = \begin{bmatrix} 3y^2+v^2 & 1+2yv \\ \\ -9+2yv & 3v^2+y^2 \end{bmatrix}.$$

At the equilibrium point $(x,y) = (0,0)$, the linearization is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\mathbf{x},$$

so the eigenvalues are $\lambda = \pm 3i$, so we cannot decide if the equilibrium point $(\theta, \dot{\theta}) = (0,0)$ is stable or unstable. Why? Because Theorem 18.4 in Section 18.2 does not give conclusions when there is an eigenvalue whose real part is zero.

So, $V(x,y) = \frac{9}{2}y^2 + \frac{1}{2}\,v^2$ gives the conclusion that $(0,0)$ is unstable, even though the method of linearization is inconclusive about stability in this problem.

18.4.5.13. Note that the Errata webpage changes the second ODE in the system to be $\dot{y} = xy + y$.

$V(x,y) = x^2 - y^2$ is indefinite, according to Definition 18.12 in Section 18.4, because for every $R > 0$ there exists $(x,y) = \left(\frac{R}{2}, 0\right)$ and $\left(0, \frac{R}{2}\right)$with $V\left(0, \frac{R}{2}\right) < 0 < V\left(\frac{R}{2}, 0\right)$.

We calculate that $\dot{V}_{\mathbf{f}}(\mathbf{x}) = x\dot{x} - y\dot{y} = 2x\left(-x + y^2\right) - 2y(xy + y) = -2(x^2+y^2)$, which implies that $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is negative definite. By Theorem 18.10(a) in Section 18.4, $(x,y) = (0,0)$ is unstable.

18.4.5.15. To find e^{tA}, it is probably easiest to use the fact that $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ is in companion form: Equivalent to the LCCHS $(\star)$ $\quad \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\mathbf{x}$ is the second order scalar ODE $\ddot{y} + 2\dot{y} + y = 0$, whose characteristic equation is $0 = s^2 + 2s + 1 = (s+1)^2$. We see that a general solution of $(\star)$ is given by

$$\mathbf{x} = c_1 \begin{bmatrix} e^{-t} \\ \frac{d}{dt}\big[e^{-t}\big] \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} te^{-t} \\ \frac{d}{dt}\big[te^{-t}\big] \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} t \\ 1-t \end{bmatrix},$$

which gives a fundamental matrix

$$Z(t) = e^{-t} \begin{bmatrix} 1 & t \\ -1 & 1-t \end{bmatrix}$$

and thus principal fundamental matrix

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = e^{-t} \begin{bmatrix} 1 & t \\ -1 & 1-t \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}.$$

Using the Maclaurin series for e^{tA} in Section 5.2.3, we note that $(e^{tA})^T = e^{(tA)^T}$.
This gives us Liapunov function $V(\mathbf{x}) \triangleq \mathbf{x}^T B \mathbf{x}$, where

$$B = \int_0^\infty e^{tA^T}\, e^{tA}\, dt = \lim_{b\to\infty} \left(\int_0^b e^{-t} \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix} e^{-t} \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix} dt \right)$$

$$= \lim_{b\to\infty} \left(\int_0^b e^{-2t} \begin{bmatrix} (1+t)^2 + t^2 & (1+t)t - t(1-t) \\ t(1+t) - (1-t)t & t^2 + (1-t)^2 \end{bmatrix} dt \right)$$

$$= \lim_{b\to\infty} \left(\int_0^b \begin{bmatrix} (2t^2+2t+1)e^{-2t} & 2t^2 e^{-2t} \\ 2t^2 e^{-2t} & (2t^2-2t+1)e^{-2t} \end{bmatrix} dt \right)$$

$$= \lim_{b\to\infty} \begin{bmatrix} \Big[\Big(\big(-t^2-t-\tfrac{1}{2}\big) + \big(-t-\tfrac{1}{2}\big) - \tfrac{1}{2}\Big)e^{-2t}\Big]_0^b & \Big[\big(-t^2-t-\tfrac{1}{2}\big)e^{-2t}\Big]_0^b \\ \Big[\big(-t^2-t-\tfrac{1}{2}\big)e^{-2t}\Big]_0^b & \Big[\Big(\big(-t^2-t-\tfrac{1}{2}\big) - \big(-t-\tfrac{1}{2}\big) - \tfrac{1}{2}\Big)e^{-2t}\Big]_0^b \end{bmatrix}$$

$$= \lim_{b\to\infty} \begin{bmatrix} \big(-b^2-2b-\tfrac{3}{2}\big)e^{-2b} + \tfrac{3}{2} & \big(-b^2-b-\tfrac{1}{2}\big)e^{-2b} + \tfrac{1}{2} \\ \big(-b^2-b-\tfrac{1}{2}\big)e^{-2b} + \tfrac{1}{2} & \big(-b^2-\tfrac{1}{2}\big)e^{-2b} + \tfrac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{3}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix},$$

because L'Hôpital's Rule in the $\frac{\infty}{\infty}$ case implies that

$$\lim_{b\to\infty} b^2 e^{-2b} = \lim_{b\to\infty} \frac{b^2}{e^{2b}} = \lim_{b\to\infty} \frac{(b^2)'}{(e^{2b})'} = \lim_{b\to\infty} \frac{2b}{2e^{2b}} = \lim_{b\to\infty} \frac{(2b)'}{(2e^{2b})'} = \lim_{b\to\infty} \frac{2}{4e^{2b}} = 0$$

and

$$\lim_{b\to\infty} be^{-2b} = \lim_{b\to\infty} \frac{b}{e^{2b}} = \lim_{b\to\infty} \frac{(b)'}{(e^{2b})'} = \lim_{b\to\infty} \frac{1}{2e^{2b}} = 0.$$

In summary, Theorem 18.9 in Section 18.4 produces a Liapunov function $V(\mathbf{x}) \triangleq \mathbf{x}^T \begin{bmatrix} \tfrac{3}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix} \mathbf{x}$.

18.4.5.17. To find e^{tA}, first we find eigenvalues and eigenvectors: $0 = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)^2 - 1$ $\Rightarrow$ the eigenvalues of A are $\lambda_1 = -3-1 = -4$ and $\lambda_2 = -3+1 = -2$.

Corresponding to eigenvalue $\lambda_1 = -4$, eigenvectors are found by

$$\left[\, A-(-4)I \mid \mathbf{0} \,\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $-R_1 + R_2 \rightarrow R_2$. Corresponding to eigenvalue $\lambda_1 = -4$ we have an eigenvector $\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Corresponding to eigenvalue $\lambda_2 = -2$, eigenvectors are found by

$$\left[\, A-(-2)I \mid \mathbf{0} \,\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

after row operations $R_1 + R_2 \rightarrow R_2$, $-R_1 \rightarrow R_1$. Corresponding to eigenvalue $\lambda_2 = -2$ we have an eigenvector $\mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A general solution of the ODE system is given by

$$\mathbf{x} = c_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which gives a fundamental matrix

$$Z(t) = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -e^{-4t} & e^{-2t} \end{bmatrix}$$

and thus principal fundamental matrix

$$e^{tA} = Z(t)\big(Z(0)\big)^{-1} = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -e^{-4t} & e^{-2t} \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} e^{-4t} + e^{-2t} & -e^{-4t} + e^{-2t} \\ -e^{-4t} + e^{-2t} & e^{-4t} + e^{-2t} \end{bmatrix}.$$

Using the Maclaurin series for e^{tA} in Section 5.2.3, we note that $(e^{tA})^T = e^{(tA)^T}$.
This gives us Liapunov function $V(\mathbf{x}) \triangleq \mathbf{x}^T B \mathbf{x}$, where

$$\begin{aligned}
B &= \int_0^\infty e^{tA^T}\, e^{tA}\, dt \\
&= \lim_{b\to\infty} \left(\int_0^b \frac{1}{2} \begin{bmatrix} e^{-4t} + e^{-2t} & -e^{-4t} + e^{-2t} \\ -e^{-4t} + e^{-2t} & e^{-4t} + e^{-2t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} e^{-4t} + e^{-2t} & -e^{-4t} + e^{-2t} \\ -e^{-4t} + e^{-2t} & e^{-4t} + e^{-2t} \end{bmatrix} dt \right) \\
&= \ldots = \frac{1}{2} \lim_{b\to\infty} \left(\int_0^b \begin{bmatrix} e^{-8t} + e^{-4t} & -e^{-8t} + e^{-4t} \\ -e^{-8t} + e^{-4t} & e^{-8t} + e^{-4t} \end{bmatrix} dt \right) \\
&= \frac{1}{2} \lim_{b\to\infty} \begin{bmatrix} -\frac{1}{8}e^{-8b} + -\frac{1}{4}e^{-4b} + \frac{1}{8} + \frac{1}{4} & \frac{1}{8}e^{-8b} - \frac{1}{4}e^{-4b} - \frac{1}{8} + \frac{1}{4} \\ \frac{1}{8}e^{-8b} - \frac{1}{4}e^{-4b} - \frac{1}{8} + \frac{1}{4} & -\frac{1}{8}e^{-8b} - \frac{1}{4}e^{-4b} + \frac{1}{8} + \frac{1}{4} \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.
\end{aligned}$$

In summary, Theorem 18.9 in Section 18.4 produces a Liapunov function $V(\mathbf{x}) \triangleq \dfrac{1}{16}\mathbf{x}^T \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$.

18.4.5.19. With respect to the origin, $V(x,y) = x^2 + y^2$ is positive definite everywhere. For the system of ODEs $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3y + \alpha x(x^2+y^2) \\ -3x + \alpha y(x^2+y^2) \end{bmatrix}$, we calculate that

$$\dot{V}_{\mathbf{f}}(\mathbf{x}) = x\dot{x} + y\dot{y} = x\left(3y + \alpha x(x^2+y^2)\right) + y\left(-3x + \alpha y(x^2+y^2)\right) = \alpha(x^2+y^2)^2.$$

If $\alpha < 0$ then $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is negative definite, so $(0,0)$ is asymptotically stable by Theorem 18.7(b) in Section 18.4.

If $\alpha > 0$ then $\dot{V}_{\mathbf{f}}(\mathbf{x})$ is positive definite, so $(0,0)$ is unstable by Theorem 18.10(b) in Section 18.4.

Finally, if $\alpha = 0$, then the system is the LCCHS $\dot{\mathbf{x}} = A\mathbf{x}$, where $\mathbf{x} = [\,x \quad y\,]^T$ and $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$. Because $\dot{V}_{\mathbf{f}}(\mathbf{x}) \equiv 0 \le 0$, the origin is stable by Theorem 18.7(a) in Section 18.4. Because all of the solutions in the phase plane are ellipses centered at the origin, the origin is not an attractor, so the origin is not asymptotically stable.

18.4.5.21. The two mass and three horizontal springs system of Example 5.5 in Section 5.1 is modeled by the system of two second order ODEs

$$\left\{ \begin{array}{c} m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) \\ \\ m_2\ddot{x}_2 = k_2(x_1 - x_2) - k_3x_3 \end{array} \right\}.$$

Define $q_1 = x_1$, $q_2 = x_2$, $p_1 = m_1\dot{x}_1$, and $p_2 = m_2\dot{x}_2$. Then

$$(1) \quad \dot{q}_1 = \dot{x}_1 = \frac{1}{m_1}\,p_1 \quad \text{and} \quad (2) \quad \dot{q}_2 = \dot{x}_2 = \frac{1}{m_2}\,p_2,$$

as well as

$$(3) \quad \dot{p}_1 = m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) = -(k_1+k_2)q_1 + k_2q_2$$

and

$$(4) \quad \dot{p}_2 = m_2\ddot{x}_2 = k_2(x_1 - x_2) - k_3x_2 = k_2q_1 - (k_2+k_3)q_2.$$

This can be written as the Hamiltonian system

$$\left\{ \begin{array}{c} \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \\ \\ \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \end{array} \right\},$$

where $\mathbf{q} = [\,q_1 \quad q_2\,]^2$, $\mathbf{p} = [\,p_1 \quad p_2\,]^2$, and

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\,(k_1+k_2)q_1^2 - k_2q_1q_2 + \frac{1}{2}\,(k_2+k_3)q_2^2 + \frac{1}{2m_1}\,p_1^2 + \frac{1}{2m_2}\,p_2^2.$$

The Hamiltonian can also be rewritten in terms of quadratic forms using matrices:

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\,\mathbf{q}^T \begin{bmatrix} k_1+k_2 & -k_2 \\ \\ -k_2 & k_2+k_3 \end{bmatrix} \mathbf{q} \;+\; \frac{1}{2}\,\mathbf{p}^T \begin{bmatrix} \frac{1}{m_1} & 0 \\ \\ 0 & \frac{1}{m_2} \end{bmatrix} \mathbf{p}.$$

The result of problem 18.4.5.20 implies that for the two mass and three horizontal springs system of Example 5.5 in Section 5.1, the equilibrium point $(\mathbf{q},\mathbf{p}) = (\mathbf{0},\mathbf{0}) = (0,0,0,0)$ is stable in $\mathbb{R}^4$, hence the equilibrium point $(x_1, x_1, \dot{x}_1, \dot{x}_2) = (0,0,0,0)$ is stable in $\mathbb{R}^4$.

Section 18.5.2

18.5.2.1. For the ODE system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -x - y^3 \end{bmatrix}$, let $\mathcal{G} = \mathbb{R}^2$. The function $V = V(x,y) \triangleq \frac{1}{2}(x^2 + y^2)$ is positive definite on $\mathcal{G}$ and has

$$\dot{V}_{\mathbf{f}} = x\dot{x} + y\dot{y} = x \cdot (y) + y \cdot \big(-x - y^3\big) = -y^4 \le 0,$$

so $V(x,y)$ is a Liapunov function on $\mathcal{G}$. So, $\mathbf{0}$ is stable.

We calculate that

$$\mathcal{S} \triangleq \{(x,y) \text{ in } \mathcal{G} : \dot{V}_{\mathbf{f}}(x,y) = 0\} = \{(x,y) : y = 0\}.$$

Which initial condition(s) $\mathbf{x}_0 = (x_0, 0)$ in $\mathcal{S}$ have $\mathbf{x}(t;\mathbf{x}_0)$ remaining in $\mathcal{S}$ for all $t > 0$? In order to have $\mathbf{x}(t;\mathbf{x}_0) = \big(x(t), y(t)\big)$ remain in $\mathcal{S}$ requires that $y(t) \equiv 0$, hence $\dot{x}(t) = y(t) \equiv 0$, hence $x(t) \equiv x_0$, hence $\dot{y} = -x(t) - y(t)^3 \equiv x_0 - 0^3 = x_0$ is constant, which implies $0 \equiv y(t) = x_0 t$. But this is true only if $x_0 = 0$. So, in $\mathcal{S}$, only the initial condition $\mathbf{x}_0 = (0,0)$ has $\mathbf{x}(t;\mathbf{x}_0)$ remaining in $\mathcal{S}$ for all $t > 0$.

So, the *only* closed, positively invariant subset of $\mathcal{S}$ is the set $\{(0,0)\}$. So, the maximal, closed, positively invariant set in $\mathcal{S}$ is $\mathcal{E} = \{(0,0)\}$.

So, every solution $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. This says that $\mathbf{0}$ is a global attractor. We already knew that $\mathbf{0}$ is stable, so $\mathbf{0}$ is asymptotically stable.

18.5.2.3. For the ODE system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -y - x\sin^2 x \\ x - y\sin^2 x \end{bmatrix}$, let $\mathcal{G} = \mathbb{R}^2$. The function $V = V(x,y) \triangleq \frac{1}{2}(x^2+y^2)$ is positive definite on $\mathcal{G}$ and has

$$\dot{V}_{\mathbf{f}} = x\dot{x} + y\dot{y} = x \cdot \big(-y - x\sin^2 x\big) + y \cdot \big(x - y\sin^2 x\big) = -(x^2+y^2)\sin^2 x \ \le 0,$$

so $V(x,y)$ is a Liapunov function on $\mathcal{G}$. So, $\mathbf{0}$ is stable.

We calculate that

$$\mathcal{S} \triangleq \{(x,y) \text{ in } \mathcal{G} : \dot{V}_{\mathbf{f}}(x,y) = 0\} = \{(x,y) : \sin x = 0 \text{ or } x = y = 0\} = \{(x,y) : x = n\pi,\ n = \text{integer}\}.$$

Which initial condition(s) $\mathbf{x}_0 = (x_0, y_0) = (n\pi, y_0)$ in $\mathcal{S}$ have $\mathbf{x}(t;\mathbf{x}_0)$ remaining in $\mathcal{S}$ for all $t > 0$? Note that the set $\mathcal{S}$ consists of isolated points, no two of which are closer than a distance of π. Solutions of ODE systems are continuously differentiable parametric curves, so, in order to have $\mathbf{x}(t;\mathbf{x}_0) = \big(x(t), y(t)\big)$ remain in $\mathcal{S}$ requires that $x(t) \equiv n\pi$ for some integer n. It follows that $0 \equiv \dot{x}(t) = -y(t) - n\pi\sin^2(n\pi) = -y(t)$, hence $0 \equiv y(t)$, hence $y_0 = 0$. But, then $0 \equiv \dot{y}(t) = x(t) - y(t)\sin^2\big(x(t)\big) = n\pi - 0 = n\pi$, hence $n = 0$.

So, in $\mathcal{S}$, only the initial condition $\mathbf{x}_0 = (0,0)$ has $\mathbf{x}(t;\mathbf{x}_0)$ remaining in $\mathcal{S}$ for all $t > 0$.

So, the *only* closed, positively invariant subset of $\mathcal{S}$ is the set $\{(0,0)\}$. So, the maximal, closed, positively invariant set in $\mathcal{S}$ is $\mathcal{E} = \{(0,0)\}$.

So, every solution $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. This says that $\mathbf{0}$ is a global attractor. We already knew that $\mathbf{0}$ is stable, so $\mathbf{0}$ is asymptotically stable.

18.5.2.5. The ODE system of Example 5.8 in Section 5.2 is $\begin{bmatrix} \dot{T} \\ \dot{M} \end{bmatrix} = \begin{bmatrix} -k_T & k_T \\ k_M & -k_M \end{bmatrix} \begin{bmatrix} T \\ M \end{bmatrix}$, where the constants k_T and k_M are positive.

Because $\begin{vmatrix} -k_T & k_T \\ k_M & -k_M \end{vmatrix} = 0$, there is a whole line through the origin consisting of equilibria. In fact, that line is $\{(\alpha,\alpha) : -\infty < \alpha < \alpha\}$. Physically, equilibria have the temperatures T and M being equal.

Let $\mathcal{G} = \mathbb{R}^2$. The function $V = V(T,M) \triangleq \frac{1}{2}(k_M T^2 + k_T M^2)$ is positive definite on $\mathcal{G}$ and has

$$\dot{V}_{\mathbf{f}} = k_M T\dot{T} + k_T M\dot{M} = k_M T \cdot (-k_T T + k_T M) + k_T M \cdot (k_M T - k_M M) = -k_M k_T (T^2 - 2TM + M^2)$$

$$= -k_M k_T (T - M)^2 \le 0,$$

So $V(x,y)$ is a Liapunov function on $\mathcal{G}$. So, $\mathbf{0}$ is stable.

Moreover, every individual equilibrium point (α, α) is also stable, by using the Liapunov function $V_\alpha(T,M) \triangleq \frac{1}{2}(k_M(T-\alpha)^2 + k_T(M-\alpha)^2)$ and the calculation that

$$\dot{V}_{\mathbf{f}} = k_M(T-\alpha)\dot{T} + k_T(M-\alpha)\dot{M} = k_M(T-\alpha)\cdot(-k_T T + k_T M) + k_T(M-\alpha)\cdot(k_M T - k_M M)$$

$$= -k_M k_T(T^2 - 2TM + M^2) - \alpha k_M k_T(-T+M) - \alpha k_M k_T(T-M)$$

$$= -k_M k_T(T-M)^2 \le 0,$$

Returning to consideration of the Liapunov function $V = V(T,M) \triangleq \frac{1}{2}(k_M T^2 + k_T M^2)$, we calculate that

$$\mathcal{S} \triangleq \{(y,v) \text{ in } \mathcal{G} : \dot{V}_{\mathbf{f}}(y,v) = 0\} = \{(y,v) : T = M\}.$$

Every point (T_0, M_0) that is in $\mathcal{S}$ is an equilibrium point, so every point in $\mathcal{S}$ is an initial condition that has $\mathbf{x}(t;\mathbf{x}_0) = (T_0, M_0)$ remaining in $\mathcal{S}$ for all $t > 0$. It follows that $\mathcal{E} = \mathcal{S}$ is the maximal, closed, positively invariant subset of $\mathcal{S}$. By Theorem 18.13 in Section 18.5, $\mathcal{S}$, the *set* consisting of *all* of the equilibria, is asymptotically stable for the system of Example 5.8 in Section 5.2.

Section 18.6.5

18.6.5.1. Define $r(t) = \sqrt{x(t)^2 + y(t)^2} \ge 0$, hence $r(t)^2 = x(t)^2 + y(t)^2$. The original ODE system can be rewritten as

$$(\star) \qquad \begin{bmatrix} \dot{x} \\ \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y + x(1-r)(2-r) \\ \\ -x + y(1-r)(2-r) \end{bmatrix}.$$

We calculate that

$$2r(t)\dot{r}(t) = \frac{d}{dt}\left[\, r(t)^2 \,\right] = 2x\dot{x} + 2y\dot{y} = 2x\left(y + x(1-r)(2-r)\right) + 2y\left(-x + y(1-r)(2-r)\right)$$

$$= 2(x^2+y^2)(1-r)(2-r) = 2r^2(1-r)(2-r),$$

so

$$(\star\star) \quad \dot{r} = r(1-r)(2-r).$$

The latter is an autonomous ODE in $\mathbb{R}^1$ so we can use the phase line to study its equilibria and their stability. Only $r = ||\,\mathbf{x}\,|| \ge 0$ is relevant to our problem $(\star)$. The phase line is shown in the figure.

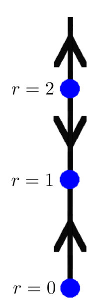


Figure 3: Answer key for problem18.6.5.1

The equilibria of $(\star\star)$, $r = 0$, $r = 1$, and $r = 2$, give periodic solutions, $x(t)^2 + y(t)^2 \equiv 0$, $x(t)^2 + y(t)^2 \equiv 1$, and $x(t)^2 + y(t)^2 \equiv 4$, of $(\star)$. From the phase line picture, we see that $x^2 + y^2 \equiv 4$ gives an unstable limit cycle and that $x^2 + y^2 \equiv 1$ gives a stable limit cycle. Note that the origin is an equilibrium point and is thus not a limit cycle.

Note also that $(\star\star)$ has no periodic solution other than the equilibrium at the origin and on the circles $r = 1$ and $r = 2$. Why not? Because if the initial value of a solution has $0 < r(0) \ne 1$ and $r(0) \ne 2$ then

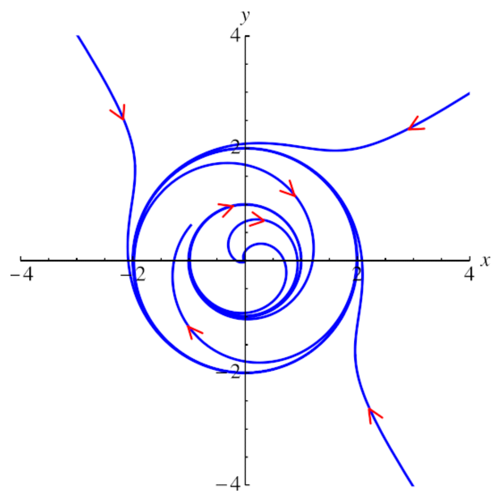


Figure 4: Answer key for problem18.6.5.1: Phase plane picture

the phase line in the figure shows that at no time $\mathsf{T} > 0$ can we have $r(\mathsf{T}) = r(0)$, which implies that at no time $\mathsf{T} > 0$ can we have $\mathbf{x}(\mathsf{T}) = \mathbf{x}(0)$. So, no solution other than on the circles $r = 1$, $r = 2$, or at the origin can return to where it started after T units of time, and thus the only possible limit cycles are on the circles $r = 1$ and $r = 2$.

18.6.5.3. Define $r(t) = \sqrt{x(t)^2 + y(t)^2} \geq 0$, hence $r(t)^2 = x(t)^2 + y(t)^2$. The original ODE system can be rewritten as

$$(\star) \quad \begin{bmatrix} \dot{x} \\ \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x + y + 2xe^{-r} \\ \\ -x - y + 2ye^{-r} \end{bmatrix}.$$

We calculate that

$$2r(t)\dot{r}(t) = \frac{d}{dt}\left[\, r(t)^2 \,\right] = 2x\dot{x} + 2y\dot{y} = 2x\left(-x + y + 2xe^{-r}\right) + 2y\left(-x - y + 2ye^{-r}\right)$$
$$= -2(x^2 + y^2) + 4(x^2 + y^2)e^{-r} = 2r^2(-1 + 2e^{-r}),$$

so

$$(\star\star) \quad \dot{r} = r(-1 + 2e^{-r}).$$

The latter is an autonomous ODE in $\mathbb{R}^1$ so we can use the phase line to study its equilibria and their stability. Only $r = ||\,\mathbf{x}\,|| \geq 0$ is relevant to our problem $(\star)$. The phase line is shown in the figure.

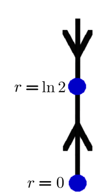


Figure 5: Answer key for problem18.6.5.3

The equilibria of $(\star\star)$, $r = 0$ and $r = \ln 2$, give periodic solutions, $x(t)^2 + y(t)^2 \equiv 0$ and $x(t)^2 + y(t)^2 \equiv (\ln 2)^2$, of $(\star)$. From the phase line picture, we see that $x^2 + y^2 \equiv (\ln 2)^2$ gives a stable limit cycle. Note that the origin is an equilibrium point and is thus not a limit cycle.

Note also that $(\star\star)$ has no periodic solution other than the equilibrium at the origin and on the circle $r = \ln 2$. Why not? Because if the initial value of a solution has $0 < r(0) \neq \ln 2$ then the phase line in the figure shows that at no time $\mathsf{T} > 0$ can we have $r(\mathsf{T}) = r(0)$, which implies that at no time $\mathsf{T} > 0$ can we have $\mathbf{x}(\mathsf{T}) = \mathbf{x}(0)$. So, no solution other than on the circle $r = \ln 2$, or at the origin can return to where it started after T units of time, and thus the only possible limit cycles are on the circles $r = \ln 2$.

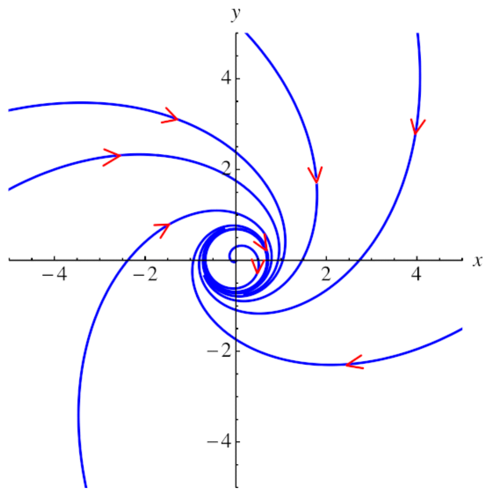


Figure 6: Answer key for problem18.6.5.3: Phase plane picture

18.6.5.5. Suppose $f(x,y)$ is twice continuously differentiable in some open set $\mathcal{O}$. Suppose that the "gradient system" $(\star)$ $\left\{ \begin{array}{c} \dot{x} = \frac{\partial f}{\partial x} \\ \\ \dot{y} = \frac{\partial f}{\partial y} \end{array} \right\}$ has a solution $\mathbf{x}(t) = [x(t) \quad y(t)]^T$ that is periodic with period T, hence in particular $\mathbf{x}(\mathsf{T}) = \mathbf{x}(0)$. Define $h(t) \triangleq f\big(x(t), y(t)\big)$. We calculate that

$$\dot{h}(t) = \dot{x}(t)\frac{\partial f}{\partial x}\big(x(t),y(t)\big)+\dot{y}(t)\frac{\partial f}{\partial y}\big(x(t),y(t)\big) = \frac{\partial f}{\partial x}\big(x(t),y(t)\big)\cdot\frac{\partial f}{\partial x}\big(x(t),y(t)\big)+\frac{\partial f}{\partial y}\big(x(t),y(t)\big)\cdot\frac{\partial f}{\partial y}\big(x(t),y(t)\big)$$

$$= \left(\frac{\partial f}{\partial x}\big(x(t),y(t)\big)\right)^2 + \left(\frac{\partial f}{\partial y}\big(x(t),y(t)\big)\right)^2 = ||\boldsymbol{\nabla} f\big(x(t),y(t)\big)||^2.$$

This implies that $\dot{h}(t) \geq 0$ for all t and that $\dot{h}(t) > 0$ unless $\boldsymbol{\nabla} f\big(x(t),y(t)\big) \equiv \mathbf{0}$.

But $\mathbf{x}(\mathsf{T}) = \mathbf{x}(0)$ implies that

$$h(\mathsf{T}) = f\big(x(\mathsf{T}), y(\mathsf{T})\big) = f\big(x(0), y(0)\big) = h(0),$$

so Rolle's Theorem in Calculus I, applied to the continuously differentiable function $h(t)$, implies that there is a τ with $0 < \tau < \mathsf{T}$ and $\dot{h}(\tau) = 0$, hence $\boldsymbol{\nabla} f\big(x(\tau), y(\tau)\big) \equiv \mathbf{0}$.

Note that $\boldsymbol{\nabla} f\big(x(\tau), y(\tau)\big) \equiv \mathbf{0}$ is equivalent to $\big(x(\tau), y(\tau)\big)$ being an equilibrium point for $(\star)$, because

$$\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} = \frac{\partial f}{\partial x}\big(x(t),y(t)\big)\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\big(x(t),y(t)\big)\hat{\boldsymbol{\jmath}}.$$

But, $\big(x(\tau), y(\tau)\big)$ being an equilibrium point for $(\star)$ implies that $\big(x(t), y(t)\big)$ is a constant solution of $(\star)$, as we wished to explain.

18.6.5.7. Suppose $z(t)$ satisfies a differential equation of the form $(\star)$ $\ddot{z} + F(\dot{z}) + z = 0$, where $F(x)$ is odd and continuously differentiable on $-\infty < x < \infty$.

(a) Take the derivative with respect to t of all terms in ODE $(\star)$ and substitute $x(t) \triangleq \dot{z}(t)$ to get

$$0 = \dddot{z} + F'(\dot{z})\ddot{z} + \dot{z} = \ddot{x} + F'(x)\dot{x} + x,$$

where $f(x) \triangleq F'(x)$, hence $x(t)$ satisfies the ODE

$$(\star\star)\ \ddot{x} + f(x)\dot{x} + x = 0.$$

Because we assume that $F(x)$ is odd and continuously differentiable on $-\infty < x < \infty$, it follows that $f(x) = F'(x)$ is even and continuous on $-\infty < x < \infty$.

(b) Because $F'(x) = f(x)$, the Fundamental Theorem of Calculus, that is, Theorem 7.4(b) in Section 7.1, implies that

$$\widetilde{F}(x) \triangleq \int_0^x f(x) = F(x) - F(0).$$

Because we assume that $F(x)$ is odd and we assume that (i) $F(x) \to \infty$, as $x \to \infty$, we know that also (ii) $F(x) \to -\infty$, as $x \to -\infty$. Because (i) and (ii) are true and we assume that (iii) there is a constant $a > 0$ such that $F(x) < 0$ for $0 < x < a$ and $F(x) > 0$ for $a < x < \infty$, it follows that there is an $\widetilde{a}$ such that $\widetilde{F}(x) < 0$ for $0 < x < \widetilde{a}$ and $\widetilde{F}(x) > 0$ for $\widetilde{a} < x < \infty$.

Further, because $\widetilde{F}(x) = F(x) - F(0)$ and (i) is true, it follows that $\widetilde{F}(x) \to \infty$, as $x \to \infty$.

Defining $g(x) \triangleq x$, it is easy to see that

$$\left\{ \begin{array}{l} \bullet\ g(x) \text{ is odd and continuously differentiable on } -\infty < x < \infty, \\ \\ \bullet\ xg(x) > 0 \quad \text{for } x > 0 \\ \\ \bullet\ G(x) \triangleq \displaystyle\int_0^x g(\xi)d\xi = \frac{1}{2}\,x^2 \to \infty \text{ as } |x| \to \infty. \end{array} \right\}.$$

So, all of the hypotheses of the Levinson-Smith Theorem, that is, Theorem 18.16 in Section 18.6, are verified. It follows that the second order nonlinear ODE $(\star\star)$ has a stable limit cycle $\mathcal{C} : \mathbf{x} = \mathbf{x}(t) = [\,x(t) \quad y(t)\,]^T$, which is periodic with some period T, and it is unique in $\mathbb{R}^2$, up to translation in time.

(c) Suppose $(\star)$ has at least two limit cycles $z_1(t)$ and $z_2(t)$ and they are not time translations of each other. Then $(\star\star)$ has two limit cycles $x_1(t) \triangleq \dot{z}_1(t)$ and $x_2(t) \triangleq \dot{z}_2(t)$. By the result in part (b), this would contradict the Levinson-Smith Theorem, that is, Theorem 18.16 in Section 18.6. So, no, $(\star)$ has at most one limit cycle.

18.6.5.9. Note that $0.01x$ has been changed to x on the Errata webpage.

The ODE is $\ddot{x} + \big(x^2 - 1\big)\,\dot{x} + x + \sin x = 0$. In terms of the hypotheses of the Levinson-Smith Theorem 18.16 in Section 18.6,

$$f(x) = x^4 - 1 \quad \text{and} \quad g(x) = x + \sin x,$$

both of which are continuously differentiable on $-\infty < x < \infty$. Further, on $-\infty < x < \infty$, $f(x)$ is even because it is a sum of even powers of x and $g(x)$ is odd because it is a sum of the odd function $\sin x$ and an odd power of x. Also, $x\,g(x) > 0$ for $x > 0$. Why? First, $g(0) = 0$ and $g'(x) = 1 + \cos x \geq 0$ for all x, so $g(x)$ is increasing on the interval $0 \leq x < \infty$ for all $x \geq 0$, hence $g(x) \geq 0$ for all $x \geq 0$. Furthermore, $g'(x) > 0$ for $0 < x < \pi$, hence $g(x) > 0$ for $0 < x < \pi$. Since $g(x)$ is increasing on the interval $\pi \leq x < \infty$, $g(x) > 0$ for all $x > 0$.

The anti-derivatives are

$$F(x) = \int_0^x f(\xi)d\xi = \int_0^x \big(\xi^2 - 1\big)d\xi = \frac{1}{3}\,x^3 - x \quad \text{and} \quad G(x) = \int_0^x g(\xi)d\xi = \int_0^x (\xi + \sin\xi)d\xi = 0.5\,x^2 + 1 - \cos x.$$

Immediately we see that $F(x) = \dfrac{1}{3}x^3 \cdot \big(1 - \dfrac{3}{x^2}\big) \to \infty$, as $x \to \infty$. Also, $G(x) = x^2\Big(0.5 + \dfrac{1 - \cos x}{x^2}\Big) \to \infty$, as $|x| \to \infty$.

Finally, for $F(x) = \dfrac{1}{3}\,x \cdot \big(x^2 - 3\big)$, it is easy to see that $a = \sqrt{3}$ satisfies the requirements that $F(x) < 0$ for $0 < x < a$ and $F(x) > 0$ for $a < x$.

So, yes, the ODE $\ddot{x} + \big(x^2 - 1\big)\,\dot{x} + x + \sin x = 0$ has a unique stable limit cycle because the ODE satisfies the hypotheses of the Levinson-Smith Theorem 18.16 in Section 18.6.

18.6.5.11. The system is $\begin{bmatrix} \dot{x} \\ \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x^3 \\ \\ -x^3y - y^3 \end{bmatrix} \triangleq \begin{bmatrix} f(x,y) \\ \\ g(x,y) \end{bmatrix}$. Using $\mu(x,y) \triangleq e^{-x}$, we calculate that

$$\frac{\partial}{\partial x}\Big[\mu(x,y)f(x,y)\Big] + \frac{\partial}{\partial y}\Big[\mu(x,y)g(x,y)\Big] = \frac{\partial}{\partial x}\Big[-e^{-x}x^3\Big] + \frac{\partial}{\partial y}\Big[-e^{-x}(x^3y+y^3)\Big]$$

$$= -3x^2e^{-x} - x^3(-e^{-x}) - x^3e^{-x} - 3y^2e^{-x} = -3(x^2+y^2)e^{-x} < 0$$

takes on only one sign in $\mathbb{R}^2$. Theorem 18.19 in Section 18.6 implies that the system has no nonconstant periodic solution in $\mathbb{R}^2$.

18.6.5.13. Define $r(t) = \sqrt{x(t)^2 + y(t)^2} \geq 0$, hence $r(t)^2 = x(t)^2 + y(t)^2$. The original ODE system can be rewritten as

$$(\star) \qquad \begin{bmatrix} \dot{x} \\ \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y + \beta\, xr^2 \\ \\ -x + \beta\, y\, r^2 \end{bmatrix}.$$

We calculate that

$$2r(t)\dot{r}(t) = 2x\dot{x} + 2y\dot{y} = 2x\left(y + \beta\, xr^2\right) + 2y\left(-x + \beta\, y\, r^2\right) = 2\beta\,(x^2+y^2)r^2 = 2\beta\, r^4,$$

that is $\dot{r} = \beta r^3$.

The latter is an autonomous ODE in $\mathbb{R}^1$ so we can use the phase line to study how its equilibria and their stability depend on the parameter β. Only $r = ||\,\mathbf{x}\,|| \geq 0$ is relevant to our problem $(\star)$. We have that for $\beta \neq 0$, the only equilibrium value is at $r = 0$. For $\beta = 0$, all values of $r \geq 0$ are equilibria; in fact, for $\beta = 0$, all of the solutions of the original ODE system,

$$(\star) \qquad \begin{bmatrix} \dot{x} \\ \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ \\ -x \end{bmatrix},$$

are circles or the origin, $(0,0)$.

The phase lines are shown in the figure. From the pictures we see that the only value of β where the nature of solutions of $(\star)$ $\begin{bmatrix} \dot{x} \\ \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y + \beta\, xr^2 \\ \\ -x + \beta\, y\, r^2 \end{bmatrix}$ change as β varies, that is, the bifurcation point, is at $\beta = 0$. For β slightly less than 0, the origin is the only periodic solution and it is stable; for $\beta = 0$, all of the circles $x^2 + y^2 = a^2$ are periodic solutions and are stable but are not limit cycles because they are not isolated; and, for β slightly greater than 0, the origin is the only periodic solution and it is unstable.

In fact, there is no limit cycle for any value of β.

Section 18.7.2

18.7.2.1. $y_1(t) \equiv 1$, $y_2(t) = y(2) + \int_2^t y_1(s)ds = 1 + \int_2^t 1ds = 1 + (t-2)$,

$$y_3(t) = y(2) + \int_2^t y_2(s)ds = 1 + \int_2^t \big(1 + (s-2)\big)ds = 1 + \Big[\, s + \frac{1}{2}\,(s-2)^2\,\Big]_2^t = 1 + (t-2) + \frac{1}{2}\,(t-2)^2,$$

$$y_4(t) = y(2) + \int_2^t y_3(s)ds = 1 + \int_2^t \Big(1 + (s-2) + \frac{1}{2}\,(s-2)^2\Big)ds = 1 + \Big[\,(s-2) + \frac{1}{2!}\,(s-2)^2 + \frac{1}{3!}\,(s-2)^3\,\Big]_2^t$$

$$= 1 + (t-2) + \frac{1}{2!}\,(t-2)^2 + \frac{1}{3!}\,(t-2)^3.$$

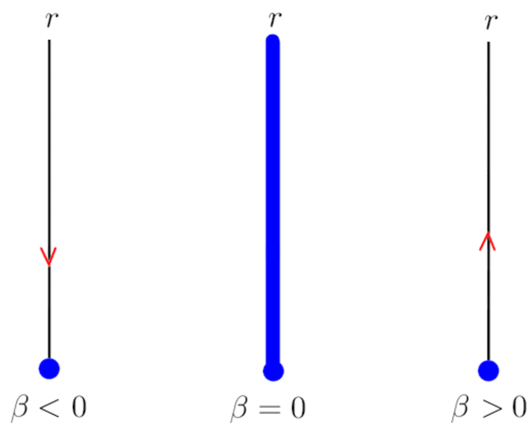


Figure 7: Answer key for problem18.6.5.13: Phase line pictures $\dot{r} = \beta r^3$as λ varies

We guess

$$y_k(t) = 1 + (t-2) + \frac{1}{2!}(t-2)^2 + \frac{1}{3!}(t-2)^3 + + ... + \frac{1}{(k-1)!}(t-2)^{k-1},$$

and then

$$y_{k+1}(t) = y(2) + \int_2^t y_k(s)ds = 1 + \int_2^t \Big(1 + (s-2) + \frac{1}{2}(s-2)^2 + + ... + \frac{1}{(k-1)!}(s-2)^{k-1}\Big)ds$$

$$= 1 + \Big[(s-2) + \frac{1}{2!}(s-2)^2 + ... + \frac{1}{k!}(s-2)^k\Big]_2^t = 1 + (t-2) + \frac{1}{2!}(t-2)^2 + ... + \frac{1}{k!}(t-2)^k.$$

This suggests

$$y_\infty(t) \triangleq \lim_{k\to\infty} y_k(t) = 1 + (t-2) + \frac{1}{2!}(t-2)^2 + ... + \frac{1}{k!}(t-2)^k + ...,$$

which equals $y(t) = e^{t-2}$, the exact solution of the IVP.

18.7.2.3. $\mathbf{x}_1(t) \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$\mathbf{x}_2(t) = \mathbf{x}(0) + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}_1(s)ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 \\ -1 \end{bmatrix} ds = \begin{bmatrix} 1 \\ -t \end{bmatrix},$$

$$\mathbf{x}_3(t) = \mathbf{x}(0) + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}_2(s)ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -s \end{bmatrix} ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} -s \\ -1 \end{bmatrix} ds$$

$$= \begin{bmatrix} 1 - \frac{1}{2}t^2 \\ -t \end{bmatrix},$$

$$\mathbf{x}_4(t) = \mathbf{x}(0) + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}_4(s)ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 - \frac{1}{2}s^2 \\ -s \end{bmatrix} ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} -s \\ -1 + \frac{1}{2}s^2 \end{bmatrix} ds$$

$$= \begin{bmatrix} 1 - \frac{1}{2}t^2 \\ -t + \frac{1}{3!}t^3 \end{bmatrix},$$

and

$$\mathbf{x}_5(t) = \mathbf{x}(0) + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}_4(s)ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 - \frac{1}{2}s^2 \\ -s + \frac{1}{3!}s^3 \end{bmatrix} ds$$

$$= \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} -s + \frac{1}{3!}\, s^3 \\ \\ -1 + \frac{1}{2}\, s^2 \end{bmatrix} ds = \begin{bmatrix} 1 - \frac{1}{2}\, t^2 + \frac{1}{4!}\, t^4 \\ \\ -t + \frac{1}{3!}\, t^3 \end{bmatrix}.$$

We guess that for $k = odd$,

$$\mathbf{x}_k(t) = \begin{bmatrix} x_{k,1}(t) \\ x_{k,2}(t) \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}\, t^2 + \frac{1}{4!}\, t^4 + ... + (-1)^{(k-1)/2} \frac{1}{(k-1)!}\, t^{k-1} \\ \\ -t + \frac{1}{3!}\, t^3 + ... + (-1)^{(k-1)/2} \frac{1}{(k-2)!}\, t^{k-2} \end{bmatrix},$$

and for $k = even$,

$$\mathbf{x}_k(t) = \begin{bmatrix} x_{k,1}(t) \\ x_{k,2}(t) \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}\, t^2 + \frac{1}{4!}\, t^4 + ... + (-1)^{(k-2)/2} \frac{1}{(k-1)!}\, t^{k-2} \\ \\ -t + \frac{1}{3!}\, t^3 + ... + (-1)^{k/2} \frac{1}{(k-1)!}\, t^{k-1} \end{bmatrix}.$$

This suggests

$$\mathbf{x}_\infty(t) = \begin{bmatrix} x_{\infty,1}(t) \\ x_{\infty,2}(t) \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}\, t^2 + \frac{1}{4!}\, t^4 + ... + (-1)^\ell \frac{1}{(2\ell)!}\, t^{2\ell} + ... \\ \\ -t + \frac{1}{3!}\, t^3 + ... + (-1)^\ell \frac{1}{(2\ell+1)!}\, t^{2\ell+1} + ... \end{bmatrix},$$

which equals $\mathbf{x}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$, the exact solution of the IVP system.

18.7.2.5. (a) Suppose a continuously differentiable function $\mathbf{x}(t)$ satisfies IVP (18.69) in Section 18.7, that is,

$$\left\{ \begin{array}{c} \dfrac{d\mathbf{x}}{dt}(t) = \mathbf{f}\big(t, \mathbf{x}(t)\big),\ t_0 - \delta < t < t_0 + \delta \\ \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{array} \right\},$$

for some $\delta > 0$. Then, by the Fundamental Theorem of Calculus, that is, Theorem 7.4(b) in Section 7.1, for all t in the interval $\mathcal{I} \triangleq (t_0 - \delta,\, t_0 + \delta)$,

$$\mathbf{x}(t) - \mathbf{x}_0 = \mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^t \frac{d\mathbf{x}}{ds}(s)\, ds = \int_{t_0}^t \mathbf{f}\big(s, \mathbf{x}(s)\big)\, ds,$$

hence

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}\big(s, \mathbf{x}(s)\big)\, ds,\ t_0 - \bar{\alpha} \le t \le t_0 + \bar{\alpha},$$

where $\delta = \bar{\alpha}$, which is integral equation (18.73) in Section 18.7.

(b) Suppose a continuously differentiable function $\mathbf{x}(t)$ satisfies integral equation (18.73) in Section 18.7, that is,

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}\big(s, \mathbf{x}(s)\big)\, ds,\ t_0 - \bar{\alpha} \le t \le t_0 + \bar{\alpha},$$

for some $\bar{\alpha} > 0$. Take the derivative with respect to t of both sides and use the Fundamental Theorem of Calculus, that is, Theorem 7.4(a) in Section 7.1, to get

$$\frac{d\mathbf{x}}{dt} = \mathbf{0} + \frac{d}{dt}\Big[\int_{t_0}^t \mathbf{f}\big(s, \mathbf{x}(s)\big)\, ds \Big] = \mathbf{f}\big(t, \mathbf{x}(t)\big),$$

or $\dot{\mathbf{x}} = f\big(t, \mathbf{x}\big)$, for short, and

$$\mathbf{x}(t_0) = \mathbf{x}_0 + \int_{t_0}^{t_0} \mathbf{f}\big(s, \mathbf{x}(s)\big)\, ds = \mathbf{x}_0 + \mathbf{0}.$$

So,

$$\left\{ \begin{array}{c} \dfrac{d\mathbf{x}}{dt}(t) = \mathbf{f}\big(t, \mathbf{x}(t)\big),\ t_0 - \delta < t < t_0 + \delta \\ \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{array} \right\},$$

where $\delta = \bar{\alpha}$, which is IVP (18.69) in Section 18.7.

18.7.2.7. A being constant implies that the general solution of the LCCHS $\dot{\mathbf{x}} = A\mathbf{x}$ can be written as $\mathbf{x}(t) = e^{tA}\mathbf{c}$. where $\mathbf{c}$ is a constant vector. Define $\mathbf{x}(t; t_1, \mathbf{x}_1)$ to be the unique solution of the IVP $\dot{\mathbf{x}} = A\mathbf{x},\ \mathbf{x}(t_1) = \mathbf{x}_1$, so

$$\mathbf{x}_1 = \mathbf{x}(t_1) = e^{t_1 A}\mathbf{c}$$

implies that

$$\mathbf{x}(t; t_1, \mathbf{x}_1) = e^{tA}(\mathbf{c}) = e^{tA}\left(\left(e^{t_1 A}\right)^{-1}\mathbf{x}_1\right) = e^{tA}\left(e^{-t_1 A}\mathbf{x}_1\right) = e^{(t-t_1)A}\mathbf{x}_1.$$

We calculate that

$$Y(t) \triangleq \frac{\partial \mathbf{x}}{\partial \mathbf{x}_1}(t; t_1, \mathbf{x}_1) = \frac{\partial}{\partial \mathbf{x}_1}\left[\, e^{(t-t_1)A}\mathbf{x}_1 \,\right] = e^{(t-t_1)A}\frac{\partial}{\partial \mathbf{x}_1}\left[\, \mathbf{x}_1 \,\right] = e^{(t-t_1)A}\ I = e^{(t-t_1)A}.$$

On the other hand, Theorem 18.24 in Section 18.7 says that $Y(t) \triangleq \dfrac{\partial \mathbf{x}}{\partial \mathbf{x}_1}(t; t_1, \mathbf{x}_1)$ satisfies (18.78), that is, the linear variational equation

$$\dot{Y} = \left[\, \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\big(t, \mathbf{x}(t; t_1, \mathbf{x}_1)\big) \,\right] Y,$$

along with [See the Errata webpage] the IC $Y(t_1) = I$.

Note that for the original system, $\dot{\mathbf{x}} = A\mathbf{x}$, we have $\mathbf{f}\big(t, \mathbf{x}\big) = A\mathbf{x}$, so

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\big(t, \mathbf{x}(t; t_1, \mathbf{x}_1)\big) = \frac{\partial}{\partial \mathbf{x}}\left[\, A\mathbf{x} \,\right]\Big|_{\mathbf{x} = \mathbf{x}(t; t_1, \mathbf{x}_1)} = A$$

hence the linear variational equation and IC is

$$\dot{Y} = AY,\ Y(t_1) = I$$

Clearly the solution of this IVP is $Y(t) = e^{(t-t_1)A}$, which agrees with our conclusion earlier in this problem.

18.7.2.9. Note that this problem concerns the ODE-IVP $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix}\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_1$, as mentioned in the Errata webpage.

(a) We are given that λ is a positive constant. We can find the general solution of the ODE system $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix}\mathbf{x}$ by noting that the ODE system is in companion form.

The e^{tA} for the corresponding LCCHS $\dot{\mathbf{x}} = A\mathbf{x}$ is of the form

$$e^{tA} = \begin{bmatrix} y_1(t;\lambda) & y_2(t;\lambda) \\ \dot{y}_1(t;\lambda) & \dot{y}_2(t;\lambda) \end{bmatrix}$$

where $y_1(t;\lambda)$ and $y_2(t;\lambda)$ satisfy

$$\left\{ \begin{array}{c} \ddot{y}_1 + \lambda y_1 = 0 \\ y_1(0;\lambda) = 1,\ \dot{y}_1(0;\lambda) = 0 \end{array} \right\}, \quad \left\{ \begin{array}{c} \ddot{y}_2 + \lambda y_2 = 0 \\ y_2(0;\lambda) = 0,\ \dot{y}_2(0;\lambda) = 1 \end{array} \right\},$$

respectively.

Define $\omega = \sqrt{\lambda} > 0$. The general solution of the ODE $\ddot{y} + \omega^2 y = 0$ is $y = c_1\cos(\omega t) + c_2\sin(\omega t)$, where c_1, c_2 are arbitrary constants, so it is easy to see that

$$y_1(t;\lambda) = \cos(\sqrt{\lambda}\,t) \quad \text{and} \quad y_2(t;\lambda) = \lambda^{-1/2}\sin(\sqrt{\lambda}\,t).$$

So, the solution of the IVP $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_1 = [a \quad b]^T$ =constant is

$$\mathbf{x}(t;\lambda) = e^{tA}[a \quad b]^T = \begin{bmatrix} \cos(\sqrt{\lambda}\,t) & \lambda^{-1/2}\sin(\sqrt{\lambda}\,t) \\ -\lambda^{1/2}\sin(\sqrt{\lambda}\,t) & \cos(\sqrt{\lambda}\,t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cos(\sqrt{\lambda}\,t) + b\lambda^{-1/2}\sin(\sqrt{\lambda}\,t) \\ -a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) + b\cos(\sqrt{\lambda}\,t) \end{bmatrix}.$$

(b) We calculate that

$$\mathbf{z}(t;\lambda) \triangleq \frac{\partial \mathbf{x}}{\partial \lambda}(t;\lambda) = \begin{bmatrix} -at\cdot\frac{1}{2}\lambda^{-1/2}\sin(\sqrt{\lambda}\,t) - b\cdot\frac{1}{2}\lambda^{-3/2}\sin(\sqrt{\lambda}\,t) + bt\cdot\frac{1}{2}\lambda^{-1}\cos(\sqrt{\lambda}\,t) \\ -a\cdot\frac{1}{2}\lambda^{-1/2}\sin(\sqrt{\lambda}\,t) - at\cdot\frac{1}{2}\cos(\sqrt{\lambda}\,t) - bt\cdot\frac{1}{2}\lambda^{-1/2}\sin(\sqrt{\lambda}\,t) \end{bmatrix}.$$

$$= \frac{1}{2}\,t\cdot\lambda^{-1}\cdot \begin{bmatrix} -a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) + b\cos(\sqrt{\lambda}\,t) \\ -a\lambda\cos(\sqrt{\lambda}\,t) - b\lambda^{1/2}\sin(\sqrt{\lambda}\,t) \end{bmatrix} - \frac{1}{2}\lambda^{-3/2}\cdot \begin{bmatrix} b\sin(\sqrt{\lambda}\,t) \\ a\lambda\sin(\sqrt{\lambda}\,t) \end{bmatrix}.$$

(c) We calculate further that

$$\mathbf{z}(t;\lambda) = \frac{1}{2}\,t\cdot\lambda^{-1}\cdot \begin{bmatrix} \cos(\sqrt{\lambda}\,t) & \lambda^{-1/2}\sin(\sqrt{\lambda}\,t) \\ -\lambda^{1/2}\sin(\sqrt{\lambda}\,t) & \cos(\sqrt{\lambda}\,t) \end{bmatrix} \begin{bmatrix} b \\ -\lambda a \end{bmatrix} - \frac{1}{2}\lambda^{-3/2}\sin(\sqrt{\lambda}\,t)\cdot \begin{bmatrix} b \\ a\lambda \end{bmatrix},$$

hence

$$\mathbf{z}(t;\lambda) = \frac{1}{2}\,t\cdot\lambda^{-1}\cdot e^{tA}A \begin{bmatrix} a \\ b \end{bmatrix} - \frac{1}{2}\lambda^{-3/2}\sin(\sqrt{\lambda}\,t)\cdot \begin{bmatrix} b \\ a\lambda \end{bmatrix}.$$

It follows that

$$\dot{\mathbf{z}}(t;\lambda) = \frac{1}{2}\lambda^{-1}\cdot e^{tA}A \begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{2}\,t\lambda^{-1}\cdot Ae^{tA}A \begin{bmatrix} a \\ b \end{bmatrix} - \frac{1}{2}\lambda^{-1}\cos(\sqrt{\lambda}\,t)\cdot \begin{bmatrix} b \\ a\lambda \end{bmatrix}$$

$$= \frac{1}{2}\lambda^{-1}\cdot \begin{bmatrix} -a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) + b\cos(\sqrt{\lambda}\,t) \\ -a\lambda\cos(\sqrt{\lambda}\,t) - b\lambda^{1/2}\sin(\sqrt{\lambda}\,t) \end{bmatrix} - \frac{1}{2}\lambda^{-1}\cos(\sqrt{\lambda}\,t)\cdot \begin{bmatrix} b \\ a\lambda \end{bmatrix} + A\left(\frac{1}{2}\,t\lambda^{-1}\cdot e^{tA}A \begin{bmatrix} a \\ b \end{bmatrix}\right)$$

$$= \frac{1}{2}\lambda^{-1}\cdot \begin{bmatrix} -a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) \\ -2a\lambda\cos(\sqrt{\lambda}\,t) - b\lambda^{1/2}\sin(\sqrt{\lambda}\,t) \end{bmatrix} + A\left(\mathbf{z}(t;\lambda) + \frac{1}{2}\lambda^{-3/2}\sin(\sqrt{\lambda}\,t)\cdot \begin{bmatrix} b \\ a\lambda \end{bmatrix}\right).$$

This implies that

$$\dot{\mathbf{z}}(t;\lambda) = A\mathbf{z}(t;\lambda) + \frac{1}{2}\lambda^{-1}\cdot \begin{bmatrix} -a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) + b\lambda^{-1/2}\sin(\sqrt{\lambda}\,t) \\ a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) - 2a\lambda\cos(\sqrt{\lambda}\,t) - b\lambda^{1/2}\sin(\sqrt{\lambda}\,t) \end{bmatrix}.$$

18.7.2.11. Separation of variables gives implicit solutions of the ODEs:

$$\arctan(y) = \int \frac{1}{1+y^2}\,dy = \int dt = t + c.$$

Substituting in the IC $y(0) = 0$ gives $0 = \arctan(0) = 0 + c$, so an implicit solution of the IVP is $\arctan(y) = t$. The explicit solution is $y = \tan(t)$, which exists only on the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$. So, $\delta = \frac{\pi}{2}$.

(b) Here, $f(t,y) = 1 + y^2$ and $\frac{\partial f}{\partial y}(t,y) = 2y$, so both $f(t,y)$ and $f(t,y)$ are continuous on *any* closed rectangle $\mathcal{R}_{\alpha,\beta}$. So, Picard's Theorem 3.6 in Section 3.2 guarantees the existence and uniqueness of a solution on *some* open time interval.

(c) To use Picard's Theorem 3.7 in Section 3.2, let $M{=}\max_{|y|\le\beta}|1{+}y^2|{=}1{+}\beta^2$ and $K{=}\max_{|y|\le\beta}|2y|{=}2\beta$. Note that $f(t,y)$ does not involve t, so we might as well take $\alpha=\bar{\alpha}$ and $\beta=\bar{\beta}$. To use Theorem 3.7, we need β and α to satisfy $M\alpha\le\beta$ and $K\alpha<1$, that is,

$$(1+\beta^2)\alpha\le\beta \text{ and } (2\beta)\alpha<1.$$

So we need α to be less than or equal to $\frac{\beta}{1+\beta^2}$ *and* α to be strictly less than $\frac{1}{2\beta}$. To find the largest value of α, we first maximize $\frac{\beta}{1+\beta^2}$ over the interval $0<\beta<\infty$. That is a Calculus I problem, and the correct conclusion is to let $\beta=1$. Because of that, we need $\alpha\le\frac{1}{1+1^2}=\frac{1}{2}$, as well as $\alpha<\frac{1}{2\beta}=\frac{1}{2\cdot 1}$. So, Picard's Theorem guarantees the existence and uniqueness of a solution on the time interval $-\bar{\alpha}<t<\bar{\alpha}$, where $\bar{\alpha}$ is any positive number strictly less than $\frac{1}{2}$, which is not as long as the time interval $-\frac{\pi}{2}<t<\frac{\pi}{2}$ on which we know the solution $y=\tan(t)$ actually exists.

Section 18.9

18.9.3.1. Substitute $x(t)=e^{\lambda t}$, where λ is a constant, into the delay equation to get

$$\lambda e^{\lambda t}=\frac{d}{dt}\big[\,e^{\lambda t}\,\big]=\frac{d}{dt}\big[\,x(t)\,\big]=a\,x(t)+b\,x\left(t-\frac{r}{2}\right)+c\,x(t-r)=a\,e^{\lambda t}+b\,e^{\lambda\left(t-\frac{r}{2}\right)}+c\,e^{\lambda(t-r)}$$

$$=ae^{\lambda t}+be^{\lambda t}e^{-r\lambda/2}+ce^{\lambda t}e^{-r\lambda}.$$

Multiply through by $e^{-\lambda t}$ to get the characteristic equation

$$\lambda=a+b\,e^{-r\lambda/2}+c\,e^{-r\lambda}.$$

18.9.3.3. Substitute $\mathbf{x}(t)=e^{\lambda t}\mathbf{v}$, where λ is a constant scalar and $\mathbf{v}$ is a constant vector, into the system of delay equations to get

$$\lambda e^{\lambda t}\mathbf{v}=\frac{d}{dt}\big[\,e^{\lambda t}\mathbf{v}\,\big]=\frac{d}{dt}\big[\,\mathbf{x}(t)\,\big]=A\mathbf{x}(t)+B\mathbf{x}(t-r)=A\,e^{\lambda t}\mathbf{v}+B\,e^{\lambda(t-r)}\mathbf{v}=Ae^{\lambda t}\mathbf{v}+Be^{\lambda t}e^{-r\lambda}\mathbf{v}.$$

Multiply through by $e^{-\lambda t}$ to get

$$\lambda\mathbf{v}=A\mathbf{v}+e^{-r\lambda}B\mathbf{v},$$

that is,

$$\left(\lambda I-A-e^{-r\lambda}B\right)\mathbf{v}=\mathbf{0}.$$

The characteristic equation is

$$\det(\lambda I-A-e^{-r\lambda}B)=0.$$