# Advanced Engineering Mathematics©CRC Press

# Answer Key

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# Contents

# Chapter 1

# Linear Algebraic Equations, Matrices, and Eigenvalues

**Section 1.1.4**

1.1.4.1. unique solution: $(x, y) = (-39, -32)$

1.1.4.3. unique solution $\mathbf{x} = \begin{bmatrix} -25/27 \\ -11/27 \\ 4/27 \end{bmatrix}$

1.1.4.5. (a) there is no solution.

(b) solutions are: $\mathbf{x} = \begin{bmatrix} -11 - 2c_1 + c_2 \\ c_1 \\ 5 \\ c_2 \end{bmatrix}$, arbitrary scalars $c_1, c_2$

1.1.4.7. Ex. Here are three such possible matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.1.4.9. In whole hours, run Plant II for 9 hours/day and Plant III for 17 hours/day, or, if trucks are more profitable than cars, run Plant II for 8 hours/day and Plant III for 18 hours a day.

1.1.4.11. The meal should consist of $\frac{4}{7} \times 100\ g \approx 57.14\ g$ of food #1, $\frac{9}{7} \times 100\ g \approx 128.57\ g$ of food #2, and $\frac{1}{7} \times 100\ g \approx 14.29\ g$ of food #3.

1.1.4.13. (a) may be true or may be false: Ex. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{bmatrix}$,

(b) may be true or may be false: Ex. $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right]$ or $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{array}\right]$, (c), must be true,

(d) must be true, (e) must be true

1.1.4.15. (a) $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & -2 & -5.5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \mathsf{RREF}\,([A \mid \mathbf{b}])$

(b) $\mathsf{RREF}\,([A \mid \mathbf{b}])$; solutions are $\mathbf{x} = \begin{bmatrix} 5-2c_1 \\ -5.5+2c_1 \\ 6 \\ c_1 \end{bmatrix}$, arbitrary constant $c_1$

(c) Both $\mathsf{RREF}\,([A \mid \mathbf{b}])$ and $\mathsf{RREF}(A)$ have rank three.

**Section 1.2.5**

1.2.5.1. <u>Ex</u>. $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

1.2.5.3. <u>Ex</u>. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

1.2.5.5. $A^2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}, A^3 = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 8 \end{bmatrix}$

1.2.5.7. Hints: Assume $A = [a_{ij}]$, where $a_{ij} = 0$ for $i > j$ and $B = [b_{jk}]$, where $b_{jk} = 0$ for $j > k$. To explain why $AB = C = [c_{ik}]$ is upper triangular, suppose $i > k$ and start by calculating

$$c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = \sum_{j \le i} a_{ij}b_{jk} + \sum_{j>i} a_{ij}b_{jk} ...$$

1.2.5.9. False. <u>Ex</u>. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

1.2.5.11. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

1.2.5.13. Hints: Suppose $A = [\,\mathbf{A}_{*1} \mid \mathbf{A}_{*2} \mid ... \mid \mathbf{A}_{*n}\,]$ and $D = diag(d_{11}, d_{22}, ..., d_{nn})$. Explain why

$$AD = \left[\; d_{11}\mathbf{A}_{*1} \mid d_{22}\mathbf{A}_{*2} \mid ... \mid d_{nn}\mathbf{A}_{*n} \;\right]:$$

Begin by using Theorem 1.9 in Section 1.2, and later use Lemma 1.2 in Section 1.2.

**Section 1.3.1**

1.3.1.1. General solution is $\mathbf{x} = c_1 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -5/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$, where $c_1, c_2$ =arbitrary constants

1.3.1.3. General Solution is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$, where $c_1$ =arbitrary constant

1.3.1.5. Hint: For some scalars $c_1, c_2$, $\alpha_1, \alpha_2$, and $\beta_1, \beta_2$, $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, $\mathbf{v}_1 = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$, and $\mathbf{v}_2 = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2$.

1.3.1.7. (a) Ex. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) solutions of $B\mathbf{x} = 0$ are $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, where $c_1, c_2$ are arbitrary constants

**Section 1.4.1**

1.4.1.1. $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h, c_1 =$ arbitrary constant, where $\mathbf{x}_p = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$,
$\mathbf{x}_h = c_1 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

1.4.1.3. no solution

1.4.1.5. (a) $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_1 - c_2 \\ -c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_h = \begin{bmatrix} c_1 - c_2 \\ -c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix}$,
$c_1, c_2$ are arbitrary constants

(b) general solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{x}_h = \begin{bmatrix} c_1 - c_2 \\ -c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix}$, where $c_1, c_2$ are arbitrary constants

(c) $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}^{(2)} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ are the basic solutions of $A\mathbf{x} = \mathbf{0}$

1.4.1.7. Hint: $A\mathbf{x} = \mathbf{b} \Longrightarrow \mathbf{b}^T = \mathbf{x}^T A^T$. Use also the assumption that $A^T\mathbf{z} = \mathbf{0}$ to help calculate $\mathbf{b}^T\mathbf{z}$.

1.4.1.9. No. One can find a $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution. [Hint: Use the elementary matrices that row reduce $A$ to its RREF.]

**Section 1.5.3**

1.5.3.1. $\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$ exists

1.5.3.3. $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} -1 & -2 & 2 \\ -2 & -4 & -3 \\ 2 & -3 & -4 \end{bmatrix}$ exists

1.5.3.5. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 1 \\ -3 & -2 & 1 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} 0 & 2 & -2 \\ -5 & -2 & -3 \\ -10 & 2 & -2 \end{bmatrix}$ exists

1.5.3.7. $AB^T$

1.5.3.9. Yes: $A^{-1} = ... = \begin{bmatrix} a_{11}^{-1} & -\frac{a_{12}}{a_{11}a_{22}} & ... & & \\ & a_{22}^{-1} & -\frac{a_{23}}{a_{22}a_{33}} & ... & \\ & & & & \\ & & \ddots & & \vdots \\ 0 & & & & a_{nn}^{-1} \end{bmatrix}$

1.5.3.11. $A^{-1} = \left[\, \mathbf{y}^{(1)} \,\vdots\, \mathbf{y}^{(3)} + \mathbf{y}^{(4)} \,\vdots\, -\mathbf{y}^{(2)} \,\vdots\, \mathbf{y}^{(3)} - \mathbf{y}^{(4)} \right]$

1.5.3.13. $A^{-1} = \left[\, -\mathbf{y}^{(2)} \,\vdots\, \mathbf{y}^{(1)} + \mathbf{y}^{(2)} \,\vdots\, -\mathbf{y}^{(1)} - \mathbf{y}^{(2)} + \mathbf{y}^{(3)} \right]$

1.5.3.15. (a) Hint: If $A\mathbf{x} = \mathbf{0}$, then $(CA)\mathbf{x} = C(A\mathbf{x}) = ...$
(b) Hint: If $CA\mathbf{x} = \mathbf{0}$, then $C^{-1}(CA\mathbf{x}) = ...$

1.5.3.17. Hints: Define $D = AC$ and $B = C^{-1}A^{-1}$. Explain why $DB = I$, and that will imply $AC$ is invertible ...

1.5.3.19. (a) must be false, (b) must be true, (c) must be false, (d) must be false

1.5.3.21. (a) yes, (b) yes, (c) yes

1.5.3.23. (a) $X = BC$, (b) $X = -BC$, (c) $X = -CB$

1.5.3.25. there exists $A^{-1} = \left[\begin{array}{c|c} A_{11}^{-1} & O \\ \hline -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{array}\right]$

**Section 1.6.3**

1.6.3.1 (a) $\begin{vmatrix} 0 & 1 & 4 \\ -1 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -(1)\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} = ... = -19$

(b) Hint: Begin by using $R_1 \leftrightarrow R_2$

1.6.3.3. <u>Ex</u>. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

1.6.3.5 (a) $|A| = -132$. Hint: Use $-R_1 + R_2 \rightarrow R_2$

(b) Hint: Use $R_2 \leftarrow \frac{1}{3}R_2$. Eventually, get that the determinant is equal to $-44$.

1.6.3.7. Hint: Begin with $-aR_1 + R_2 \rightarrow R_2$

1.6.3.9. (a) Hint: Begin with $B = \alpha A = A(\alpha I) \Longrightarrow B^{-1} = (\alpha I)^{-1}A^{-1} = ...$ exists.

(b) $adj(\alpha A) = \alpha^{n-1} adj\,(A)$. [Hint: Begin with $adj(B) \cdot B = |B| = |\alpha\, A| = \alpha^n |A|$.]

1.6.3.11. $x_1 = -\dfrac{1}{2}\,s - \dfrac{1}{2}\,t$, $\quad x_2 = \dfrac{5}{2} + \dfrac{5}{4}\,s + \dfrac{3}{4}\,t$, $\quad$ and $x_3 = \dfrac{-4s - 20 - 4t}{-4} = 5 + s + t$

1.6.3.13. (a) the system has exactly one solution for $|s| \neq 2$, that is, $2 \neq s \neq -2$

(b) For $|s| \neq 2$, $x_1 = \dfrac{3}{s-2}$ and $x_2 = \dfrac{-6}{s-2}$ .

For $|s| \neq 2$, the solution is $\mathbf{x} = \dfrac{1}{s-2}\begin{bmatrix} 3 \\ -6 \end{bmatrix}$ .

(c) For $s = 2$, there does not exist a solution

For $s = -2$, there are $\infty-$ly many solutions $\quad \mathbf{x} = \begin{bmatrix} -\frac{3}{2} + \frac{1}{2}\,c_1 \\ c_1 \end{bmatrix}$, $c_1 =$ arbitrary constant

1.6.3.15. $x_2 = \frac{1}{2}\,b_1 + \frac{1}{2}\,b_3$

1.6.3.17. If $k$ is neither $0, -6$, nor 6, then the matrix is invertible.

1.6.3.19 (a) Hint: Suppose $R_i = R_j$ in matrix A. Compare $|A|$ and $|B|$, where $B$ is obtained from $A$ by $-R_i + R_j \rightarrow R_j$.

(b) Hint: If $A$ has two equal columns, then $A^T$ has two equal rows.

(c) (i) Hint: Let $B$ be obtained from $A$ by replacing the $j-$th row of $A$ by the $i-th$ row of $A$, and calculate the expansion of $|B|$ along the $j-th$ row of $B$.

(ii) Hint: Let $C$ be obtained from $A$ by replacing the $j-$th column of $A$ by the $i-th$ column of $A$, and calculate the expansion of $|C|$ along the $j-th$ row of $C$.

1.6.3.21. Hints: Define $C \triangleq AB$ and explain why $|C| \neq 0$.

1.6.3.23 (a), (c)

1.6.3.25. (a) Hint: Use $adj(AB) = |AB|\,(AB)^{-1} = |A|\;|B|\,B^{-1}A^{-1}$, $\quad$ (b) no, not necessarily

1.6.3.27. HInt: First, explain why the last $n-r$ columns of $A$ are the last $n-r$ columns of the $n\times n$ identity matrix $I_n$. After that, expand the determinant of $A$ along the last column.

1.6.3.29. Hint: Use $|A| = \left|\begin{array}{c|c} A_{11} & O \\ \hline O & I \end{array}\right| \left|\begin{array}{c|c} I & A_{11}^{-1}A_{12} \\ \hline O & A_{22} \end{array}\right|$ and the results of problems 1.6.3.27 and 1.6.3.28.

**Section 1.7.3**

1.7.3.1. $\mathbf{b}$ is in that Span $\iff 0 = \frac{1}{2}\,b_1 - b_2 + b_3$

1.7.3.3. $t = -2 \pm \sqrt{7}$

1.7.3.5. (a) not linearly independent, does not span $\mathbb{R}^3$, and is not a basis
(b) linearly independent, spans, and is a basis
(c) linearly independent, does not span, and is not a basis
(d) not linearly independent, does span, and is not a basis

1.7.3.7. (a) $2 \le rank(A) \le 4$, (b) $1 \le \nu(A) \le 3$

1.7.3.9. (a) 2, (b) 1, (c) $\left\{ \begin{bmatrix} -.5 \\ 1.5 \\ 1 \end{bmatrix} \right\}$

1.7.3.11. (a) <u>Ex</u>. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has $\nu(A) = 1$ and $A^T$ has $\nu(A^T) = 0$

(b) Suppose $m = n$. Hints: Use $\nu(A^T) = n - rank(A^T)$, $\nu(A) = n - rank(A)$, and Theorem 1.44 in Section 1.7.

# Chapter 2

# Matrix Theory

**Section 2.1.6**

2.1.6.1. eigenvalues & corresponding eigenvectors are (1) $\lambda_1 = -3$ & $\mathbf{v}_1 = c_1 \begin{bmatrix} -7 \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$, and (2) $\lambda_2 = 5$ & $\mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$

2.1.6.3. eigenvalues & corresponding eigenvectors are (1) $\lambda_1 = \sqrt{5}$ & $\mathbf{v}_1 = c_1 \begin{bmatrix} -1+\sqrt{5} \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$, and (2) $\lambda_2 = -\sqrt{5}$ & $\mathbf{v}_2 = c_1 \begin{bmatrix} -1-\sqrt{5} \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$

2.1.6.5. eigenvalues & corresponding eigenvectors are (1) $\lambda_1 = 0$ & $\mathbf{v}_1 = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$, (2) $\lambda_2 = 2$ & $\mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$, and (3) $\lambda_3 = 4$ & $\mathbf{v}_3 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$

2.1.6.7. eigenvalues & corresponding eigenvectors are (1) the defective eigenvalue $\lambda_1 = \lambda_2 = -3$ & $\mathbf{v}_1 = c_1 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$, and (2) $\lambda_3 = -1$ & $\mathbf{v}_3 = c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$

2.1.6.9. eigenvalues & corresponding eigenvectors are (1) $\lambda_1 = 1$ & $\mathbf{v}_1 = c_1 \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$, (2) $\lambda_2 = 4+\sqrt{3}$ & $\mathbf{v}_2 = c_1 \begin{bmatrix} 0 \\ \frac{-1+\sqrt{3}}{2} \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$, and (3) $\lambda_3 = 4-\sqrt{3}$ & $\mathbf{v}_3 = c_1 \begin{bmatrix} 0 \\ \frac{-1-\sqrt{3}}{2} \\ 1 \end{bmatrix}$, any constant $c_1 \neq 0$

2.1.6.11. $\lambda = \dfrac{19 \pm \sqrt{177}}{2}$ are the values of $\lambda$ for which $A\mathbf{x} = \lambda B\mathbf{x}$ has a non-trivial solution for $\mathbf{x}$

2.1.6.13. The eigenvalues are $9, -6, -6$. Their algebraic and geometric multiplicities are $\alpha(9) = 1 = \mu(9)$, $\alpha(-6) = 2 = \mu(6)$. No, $A$ cannot have another eigenvalue.

2.1.6.15. (a) No, two unequal eigenvalues cannot have the same eigenvector ...

(b) Yes, a nonzero vector $\mathbf{x}$ be an eigenvector for two unequal eigenvalues $\lambda_1$ and $\lambda_2$ corresponding to two different matrices $A$ and $B$, respectively.

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

2.1.6.17. (a) $\gamma \triangleq \lambda + \beta$ is an eigenvalue of $C$

(b) Hints: Having the $\mathbf{x} \neq \mathbf{0}$ with $(\lambda + \beta)\mathbf{x} = (A + B)\mathbf{x} = C\mathbf{x} = A^2\mathbf{x} = \lambda^2\mathbf{x}$ implies a quadratic equation for $\lambda$.

2.1.6.19. (a) yes, (b) $A\mathbf{x} = 0$ would have $\infty-$many solutions

2.1.6.21. Hints: First, explain why the $n$ distinct eigenvalues of $A$ must each have algebraic multiplicity of one. Then do row reduction, for example, on $[\, A - a_{11}I \,|\, \mathbf{0}\,]$.

2.1.6.23. (a) Hint: If $(I - A^{-1})$ were not invertible, then there would be an $\mathbf{x} \neq \mathbf{0}$ with $(I - A^{-1})\mathbf{x} = \mathbf{0}$ ...

(b) Hint: $A^{-1}\left(I - A^{-1}\right)^{-1} = \left(\left(I - A^{-1}\right)A\right)^{-1}$

(c) Hint: Use Theorem 1.23(c) in Section 1.5.

2.1.6.25. (a) because $B$ is invertible and $\mathbf{x} \neq \mathbf{0}$

(c) Hint: Explain why $\{\mathbf{x}, \mathbf{y}\}$ is a linearly dependent set.

2.1.6.27. (a) Hint: Use $|\, A - 0 \cdot I \,| = \mathcal{P}(0)$.

(b) Hint: Use $|\, A - 1 \cdot I \,| = \mathcal{P}(1)$.

(c) Hints: Use the results of parts (a) and (b).

2.1.6.29. (a) $\lambda_1 = 0$, $\lambda_2 = -\frac{1}{2}$

(b) $\mathbf{w}_1 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $c_1 \neq 0$, gives non-trivial solns. corr. to $\lambda_1$; $\mathbf{w}_2 = c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $c_2 \neq 0$, gives non-trivial solns. corr. to $\lambda_2$

**Section 2.2.3**

2.2.3.1. Ex. $P = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \,] = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

2.2.3.3. Ex. $P = \begin{bmatrix} -3 & 0 \\ 1 & 1 \end{bmatrix}$

2.2.3.5. Ex. $P = \begin{bmatrix} -1 + \sqrt{2} & 1 + \sqrt{2} \\ 1 & 1 \end{bmatrix}$

2.2.3.7. Ex. $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

2.2.3.9. Ex. $P = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.2.3.11. (a) Ex. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\{\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \mathbf{p}^{(3)}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors of $A$

2.2.3.13. (a) must be true, (b) may be true and may be false, (c) must be false, (d) must be false, (e) must be true, (f) may be true and may be false, (g) may be true and may be false

2.2.3.15. Ex. $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

2.2.3.17. $|A| = 2 \cdot 3 \cdot 3 = 18$

2.2.3.19. Hint: $S^2 = (P\ diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ P^{-1})(P\ diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})\ P^{-1}) = ...$

2.2.3.21. Ex. $\begin{bmatrix} 17 & 40 \\ 2 & 19 \end{bmatrix}$

2.2.3.23. (a) $\mathcal{P}_A(\lambda) = (2-\lambda)(-2-\lambda)(\sqrt{3}-\lambda)$

(b) the set of these three given eigenvectors, $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, is also a linearly independent set of eigenvectors of $A^2$

(c) $\mathcal{P}_{A^2}(\lambda) = (4-\lambda)(4-\lambda)(3-\lambda) = 48 - 40\lambda + 11\lambda^2 - \lambda^3$

2.2.3.25. Hint: Multiply

$$\mathbf{0} = c_{1,1}\mathbf{x}^{1,1} + ... + c_{1,m_1}\mathbf{x}^{1,m_1} + c_{2,1}\mathbf{x}^{2,1} + ... + +c_{2,m_2}\mathbf{x}^{2,m_2} + ... + c_{p,1}\mathbf{x}^{p,1} + ... + c_{p,m_p}\mathbf{x}^{p,m_p}$$

on the left by $(A - \mu_2 I)(A - \mu_3 I) \cdots (A - \mu_p I)$.

2.2.3.27. The eigenvalues are $\lambda = 2, 4 \pm \sqrt{15}$. Because the $3 \times 3$ matrix has three distinct eigenvalues it has a set of three linearly independent eigenvectors.

## Section 2.3.4

2.3.4.1. $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

2.3.4.3. $\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \ \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\}$

2.3.4.5. $\mathcal{S} = \left\{ \frac{1}{5} \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \ \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$

2.3.4.7. Ex. $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2.3.4.9. Yes. Begin by calculating

$$\langle \mathbf{x}, \mathbf{x} + \mathbf{y} - \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{z} \rangle = ||\, \mathbf{x} \,||^2 + \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \mathbf{z} \rangle,$$

$$\langle \mathbf{y}, \mathbf{y} + \mathbf{z} - \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle = ||\, \mathbf{y} \,||^2 + \langle \mathbf{y}, \mathbf{z} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle,$$

and similarly for

$$\langle \mathbf{z}, \mathbf{z} + \mathbf{x} - \mathbf{y} \rangle = \ldots$$

2.3.4.11. (a) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, where $\mathbf{v}_1 = \mathbf{a}_1$, $\mathbf{v}_2 = \mathbf{a}_2 - \frac{3}{2}\mathbf{a}_1$, and $\mathbf{v}_3 = \mathbf{a}_3 - 2\mathbf{a}_1$
(b) $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, where $\mathbf{q}_1 = \frac{1}{\sqrt{2}}\mathbf{a}_1$, $\mathbf{q}_2 = \sqrt{2}(\mathbf{a}_2 - \frac{3}{2}\mathbf{a}_1)$, and $\mathbf{q}_3 = \mathbf{a}_3 - 2\mathbf{a}_1$

2.3.4.13. Hint: Begin by explaining why, for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, we have

$$||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + 2\langle \mathbf{x}, \, \mathbf{y} \rangle + ||\mathbf{y}||^2.$$

2.3.4.15. $\mathbf{v}_n = \mathbf{0}$, so the Gram-Schmidt process fails at this step.

2.3.4.17. Hint: begin by calculating $P^2 = (\mathbf{q}\mathbf{q}^T)(\mathbf{q}\mathbf{q}^T) = \mathbf{q}(\mathbf{q}^T\mathbf{q})\mathbf{q}^T = \ldots$

2.3.4.19. Begin by calculating $P^2 = (P_1P_2)(P_1P_2) = P_1(P_2P_1)P_2 = \ldots$ and
$P^T = (P_1P_2)^T = P_2^T P_1^T = \ldots$

## Section 2.4.4

2.4.4.1. No. In using the G.-S. process, we arrive at $\mathbf{v}_3 = \mathbf{0}$. The underlying cause of the G.-S. process's failure is that the given set of three vectors is linearly dependent.

2.4.4.3. (1) $a = \frac{1}{\sqrt{2}}$ and $b = -\frac{1}{\sqrt{2}}$; (2) $a = -\frac{1}{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$

2.4.4.5. Ex. 1 $Q_1 = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{13}{\sqrt{182}} & 0 \\ \frac{2}{\sqrt{14}} & -\frac{2}{\sqrt{182}} & \frac{9}{\sqrt{117}} \\ \frac{3}{\sqrt{14}} & -\frac{3}{\sqrt{182}} & -\frac{6}{\sqrt{117}} \end{bmatrix}$; Ex. 2 $Q_2 = \begin{bmatrix} \frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{5}} & \frac{6}{\sqrt{70}} \\ \frac{3}{\sqrt{14}} & 0 & -\frac{5}{\sqrt{70}} \end{bmatrix}$

2.4.4.7. Ex. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.4.4.9. Hint: Calculate $Q^TQ = (Q_1Q_2)^T(Q_1Q_2) = \left(Q_2^TQ_1^T\right)(Q_1Q_2) = ...$

2.4.4.11. The hint and a little further work: Rewrite $\langle Q\mathbf{x}, Q\mathbf{y}\rangle = (Q\mathbf{x})^T(Q\mathbf{y}) = (\mathbf{x}^TQ^T)(Q\mathbf{y}) = ...$

2.4.4.13. Hint: Use the result of problem 2.4.4.12.

2.4.4.15. Hint: Use $\mathbf{x} = \langle \mathbf{x}, \mathbf{q}_1\rangle\mathbf{q}_1 + ... + \langle \mathbf{x}, \mathbf{q}_n\rangle\mathbf{q}_n$ to rewrite $\langle \mathbf{x}, \mathbf{y}\rangle = \Big\langle \langle \mathbf{x}, \mathbf{q}_1\rangle\mathbf{q}_1 + ... + \langle \mathbf{x}, \mathbf{q}_n\rangle\mathbf{q}_n, \mathbf{y}\Big\rangle =$ ...

2.4.4.17. Hints: First explain why $Q$ is symmetric, and then calculate
$Q^TQ = Q^2 = \left(I - 2\mathbf{q}\mathbf{q}^T\right)\left(I - 2\mathbf{q}\mathbf{q}^T\right) = ...$

2.4.4.19. $P_A = I_2$

2.4.4.21. $P_A = \frac{1}{45}\begin{bmatrix} 25 & -20 & 10 \\ -20 & 25 & 10 \\ 10 & 10 & 40 \end{bmatrix}$

**Section 2.5.2**

2.5.2.1. there is only one l.s.s.: $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{11}\begin{bmatrix} 2 \\ 7 \end{bmatrix}$

2.5.2.3. there is only one l.s.s.: $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{12}\begin{bmatrix} 11 \\ 18 \end{bmatrix}$

2.5.2.5. there is only one l.s.s.: $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{11}\begin{bmatrix} 29 \\ 14 \end{bmatrix}$

2.5.2.7. Hint: Use Theorem 1.21 in Section 1.5.

2.5.2.9. Take the hint and begin by defining $\mathbf{y} \triangleq [\frac{1}{m} \quad \frac{1}{m} \quad ... \quad \frac{1}{m}]^T$. Use the assumption that at least two of the $x_i$'s are distinct to help explain why the set of vectors $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent. Then, use the Cauchy-Schwarz inequality:

$$|\overline{x}| = \left|\frac{1}{m}(x_1 + ... + x_m)\right| = |\langle \mathbf{x}, \mathbf{y}\rangle| < ||\mathbf{x}|| \;\; ||\mathbf{y}|| \, ...$$

2.5.2.11. $f(x) = \mu x + \beta \approx 1.028880866\,x + 0.0108303249$

2.5.2.13. First, find $f(x) = \mu x + \beta \approx 0.9630541872\,x - 0.0743021346$.
(a) $\approx 2.525944171$, that is, about a $B^-$, in Circuits
(b) $\approx 2.880733163$, that is, between a $B^-$ and a $B$ but closer to a $B$.

2.5.2.15. Yes, there can be infinitely many solutions, depending upon the matrix $A^TA$.
Ex. For the system

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 3 & -3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix},$$

there are infinitely many l.s.s. given by

$$\mathbf{x} = \begin{bmatrix} -\frac{3}{7} + t \\ t \end{bmatrix}, \ -\infty < t < \infty.$$

2.5.2.17. $y = Ae^{\alpha x} \approx 1.000051019\,e^{0.4992600699\,x}$

**Section 2.6.3**

2.6.3.1. Ex. $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ should diagonalize $A$

2.6.3.3. Ex. $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$ should diagonalize $A$

2.6.3.5. Ex. $Q = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 & \sqrt{2} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{2} & \sqrt{3} \end{bmatrix}$ should diagonalize $A$

2.6.3.7. Ex. $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ should diagonalize $A$

2.6.3.9. Ex. $Q = \frac{1}{\sqrt{306}} \begin{bmatrix} -\sqrt{18} & 4 & -4\sqrt{17} \\ 4\sqrt{18} & 1 & -\sqrt{17} \\ 0 & 17 & \sqrt{17} \end{bmatrix}$ should diagonalize $A$

2.6.3.11. For every vector $\mathbf{x} = [\,x_1 \quad \dots \quad x_n\,]^T$ in $\mathbb{C}^n$, calculate $\mathbf{x}^T\overline{\mathbf{x}} = x_1\overline{x_1} + \cdots + x_n\overline{x_n} = ...$

2.6.3.13. Hint: Calculate $B^2 = \big(A(A^TA)^{-1}A^T\big)^2 = \big(A(A^TA)^{-1}A^T\big)\big(A(A^TA)^{-1}A^T\big) = ...$

2.6.3.15. Hints: Use the spectral decomposition (2.34) in Section 2.6 to get $A = \sum_{i=1}^{n} \lambda_i \mathbf{q}_i\mathbf{q}_i^T$ and we define

$$\sqrt{A} \triangleq \sum_{i=1}^{n} \sqrt{\lambda_i}\ \mathbf{q}_i\mathbf{q}_i^T.$$

We calculate that

$$\left(\sqrt{A}\right)^2 = \left(\sum_{i=1}^{n}\sqrt{\lambda_i}\ \mathbf{q}_i\mathbf{q}_i^T\right)\left(\sum_{j=1}^{n}\sqrt{\lambda_j}\ \mathbf{q}_j\mathbf{q}_j^T\right) = ...$$

2.6.3.15. Hint: For all $\mathbf{x} = [\,x_1 \quad x_2\,]^T$ in $\mathbb{R}^2$, we have $\mathbf{x}^T A\mathbf{x} = ... = (x_1 + \alpha x_2)^2 + (1-\alpha^2)x_2^2$.

2.6.3.17. Define $\langle \mathbf{x},\mathbf{y}\rangle_W \triangleq \langle W\mathbf{x},\mathbf{y}\rangle \triangleq \mathbf{x}^T W^T\mathbf{y}$ and $||\,\mathbf{x}\,||_W \triangleq \sqrt{\langle \mathbf{x},\mathbf{x}\rangle_W}$, where $W$ is a real, symmetric, positive definite $n\times n$ matrix.

Hint: *Basically, the method of establishing these results is to apply Theorems 2.3.1 and 2.3.2's properties concerning* $\langle \mathbf{x},\mathbf{y}\rangle$ *and* $||\,\mathbf{x}\,||$ *to get properties for* $\langle \mathbf{x},\mathbf{y}\rangle_W$ *and* $||\,\mathbf{x}\,||_W$.

2.6.3.19. Let $W = diag(b_1^{-2},...,b_m^{-2}) = W^T$. Then the **relative squared error** is

$$\sum_{i=1}^{m}\left(\frac{(A\mathbf{x})_i - b_i}{b_i}\right)^2 = ... = \big\langle diag(b_1^{-2},...,b_m^{-2})(A\mathbf{x}-\mathbf{b}),(A\mathbf{x}-\mathbf{b})\big\rangle = ...$$

2.6.3.21. Exs. $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

**Section 2.7.7**

2.7.7.1. $Q = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & -1 \\ \sqrt{3} & 1 \\ 0 & 2 \end{bmatrix}$ and $R = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}$

2.7.7.3. $Q = \frac{1}{\sqrt{42}}\begin{bmatrix} \sqrt{14} & 1 \\ \sqrt{14} & -5 \\ \sqrt{14} & 4 \end{bmatrix}$ and $R = \frac{1}{\sqrt{3}}\begin{bmatrix} 3 & -1 \\ 0 & \sqrt{14} \end{bmatrix}$

2.7.7.5. $Q = \left[\ \frac{1}{||\mathbf{a}_1||}\,\mathbf{a}_1 \ \vdots \ ... \ \vdots \ \frac{1}{||\mathbf{a}_n||}\,\mathbf{a}_n\ \right]$ and $R = diag(\,||\mathbf{a}_1||,...,||\mathbf{a}_n||\,)$

2.7.7.7. $Q = diag\Big(sgn(a_{11}),...,sgn(a_{nn})\Big)$ and $R = \begin{bmatrix} |a_{11}| & a_{12}sgn(a_{11}) & . & . & . & a_{1n}sgn(a_{11}) \\ 0 & |a_{22}| & & & & a_{2n}sgn(a_{22}) \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ 0 & . & . & & . & |a_{nn}| \end{bmatrix}$

2.7.7.9. Hint: Explain why $A\mathbf{x} = Q(R\mathbf{x}) = Q(Q^T\mathbf{b})...$ and then use $\mathbf{b} = Q\mathbf{c}$ to see why $A\mathbf{x} = ... = \mathbf{b}$.

2.7.7.11. there is only one l.s.s.: $\mathbf{x}^\star = \frac{1}{45}\begin{bmatrix} 41 \\ \\ 36 \end{bmatrix}$

2.7.7.13. Hint: Use Corollary 1.3 in Section 1.7.

2.7.7.15. $Q = \left[\frac{1}{\sqrt{2}}\mathbf{a}_1 \quad \frac{1}{\sqrt{6}}(\mathbf{a}_1 + 2\mathbf{a}_2)\right]$, $\quad R = \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ & \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix}$

2.7.7.17. $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$, $V^T = I_2$

2.7.7.19. $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ -\sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$, $V^T = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$

2.7.7.21. Hint: Use $\mathbf{x}^\star \triangleq V_1 S^{-1} U_1^T \mathbf{b} = [\,\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_r\,] \begin{bmatrix} \sigma_1^{-1} & 0 & . & . & . & 0 \\ 0 & \sigma_2^{-1} & & & & . \\ . & & & . & & . \\ . & & & & . & . \\ 0 & . & . & . & & \sigma_r^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_r^T \end{bmatrix} \mathbf{b}$

2.7.7.23. Hint: Calculate $A = \begin{bmatrix} \frac{3}{\sqrt{2}} & -1 & -1 \\ & & \\ \frac{3}{\sqrt{2}} & 1 & 1 \\ & & \\ 0 & 0 & 0 \end{bmatrix}$ and use the result of Example 2.34 in Section 2.7.

2.7.7.25. Hints: Without loss of generality suppose that $A = QDQ^T$ as in Theorem 2.23, where $D = diag(\lambda_1, ..., \lambda_n)$, where $\lambda_1 \geq ... \geq \lambda_n > 0$.

2.7.7.27. <u>Ex</u>. (a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $U = \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 \\ & & \\ \sqrt{2+\sqrt{2}} & 0 & -\sqrt{2-\sqrt{2}} \\ & & \\ \sqrt{2}/\sqrt{2+\sqrt{2}} & 0 & \sqrt{2}/\sqrt{2-\sqrt{2}} \end{bmatrix}$, $\quad \Sigma = \begin{bmatrix} \sqrt{2+\sqrt{2}} & 0 & 0 \\ & & \\ 0 & 2 & 0 \\ & & \\ 0 & 0 & \sqrt{2-\sqrt{2}} \end{bmatrix}$,

and $V^T = \begin{bmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ & & \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ & & \\ -1/2 & 1/2 & 1/\sqrt{2} \end{bmatrix}$

2.7.7.29. (a) <u>Ex</u>. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $V^T = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 & -1 & 5 \\ \sqrt{6} & 2\sqrt{6} & 0 \\ -2\sqrt{5} & \sqrt{5} & \sqrt{5} \end{bmatrix}$

**Section 2.8.3**

2.8.3.1. $L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}$, $U = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \\ 0 & 0 & -\frac{4}{3} \end{bmatrix}$

2.8.3.5. $L = \begin{bmatrix} \sqrt{3} & 0 \\ -\frac{2}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$

2.8.3.7. $L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & -\sqrt{\frac{2}{3}} & \frac{2}{\sqrt{3}} \end{bmatrix}$

**Section 2.9.2**

2.9.2.1. the Rayleigh quotient gives maximum eigenvalue being $\frac{1+\sqrt{13}}{2}$ and minimum eigenvalue being $\frac{1-\sqrt{13}}{2}$

2.9.2.3. Mathematica™, as in Example 2.37 in Section 2.9, gives maximum value of $+3$ and minimum value of $-1$.

2.9.2.5. $\mathcal{R}_A([1\ \ 1\ \ 1\ \ 1\ \ 1]^T) = 4, \quad \mathcal{R}_A([1\ \ -1\ \ 1\ \ -1\ \ 1]^T) = -\frac{4}{5}$

2.9.2.7. Hint: for each index $i = 1, ..., n$, use $\mathcal{R}_A(\mathbf{e}^{(i)}) = ... = a_{ii}$

2.9.2.9. (a) Hints: Use the facts that $\{t\mathbf{q}_1, (1-t)\mathbf{q}_n\}$ is an orthogonal set of vectors and $y$ is real, along with the Pythagorean theorem to calculate $||\mathbf{x}(t)||^2$.
(c) Hint: Use the Intermediate Value Theorem.

**Section 2.10.8**

2.10.8.1. Hints: Define the function $q(x) \equiv 1$ and suppose $p$ is a polynomial. On the space $\mathcal{P}_n$ consisting of all polynomials of degree less than or equal to $n$ with real coefficients, define the inner product

$$\langle p, q \rangle \triangleq \int_{-1}^{1} p(x)q(x)dx \tag{2.78}$$

and then use the Cauchy-Schwarz inequality:

$$\left| \int_{-1}^{1} p(x) \cdot dx \right| = |\langle p, q \rangle| \le ||p|| \cdot ||q|| = \left( \int_{-1}^{1} |p(x)|^2 \, dx \right)^{1/2} \left( \int_{-1}^{1} |1|^2 \, dx \right)^{1/2} = ...$$

2.10.8.3. Hints: For any unit vector in $\mathcal{V}$, $||\mathbf{x}|| = ||BA\mathbf{x}|| \le ||BA||\ ||\mathbf{x}|| \le ||B||\ ||A||\ ||\mathbf{x}||$...

2.10.8.5. Hints: For *all* $\mathbf{x}, \mathbf{y}$ in $\mathbb{C}^n$, $\langle \mathbf{x}, A^*\mathbf{y} \rangle = \langle A\mathbf{x}, \mathbf{y} \rangle \triangleq (A\mathbf{x})^T \overline{\mathbf{y}} = \mathbf{x}^T A^T \overline{\mathbf{y}} = ...$

2.10.8.7. Hints: Begin with $|\langle \mathbf{x}_k, \mathbf{y}_k\rangle - \langle \mathbf{x}_\infty, \mathbf{y}_\infty\rangle| = |(\langle \mathbf{x}_k, \mathbf{y}_k\rangle - \langle \mathbf{x}_k, \mathbf{y}_\infty\rangle) + (\langle \mathbf{x}_k, \mathbf{y}_\infty\rangle - \langle \mathbf{x}_\infty, \mathbf{y}_\infty\rangle)| \le$ ...

2.10.8.9. Hint: Begin by explaining why $||\, A \,|| \le |\lambda_1| + \cdots + |\lambda_n|$.

2.10.8.11. Hint: For *all* $\mathbf{x}, \mathbf{y}$ in $\mathcal{H}$,

$$\left\langle \mathbf{x}, \left(A^{-1}\right)^* \mathbf{y} \right\rangle = \langle A^{-1}\mathbf{x}, \mathbf{y}\rangle = ...$$

2.10.8.13. HInt: For each $\mathbf{x} = [\, x_1 \quad \dots \quad x_n \,]$ in $\mathbb{C}^n$,

$$||\, A\mathbf{x} \,||^2 = \left|\left| \begin{bmatrix} \sum_{k=1}^n a_{1k}x_k \\ \vdots \\ \sum_{k=1}^n a_{nk}x_k \end{bmatrix} \right|\right|^2 = \sum_{j=1} \left| \sum_{k=1}^n a_{jk}x_k \right|^2 = ...$$

Next, use the fact that or each index $j = 1, ..., n$, $\mathbf{A}_{*j}$ satisfies

$$\sum_{k=1}^n a_{jk}x_k = \mathbf{A}_{*j} \bullet \mathbf{x},$$

and use the Cauchy-Schwarz inequality ...

# Chapter 3

# Scalar ODEs I: Homogeneous Problems

**Section 3.1.4**

3.1.4.1. $y = -e^{-2t} + c\, e^{-t}$, where $c =$arb. const.

3.1.4.3. $y = t^4 + c\, t^3$, where $c =$arb. const.

3.1.4.5. $y = 3 \cdot t^2 + t^2 \ln\, t = t^2(3 + \ln\, t)$

3.1.4.7. $y = -\dfrac{1}{2}\, t^{-3} e^{-2t} + \big( -1 + \dfrac{1}{2}\, e^{-2} \big)\, t^{-3}$

3.1.4.9. $y = -t - e \cdot te^{-t} = -t(1 + e^{-(t-1)})$

3.1.4.11. $A = 100(1 - 0.02t) - 90 \cdot (1 - 0.02t)^3$

3.1.4.13. $y = \dfrac{2}{3}\, t - \dfrac{2}{9} - \dfrac{4}{9}\, e^{-3(t-1)}$

3.1.4.15. $y = (1 + t^2)^{-1} \Big( -7 - \ln 2 + \ln t + \dfrac{1}{2}\, t^2 \Big)$

3.1.4.17. $y = -t^{-1} e^{-2t} + e^{-1} t^{-1} e^{-t} = -t^{-1} e^{-2t} + t^{-1} e^{-(t+1)}$

3.1.4.19. the steady state solution is $y_S(t) = \dfrac{1}{2} \big( -\cos t + \sin t \big)$

3.1.4.21. the steady state solution is $y_S(t) = \dfrac{1}{5} \big( \cos 2t + 2 \sin 2t \big)$

3.1.4.23. (a) $y = \dfrac{2}{\alpha} + \Big( 1 - \dfrac{2}{\alpha} \Big) e^{-\alpha t}$, (b) $\alpha \approx 0.64379776$

3.1.4.25. the temperature at 1:00 pm. was $\approx 499.8371056 \approx 500^\circ\, C$.

3.1.4.27. the person died about 1.960430011 *hours* before 11 am., that is, at about 9:02 am.

3.1.4.29. IVP: $\dfrac{d}{dt}[\, mv\,] = mg - 4v$, $v(0) = 0$; the solution of the IVP is $v = \dfrac{9.81}{8}\big(1 - e^{-8t}\big)$; and the steady state velocity is $v_S = \dfrac{9.81}{8} \approx 1.22625\, m/s$

3.1.4.31. (a) Let $A$ be the number of acres occupied by the plant and $t$ be measured in years. The ODE is $\dot{A} = -10 + kA$, where $k$ is a positive constant.

(b) $A = \dfrac{10}{k} + c\,e^{kt}$

3.1.4.33. the velocity is $v = -32t - u\ln\left(1 - \frac{t}{200}\right) + v_0$; the velocity when the rocket stops burning is $v(190) = -6080 + u\ln 20 + v_0$

3.1.4.35. $y(t) = 3e^{-t^2} + \displaystyle\int_0^t e^{-(t^2-s^2)}ds$

3.1.4.37. $y(t) = \exp\left(-\int_0^t e^{s^2}ds\right)\cdot\left(y(0)+\int_0^t \exp\left(\int_0^s e^{u^2}du\right)ds\right)$

**Section 3.2.4**

3.2.4.1. $C = \phi(t,y) = -\frac{1}{2}\,e^{2t} + ty - \frac{1}{2}t^2\ln y - \frac{1}{3}\,e^{-3y}$, where $C$ is an arbitrary constant

3.2.4.3. (a) $y_1(t) = 1 + \sqrt{2}\,\sqrt{1+t^2} = 1 + \sqrt{2(1+t^2)}$, (b) $y_2(t) \equiv 1$

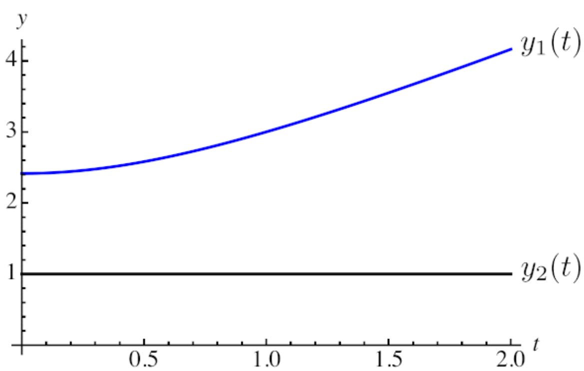


Figure 3.1: Problem 3.2.4.3: Solutions for two different ICs

3.2.4.5. the sarcophagus was buried about 3820 years ago

3.2.4.7. about July 16, 2026

3.2.4.9. between April 26, 2020 and May 5, 2035

3.2.4.11. (a) $y(t) = \dfrac{2}{1 - 2\frac{1}{12}\,t} = \dfrac{2}{1 - \frac{1}{6}\,t}$, (b) the particle reaches position $y = 8$ at time $t = 4.5$
(c) the particle reaches position $y = 8$ at time $t = 5.988$, (d) the particle lives for $0 \le t < 6$

3.2.4.13. $4 = -x^2 - xy + 3y + y^3$

3.2.4.15. $C = \phi(t,y) = t\sin y + y\sin t - 4t + \dfrac{1}{2}\,y^2$, where $C$ is an arbitrary constant

3.2.4.17. (a) Assuming $A = 0.005$ and $k = 0.01$, it would take about 2 hours and 9 minutes for the person's blood alcohol concentration to be within the legal limit.

(b) Ex: 1: Assuming $A = 0.005$ and $k = 0.007$, it would take about 3 hours and 4 minutes for the person's blood alcohol concentration to be within the legal limit.

Ex: 2: Assuming $A = 0.01$ and $k = 0.01$, it would take about 2 hours and 42 minutes for the person's blood alcohol concentration to be within the legal limit.

(c) add a positive constant, $b$, to the right hand side of the ODE

3.2.4.19. Exs: $y_1(t) = 1 + \left(\frac{t-2}{2}\right)^2$, $y_2(t) \equiv 1$

3.2.4.21. Exs: $y_1(t) = \left(\frac{t}{3}\right)^3$, $y_2(t) \equiv 0$

3.2.4.23. (a) Hint: get solution $y(t) = -\dfrac{1}{t - \frac{1}{3}} \cdot \dfrac{3}{3} = \dfrac{3}{1-3t}$,

(b) Hints: $f(t,y) \triangleq y^2$ is continuous and has continuous partial derivative with respect to $y$ everywhere.

(c) Hints: Use Picard's Theorem 3.7 to get the criterion $2(3+0)\bar{\alpha} < 1$. The theoretically guaranteed time open interval of existence is *at most* $-\dfrac{1}{6} < t < \dfrac{1}{6}$ but the actual time interval of existence is $\left(-\infty, \dfrac{1}{3}\right)$. In fact, we need

$$\bar{\alpha} < g(3) = \frac{1}{12},$$

which is more restrictive than $\bar{\alpha} < \frac{1}{6}$.

**Section 3.3.8**

3.3.8.1. $y(t) = c_1 e^{-5t} + c_2 e^{-3t}$, $c_1, c_2 =$ arbitrary consts., time constant is $\tau = \frac{1}{3}$

3.3.8.3. $y(t) = c_1 e^{-5t/2} + c_2 e^{3t/2}$, $c_1, c_2 =$ arbitrary consts., there is no time constant

3.3.8.5. $y(t) = c_1 e^{-4t}\cos(\sqrt{2}\,t) + c_2 e^{-4t}\sin(\sqrt{2}\,t)$, where $c_1, c_2 =$arb. consts., time constant is $\tau = \dfrac{1}{4}$

3.3.8.7. $y(t) = 2e^{-t} + 3e^{3t}$

3.3.8.9. $y(t) = 7e^{-t/2}\left(\cos(\sqrt{2}\,t) + \dfrac{1}{2\sqrt{2}}\sin(\sqrt{2}\,t)\right)$

3.3.8.11. $y(t) = \dfrac{1}{7}\left(5e^{-5t} + 2e^{2t}\right)$

3.3.8.13. $y(t) = \left(-1 + \dfrac{3}{2}\,t\right)e^{-t/2}$

3.3.8.15. $y(t) = e^{-t/2}\left(-2\cos\left(\dfrac{\sqrt{3}}{2}\,t\right) - 2\sqrt{3}\sin\left(\dfrac{\sqrt{3}}{2}\,t\right)\right) = 4e^{-t/2}\cos\left(\dfrac{\sqrt{3}}{2}\,t - \dfrac{4\pi}{3}\right)$

3.3.8.17. $y(t) = e^{-t}\left(-2\cos 2t + 2\sin 2t\right) = 2\sqrt{2}\,e^{-t}\cos\left(2t - \frac{3\pi}{4}\right)$

3.3.8.19. $y(t) = e^{-2t}\Big( -\cos t + \sqrt{3}\,\sin t\Big) = 2\,e^{-2t}\cos\left(t - \frac{2\pi}{3}\right)$

3.3.8.21. $y(t) = e^{-2t}\left(-2\,\cos\left(\sqrt{8}\,t\right) - \sqrt{2}\,\sin\left(\sqrt{8}\,t\right)\right) = \sqrt{6}\,e^{-2t}\cos\left(\sqrt{8}\,t - \pi - \arctan\frac{1}{\sqrt{2}}\right)$

3.3.8.23. Note that $m = 1$. (a) Ex. $b = 3, k = 2$ would be in the overdamped case,
(b) Ex. $b = 3, k = 3$ would be in the underdamped case,
(c) Ex. $b = 2, k = 1$ would be in the critically damped case

3.3.8.25. only pair (e) could conceivably give the graphs shown in Figure 3.15 in the textbook

3.3.8.27. $\dfrac{1}{e}$ meters

3.3.8.29. $b = 6$ & $k = 10$

3.3.8.31. $C \geq \dfrac{1}{50}$

3.3.8.33. $p \approx 0.2506287350$ & $q \approx 9.885308092$

3.3.8.35. (a) a1, (b) b3

3.3.8.39. Hint: after some work, find that $y(t) = y_0 e^{-\alpha t}\left(1 + \alpha t\right)$, and then use that to analyze whether $\dot{y}(t)$ can equal zero

3.3.8.41. (b) HInt: calculate the Wronskian, (c) there is no solution of this IVP
(d) Hint: examine the hypotheses of Theorem 3.8

3.3.8.43. $y(t) = c\,\cosh\big(\omega(L - t)\big)$, where $c$ is an arbitrary constant

3.3.8.45. all $\alpha$ *except* $\alpha = \pm\sqrt{3}$

3.3.8.47. Hint: Begin by using Calculus to find the relative maxima and minima of $y(t) = Ae^{\alpha t}\cos(\nu t - \delta)$

**Section 3.4.4**

3.4.4.1. $y(t) = c_1 e^{t} + c_2 e^{-t}\cos t + c_3 e^{-t}\sin t$, where $c_1, c_2, c_3$ =arbitrary constants

3.4.4.3. $y(t) = c_1 e^{-t} + c_2\,t\,e^{-t} + c_3\,t^2\,e^{-t}$, where $c_1, c_2, c_3$ =arbitrary constants

3.4.4.5. $y(t) = \dfrac{1}{120}\,(5e^{-3t} - 8 + 3e^{5t})$; there is no time constant

3.4.4.7. $y(t) = \dfrac{1}{4\sqrt{3}}\,e^{-\sqrt{3}\,t} - \dfrac{1}{4\sqrt{3}}\,e^{\sqrt{3}\,t} + \dfrac{1}{2}\sin t$

3.4.4.9. $y(t) = c_1 e^{4t} + c_2\,t\,e^{4t} + c_3\cos t + c_4\sin t$, where $c_1, c_2, c_3, c_4$ =arbitrary constants

3.4.4.11. (a) $y(t) = c_1 e^{-2t} + c_2\,e^{t}\cos(\sqrt{3}\,t) + c_3\,e^{-t}\sin(\sqrt{3}\,t)$, where $c_1, c_2, c_3$ =arbitrary constants

(b) $y(t) = c_1\, e^{\sqrt[3]{2}\, t} + c_2\, e^{-\sqrt[3]{2}\, t/2} \cos\left(\dfrac{\sqrt[3]{2} \cdot \sqrt{3}t}{2}\right) + c_3\, e^{-\sqrt[3]{2}t/2} \sin\left(\dfrac{\sqrt[3]{2} \cdot \sqrt{3}t}{2}\right)$, where $c_1, c_2, c_3$ =arbitrary constants

**Section 3.5.1**

3.5.1.1. $y(r) = c_1 r^{-2+\sqrt{6}} + c_2 r^{-2-\sqrt{6}}$, where $c_1, c_2$ =arbitrary constants

3.5.1.3. $y(r) = c_1 \cos(2 \ln r) + c_2 \sin(2 \ln r)$, where $c_1, c_2$ =arbitrary constants

3.5.1.5. $y(r) = c_1 r^{-2} + c_2 r^{-2} \ln r$, where $c_1, c_2$ =arbitrary constants

3.5.1.7. $y(r) = \frac{11}{3}(-e^2 r^{-1} + e^{-1} r^2)$

3.5.1.9. $y(r) = -2r \cos(2 \ln r) + r \sin(2 \ln r)$

3.5.1.11. (a) *Case 1*: If $m = 0$, $y(r) = c_1 + c_2 \ln r$, where $c_1, c_2$ =arbitrary constants
(b) *Case 2*: If integer $m \geq 1$, $y(r) = c_1 r^{2m} + c_2 r^{-2m}$, where $c_1, c_2$ =arbitrary constants

# Chapter 4

# Scalar ODEs II: Non-homogeneous Problems

**Section 4.1.5**

4.1.5.1. $y(t) = c_1 e^{-3t} + c_2 e^{-2t} + \dfrac{3}{2}\, e^{-t}$, where $c_1, c_2$ =arbitrary constants

4.1.5.3. $y(t) = -2\, t\, e^{-3t} + c_1 e^{-3t} + c_2 e^{-2t}$, where $c_1, c_2$ =arbitrary constants

4.1.5.5. $y(t) = \dfrac{5}{3}\, e^{-2t} - \dfrac{1}{2}\, t\, e^{-t} + c_1 e^{-t} + c_2 e^{t}$, where $c_1, c_2$ =arbitrary constants

4.1.5.7. $y(t) = -\dfrac{5}{7}\, t\, e^{-4t} + c_1 e^{-4t} + c_2 e^{3t}$, where $c_1, c_2$ =arbitrary constants

4.1.5.9. $y(t) = -e^{-2t}\cos(e^t) + \big(\cos(1) + \sin(1)\big)e^{-2t} - \sin(1)e^{-t}$

4.1.5.11. $y(t) = -\dfrac{1}{9}\, te^t + \dfrac{1}{6}\, t^2 e^t + \dfrac{17}{27}\big(e^{-2t} - e^t\big)$

4.1.5.13. $y(t) = \dfrac{1}{3}\, e^{-2t} - \dfrac{1}{2}\, e^{-t} + \dfrac{1}{6}\, e^{t}$

4.1.5.15. $y(t) = 2 - 2e^{-t/2}\left(\cos\left(\dfrac{\sqrt{19}}{2}\, t\right) + \dfrac{1}{\sqrt{19}}\, \cos\left(\dfrac{\sqrt{19}}{2}\, t\right)\right)$

4.1.5.17. The steady state solution is $y_S(t) = \frac{1}{17}(-4\cos 2t + \sin 2t)$, whose amplitude is $\dfrac{1}{\sqrt{17}}$.

4.1.5.19. The steady state solution is $y_S(t) = \frac{1}{8}(\cos t + \sin t)$, whose amplitude is $\dfrac{1}{\sqrt{32}}$.

4.1.5.21. The steady state solution is $y_S(t) = \dfrac{3}{5}$, whose amplitude is $\dfrac{3}{5}$.

4.1.5.23. The steady state solution is $y_S(t) = \dfrac{f_0}{12}\big(-\cos(\sqrt{2}\, t) + \sqrt{2}\, \sin(\sqrt{2}\, t)\big)$, whose amplitude is $\dfrac{|f_0|}{4\sqrt{3}}$.

4.1.5.25. $y(t)=\frac{1}{2}(1+t)+\frac{1}{2}(1+\pi)e^{t-\pi}\cos t-\frac{\pi}{2}e^{t-\pi}\sin t$

4.1.5.27. $y(t)=-\frac{3}{5}\,t\,e^{-2t}+c_1e^{-2t}+c_2\cos t+c_3\sin t$, where $c_1,c_2,c_3$ =arbitrary constants

4.1.5.29. $y(t)=-\frac{1}{10}(-2t+3)-\frac{2}{17}(-\cos 2t-4\sin 2t)+c_1e^{-t}\cos 2t+c_2e^{-t}\sin 2t$, where $c_1,c_2$ =arbitrary constants

4.1.5.31. Ex. 1: $(\star)\quad \ddot{y}+5\dot{y}+6y=2e^{-t}$, Ex. 2: $(\star\star)\quad \ddot{y}+4\dot{y}+3y=-e^{-2t}$
In fact, there are infinitely many examples based on each of $(\star)$ and $(\star\star)$, for example, $5\ddot{y}+25\dot{y}+30y=10e^{-t}$.

**Section 4.2.5**

4.2.5.1. $y(t)=-\frac{3}{4}\,t\sin 2t+c_1\cos 2t+c_2\sin 2t$, where $c_1,c_2$ =arbitrary constants

4.2.5.3. $y(t)=\frac{1}{2}\,t\sin 4t-\cos 4t+\frac{3}{4}\sin 4t$

4.2.5.5. the steady state solution is $y_S(t)=\frac{1}{5}(-2\cos t+\sin t)=\frac{1}{\sqrt{5}}\cos\Big(t-\pi+\arctan\frac{1}{2}\Big)$, whose amplitude is $\frac{1}{\sqrt{5}}$

4.2.5.7. the steady state solution is $y_S(t)=\frac{1}{17}(-4\cos 2t-\sin 2t)=\frac{1}{\sqrt{17}}\cos\Big(2t-\pi-\arctan\frac{1}{4}\Big)$, whose amplitude is $\frac{1}{\sqrt{17}}$

4.2.5.9. the steady state solution is $y_S(t)=2\cos 3t+12\sin 3t=2\sqrt{37}\cos\Big(3t-\arctan 6\Big)$, whose amplitude is $2\sqrt{37}$

4.2.5.11. the steady state solution is $y_S(t)=\frac{1}{10}(-2\cos 2t+\sin 2t)=\frac{1}{2\sqrt{5}}\cos\Big(2t-\pi+\arctan\frac{1}{2}\Big)$, whose amplitude is $\frac{1}{2\sqrt{5}}$

4.2.5.13. (a) the frequency of the beats is $\zeta=\frac{|2\sqrt{2}-3|}{2}=\frac{3-2\sqrt{2}}{2}$
(b) the maximum amplitude of the motion given is 10

4.2.5.15. $y_S(t)=\frac{f_0}{(k-m\omega^2)^2+(b\omega)^2}\Big(-b\omega\cos\omega t+(k-m\omega^2)\sin\omega t\Big)$

4.2.5.17. (a) $\omega=\frac{2\pi}{4}=\frac{\pi}{2}$, (b) the steady state solution appears to have $Amplitude\approx 1.7$,
(c) there are two possibilities for the natural frequency: $\omega_0^+=\sqrt{k^+}\approx\sqrt{4.013394149}\approx 2.0$ and $\omega_0^-=\sqrt{k^-}\approx\sqrt{0.9214080517}\approx 0.96$, to the two significant digits of accuracy of the data we read from the graph

4.2.5.19. as $d \to 0^+$, the $Amplitude\ of\ the\ steady\ state\ solution\ \to +\infty$

4.2.5.21. pure resonance vibrations occur if the machine spins at $\approx 9.862471105\,Hz$, that is, revolutions per second

4.2.5.23. $\omega = 2, \quad b = 0, \quad m = \dfrac{3}{2}, \quad f_0 = 12$

4.2.5.25. (a) Ex: $b = 1$, $f_0 = 1$, & $\omega = 1$, (b) Ex: $b = 0$, $f_0 = 1$, & $\omega = 3$, (c) Ex: $b = 0$, $f_0 = 1$, & $\omega = 2$

4.2.5.27. Hint: $\omega_0^2 = \dfrac{k}{m}$, so $k - m\omega_0^2 = 0$

4.2.5.29. (a) $\ddot{y} + y = 1.5 - 0.52e^{-t/5}$
(b) $0.3 = (D^2+1)\Big(D + \frac{1}{5}\Big)[\,y\,] = \dddot{y} + \frac{1}{5}\,\ddot{y} + \dot{y} + \frac{1}{5}\,y$
(c) $0 = (D^2+1)\Big(D + \frac{1}{5}\Big)D[\,y\,] = y^{(iv)} + \frac{1}{5}\,\dddot{y} + \ddot{y} + \frac{1}{5}\,\dot{y}$

4.2.5.31. (a) the steady state solution is $y_S(t) = \frac{f_0}{\omega^2+\delta^2}\,(-\delta\cos\omega t + \omega\sin\omega t)$
(b) the steady state solution is $y_S(t) = \frac{f_0}{\omega^2+\delta^2}\,(-\omega\cos\omega t + \delta\sin\omega t)$
(c) the steady state solution is $y_S(t) = \frac{f_0}{\omega^2+\delta^2}\Big((a\delta - b\omega)\cos\omega t + (a\omega + b\delta)\sin\omega t\Big)$

**Section 4.3.2**

4.3.2.1. $y(x) = -\frac{1}{2}\,x\cos(2x) + \frac{1}{4}\,\sin(2x)\ln|\sin 2x| + c_1\cos 2x + c_2\sin 2x$, where $c_1, c_2$ =arbitrary constants

4.3.2.3. $y(t) = -1 + \sin(t)\ln|\tan t + \sec t| + c_1\cos t + c_2\sin t$, where $c_1, c_2$ =arbitrary constants

4.3.2.5. $y(x) = \frac{1}{2}\,x^3 + c_1 x + c_2 x^2$, where $c_1, c_2$ =arbitrary constants

4.3.2.7. $y(x) = c_1 x^2 + c_2 x^4 + \frac{1}{2}\,x^2 e^{-x} + \frac{1}{2}\,x^4 \int_1^x s^{-2}e^{-s}ds$, where $c_1, c_2$ =arbitrary constants

4.3.2.9. $y(r) = c_1 r^3 + c_2 r^4 - \frac{1}{4} r^3 \sin 2r$, where $c_1, c_2$ =arbitrary constants

4.3.2.11. $y(r) = \widetilde{c}_1\, r^2 + \widetilde{c}_2\, r^3 - r^2\ln|r|$, where $\widetilde{c}_1, \widetilde{c}_2$ =arbitrary constants

4.3.2.13. $y(t) = -t\,e^{-t} + \widetilde{c}_1 + \widetilde{c}_2\,e^{-t}$, where $\widetilde{c}_1, \widetilde{c}_2$ =arbitrary constants.

4.3.2.15. $y(t) = \Big(e^2 + \frac{2}{5} - 2\Big(e^2 + \frac{1}{3}\Big)t + \frac{4}{15}\,t^{5/2}\Big)\,e^{-2t}$

4.3.2.17. $y(r) = \frac{1}{5}\,r^2 + \frac{14}{5}\,\cos(\ln r) - \frac{7}{5}\,\sin(\ln r)$

4.3.2.19. $y(x) = -6 - 6x - 3x^2 - x^3 + c_1 e^x + c_2\big(1 + x + \frac{1}{2}x^2\big)$, where $c_1, c_2$ =arbitrary constants

## Section 4.4.1

4.4.1.1. $-\dfrac{5}{s-3}+\dfrac{2}{s^2+4}$

4.4.1.3. $\dfrac{1}{s}\left(1+\dfrac{a}{s}+\left(\dfrac{a}{s}\right)^2+\left(\dfrac{a}{s}\right)^3\right)$

4.4.1.5. $\dfrac{s-\frac{1}{2}}{s^2+s+1}$

4.4.1.7. $-\dfrac{4}{5}\,e^{-2t}+\dfrac{1}{5}(4\cos t-3\sin t)$

4.4.1.9. $e^{-2t}\big(-\cos(\sqrt{3}\,t)+2\sqrt{3}\,\sin(\sqrt{3}\,t)\big)$

4.4.1.11. $\dfrac{17}{25}\,e^{-3t}+\dfrac{3}{5}\,t\,e^{-3t}+e^{-t}\left(-\dfrac{17}{25}\,\cos t+\dfrac{19}{25}\,\sin t\right)$

4.4.1.13. $y(t)=\ -\dfrac{5}{2}\,e^{2t}+\dfrac{3}{2}\,e^{4t}$; there is no steady state solution

4.4.1.15. $y(t)=\dfrac{5}{7}\,e^{-5t}+\dfrac{2}{7}\,e^{2t}$; there is no steady state solution

4.4.1.17. $y(t)=-\frac{1}{5}\cos(3t)-\frac{4}{15}\sin(3t)+\frac{1}{5}\,e^{-t}+te^{-t}$

4.4.1.19. the steady state solution is $y_S(t)=\frac{1}{5}\,\mathcal{L}^{-1}\left[\frac{-2s+1}{s^2+1}\right]=\frac{1}{5}(-2\cos t+\sin t)$

4.4.1.21. beats phenomenon case

4.4.1.23. $\mathcal{L}\,[\,p_n(t)\,]=\dfrac{1-\left(\frac{1}{s}\right)^{n+1}}{s-1}$

## Section 4.5.8

4.5.8.1. $\dfrac{7}{s}+\dfrac{3}{s+2}-\left(\dfrac{4}{s}+\dfrac{1}{s^2}\right)e^{-4s}$

4.5.8.3. $-\dfrac{1}{s^2}+\dfrac{4}{s}+\dfrac{2}{s^2}\,e^{-2s}$

4.5.8.5. $\dfrac{s}{(s+1)(s^2+4)}$

4.5.8.7. only (a) is correct

4.5.8.9. $\dfrac{1}{2}\big(-1+e^{2(t-1)}\big)\mathit{step}(t-1)$

4.5.8.11. $\dfrac{1}{4}\,t\sin(2t)$

4.5.8.13. $\dfrac{1}{4}\sin 2t-\dfrac{1}{2}t\cos 2t$

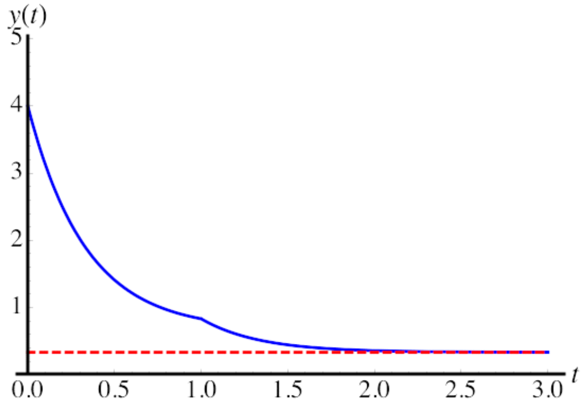


Figure 4.1: Answer for problem 4.5.8.15

4.5.8.15. $y(t) = \frac{1}{3}\Big(10e^{-3t} + 2 + \big(-1 + e^{-3(t-1)}\big)\, step(t-1)\Big)$

4.5.8.17. $y(t) = -\frac{5}{2}\sin(2t) + \frac{1}{4}\Big(1 - \cos\big(2(t-c)\big)\Big) step(t-c)$ For the case when $c = \sqrt{2}$, the graph of the solution is given below. The amplitude of the steady state solution is $\frac{1}{4} + \sqrt{\big(-\frac{1}{4}\cos(2\sqrt{2})\big)^2 + \big(-\frac{5}{2} - \frac{1}{4}\sin(2\sqrt{2})\big)^2}$, as indicated by the dashed horizontal lines in the figure.

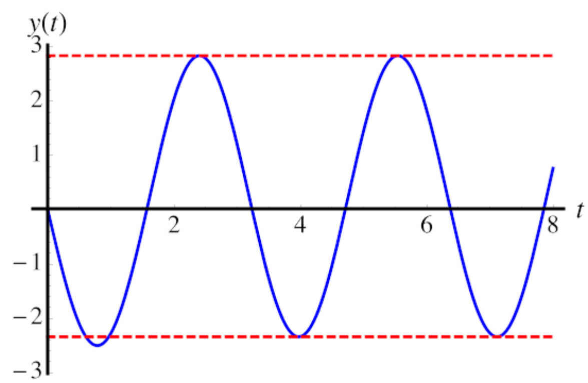


Figure 4.2: Answer for problem 4.5.8.17

4.5.8.19. $y(t) = e^{-3t} + e^{-3(t-2)} step(t-2)$, whose graph is given below

4.5.8.21. $y(t) = a\cos(t) + b\sin(t) + \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t)$. The graph of the solution, when $a = \frac{1}{2}$ and $b = -1$, is given below, with the red dashed line showing the "envelope" $y = \pm\frac{1}{2}\,t$.

4.5.8.23. $y(t) = \int_0^t \frac{1}{2}\sin\big(2(t-u)\big) f(u)\, du$

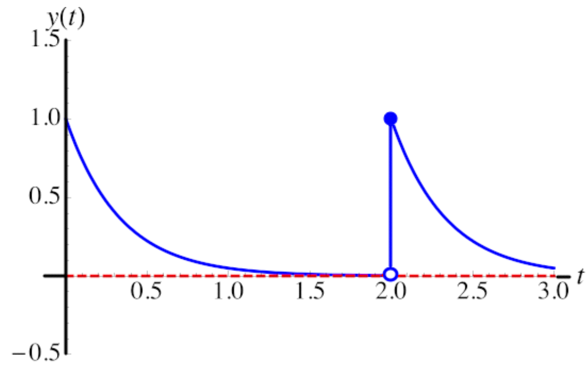


Figure 4.3: Answer for problem 4.5.8.19

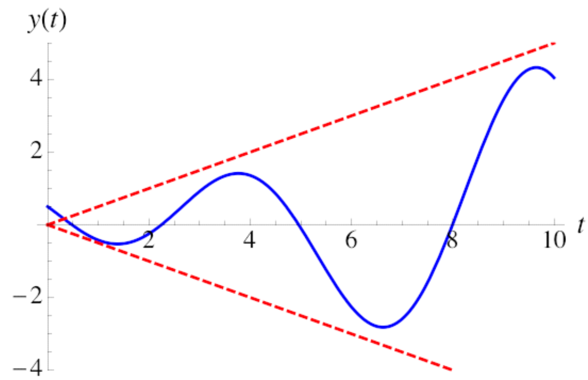


Figure 4.4: Answer for problem 4.5.8.21

4.5.8.25. $y(t) = \frac{1}{25}\, e^{-5t} \cdot \begin{cases} 1 + (5t-1)e^{5t}, & 0 \le t \le \pi \\ 2 + (5\pi - 1)e^{5\pi} + (5(t-\pi)-1)e^{5t}, & \pi \le t \le 2\pi \\ 3 + (5\pi-1)e^{5\pi} + (5\pi-1)e^{10\pi} + (5(t-2\pi)-1)e^{5t}, & 2\pi \le t \le 3\pi \\ 4 + (5\pi-1)e^{5\pi} + (5\pi-1)e^{10\pi} + (5\pi-1)e^{15\pi} + (5(t-3\pi)-1)e^{5t}, & 3\pi \le t \le 4\pi \\ \vdots & \vdots \end{cases}$

In general, for $k\pi \le t \le (k+1)\pi$, $y(t) = \frac{1}{25} e^{-5t} \left( k + 1 + (5(t-k\pi)-1)e^{5t} + (5\pi - 1) \sum_{j=1}^{k} e^{5k\pi} \right)$. The solution is graphed below.

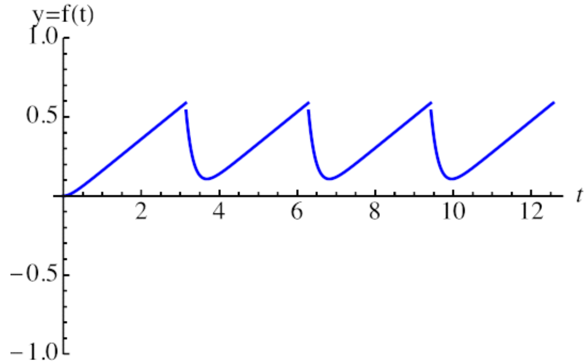


Figure 4.5: Answer for problem 4.5.8.25

**Section 4.6.4**

4.6.4.1. $y_k = c_1 + c_2\,(-1)^k$, where $c_1, c_2$ are arbitrary constants

4.6.4.3. $y_k = c_1\,\cos\left(\frac{\pi}{2}\,k\right) + c_2\,\sin\left(\frac{\pi}{2}\,k\right)$, where $c_1, c_2$ are arbitrary constants

4.6.4.5. $y_k = c_1 + c_2\,\cos\left(\frac{2\pi}{3}\,k\right) + c_3\,\sin\left(\frac{2\pi}{3}\,k\right)$, where $c_1, c_2, c_3$ are arbitrary constants

4.6.4.7. $y_k = y_k^{(h)} + y_k^{(p)} = c_1 + c_2(-1)^k + \frac{1}{2}\,k - \frac{1}{4}\,k^2$, where $c_1, c_2$ =arbitrary constants

4.6.4.9. $y_k = 2^{k/2}\left(2\,\cos(\omega k) + \frac{4}{\sqrt{7}}\,\sin(\omega k)\right)$, where $\omega = \pi - \arctan(\sqrt{7})$

4.6.4.11. $y_k = 13^{k/2}\,(-2\,\cos(\omega k) - \sin(\omega k))$, where $\omega = \pi - \arctan(3/2)$

4.6.4.13. $\det(A_k) = y_k = \cos\left(\frac{\pi}{3}\,k\right) + \frac{1}{\sqrt{3}}\,\sin\left(\frac{\pi}{3}\,k\right) = \frac{2}{\sqrt{3}}\sin\left(\frac{\pi}{3}(2-k)\right)$, for $k \geq 3$

4.6.4.15. $\det(A_k) = 1\cdot(-1)^k + 1\cdot k\,(-1)^k = (1+k)(-1)^k$, $k \geq 3$

4.6.4.17. $\det(A_k) = 2^k \cdot \left(\cos(\omega\,k) - \frac{3}{\sqrt{7}}\,\sin(\omega\,k)\right)$, $k \geq 3$, where $\omega = \pi - \arctan\left(\frac{\sqrt{7}}{3}\right)$

4.6.4.19. After running through a cascade of 1024 centrifuges, the mixture contains $\approx 4.19\%$ of $U^{235}F_6$.

4.6.4.21. (a) Hint: Use $-a_{2,k}R_1 + R_3 \to R_3$ and $-a_{1,k}R_2 + R_3 \to R_3$ to begin.
(b) Hint: Use $-a_{n-1,k}R_1 + R_n \to R_n$, $-a_{n-2,k}R_2 + R_n \to R_n$,..., $-a_{1,k}R_{n-1} + R_n \to R_n$ to begin.

**Section 4.7.5**

4.7.5.1. $x[n] = \mathcal{Z}^{-1}[\,X(z)\,] = \frac{\alpha}{\alpha+1}\cdot(-1)^n + \frac{1}{1+\alpha}\cdot\alpha^n$

4.7.5.3. the steady state solution is $x_S[n] = -\frac{1}{3}\cos\frac{\pi n}{3} + \frac{1}{\sqrt{3}}\,\sin\frac{\pi n}{3} + \frac{1}{3}\,(-1)^n$

4.7.5.5. the steady state solution is $x_S[n] = \frac{1}{\alpha^2+1}\left(\cos\frac{\pi n}{2} + \alpha\sin\frac{\pi n}{2}\right)$

4.7.5.9. $x[n] = \frac{1}{6}\cdot 2^n + \frac{1}{6}\cdot(-1)^n - \frac{1}{3}\cdot\cos\frac{\pi n}{3}$

# Chapter 5

# Linear Systems of ODEs

**Section 5.1.3**

5.1.3.1. $\dot{\mathbf{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21}-a_{12} & a_{22} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 \\ a_{51} & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

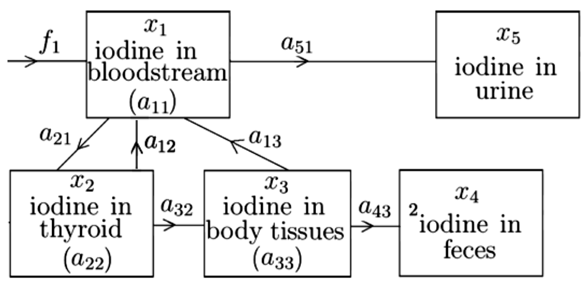


Figure 5.1: Problem 5.1.3.1: Modification of Example 5.6

5.1.3.3. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies $\ddot{\mathbf{x}} = \begin{bmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} & \dfrac{k_3}{m_2} \\ 0 & \dfrac{k_3}{m_3} & -\dfrac{k_3+k_4}{m_3} \end{bmatrix} \mathbf{x} \triangleq A\mathbf{x}$

5.1.3.5. $\left\{ \begin{array}{l} \dot{A}_1 = 8 - \dfrac{6}{V_1(0)}\,A_1 + \dfrac{2}{V_2(0)}\,A_2 \\ \\ \dot{A}_2 = \qquad \dfrac{5}{V_1(0)}\,A_1 - \dfrac{5}{V_2(0)}\,A_2 \end{array} \right\}$

5.1.3.7. Let $T_1, T_2, M$ be the temperatures, respectively, of the two objects and the medium.

$$\frac{d}{dt}\begin{bmatrix} T_1 \\ T_2 \\ M \end{bmatrix} = \begin{bmatrix} -k_{T,1} & 0 & k_{T,1} \\ 0 & -k_{T,2} & k_{T,2} \\ k_M & k_M & -2k_M \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ M \end{bmatrix}$$

## Section 5.2.5

5.2.5.1. $\mathbf{x}(t) = c_1 e^{-3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{7t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, where $c_1, c_2 =$ arbitrary constants

5.2.5.3. $\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -3 \\ 12 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, where $c_1, c_2, c_3 =$ arbitrary constants

5.2.5.5. $\mathbf{x}(t) = c_1 e^{-4t}\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$, where $c_1, c_2 =$ arbitrary constants, time constant is $\tau = 1$

5.2.5.7. $\mathbf{x}(t) = c_1 e^{at}\begin{bmatrix} a-c \\ b \end{bmatrix} + c_2 e^{ct}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $c_1, c_2 =$ arbitrary constants;

If $a < 0$ and $c < 0$, the time constant is $\tau = \dfrac{1}{\min\{|a|, |c|\}}$.

5.2.5.9. Ex: $Z(t) = \begin{bmatrix} e^{-t} & e^{2t} & e^{3t} \\ -3e^{-t} & 0 & e^{3t} \\ 12e^{-t} & 0 & 0 \end{bmatrix}$

5.2.5.11. $e^{tA} = \frac{1}{2\sqrt{3}}\begin{bmatrix} (-1+\sqrt{3})e^{-3t} + (1+\sqrt{3})e^{3t} & e^{-3t} - e^{3t} \\ & \\ 2e^{-3t} - 2e^{3t} & (1+\sqrt{3})e^{-3t} + (-1+\sqrt{3})e^{3t} \end{bmatrix}$

5.2.5.13. $e^{tA} = \frac{1}{8}\begin{bmatrix} -e^{-2t} + 6 + 3e^{2t} & 2e^{-2t} - 2e^{2t} & -3e^{-2t} + 6 - 3e^{2t} \\ -3e^{-2t} + 6 - 3e^{2t} & 6e^{-2t} + 2e^{2t} & -9e^{-2t} + 6 + 3e^{2t} \\ e^{-2t} + 2 - 3e^{2t} & -2e^{-2t} + 2e^{2t} & 3e^{-2t} + 2 + 3e^{2t} \end{bmatrix}$

5.2.5.15. $e^{tA} = \frac{1}{2}\, e^{-at}\begin{bmatrix} e^{-bt} + e^{bt} & -e^{-bt} + e^{bt} \\ & \\ -e^{-bt} + e^{bt} & e^{-bt} + e^{bt} \end{bmatrix}$

5.2.5.17. (a) Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -3 & 2 \end{bmatrix}$, (b) $\mathbf{x}(t) = c_1 e^{t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ where $c_1, c_2, c_3 =$ arbitrary constants

5.2.5.19. (a) $X(t) = \begin{bmatrix} -\frac{1}{2}e^{-2t} & -\frac{1}{3}e^{-3t} \\ & \\ e^{-2t} & e^{-3t} \end{bmatrix}$, $e^{tA} = \begin{bmatrix} -2e^{-3t} + 3e^{-2t} & -e^{-3t} + e^{-2t} \\ & \\ 6e^{-3t} - 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix}$ is the same conclusion as in Example 5.13

5.2.5.21. $X(t) = \begin{bmatrix} t & t^2 \\ \\ 1 & 2t \end{bmatrix}$

5.2.5.23. Hint: use Theorem 1.9 in Section 1.2 of the textbook

5.2.5.25. Yes

5.2.5.29. $\dfrac{1}{16} \begin{bmatrix} 3 & 1 \\ \\ 1 & 3 \end{bmatrix}$

5.2.5.33. (a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \star & \star & \star \end{bmatrix}$

5.2.5.35. Yes. Hint: use the Maclaurin series formula $e^{tA} \triangleq I + tA + \dfrac{t^2}{2!}A^2 + \dfrac{t^3}{3!}A^3 + ...$

**Section 5.3.6**

5.3.6.1. $\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} -\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix}$, where $c_1, c_2$ =arbitrary constants

5.3.6.3. $\mathbf{x}(t) = c_1 e^{t} \begin{bmatrix} -\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -\cos 2t - \sin 2t \\ 2\sin 2t \end{bmatrix}$, where $c_1, c_2$ =arbitrary constants

5.3.6.5. $\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 3\cos 2t - \sin 2t \\ 0 \\ 2\cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 3\sin 2t + \cos 2t \\ 0 \\ 2\sin 2t \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, where $c_1, c_2, c_3$ =arbitrary constants

5.3.6.7. <u>Ex</u>: $X(t) = e^{-t} \begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{bmatrix}$

5.3.6.9. <u>Ex</u>: $X(t) = e^{-2t} \begin{bmatrix} \frac{5}{2} & \frac{5}{2}t - \frac{1}{4} \\ \\ -1 & -t \end{bmatrix}$

5.3.6.11. <u>Ex</u>: $X(t) = e^{-t} \begin{bmatrix} 1 & t \\ \\ -1 & 1-t \end{bmatrix}$

5.3.6.13. <u>Ex</u>: $Z(t) = e^{at} \begin{bmatrix} \sin bt & -\cos bt \\ \cos bt & \sin bt \end{bmatrix}$

5.3.6.15. <u>Ex</u>: $Z(t) = e^{-3t} \begin{bmatrix} \cos(\sqrt{5}\,t) - \sqrt{5}\,\sin(\sqrt{5}\,t) & \sin(\sqrt{5}\,t) + \sqrt{5}\,\cos(\sqrt{5}\,t) \\ \\ 2\cos(\sqrt{5}\,t) & 2\sin(\sqrt{5}\,t) \end{bmatrix}$,

$$e^{tA} = \frac{1}{\sqrt{5}}\, e^{-3t} \begin{bmatrix} \sqrt{5}\cos(\sqrt{5}\,t) + \sin(\sqrt{5}\,t) & -3\sin(\sqrt{5}\,t) \\ \\ 2\sin(\sqrt{5}\,t) & \sqrt{5}\cos(\sqrt{5}\,t) - \sin(\sqrt{5}\,t) \end{bmatrix}$$

5.3.6.17. $e^{tA} = \begin{bmatrix} e^{3t}\cos(\sqrt{8}\,t) & 0 & -\frac{1}{\sqrt{2}}\, e^{3t}\sin(\sqrt{8}\,t) \\ \\ 0 & e^{-t} & 0 \\ \\ \sqrt{2}\, e^{3t}\sin(\sqrt{8}\,t) & 0 & e^{3t}\cos(\sqrt{8}\,t) \end{bmatrix}$

5.3.6.19. $e^{tA} = e^{-t} \begin{bmatrix} 11t+1 & 11t \\ \\ -11t & -11t+1 \end{bmatrix}$

5.3.6.21. $e^{tA} = e^{-t} \begin{bmatrix} -1+2\cos t & -\sin t & -1+\cos t \\ 2\sin t & \cos t & \sin t \\ 2-2\cos t & \sin t & 2-\cos t \end{bmatrix}$

5.3.6.23. $\mathbf{x}(t) = e^{-t} \begin{bmatrix} 2\sin 2t + \pi\, \cos 2t \\ 2\cos 2t \;\; - \pi\, \sin 2t \end{bmatrix}$

5.3.6.25. $\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \mathbf{x}(t) = e^{-t} \begin{bmatrix} \cos 2t + \frac{1}{2}\, \sin 2t \\ \\ -\frac{5}{2}\, \sin 2t \end{bmatrix}$

5.3.6.27. $\mathbf{x}(t) = (c_1\cos 2t + d_1 \sin 2t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + (c_2 \cos t + d_2 \sin t) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, where $c_1, c_2, d_1, d_2$ are arbitrary constants

5.3.6.29. the system is asymptotically stable

5.3.6.31. the system is asymptotically stable

5.3.6.33. the system is asymptotically stable

5.3.6.35. (a) must be false, (b) may be true and maybe false, (c) must be true, (d) must be true,
(e) may be true

**Section 5.4.1**

5.4.1.1. $\mathbf{x}(t) = \begin{bmatrix} e^{-2t} & e^{-3t} \\ \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix} \mathbf{c} + \begin{bmatrix} \frac{5}{6} \\ \\ -1 \end{bmatrix}$, where $\mathbf{c}$ is a vector of arbitrary constants

5.4.1.3. $\mathbf{x}(t) = e^{-3t} \begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\sin t \end{bmatrix} \mathbf{c} + e^{-3t} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, where $\mathbf{c}$ is a vector of arbitrary constants.

5.4.1.5. $\mathbf{x}(t) = \begin{bmatrix} -\frac{14}{17} \\ \\ t\,e^{-t} \\ \\ -\frac{21}{17} \end{bmatrix} + \begin{bmatrix} -e^{3t}\sin(\sqrt{8}\,t) & e^{3t}\cos(\sqrt{8}\,t) & 0 \\ \\ 0 & 0 & e^{-t} \\ \\ \sqrt{2}\,e^{3t}\cos(\sqrt{8}\,t) & \sqrt{2}\,e^{3t}\sin(\sqrt{8}\,t) & 0 \end{bmatrix}\mathbf{c}$, where $\mathbf{c}$ is a vector of arbitrary constants

5.4.1.7. $\mathbf{A}(t) = \begin{bmatrix} c_1\,e^{-t/10} + 50 \\ 1.5\,c_1\,e^{-t/10} + c_2\,e^{-t/6} + 30 \end{bmatrix}$, where $c_1, c_2$ are arbitrary constants

5.4.1.9. (a) Ex: $X(t) = \begin{bmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{bmatrix}$, (b) $\mathbf{x}(t) = \begin{bmatrix} \frac{13}{2}\,t^2 - \frac{3}{2}\,t^4 - 5t \\ \\ 13\,t - 6t^3 - 8 \end{bmatrix}$

5.4.1.13. Hint: Suppose $\{y_1(t), y_2(t)\}$ is a complete set of basic solutions of the corresponding linear homogeneous ODE, $\ddot{y} + p(t)\dot{y} + q(t)y = 0$ on some open interval containing $t = 0$.

5.4.1.15. $\mathbf{x}(t) = \frac{1}{24}\begin{bmatrix} 8\cos t + 11\sin t + 16\cos 2t - 16\sin 2t + 15\sin 3t \\ \\ \cos t + 6\sin t + 8\cos 2t - 9\cos 3t + 6\sin 3t \end{bmatrix}$,

**Section 5.5.2**

5.5.2.1. $\mathbf{x}(t) = \frac{1}{32}\begin{bmatrix} 19e^{-5t} + 75e^{3t} \\ -133e^{-5t} + 75e^{3t} \end{bmatrix} + \frac{1}{16}\begin{bmatrix} e^{-t} \\ -3e^{-t} \end{bmatrix}$, which agrees with the final conclusion of Example 5.23 in Section 5.4

5.5.2.3. $\mathbf{x}(t) = e^{-3t}\begin{bmatrix} -2 \\ \\ -1 \end{bmatrix} + e^{tA}\mathbf{c} = e^{-3t}\begin{bmatrix} -2 \\ \\ -1 \end{bmatrix} + \begin{bmatrix} 3e^{-2t} - 2e^{-t} & -3e^{-2t} + 3e^{-t} \\ \\ 2e^{-2t} - 2e^{-t} & -2e^{-2t} + 3e^{-t} \end{bmatrix}\mathbf{c}$, where $\mathbf{c}$ is a vector of arbitrary constants

5.5.2.5. $\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t) = \frac{1}{10}\begin{bmatrix} 3\cos t - \sin t \\ \\ -4\cos t - 2\sin t \end{bmatrix} + e^{-t}\begin{bmatrix} \cos 2t & \sin 2t \\ \\ -\sin 2t & \cos 2t \end{bmatrix}\mathbf{c}$, where $\mathbf{c}$ is a vector of arbitrary constants

5.5.2.7. $\mathbf{x}(t) = \frac{1}{16}\begin{bmatrix} 4e^{-t} - 42e^{-3t} + 102e^{t} \\ \\ 4e^{-t} + 42e^{-3t} + 34e^{t} \end{bmatrix}$

5.5.2.9. $\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}(t)_h = e^{-t}\begin{bmatrix} -2t + 1 \\ -2t \end{bmatrix} + c_1 e^{-2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $c_1, c_2$ =arbitrary constants

**Section 5.6.2**

5.6.2.1. $\mathcal{V}(A, \mathbf{b}) = Span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$, and yes, the system can be driven from $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ to $\mathbf{0}$

5.6.2.3. the system is completely controllable

5.6.2.5. the system is not completely controllable

5.6.2.7. the system is not completely controllable

**Section 5.7.7**

5.7.7.1. $\mathbf{x}_k = c_1(-\sqrt{13})^k \begin{bmatrix} 3-\sqrt{13} \\ 2 \end{bmatrix} + c_2(\sqrt{13})^k \begin{bmatrix} 3+\sqrt{13} \\ 2 \end{bmatrix}$, where $c_1, c_2 =$ arbitrary constants

5.7.7.3. $\mathbf{x}_k = c_1(-1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 4^k \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, where $c_1, c_2 =$ arbitrary constants

5.7.7.5. $\mathbf{x}_k = c_1 \begin{bmatrix} -\sin\frac{\pi k}{2} \\ \\ \cos\frac{\pi k}{2} \end{bmatrix} + c_2 \begin{bmatrix} \cos\frac{\pi k}{2} \\ \\ \sin\frac{\pi k}{2} \end{bmatrix}$ where $c_1, c_2 =$ arbitrary constants

5.7.7.7. (a) the equivalent impedance is $Z = \frac{683}{16}$
(b) for $k = 1, ..., 4$, $V_k = \frac{V_0}{2049}\big(2^k + 2048 \cdot 2^{-k}\big)$, and $V_5 = \frac{32}{683} V_0$

5.7.7.9. (a) the equivalent impedance is $Z = 89$
(b) for $k = 1, ..., 4$. $V_k = -\frac{V_0}{178\sqrt{5}}\big((89(1-\sqrt{5})+110)\lambda_1^k - (89(1+\sqrt{5})+110)\lambda_2^k\big)$, and $V_5 = \frac{V_0}{89}$

5.7.7.11. (a) the equivalent impedance is $Z = 153$
(b) for $k = 1, ..., 3$. $V_k = -\frac{V_0}{306\sqrt{3}}\big((153(1-\sqrt{3})+112)(2+\sqrt{3})^k - (153(1\sqrt{3})+112)(2-\sqrt{3})^k\big)$, and
$V_5 = \frac{V_0}{153}$

5.7.7.13. $\mathbf{x}_k = -\frac{2}{19}\left(\frac{1}{2}\right)^k \begin{bmatrix} 17 \\ 5 \end{bmatrix} + c_1(-\sqrt{5})^k \begin{bmatrix} 1-\sqrt{5} \\ 1 \end{bmatrix} + c_2(\sqrt{5})^k \begin{bmatrix} 1+\sqrt{5} \\ 1 \end{bmatrix}$, where $c_1, c_2 =$ arbitrary constants

5.7.7.15. $\mathbf{x}_k = \frac{1}{53}\left(\frac{1}{2}\right)^k \begin{bmatrix} -106 \\ 56 \\ -34 \end{bmatrix} + c_1 \begin{bmatrix} -10 \\ 9 \\ 2 \end{bmatrix} + c_2 17^{k/2} \begin{bmatrix} 0 \\ \cos\left(k\arctan\frac{1}{4}\right) + \sin\left(k\arctan\frac{1}{4}\right) \\ 2\cos\left(k\arctan\frac{1}{4}\right) \end{bmatrix}$

$$+c_3 17^{k/2} \begin{bmatrix} 0 \\ -\cos\left(k\arctan\frac{1}{4}\right) + \sin\left(k\arctan\frac{1}{4}\right) \\ 2\sin\left(k\arctan\frac{1}{4}\right) \end{bmatrix},$$

where $c_1, c_2, c_3 =$arbitrary constants

5.7.7.17. (a) The system may be neutrally stable. (b) The system may be unstable.
(c) It must be true that, "the system may be neutrally stable, depending upon more information concerning $A$."

(d) It must be false to claim that, "($\star$) has solutions $\mathbf{x}_k$ that are periodic in $k$ with period 6, that is, $\mathbf{x}_{k+6} \equiv \mathbf{x}_k$."

5.7.7.23. the system is unstable

5.7.7.25. the system is neutrally stable

5.7.7.27. the system is asymptotically stable

**Section 5.8.6**

5.8.6.1. (a) Hint: Calculate that $A(t)Z(t) = \begin{bmatrix} e^{t-\cos t}(1+\sin t) & 0 \\ e^t & 0 \end{bmatrix} = ... = \dot{Z}(t)$, and do the calculations to show that $\det\big(Z(t)\big)$ is never zero.

(b) the principal fundamental matrix at $t = 0$ is $X(t) = \begin{bmatrix} e^{1+t-\cos t} & 0 \\ e\cdot(e^t-1) & 1 \end{bmatrix}$, the period is $\mathsf{T} = 2\pi$, the characteristic multipliers are $\mu_1 = 1$ and $\mu_2 = e^{2\pi}$, $D = diag(\mu_1, \mu_2) = (1, e^{2\pi})$, $Q = \begin{bmatrix} 0 & 1 \\ 1 & e \end{bmatrix}$, $E = diag\big(0, 1\big)$, and $C = \begin{bmatrix} 1 & 0 \\ e & 0 \end{bmatrix}$.

The Floquet representation is $X(t) = P(t)e^{tC} = \begin{bmatrix} e^{1-\cos t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ e\cdot(e^t-1) & 1 \end{bmatrix}$.

5.8.6.3. (a) The $1\times1$, principal fundamental matrix at $t = 0$ is $X(t) \triangleq [\, y_1(t)\,] = \left[\exp\left(\int_0^t p(s)ds\right)\right]$.
The Floquet representation, a product of $1\times 1$ matrices, is

$$X(t) = P(t)e^{tC} = \left[\exp\int_T^t p(s)ds\right]\left[\exp\Big(\int_0^T p(s)ds\Big)\right].$$

If $\int_0^{\mathsf{T}} p(t)dt < 0$, then the homogeneous ODE is asymptotically stable.

(b) For the scalar, linear, nonhomogeneous ODE $\dot{y} - p(t)y = f(t)$, the solutions are $y(t) = cy_1(t) + \exp\left(\int_0^t p(s)ds\right)\int_0^t \nu(s)f(s)ds \triangleq cy_1(t) + y_p(t)$, where $c$ is an arbitrary constant.

The non-homogeneous ODE has a $\mathsf{T}$-periodic solution if, and only if, $\int_0^{\mathsf{T}} p(s)ds \neq 0$.

If $\int_0^{\mathsf{T}} p(t)dt < 0$, then the non-homogeneous ODE has a $\mathsf{T}$-periodic solution.

5.8.6.5. the characteristic multipliers $\mu$ satisfy $0 = |\,X(\mathsf{T};\lambda) - \mu I\,| = ... = 1 - \big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)\mu + \mu^2$.
The assumption $|\dot{y}_2(\mathsf{T};\lambda) + y_1(\mathsf{T};\lambda)| < 2$, implies that $\mu_{1,2}$ are not real and have moduli

$$|\,\mu_{1,2}\,| = \frac{1}{2}\sqrt{\big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)^2 + 4 - \big(y_1(\mathsf{T};\lambda) + \dot{y}_2(\mathsf{T};\lambda)\big)^2} = \frac{1}{2}\sqrt{4} = 1,$$

hence the system is neutrally stable.

5.8.6.7. (a) Hint: construct the Floquet representation $X(t) = ... = P(t)e^{tO}$
(b) Hints: the result of problem 5.2.5.27 implies that $X(-t) \equiv X(t)$, and substitute $t = \frac{\mathsf{T}}{2} - \mathsf{T}$
(c) HInts: Lemma 5.6 states that $X(t + \mathsf{T}) \equiv X(t)X(\mathsf{T})$ and substitute $t = -\frac{\mathsf{T}}{2}$
(d) Hints: proceed from the right hand side of the result of part (c) and then use the result of part (b)

(e) HInts: $X(t)$ is invertible for all $t$ [Why?] and then use the result of part (d); after that, use the result of part (a)

5.8.6.9. a Floquet representation is given by $X(t) = P(t)e^{tC} = \left( \begin{bmatrix} \dfrac{2}{2+\sin t} & 0 \\ 4\ln\left(\dfrac{2+\sin t}{2}\right) & 1 \end{bmatrix} \right) \left(e^{-t} I\right)$

5.8.6.11. Hints: First, use the product rule to help get $\dot{X}(t) = \dot{P}(t)e^{tC} + P(t)Ce^{tC}$; after that, get $\dot{X}(t) = ... = e^{t\Omega}A_0 e^{tC}$ and $A(t)X(t) = ... = e^{t\Omega}A_0 e^{tC}$

*Note*: At *no* time should you assume that $A_0\Omega = \Omega A_0$ or that $e^{tC}$ could be rewritten as $e^{tA_0}e^{-tC}$ or as $e^{-tC}e^{tA_0}$.

5.8.6.13. Hint: Study this problem in nine cases:

(1) $\delta+\epsilon>0$ and $\delta-\epsilon>0$, (2) $\delta+\epsilon>0$ and $\delta-\epsilon=0$, (3) $\delta+\epsilon>0$ and $\delta-\epsilon<0$,

(4) $\delta+\epsilon=0$ and $\delta-\epsilon>0$, (5) $\delta+\epsilon=0$ and $\delta-\epsilon=0$, (6) $\delta+\epsilon=0$ and $\delta-\epsilon<0$,

(7) $\delta+\epsilon<0$ and $\delta-\epsilon>0$, (8) $\delta+\epsilon<0$ and $\delta-\epsilon=0$, (9) $\delta+\epsilon<0$ and $\delta-\epsilon<0$.

For example, in Cases (1), (2), and (3), we assume $\delta+\epsilon>0$ and denote $\omega \triangleq \sqrt{\delta+\epsilon}$. Let $y_1(t;\delta,\epsilon)$ solve $\ddot{y}+(\delta+\epsilon)y=0,\ 0<t<\pi$ with initial data $y_1(0;\delta,\epsilon)=1, \dot{y}_1(0;\delta,\epsilon)=0$ and let $y_2(t;\delta,\epsilon)$ solve $\ddot{y}+(\delta+\epsilon)y=0,\ 0<t<\pi$ with initial data $y_2(0;\delta,\epsilon)=0, \dot{y}_2(0;\delta,\epsilon)=1$.
*Case 1*: Further, assume $\delta-\epsilon>0$ and denote $\nu=\sqrt{\delta-\epsilon}$. The general solution of the ODE (b) $\ddot{y}+(\delta-\epsilon)y=0$ on the interval $\pi<t<2\pi$ can be written as $y(t;\delta,\epsilon) = \mathit{constant}\cdot\cos\big(\nu(t-\pi)\big) + \mathit{constant}\cdot\sin\big(\nu(t-\pi)\big)$.

Use continuity at $t=\pi$ of the function $y_1(t)=\cos(\omega t)$ and its first derivative, $\dot{y}_1(t) = -\omega\sin(\omega t)$.

After a lot of calculations, find that the characteristic multipliers are

$$\mu = \cos(\omega\pi)\cos(\nu\pi) - \frac{\delta}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sin(\nu\pi) \pm \sqrt{\left(\cos(\omega\pi)\cos(\nu\pi) - \frac{\delta}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sin(\nu\pi)\right)^2 - 1}\,.$$

*Case 2*: the characteristic multipliers are $\mu = \cos(\omega\pi) \pm \frac{1}{2}\sqrt{\Big(2\cos(\omega\pi) - \pi\omega\sin(\omega\pi)\Big)^2 - 4}$

*Case 3*: the characteristic multipliers are

$$\mu = \cos(\omega\pi)\cosh(\nu\pi) - \frac{\epsilon}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sinh(\nu\pi) \pm \sqrt{\left(\cos(\omega\pi)\cosh(\nu\pi) - \frac{\epsilon}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sinh(\nu\pi)\right)^2 - 1}$$

*Case 4*: the characteristic multipliers are $\mu = \dfrac{2\cos(\nu\pi) - \pi\nu\sin(\nu\pi) \pm \sqrt{\Big(2\cos(\nu\pi) - \pi\nu\sin(\nu\pi)\Big)^2 - 4}}{2}$

*Case 5*: the characteristic multipliers are $\mu = 1 \pm 0$

*Case 6*: the characteristic multipliers are $\mu = \dfrac{2\cosh(\nu\pi) + \pi\nu\sinh(\nu\pi) \pm \sqrt{\Big(2\cosh(\nu\pi) + \pi\nu\sinh(\nu\pi)\Big)^2 - 4}}{2}$

*Case 7*: the characteristic multipliers are

$$\mu = \cosh(\omega\pi)\cos(\nu\pi) - \frac{\delta}{\sqrt{\delta^2-\epsilon^2}}\sinh(\omega\pi)\sin(\nu\pi) \pm \sqrt{\left(\cosh(\omega\pi)\cos(\nu\pi) - \frac{\delta}{\sqrt{\delta^2-\epsilon^2}}\sinh(\omega\pi)\sin(\nu\pi)\right)^2 - 1}$$

*Case 8*: the characteristic multipliers are $\mu = \cos(\omega\pi) \pm \frac{1}{2}\sqrt{\Big(2\cosh(\omega\pi) + \pi\omega\sinh(\omega\pi)\Big)^2 - 4}$
*Case 9*: the characteristic multipliers are

$$\mu = \cos(\omega\pi)\cosh(\nu\pi) - \frac{\epsilon}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sinh(\nu\pi) \pm \sqrt{\left(\cos(\omega\pi)\cosh(\nu\pi) - \frac{\epsilon}{\sqrt{\delta^2-\epsilon^2}}\sin(\omega\pi)\sinh(\nu\pi)\right)^2 - 1}$$

5.8.6.15. Hint: let $z(t) \triangleq \mathbf{y}^T(t)\mathbf{x}(t)$ and calculate $\dot{z}(t) = ... = \mathbf{y}^T(t)\mathbf{f}(t)$ and then $z(\mathsf{T}) - z(0) = \int_0^{\mathsf{T}} \mathbf{y}^T(s)\mathbf{f}(s)\,ds$

5.8.6.17. the system is neutrally stable

# Chapter 6

# Geometry, Calculus, and other tools

**Section 6.1.4**

6.1.4.1. (a) $-2$,    (b) $-\hat{\boldsymbol{\imath}}-\hat{\boldsymbol{\jmath}}+2\hat{\boldsymbol{k}}$,    (c) the angle between $\mathbf{A}$ and $\mathbf{B}$ is $\theta = \cos^{-1}\left(\frac{-2}{\sqrt{10}}\right) \approx 2.25551553$. [In Calculus, angles are measured in radians.]

6.1.4.3. $x = 2t, y = 3 - 2t, z = 5 - 6t,\ -\infty < t < \infty$ are parametric equations of the line

6.1.4.5. $x = 2 - t, y = 4 + 2t, z = 1 + 3t,\ -\infty < t < \infty$ are parametric equations of the line

6.1.4.7. $(x - 0) - 3(y - 1) - 4(z - 1) = 0$ or $x - 3y - 4z = -7$ or other versions of the same

6.1.4.9. $3(x - 2) - 2(y - 4) + (z - 1) = 0$ or $3x - 2y + z = -1$ or other versions of the same

6.1.4.11. $-(x - 2) + (y - 4) + (z - 1) = 0$ or $-x + y + z = 3$ or other versions of the same

6.1.4.13. $\mathbf{v} = \pm\dfrac{1}{\sqrt{21}}[-1 \quad 2 \quad 4]^T$.

6.1.4.15. $\theta = 0$

6.1.4.17. Note that the instructions for this problem have been changed on the Errata page.
Hints: Express the solution of the system in the form $\mathbf{r}(t) = \mathbf{r}_0 + g(t)\mathbf{v}(0) + h(t)\mathbf{a}(0)$, and eventually get

$$(1) \quad \ddot{g}(t)\mathbf{v}(0)+\ddot{h}(t)\mathbf{a}(0) = -\dot{g}(t)\left(\frac{q}{m}\right)\mathbf{v}(0)\times\mathbf{B}_0-\dot{h}(t)\left(\frac{q}{m}\right)\mathbf{a}(0)\times\mathbf{B}_0 = -\dot{g}(t)\mathbf{a}(0)-\dot{h}(t)\left(\frac{q}{m}\right)\mathbf{a}(0)\times\mathbf{B}_0.$$

Take the dot product of (1) with $\mathbf{a}(0)$ and use the result of problem 6.8.4.15. Take the dot product of (1) with $\mathbf{v}(0)$ and use again the result of problem 6.8.4.15.

**Section 6.2.6**

6.2.6.1. $(r, \theta) = (2\sqrt{2}, \dfrac{11\pi}{6})$.

6.2.6.3. (a) $(r, \theta, z) = (2, \dfrac{5\pi}{6}, 5)$,    (b) $(r, \theta, z) = (2, \dfrac{2\pi}{3}, 4)$.

6.2.6.5. (a) $(\rho, \phi, \theta) = (2, \dfrac{3\pi}{4}, \dfrac{4\pi}{3})$, (b) $(\rho, \phi, \theta) = (2, \dfrac{3\pi}{4}, \dfrac{5\pi}{6})$.

6.2.6.7. $\hat{\mathbf{e}}_r = \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi$, $\hat{\mathbf{e}}_\theta = \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi$

**Section 6.3.4**

6.3.4.1. $\mathcal{C}:\ \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (1-2t)\hat{\boldsymbol{\imath}} + (2-2t)\,\hat{\boldsymbol{\jmath}} + (4-t)\hat{\boldsymbol{k}},\ 0 \le t \le 1$

6.3.4.3. $\mathcal{C}_1:\ \mathbf{r} = 4\cos t\,\hat{\boldsymbol{\imath}} + (4 + 4\sin t)\hat{\boldsymbol{\jmath}},\ 0 \le t \le 2\pi,$
or $\mathcal{C}_2:\ \mathbf{r} = 8\sin\theta(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) = 4\sin 2\theta\,\hat{\boldsymbol{\imath}} + 4(1-\cos 2\theta)\,\hat{\boldsymbol{\jmath}}, 0 \le \theta \le \pi.$

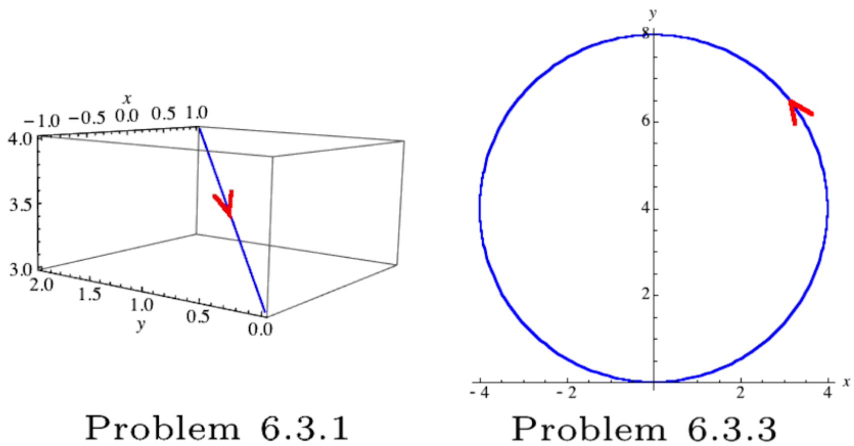


Figure 6.1: (a) Answer for problem 6.3.4.1 and (b) Answer for problem 6.3.4.3

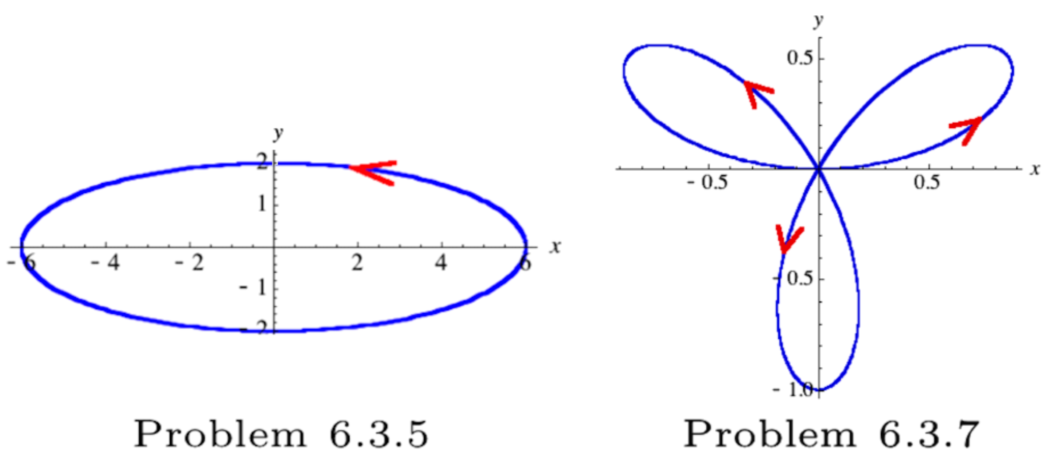


Figure 6.2: (a) Answer for problem 6.3.4.5 and (b) Answer for problem 6.3.4.7

6.3.4.5. ellipse with center at $(0,0)$, major axis of length 12 on the $x-$axis and minor axis of length 4 on the $y-$axis...see sketched figure

6.3.4.7. sketched figure is $\mathcal{C}:\ \mathbf{r} = \sin 3\theta(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}), 0 \le \theta \le 2\pi$ is called a “three-leaved rose"

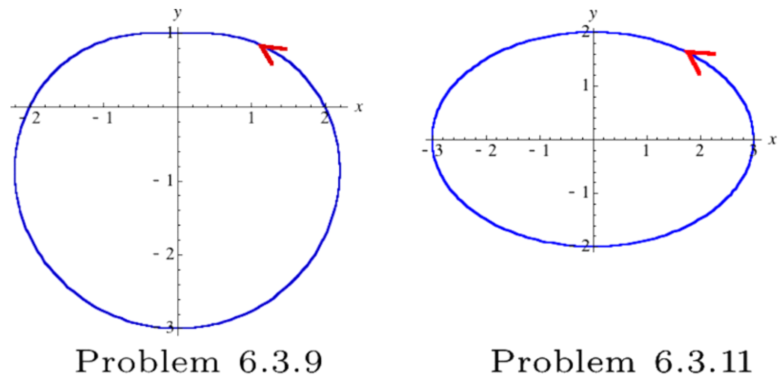


Figure 6.3: (a) Answer for problem 6.3.4.9 and (b) Answer for problem 6.3.4.11

6.3.4.9. see sketched figure

6.3.4.11. $\mathcal{C}:\ \mathbf{r} = 3\cos t\,\hat{\boldsymbol{\imath}} + 2\sin t\,\hat{\boldsymbol{\jmath}}, 0 \le t \le 2\pi$ gives a simple, closed curve

6.3.4.13. $\mathcal{C}:\ \mathbf{r} = (6 + 6\cos t)\hat{\boldsymbol{\imath}} + 2\sin t\,\hat{\boldsymbol{\jmath}}, 0 \le t \le 2\pi$ gives a simple, closed curve.

6.3.4.15. Equation (6.26) $\mathcal{C}_1,\ 0 \le \theta \le \pi$, is the circle of radius 4, center at $(2, 0)$, traversed once counter-clockwise.

$\mathcal{C}_2,\ 0 \le \theta \le 2\pi$ is the circle of radius 4, center at $(2, 0)$, traversed counter-clockwise twice

6.3.4.17. $x = \frac{\sqrt{3}}{2} + \frac{1}{2}\,t,\ y = 1 + \frac{6}{\pi}\,t,\ -\infty < t < \infty$ are parametric equations of the desired tangent line. Other choices are possible.

6.3.4.19. $\mathbf{r} = (-1)^n\hat{\boldsymbol{\imath}} + \left(n + \frac{1}{2}\right)\pi\,\hat{\boldsymbol{\jmath}}$

6.3.4.21. sphere of radius 2 and center at $(0, 0, 2)$; see the figure

6.3.4.23. circular cylindrical surface of radius $\frac{a}{2}$ and axis of symmetry being the line satisfying both $x = \frac{a}{2}$ and $y = 0$; see the figure

6.3.4.25. $\mathcal{S}_1 : \mathbf{r}(\phi, \theta) = \dfrac{1}{2}\,\sin\phi\,\cos\theta\,\hat{\boldsymbol{\imath}} + \dfrac{1}{2}\,\sin\phi\,\sin\theta\,\hat{\boldsymbol{\jmath}} + \dfrac{1}{2}(-1 + \cos\phi)\,\hat{\boldsymbol{k}},\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$

$$\mathcal{S}_2 : \mathbf{r}(\phi, \theta) = -\cos\phi\left(\sin\phi\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\phi\sin\theta\,\hat{\boldsymbol{\jmath}} + \cos\phi\,\hat{\boldsymbol{k}}\right) = -\tfrac{1}{2}\left(\sin 2\phi\,\cos\theta\,\hat{\boldsymbol{\imath}} + \sin 2\phi\,\sin\theta\,\hat{\boldsymbol{\jmath}} + (1 + \cos 2\phi)\hat{\boldsymbol{k}}\right),$$

$$0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi.$$

6.3.4.27. circular paraboloid with axis being the positive $z-$ axis.
$\mathcal{S}_1 : \mathbf{r}(x, y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + (x^2 + y^2)\hat{\boldsymbol{k}}, -\infty < x < \infty,\ -\infty < y < \infty.$
$\mathcal{S}_2 : \mathbf{r}(r, \theta) = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} + r^2\,\hat{\boldsymbol{k}},\ 0 \le r < \infty,\ 0 \le \theta \le 2\pi$

6.3.4.29. elliptic cone with axis being the positive $z-$ axis.
One parametrization is $\mathcal{S}_1 : \mathbf{r}(x, y) = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + \sqrt{x^2 + \frac{1}{2}\,y^2}\,\hat{\boldsymbol{k}}, -\infty < x < \infty,\ -\infty < y < \infty$
$\mathcal{S}_2 : \mathbf{r}(r, \theta) = u\cos v\,\hat{\boldsymbol{\imath}} + \sqrt{2}\,u\sin v\,\hat{\boldsymbol{\jmath}} + u\,\hat{\boldsymbol{k}},\ 0 \le u < \infty,\ 0 \le v \le 2\pi$

6.3.4.31. $\omega = \frac{\sqrt{3}}{2}$.

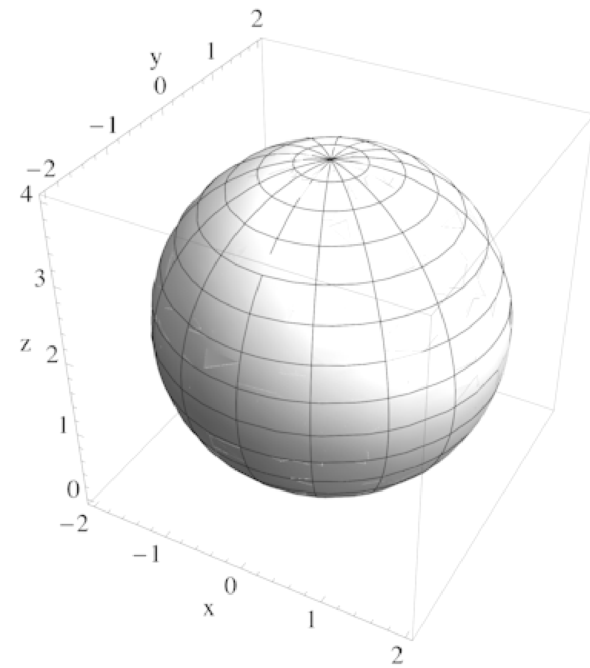


Figure 6.4: Answer for problem 6.3.4.21

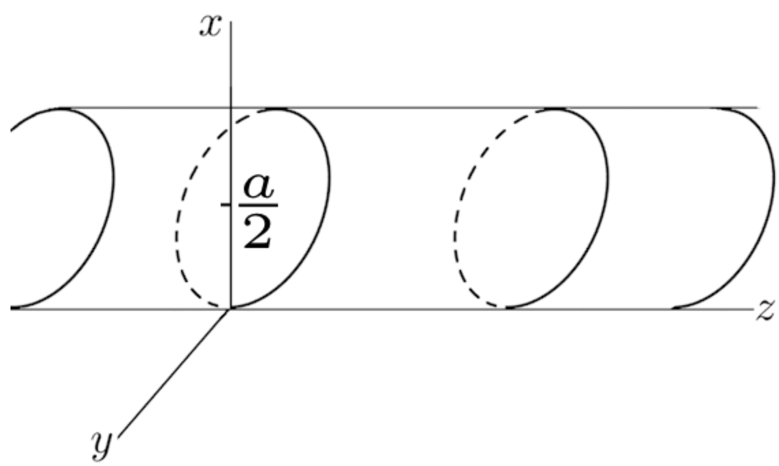


Figure 6.5: Answer for problem 6.3.4.23

6.3.4.33. For $0 \le \theta \le 2\pi$, $-\infty < z < \infty$, $\mathcal{S}:\ \mathbf{r} = \mathbf{r}(x,y) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$

6.3.4.35. surface is the sphere with radius 1, center at $(x,y,z) = (0,0,1)$. It lies in the half-space $z \ge 0$, because $2z = x^2 + y^2 + z^2 \ge 0$.

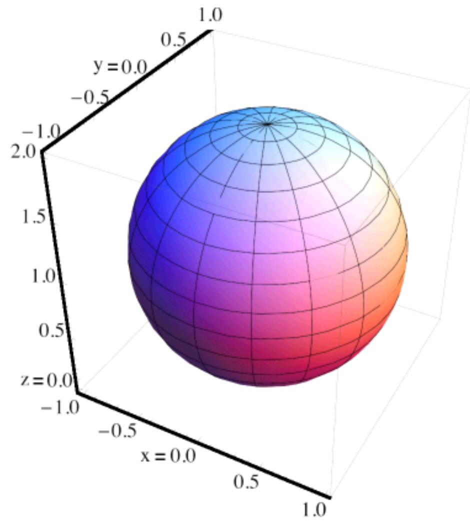


Figure 6.6: Answer 6.3.4.35

**Section 6.4.5**

6.4.5.1. $\dfrac{\partial^2 f}{\partial x \partial y} = 2y\cos(xy^2+3) - 2xy^3\sin(xy^2+3)$

6.4.5.3. $\left\{ \begin{array}{l} x = -1+t \\ y = -2 \\ z = 2\sqrt{2} - \sqrt{2}\,t \end{array} \right\}$, $-\infty < t < \infty$

6.4.5.5. $(D_{\hat{\mathbf{u}}}f)(1,2) \triangleq \boldsymbol{\nabla} f(1,2) \bullet \dfrac{1}{\sqrt{5}}(-\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}}) = \dfrac{1}{\sqrt{5}}\left(-2e^1\,\hat{\boldsymbol{\imath}} + e^1\,\hat{\boldsymbol{\jmath}}\right) \bullet (-\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}}) = \dfrac{4e}{\sqrt{5}}$

6.4.5.7. (a) $\boldsymbol{\nabla} f(x,y,z) = -\dfrac{1}{(x^2+y^2+z^2)^{3/2}}\left(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}\right)$, (b) $(D_{\hat{\mathbf{u}}}f)(x,y,z) = -1$

6.4.5.9. (a) $z = 3 - y^2 - 1^2$, that is, $z = 2 - y^2$, (b) equations of the tangent line to this hyperbola at that point are $\left\{ \begin{array}{l} x = 1 \\ y = 0.8 + t \\ z = 1.36 - 1.6\,t \end{array} \right\}$, $-\infty < t < \infty$

(c) Pictorially, $\dfrac{\partial f}{\partial y}(1, 0.8)$ is the slope of part (b)'s tangent line to the curve $z = 2 - y^2$ in the plane $x = 1$ at the point $(x,y,z) = (1, 0.8, 1.36)$; see figure.

6.4.5.11. $-8\ln 3$

6.4.5.13. (a) $\dfrac{\partial g}{\partial v}(-2,1) = e + 4e^{-4} + 2e - 4e^{-4} = 3e$, (b) $\dfrac{\partial g}{\partial v}(-2,1) = 2e + 8e^{-4} + 2e - 4e^{-4} = 4e + 4e^{-4}$

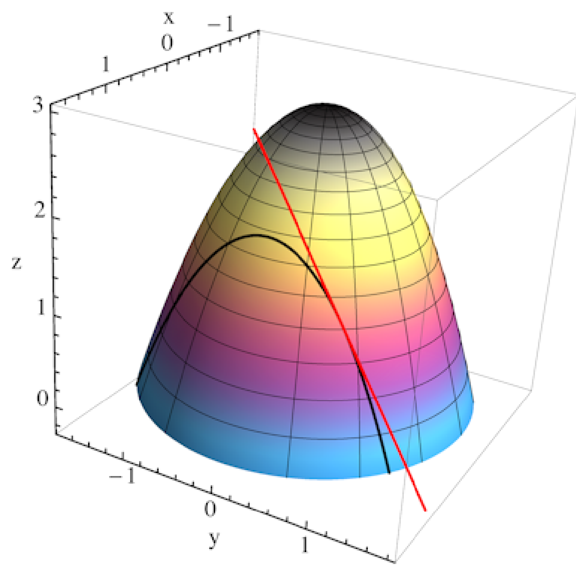


Figure 6.7: Answer for problem 6.4.5.9

6.4.5.15. Hint: The multivariable chain rule gives $\frac{\partial z}{\partial r} = \cos\theta\, \frac{\partial z}{\partial x} + \sin\theta\, \frac{\partial z}{\partial y}$

6.4.5.17. $\ln\big((1.02)^2 + (2.99)^3\big) = f(1.02, 2.99) \approx \ln 28 - \frac{0.23}{28}$

6.4.5.19. potential function $f(x,y,z) = xyz - x^2 + y\cos z + c$, $c =$arb. const.

6.4.5.21. potential function $f(x,y,z) = x\cos y + yz - z + c$, $c =$arb. const.

6.4.5.23. Hint: $\rho^2 = ||\mathbf{r}||^2 = \mathbf{r} \bullet \mathbf{r} \Longrightarrow ...\dfrac{d\rho}{dt} = \rho^{-1}(\mathbf{r} \bullet \mathbf{v})$; eventually, use the fact that the vector triple product satisfies $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\,\mathbf{B} - (\mathbf{A} \bullet \mathbf{B})\,\mathbf{C}$

**Section 6.5.1**

6.5.1.1. $14(x-1) - 4(y-2) + (z-8) = 0$, or $14x - 4y + z = 14$

6.5.1.3. $(x-1) + 7(y+1) - (z+3) = 0$, or $x + 7y - z = -3$.

6.5.1.5. $\mathbf{n} = \dfrac{1}{\sqrt{83}}\,(3\,\hat{\boldsymbol{\imath}} + 7\,\hat{\boldsymbol{\jmath}} + 5\,\hat{\boldsymbol{k}})$

6.5.1.7. (a) $\mathbf{n} = -\,\frac{\partial g}{\partial x}(x_0, y_0)\,\hat{\boldsymbol{\imath}} - \frac{\partial g}{\partial y}(x_0, y_0)\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$
(b) Yes, this method produces a normal vector which agrees with (6.53) except for a factor of $-1$.

6.5.1.9. Yes. Hints: At any point $(x,y,z)$ on the surface $0 = \phi(x,y,z) \triangleq x^2 - y^2 + z^2 - 4$, a normal vector is given by $\widehat{\mathbf{n}}_1 = 2x\,\hat{\boldsymbol{\imath}} - 2y\,\hat{\boldsymbol{\jmath}} + 2z\,\hat{\boldsymbol{k}}$;
At any point $(x,y,z)$ on the surface $0 = \psi(x,y,z) \triangleq z - \dfrac{1}{xy^2}$, a normal vector is given by $\widehat{\mathbf{n}}_2 = \frac{1}{x^2y^2}\,\hat{\boldsymbol{\imath}} + \frac{2}{xy^3}\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$

6.5.1.11. See figure. Also,

Examples of normal vectors to level sets:

(1) for $k=1$, at the point $(\frac{1}{\sqrt{2}},0)$, $2\sqrt{2}\,\hat{\boldsymbol{\imath}}$ is a normal vector,

(2) for $k=4$, at the point $(1,-\sqrt{2})$, $4\,\hat{\boldsymbol{\imath}}-2\sqrt{2}\,\hat{\boldsymbol{\jmath}}$ is a normal vector,

(3) for $k=16$, at the point $(-\sqrt{2},2\sqrt{3})$, $-4\sqrt{2}\,\hat{\boldsymbol{\imath}}+4\sqrt{3}\,\hat{\boldsymbol{\jmath}}$ is a normal vector.

6.5.1.13. See figure. Also,

Examples of normal vectors to level sets:

(1) for $k=\frac{1}{2}$, at the point $(1,1)$, $\frac{1}{2}\,\hat{\boldsymbol{\jmath}}$ is a normal vector, and $\widehat{\mathbf{n}}=\hat{\boldsymbol{\jmath}}$ is a unit normal vector,

(2) for $k=-1$, at the point $(-3,\sqrt{3})$, $\frac{1}{6}\left(\hat{\boldsymbol{\imath}}-\sqrt{3}\,\hat{\boldsymbol{\jmath}}\right)$ is a normal vector, and $\widehat{\mathbf{n}}=\frac{1}{2}\left(\hat{\boldsymbol{\imath}}-\sqrt{3}\,\hat{\boldsymbol{\jmath}}\right)$ is a unit normal vector

(3) for $k=2$, at the point $(5,-\sqrt{15})$, $\frac{1}{10}\left(\hat{\boldsymbol{\imath}}-\sqrt{15}\,\hat{\boldsymbol{\jmath}}\right)$ is a normal vector, and $\widehat{\mathbf{n}}=\frac{1}{4}\left(\hat{\boldsymbol{\imath}}-\sqrt{15}\,\hat{\boldsymbol{\jmath}}\right)$ is a unit normal vector.

6.5.1.15. (a) $\frac{1}{\sqrt{2}}\,(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}})$, (b) $k=6$: level curve can be rewritten as $\dfrac{x^2}{(\sqrt{6})^2}+\dfrac{y^2}{(\sqrt{3})^2}=1$

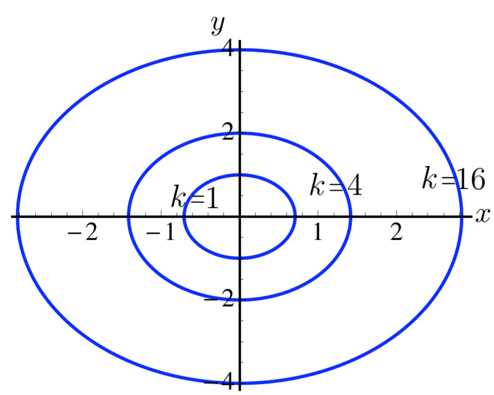


Figure 6.8: Answer 6.5.1.11

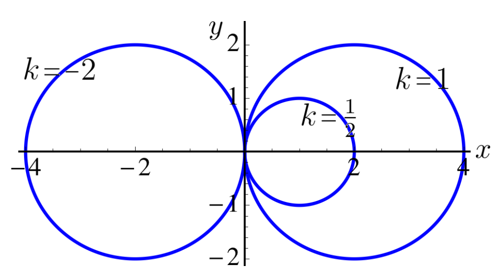


Figure 6.9: Answer 6.5.1.13

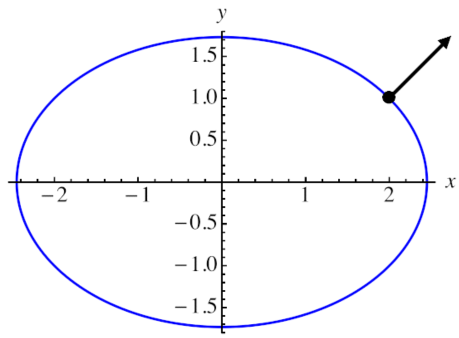


Figure 6.10: Answer 6.5.1.15(b)

**Section 6.6.5**

6.6.5.1. $\sqrt{61}$

6.6.5.3. 24

6.6.5.5. see sketch of parallelopiped; $Volume = 6$

6.6.5.7. see sketch

6.6.5.9. see sketch

6.6.5.11. $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

6.6.5.13. 16

6.6.5.15. (a) <u>Exs</u>. $A_1 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ implements $R_1 \leftrightarrow R_2$, $A_2 \triangleq \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ implements $-2R_1 + R_2 \rightarrow R_2$,

and $A_3 \triangleq \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ implements $2R_1 \rightarrow R_1$.

(b) $|\,\det(A_1)\,| = 1$, $|\,\det(A_2)\,| = 1$, and $|\,\det(A_3)\,| = 2$, respectively, are the factors by which $A_1$ ,$A_2$, and $A_3$ multiply volume.

(c) Hint: Distinguish betweeen, on one hand, the elementary row operation of adding a multiple of a row into another row or interchanging rows, and on the other hand, the operation of multiplying a row by $k$

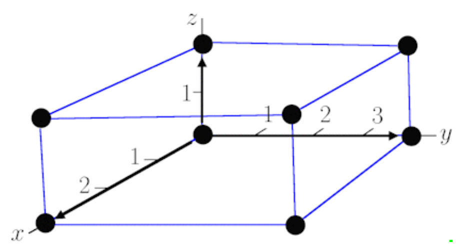


Figure 6.11: Answer for problem 6.6.5.3

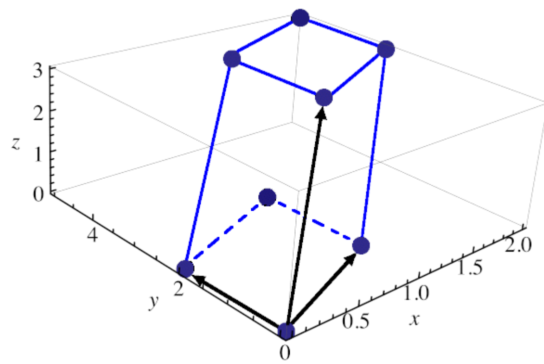


Figure 6.12: Answer for problem 6.6.5.5

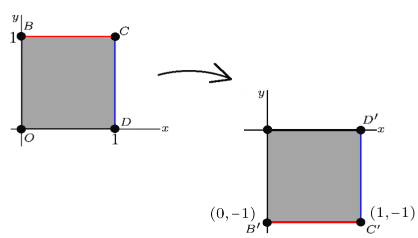


Figure 6.13: Problem 6.6.5.7

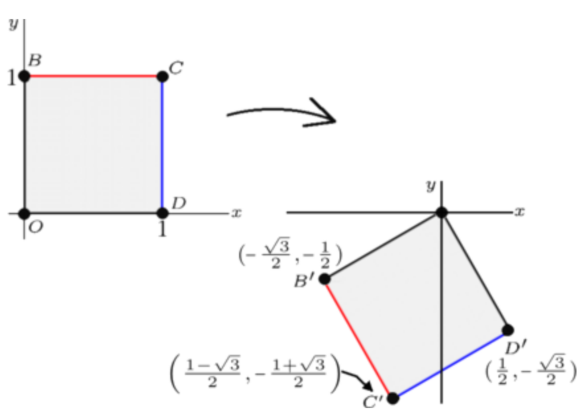


Figure 6.14: Problem 6.6.5.9

## Section 6.7

6.7.6.1. $\boldsymbol{curl}\left(x^2y\,\hat{\boldsymbol{\imath}}+x^2z\,\hat{\boldsymbol{\jmath}}+x^3\,\hat{\boldsymbol{k}}\right)=-x^2\hat{\boldsymbol{\imath}}-3x^2\hat{\boldsymbol{\jmath}}+(2xz-x^2)\hat{\boldsymbol{k}}$

6.7.6.3. (a) $div(\mathbf{F})=-y^2\sin(xy^2)+\cos(y+z^2)-\frac{1}{x-z}$

(b) $curl(\mathbf{F})=-2z\cos(y+z^2)\,\hat{\boldsymbol{\imath}}-\dfrac{1}{x-z}\,\hat{\boldsymbol{\jmath}}+2xy\sin(xy^2)\,\hat{\boldsymbol{k}}$

6.7.6.5. (a) $div(\mathbf{F})=\dfrac{-2\,(xz+xy+yz)}{\left(x^2+y^2+z^2\right)^2}$

(b) $curl(\mathbf{F})=\dfrac{1}{\left(x^2+y^2+z^2\right)^2}\Big(\left(x^2-y^2+z^2+2xz\right)\hat{\boldsymbol{\imath}}+\left(x^2+y^2-z^2+2xy\right)\hat{\boldsymbol{\jmath}}+\left(-x^2+y^2+z^2+2yz\right)\hat{\boldsymbol{k}}\Big)$

I think it's more difficult to do the problem by first rewriting $x,y,z,\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ in terms of spherical coordinates.

6.7.6.7. $\mathbf{v}=U\left(\left(1-\dfrac{a^2}{r^2}\right)\cos(\theta-\alpha)\,\hat{\mathbf{e}}_r-\left(1+\dfrac{a^2}{r^2}\right)\sin(\theta-\alpha)\,\hat{\mathbf{e}}_\theta\right)$

6.7.6.9. (i) (a) In rectangular coordinates: $div(\mathbf{F})=4(x^2+y^2)$
(b) In cylindrical coordinates: $div(\mathbf{F}=4r^2$.

(ii) (a) In rectangular coordinates: $curl(\mathbf{F})=-4y\hat{\boldsymbol{\imath}}+6x\hat{\boldsymbol{\jmath}}$

(b) In cylindrical coordinates: Using Table 6.1 on p. 407, $curl(\mathbf{F})=r\sin 2\theta\,\hat{\mathbf{e}}_r+r(5+\cos 2\theta)\hat{\mathbf{e}}_\theta$.
Again, it makes sense to take the curl in rectangular coordinates and then convert it to be in cylindrical coordinates.

6.7.6.11. (i) (a) In rectangular coordinates: $div(\mathbf{F})=3x^2+3y^2-z^2-4y+2z$
(b) In spherical coordinates: $div(\mathbf{F})=\rho^2(3-4\cos^2\phi)+\rho\,(-4\sin\phi\,\sin\theta+2\cos\phi)$.
Note that because $\mathbf{F}$ is given in terms of rectangular coordinates and rectangular coordinates basis vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, it makes sense to take the divergence in rectangular coordinates and then convert it to be in spherical coordinates.
(ii) (a) In rectangular coordinates: $curl(\mathbf{F})=-4yz\,\hat{\boldsymbol{\imath}}+4xz\,\hat{\boldsymbol{\jmath}}-4x\,\hat{\boldsymbol{k}}$
(b) In spherical coordinates: Using Table 6.2 on p. 407,

$$curl(\mathbf{F})=4\rho^2\sin\phi\,\cos\phi\,\hat{\mathbf{e}}_\theta-4\rho\sin\phi\cos\theta\,(\cos\phi\,\hat{\mathbf{e}}_\rho-\sin\phi\,\hat{\mathbf{e}}_\phi\,)\,.$$

Again, it makes sense to take the curl in rectangular coordinates and then convert it to be in spherical coordinates.

6.7.6.13. $div(\mathbf{F})=\rho^{-2}$

6.7.6.15. $\sqrt{3}(x+\sqrt{3})-(y-1)+(z-4)=0$, or $\sqrt{3}\,x-y+z=0$

6.7.6.17. Hint: Using the product rule, the left hand side of (6.66)(6) is

$$\boldsymbol{\nabla}\bullet(\mathbf{F}\times\mathbf{G})=\frac{\partial(F_yG_z-F_zG_y)}{\partial x}-\frac{\partial(F_xG_z-F_zG_x)}{\partial y}+\frac{\partial(F_xG_y-F_yG_x)}{\partial z}$$

6.7.6.19. (a) If $\nabla f$ is continuously differentiable, then

$$\nabla\times(\nabla f) = \left(\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial z}\right] - (\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial y}\right]\right)\hat{\imath} + \left(\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial x}\right] - (\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial z}\right]\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] - (\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\right)\hat{\boldsymbol{k}} = \mathbf{0}.$$

(b) Using the result of part (a), along with Theorem 6.6(5), we have

$$\nabla \times (f\nabla g) = ... = f\cdot\mathbf{0} + (\nabla f)\times(\nabla g) = ...$$

6.7.6.21. potential function $u(r,\phi,\theta) = H_0\left(\rho + \alpha\left(\dfrac{a^3}{\rho^2}\right)\right)\cos\phi + c$, $c$ =arb. const.

6.7.6.23. potential function $f(r,\theta,z) = (Ar^2 + r)(2\pi\theta - \theta^2) + \dfrac{1}{3}\sin(3z) + c$, arb. const. $c$

6.7.6.25. potential function $f(r,\theta,z) = \dfrac{rz^2}{2} + A^2 r\cos\theta + Ar^2\sin\theta + c$, $c$ =arb. const.

6.7.6.27. (a) $\mathbf{F} = ... = -k\,\dfrac{\mu\rho + 1}{\rho^2}\,e^{-\mu\rho}\,\hat{\mathbf{e}}_\rho$ , (b) $\nabla^2 V(\mathbf{r}) = ... = \mu^2 V(\mathbf{r})$

6.7.6.29. Hint: Use $\nabla\bullet\left(\dfrac{1}{h_v h_w}\,\hat{\mathbf{e}}_u\right) = \mathbf{0}$ to get $\nabla\bullet\mathbf{F} = ... = h_v h_w F_u\cdot\mathbf{0} + h_w h_u F_v\cdot\mathbf{0} + h_u h_v F_w\cdot\mathbf{0}$

$$+\Big(\nabla\,(h_v h_w F_u)\Big)\bullet\left(\frac{1}{h_v h_w}\,\hat{\mathbf{e}}_u\right) + \Big(\nabla\,(h_w h_u F_v)\Big)\bullet\left(\frac{1}{h_w h_u}\,\hat{\mathbf{e}}_v\right) + \Big(\nabla\,(h_u h_v F_w)\Big)\bullet\left(\frac{1}{h_u h_v}\,\hat{\mathbf{e}}_w\right) = ...$$

6.7.6.31. $\nabla^2 f = ... = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}\left[h_v h_w\cdot\frac{1}{h_u}\frac{\partial f}{\partial u}\right] + \frac{\partial}{\partial v}\left[h_w h_u\cdot\frac{1}{h_v}\frac{\partial f}{\partial v}\right] + \frac{\partial}{\partial w}\left[h_u h_v\cdot\frac{1}{h_w}\frac{\partial f}{\partial w}\right]\right) = ...$

6.7.6.33. (a) $\nabla f(x,y,z) = \left(A + A^T\right)\mathbf{x}$, (b) $\nabla f(x,y,z) = 2A\mathbf{x}$

## Section 6.8

6.8.4.1. $\approx -0.0294726\,\hat{\jmath}\,\mathrm{m/s^2}$

6.8.4.3. acceleration of the particle relative to the inertial reference frame is

$$\mathbf{a}(t) = -\frac{6\rho(t)}{(1+(\rho(t))^3)^3}\,(\mathbf{r}(t)\bullet\mathbf{R})\,\mathbf{R}\,.$$

6.8.4.5. $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \approx 1.0980\,\hat{\jmath}\,\mathrm{m/s^2}$

6.8.4.7. maximum period of rotation should be $\frac{2\pi}{\omega} \approx 10.9877\,\mathrm{s}$, or about 11 seconds

6.8.4.9. $3\,\hat{\imath} - 6\,\hat{\jmath} - 15\,\hat{\boldsymbol{k}}$.

6.8.4.11. $e^{t\Omega} = \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix}$.

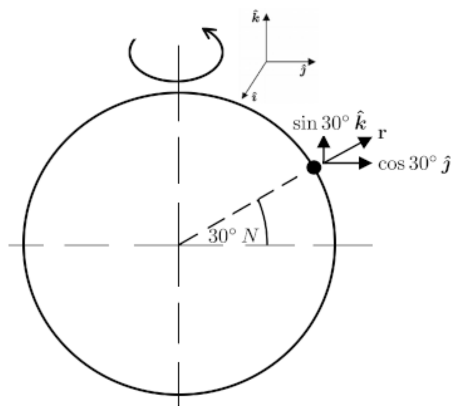


Figure 6.15: Problem 6.8.4.1

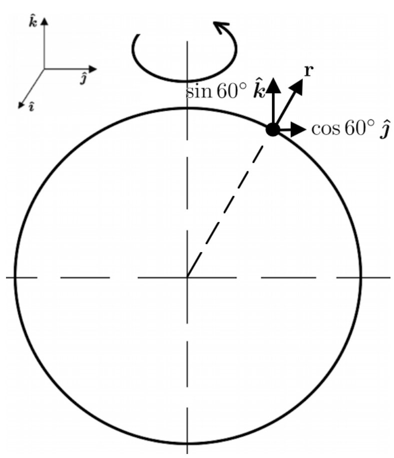


Figure 6.16: Problem 6.8.4.5

6.8.4.13. $e^{t\Omega} = \frac{1}{3}\begin{bmatrix} 1+2\cos(\sqrt{3}\,t) & 1-\cos(\sqrt{3}\,t)-\sqrt{3}\sin(\sqrt{3}\,t) & 1-\cos(\sqrt{3}\,t)+\sqrt{3}\sin(\sqrt{3}\,t) \\ 1-\cos(\sqrt{3}\,t)+\sqrt{3}\sin(\sqrt{3}\,t) & 1+2\cos(\sqrt{3}\,t) & 1-\cos(\sqrt{3}\,t)-\sqrt{3}\sin(\sqrt{3}\,t) \\ 1-\cos(\sqrt{3}\,t)-\sqrt{3}\sin(\sqrt{3}\,t) & 1-\cos(\sqrt{3}\,t)+\sqrt{3}\sin(\sqrt{3}\,t) & 1+2\cos(\sqrt{3}\,t) \end{bmatrix}$

6.8.4.15. Hints: $LHS = \mathbf{A}\bullet(\mathbf{B}\times\mathbf{C}) = (a_1\hat{\boldsymbol{\imath}}+a_2\hat{\boldsymbol{\jmath}}+a_3\hat{\boldsymbol{k}})\bullet\Big((b_1\hat{\boldsymbol{\imath}}+b_2\hat{\boldsymbol{\jmath}}+b_3\hat{\boldsymbol{k}})\times(c_1\hat{\boldsymbol{\imath}}+c_2\hat{\boldsymbol{\jmath}}+c_3\hat{\boldsymbol{k}})\Big) = \ldots$

and $RHS = \mathbf{C}\bullet(\mathbf{A}\times\mathbf{B}) = \det[\,\mathbf{C} \;\vdots\; \mathbf{A} \;\vdots\; \mathbf{B}\,] = \ldots$

6.8.4.17. Hint: Define a scalar function of $t$ by $y(t) \triangleq \mathbf{e}_i(t)\bullet\mathbf{e}_j(t)$. We calculate that $\dot{y}(t) = \ldots ==$ $\Big(\boldsymbol{\omega}(t)\times\mathbf{e}_i(t)\Big)\bullet\mathbf{e}_j(t)+\mathbf{e}_i(t)\bullet\Big(\boldsymbol{\omega}(t)\times\mathbf{e}_i(t)\Big)$; eventually, conclude that $\dot{y}(t)\equiv 0$.

6.8.4.19. Hint: $\frac{d}{dt}\Big[\mathbf{x}(t)\times\mathbf{z}(t)\Big] = \ldots = \frac{d}{dt}\Big[\,(x_2z_3-x_3z_2)\hat{\boldsymbol{\imath}}+(x_3z_1-x_1z_3)\hat{\boldsymbol{\jmath}}+(x_1z_2-x_2z_1)\hat{\boldsymbol{k}}\,\Big] = \ldots$

# Chapter 7

# Integral Theorems, Multiple Integrals, and Applications

### Section 7.1.2

7.1.2.1. $-\frac{\sqrt{2}}{2}$

7.1.2.3. $= \frac{3}{2}\ \ln 26$

7.1.2.5. $-2 + 2\cos\left(\frac{\sqrt{\pi}}{2}\right) + \sqrt{\pi}\sin\left(\frac{\sqrt{\pi}}{2}\right)$

7.1.2.7. the improper integral is convergent and converges to $\dfrac{1}{3}$

7.1.2.9. the improper integral is convergent and converges to $\dfrac{1}{\sqrt{2}}$

7.1.2.11. the improper integral is divergent

7.1.2.13. (b) for all real constants $p$, $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$ diverges.

(a) for no real constant $p$, does $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$ converge.

7.1.2.15. Hint: Begin with $\displaystyle\int_0^1 x\,f''(x)\,dx = \big[x\,f'(x)\big]_0^1 - \int_0^1 f'(x)\,dx = ...$

### Section 7.2.5

7.2.5.1. (a) $||\mathbf{r}_1 - \mathbf{r}_0||$

7.2.5.3. $\dfrac{5+e^2}{2}$

7.2.5.5. $\dfrac{(\pi-2)\sqrt{5}}{8}$

7.2.5.7. $(\overline{x},\overline{y}) = \dfrac{2a}{\pi + a^2(\pi-1)} \cdot \left(1 + \dfrac{2a^2}{3},\, 1 + \dfrac{2a^2}{3}\right)$

7.2.5.9. $(\overline{x},\overline{y}) = \dfrac{4a}{\pi(2+0.1\pi)} \cdot (0.8+0.1\pi, 1.2) \approx a \cdot (0.613005, 0.6602324)$

7.2.5.11. (a) $-\frac{a^4}{3}$, (b) $-\dfrac{a^4}{4}\Big(0-1\Big) = \dfrac{a^4}{4}$, (c) 0

7.2.5.13. $-\pi a^2$

7.2.5.15. (a) potential function $f(x,y,z) = xy^2 + ye^{3z} + z$, (b) $e^3$

7.2.5.17. $\left(\dfrac{1}{\sqrt{6}} - \dfrac{1}{\sqrt{14}}\right) mMG$

7.2.5.19. $\ln \frac{3}{2}$

7.2.5.21. $w\ell\,(-\cos\alpha + 1)$

**Section 7.3.7**

7.3.7.1. Using rectangles with $\triangle x = 2$ and $\triangle y = 5$, $Volume \approx 80$.

7.3.7.3. $\dfrac{e^2}{16}\left(11e^{10} - 20e^4 - 3e^2 + 4\right)$

7.3.7.5. $\ln \dfrac{6}{5}$

7.3.7.7. 5

7.3.7.9. $2\pi - 4$

7.3.7.11. $\dfrac{544}{15}$

7.3.7.13. $\dfrac{1}{2}\left(1 - e^{-4}\right)$

7.3.7.15. $\dfrac{2-\sqrt{2}}{3}\,\varrho_0$

7.3.7.17. 0

7.3.7.19. $\pi a^2$

7.3.7.21. 0

7.3.7.23. $\dfrac{64}{15}$

7.3.7.25. Hint: Use Corollary 7.4

7.3.7.27. $\dfrac{16}{3}$

7.3.7.29. $\dfrac{5}{2}$

7.3.7.31. Using rectangles with $\triangle x = 2$ and $\triangle y = 1$, (a) $Volume \approx 4\,\text{mile}^3$, (b) average height $\approx \frac{1}{3}$ mile

7.3.7.33. $\dfrac{27}{8}$

7.3.7.35. $(\overline{x}, \overline{y}) = \dfrac{1}{2075}(3934\ cm, 3960\ cm) \approx (1.8959\ cm, 1.90843\ cm)$, $I_0 = \frac{8771}{50}\ g - cm^2 = 175.42\ g - cm^2$

7.3.7.37. $\dfrac{16}{3}$

**Section 7.4.3**

7.4.3.1. Example: $\displaystyle\int_{\underline{0}}^{\underline{2}} \int_{\underline{0}}^{\underline{1-\frac{y}{2}}} \int_{\underline{0}}^{\underline{3-3x-\frac{3y}{2}}} \underline{f(x,y,z)}\ \underline{dz}\ \underline{dx}\ \underline{dy}$; five other final conclusions can be correct.

7.4.3.3. $\overline{y} = \dfrac{3}{4}$

7.4.3.5. $\displaystyle\int_{\underline{0}}^{\underline{2}} \int_{\underline{0}}^{\underline{1-\frac{x}{2}}} \int_{\underline{0}}^{\underline{4-2x-4y}} \underline{f(x,y,z)}\ \underline{dz}\ \underline{dy}\ \underline{dx}$

7.4.3.7. $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{16\sqrt{3}}{15\pi},\ \dfrac{16\sqrt{3}}{15\pi},\ 2\right) \approx (0.5880841551,\ 0.5880841551,\ 2)$

7.4.3.9. $24\pi$

7.4.3.11. $\displaystyle\iiint_{\mathcal{V}} f(x,y,z)dV = \int_0^{12} \int_0^{4-\frac{x}{3}} \int_{3-\frac{x}{4}-\frac{3y}{4}}^{4} f(x,y,z)\, dz\, dy\, dx$

7.4.3.13. $\overline{z} = 5$

7.4.3.15. $I_0 = \dfrac{4\pi}{15}\varrho_0$

7.4.3.17. $\dfrac{65\pi k}{4}$

7.4.3.19. Hint: Use Example 7.25's alternate cylindrical coordinates $(x, \mathrm{r}, \varphi)$ in which the $x-$axis is the axis of revolution, the position vector being given by $\mathbf{r} = x\,\hat{\boldsymbol{\imath}} + \mathrm{r}\cos\varphi\,\hat{\boldsymbol{\jmath}} + \mathrm{r}\sin\varphi\,\hat{\boldsymbol{k}}$.

7.4.3.21. Hint: Define another alternate cylindrical coordinates $(\mathrm{r}, y, \vartheta)$ in which the $y-$axis is the axis of revolution, so the position vector is given by $\mathbf{r} = \mathrm{r}\cos\vartheta\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + \mathrm{r}\sin\vartheta\,\hat{\boldsymbol{k}}$

## Section 7.5.3

7.5.3.1. $dS = a\sqrt{1+\cos^2\theta\cos^2 z}\; d\theta dz$

7.5.3.3. $\frac{9\pi}{2}\,(5-2\sqrt{3})$

7.5.3.5. $(\overline{x},\overline{y},\overline{z}) = (0,2,0)$

7.5.3.7. $\overline{z} = 2$

7.5.3.9. $\iint_{\mathcal{S}} z\,dS = \pi a^3, \quad \overline{z} = \dfrac{a}{2}$

7.5.3.11. (a) $Area(\mathcal{S}) = 4\pi\sqrt{5}$, (b) $\displaystyle\iint_{\mathcal{S}} z\,dS = 20\pi\sqrt{5}c$

7.5.3.13. $\dfrac{\pi a^4}{2}$

7.5.3.15. Hints: The top, $\mathcal{S}_+$ can be parametrized using $\mathbf{r}_+ = r\,\cos\theta\,\hat{\boldsymbol{\imath}} + r\,\sin\theta\,\hat{\boldsymbol{\jmath}} + H\,\hat{\boldsymbol{k}}$;
the side can be parametrized using $\widetilde{\mathbf{r}} = a\,\cos\theta\,\hat{\boldsymbol{\imath}} + a\,\sin\theta\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}$

7.5.3.17. Using rectangles with $\triangle x = 2$ and $\triangle y = 1$ and sampling at midpoints gives $\displaystyle\iint_{\mathcal{S}} \mathbf{F}\bullet d\mathbf{S} \approx 24$.

7.5.3.19. (a) $Area(\mathcal{S}) = \displaystyle\iint_{\mathcal{S}} dS = \iint_{\mathcal{D}} |y(t)|\cdot\sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\,dt\,d\varphi = 2\pi\int_{\alpha}^{\beta} |y(t)|\cdot\sqrt{\big(\dot{x}(t)\big)^2 + \big(\dot{y}(t)\big)^2}\,dt$

(b) $Area(\mathcal{S}) = \dfrac{4\pi}{3}\left(-\dfrac{28\sqrt{2}}{3} + 42\sqrt{13} - 4\ln\big(1+\sqrt{2}\big) + 4\ln\left(\dfrac{3+\sqrt{13}}{2}\right)\right)$, with some help from Mathematica and the fact that $arcsinh(z) = \ln(z+\sqrt{1+z^2})$, for real $z > 0$.

## Section 7.6.4

7.6.4.1. $-\dfrac{8\pi a^3}{3}$

7.6.4.3. $-36$

7.6.4.5. Hint: Use $\nabla^2\phi = \boldsymbol{\nabla}\bullet\boldsymbol{\nabla}\phi$.

7.6.4.7. Hint: Look at the hypotheses, specifically some of the "technical" hypotheses of the Divergence Theorem.

7.6.4.9. $-2\pi$

7.6.4.11. $\dfrac{\pi}{2}$

7.6.4.13. Use Stokes' Theorem.

7.6.4.15. Hints: The gravitational force **F** can be shown to satisfy

$$\hat{\boldsymbol{\imath}} \bullet \mathbf{F} = -mG\varrho_0 \int_0^{2\pi} \int_0^{\pi} \int_a^b \rho \sin\phi\, \cos\theta\, \frac{1}{(\rho^2+\rho_0^2-2\rho_0\rho\cos\phi)^{3/2}}\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

and

$$\hat{\boldsymbol{k}} \bullet \mathbf{F} = -mG\varrho_0 \int_0^{2\pi} \int_0^{\pi} \int_a^b (\rho\cos\phi - \rho_0) \frac{1}{(\rho^2+\rho_0^2-2\rho_0\rho\cos\phi)^{3/2}}\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

**Section 7.7.2**

7.7.2.1. $\sqrt{\dfrac{35}{12}}$

7.7.2.5. $k(x) = \dfrac{1}{\sqrt{\pi}\,\sigma} \exp\left(-\dfrac{x^2}{4\sigma^2}\right) = 2 \cdot \dfrac{1}{\sqrt{2\pi}\,\sqrt{2}\,\sigma} \exp\left(-\dfrac{x^2}{2(\sqrt{2}\,\sigma)^2}\right)$

7.7.2.7. ... $\dfrac{1}{\overline{x^2} - \left(\overline{x}\right)^2} = \dfrac{1}{E[\,X^2\,] - (E[\,X\,])^2}$

# Chapter 8

# Numerical Methods I

**Section 8.1.8**

8.1.8.1. (a) Newton's Method gave approximate solution $x_8 = 2.370476901$, with $f(x_8) = -3.677613769E-16$.

(b) The Bisection Method gave approximate solution $x_{26} = 2.370476902$, with $f(x_{24}) = -1.7873903989E-09$.

8.1.8.3. (a) Newton's Method gave approximate solution $x_5 = -2.226394727$, with $f(x_5) = 0.000000000E+00$.

(b) The Bisection Method gave approximate solution $x_{25} = -2.226394721$, with $f(x_{25}) = 1.841991815E-08$.

8.1.8.5. (1) For initial guess $x_0 = 0.5$, in 7 steps Newton's method gives approximate solution 1.894215957. It appears to have quadratic convergence.
(2) For initial guess $x_0 = -2$, in 5 steps Newton's method gives approximate solution $-1.541814739$. It appears to have quadratic convergence.

8.1.8.7. In 9 steps the Secant method gives approximate solution 0.2591432818.

8.1.8.9. With initial guesses 4, 7, 10, 13, get approximate solutions 3.92660231, 7.06858275, 10.2101761, 13.3517688, respectively.

8.1.8.11. (a) After 30 steps, the method of successive approximations gives approximate solution of about 0.259143282 and appears to converge.

(b) After 8 steps, Aitken's method gives approximate solution of about 0.259143282 but after that bounces around that value.

8.1.8.13. Newton's method for the equation gives approximate solutions 0.988590027, 2.01623831, 2.96951504, 4.02486196, 4.96041337, 6.02137429, 6.96387382, 8.00617912, 8.97750897, 9.98413271, 10.9947678, 11.9624699,...

This gives approximate solutions for $\lambda = x^2$ of 9.77310241, 4.06521693, 8.81801955, 16.1995138, 24.6057008, 36.2569483, 48.4955386, 64.0989040, 80.5956673, 99.6829060, 120.884919, 143.100686,...

**Section 8.2.5**

8.2.5.1. (a) Newton's method is $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \dfrac{1}{1-3x_k^2y_k^2}\begin{bmatrix} x_k^2 & 1-2x_ky_k \\ -1-2x_ky_k & y_k^2 \end{bmatrix}\begin{bmatrix} x_ky_k^2 - y_k + 1 \\ x_k^2y + x_k - 0.5 \end{bmatrix}$.

(b) $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.125 \end{bmatrix}$

8.2.5.3. approximate solutions: $(x,y) \approx (5.0000000,\ 2.0000000)$ and $(x,y) \approx (-3.0000000,\ 6.0000000)$.

8.2.5.5. approximate solutions: $(x,y) \approx (0.0000000,\ -2.8284271)$, $(x,y) \approx (1.0000000,\ 0)$, and $(x,y) \approx (0.0000000,\ 2.8284271)$ .

8.2.5.7. approximate solutions: $(x,y){\approx}(-4.7123890, -3.6651914)$, $(x,y){\approx}(-42.411501, -28.797933)$, and $(x,y){\approx}(29.845130, 19.373155)$ .

8.2.5.9. approximate solutions: $(x,y){\approx}(-4.7123890, -3.6651914)$, $(x,y){\approx}(-42.411501, -28.797933)$, and $(x,y){\approx}(29.845130, 19.373155)$ .

8.2.5.11. approximate solutions: $(x,y){\approx}(4.0000000, 1.0000000)$, $(x,y){\approx}(5.8987177, 0.050641131)$, and $(x,y){\approx}(-4.0000000, 5.0000000)$.

**Section 8.3.4**

8.3.4.1. $L \approx S_8 \approx 12.35174232614476$

8.3.4.3. for example, Boole's rule

$$U_{12} = \tfrac{2h}{45}\big(7f_0 + 32f_1 + 12f_2 + 32f_3 + 14f_4 + 32f_5 + 12f_6 + 32f_7 + 14f_8 + 32f_9 + 12f_{10} + 32f_{11} + 7f_{12}\big)$$

8.3.4.5. (a) $I = \displaystyle\int_{-\pi/4}^{\pi/3} \tan\theta\, d\theta \approx T_4 \approx 0.3786153194106639$

(b) error bound $\left|\int_{-\pi/4}^{\pi/3} \tan\theta\, d\theta - T_4\right| \le \frac{4802\pi^3}{331776} \approx 0.4487730897.$

(c) $I = \frac{1}{2}\ln 2 \approx 0.3465735903$

(d) Yes, the absolute error $|\, I - T_4\,| \approx |0.3465735903 - 0.3786153194106639| \approx 0.032041729130691254$ is well less than the error bound of 0.4487730897

8.3.4.7. The change of the position is (a) $x(4) - x(0) \approx S_8 = 63.833...$, (b) $x(4) - x(0) \approx U_8 = 63.844...$

**Section 8.4.3**

8.4.3.1. 75.8 is an upper bound on the condition number.

8.4.3.3. (a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.000 \\ 1.000 \end{bmatrix}$, without partial pivoting, (b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.000 \\ 0.9999 \end{bmatrix}$, with partial pivoting

True solution: $\begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} 1.0000500025 \\ 0.9999499975 \end{bmatrix}$

8.4.3.5. (a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.000 \\ 1.000 \end{bmatrix}$, with partial pivoting, (b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$, with implicit partial pivoting

True solution: $\begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} 1.0000500025 \\ 0.9999499975 \end{bmatrix}$

8.4.3.7. SOR method using initial guess $x = 3$, $y = 2$: (i) With relaxation parameter $\omega = 1$, that is, using the Gauss-Seidel iteration, after 10 steps the method converged to $x = 1.000010000100001$, $y = 0.9999899998999990$. (ii) Instead, with relaxation parameter $\omega = 0.9$, after 38 steps the method again converged to $x = 1.000010000100001$, $y = 0.9999899998999990$.

8.4.3.9. Hint: Use the result of problem 2.4.4.12.

**Section 8.5.7**

8.5.7.1. ... $A_2 \triangleq Q_2 A_1 Q_2 = \begin{bmatrix} 1 & -\sqrt{2} & \frac{\sqrt{2}}{\sqrt{11}} & \frac{3}{\sqrt{11}} \\ -\sqrt{2} & \frac{1}{2} & -\frac{13}{2\sqrt{11}} & -\frac{3}{\sqrt{22}} \\ 0 & -\frac{\sqrt{11}}{2} & -\frac{5}{22} & \frac{9}{11\sqrt{2}} \\ 0 & 0 & -\frac{12\sqrt{2}}{\sqrt{11}} & \frac{19}{11} \end{bmatrix}$

$$\approx \begin{bmatrix} 1.000000000 & -1.414213562 & 0.426401433 & 0.904534034 \\ -1.414213562 & 0.500000000 & -1.959823740 & -0.639602149 \\ 0.000000000 & -1.658312395 & -0.227272727 & 0.578541912 \\ 0.000000000 & 0.000000000 & 1.542778432 & 1.727272727 \end{bmatrix}$$

8.5.7.3. ... $A_2 \triangleq Q_2 A_1 Q_2 = \begin{bmatrix} 1 & -\sqrt{2} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\sqrt{2} & -\frac{1}{2} & \frac{5}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{3\sqrt{3}}{2} & \frac{5}{6} & -\frac{1}{3\sqrt{2}} \\ 0 & 0 & \frac{\sqrt{2}}{3} & \frac{5}{3} \end{bmatrix}$

$$\approx \begin{bmatrix} 1.000000000 & -1.414213562 & -0.816496581 & 0.577350269 \\ -1.414213562 & -0.500000000 & 1.443375673 & -0.408248290 \\ 0.000000000 & -2.598076211 & 0.833333333 & -0.235702260 \\ 0.000000000 & 0.000000000 & 0.471404521 & 1.666666667 \end{bmatrix}$$

8.5.7.5. ... $A_2 = R_1 Q_1 = \dfrac{1}{990\sqrt{11}} \begin{bmatrix} 2250\sqrt{11} & 258\sqrt{110} & -264\sqrt{10} \\ -270\sqrt{110} & -1457\sqrt{11} & -1111 \\ 0 & -121 & 253\sqrt{11} \end{bmatrix}$

$$\approx \begin{bmatrix} 2.272727273 & 0.824108724 & -0.254256690 \\ -0.862439362 & 1.471717172 & -0.338362731 \\ 0 & -0.036851387 & 0.255555556 \end{bmatrix}$$

8.5.7.7. With initial guess $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we calculate $M_5 = \widehat{x}_{1,5} = 3.9941520468$ and $\mathbf{x}_5 = M_5^{-1}\widehat{\mathbf{x}}_1 = \begin{bmatrix} 1.0000000000 \\ 0.9985358712 \end{bmatrix}$

## Section 8.6.1

8.6.1.1. $(\delta\Delta\delta)f_i = f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}$

8.6.1.3. error is of order one, that is, $|f''(x_i) - \frac{1}{3h^2}\left(f_{i+1} - 3f_{i-1} + 2f_{i-2}\right)| = \mathcal{O}(h)$

8.6.1.5. $\alpha = -2$ gives error of order one, that is, $|f'(x_i) - \frac{1}{-2h}\left(f_{i+1} - 3f_i + 2f_{i-1}\right)| = \mathcal{O}(h)$

8.6.1.7. $\alpha = 3$ gives error of order one, that is, $|f''(x_i) - \frac{1}{3h^2}\left(2f_{i+1} - 3f_i + f_{i-2}\right)| = \mathcal{O}(h)$

8.6.1.9. Hint: Use, for example, $f_{i+1} = f_i + hf'(x_i) + h^2\,\frac{f''(x_i)}{2} + \frac{f'''(\xi)}{3!}\,h^3$

## Section 8.7.7

8.7.7.1. (a) $\left\{\begin{array}{l} y(0) = y_0 = y(t_0) = \frac{1}{2}, \\ y(0.5) \approx y_1 = 0.4296875. \end{array}\right\}$, (b) $\left\{\begin{array}{l} y(0) = y_0 = y(t_0) = \frac{1}{2}, \\ y(-0.5) \approx y_1 = 0.625, \\ y(-1.0) \approx y_2 = 0.6953125. \end{array}\right\}$ (c) see figure on next page

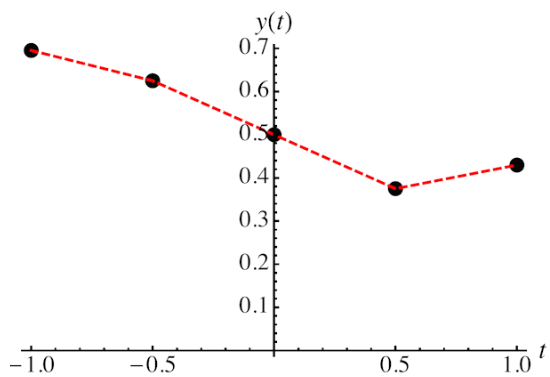


Figure 8.1: Answer for problem 8.7.7.1(c)

8.7.7.3. (a) $y(1.6) \approx y_3 \approx 4.376$, (b) $y(1.6) \approx y_1 \approx 4.4214$, (c) $y(t) = t + 1 + e^{-1+t}$
(d) The exact value of interest is $y(1.6) = 1.6 + 1 + e^{-1+1.6} \approx 4.4221188$ ...

8.7.7.5. (a) $y(1.6) \approx y_3 \approx 0.759950018$, (b) $y(1.6) \approx y_1 =\approx= 1.591082477$, (c) $y(t) = e^{-t} + \big(3e - 1\big)e^{1-2t}$.

(d) The exact value of interest is $y(1.6) = e^{-1.6} + \big(3e - 1\big)e^{1-3.2} \approx 0.994675995$ ...

8.7.7.7. Taylor's Method of order two for this ODE is $y_{i+1} = y_i + h \cdot (y_i^2 - t_i) + \dfrac{h^2}{2} \cdot \big(-1 - 2t_i y_i + 2y_i^3\big)$

8.7.7.9. Taylor's Method of order two for this ODE is

$$y_{i+1} = y_i + h \cdot y_i e^{\cos(t_i - y_i)} + \frac{h^2}{2} \cdot y_i e^{\cos(t_i - y_i)} \cdot \Big(\big(1 + y_i \sin(t_i - y_i)\big)e^{\cos(t_i - y_i)} - \sin(t_i - y_i)\Big)$$

8.7.7.11. (a) Taylor's Method of order two for this ODE is $y_{i+1} = y_i + h \cdot (-2y_i + \cos 3t_i) + \dfrac{h^2}{2} \cdot \Big(4y_i - 2\cos 3t_i - 3\sin 3t_i\Big)$, (b) $y(2.0) \approx 0.11927931$, (c) $y = \frac{11}{13}\,e^{-2t} + \frac{1}{13}\left(2\cos 3t + 3\sin 3t\right)$,

(d) The exact solution's value $y(2.0) \approx 0.0987358543$ is pretty well approximated in (b), where $h = 0.25$ and $y(2.0) \approx y_8 \approx 0.11927931$.

8.7.7.13. Hint: The characteristic equation of the difference equation has roots $r = -h \pm \sqrt{h^2+1}$.

8.7.7.15. Hint: The system of difference equations that are Euler's Method for the system has general solution

$$\mathbf{x}_k = (1+h^2)^{k/2}\left(c_1\begin{bmatrix}\sin\big(k\arctan(h)\big)\\ \cos\big(k\arctan(h)\big)\end{bmatrix} + c_2\begin{bmatrix}-\cos\big(k\arctan(h)\big)\\ \sin\big(k\arctan(h)\big)\end{bmatrix}\right),\ k = 0, 1, 2, ...,$$

where $c_1, c_2$=arbitrary constants.

8.7.7.17. In a sense, the Modified Euler's Method is a generalization of the Trapezoidal Rule for approximating a definite integral of $f(t)$.

8.7.7.19. Hint: Calculate, for example, that

$$k_2 = hf_i + \frac{h^2}{2}\frac{\partial f}{\partial t}(t_i, y_i) + \frac{h^2}{2}\, f_i\,\frac{\partial f}{\partial y}(t_i, y_i) + \frac{h^3}{8}\frac{\partial^2 f}{\partial t^2}(t_i, y_i) + \frac{h^3}{4}\, f_i\,\frac{\partial^2 f}{\partial t \partial y}(t_i, y_i) + \frac{h^3}{8} f_i^2\,\frac{\partial^2 f}{\partial y^2}(t_i, y_i) + \mathcal{O}\left(h^4\right).$$

**Section 8.8.4**

8.8.4.1. (a) $y_{i-1} + \left(-2 + h^2\cdot 2\sqrt{2}\,\cos\left(\frac{2\pi i h}{3}\right)\right)y_i + y_{i+1} = 0$, for $i = 1, 2, 3$

(b) The approximate solution is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \approx \begin{bmatrix} -.3111164480 \\ -.5746031117 \\ -.7873015559 \end{bmatrix}$. The piecewise linear approximate solution is the red, dashed graph in the figure.

(c) There is no exact solution of this non-constant coefficients linear second order ODE. But we did use the Mathematica command *NDSolve* for the ODE-BVP to find a more accurate approximate solution and it agrees well with the coarse approximation we found.

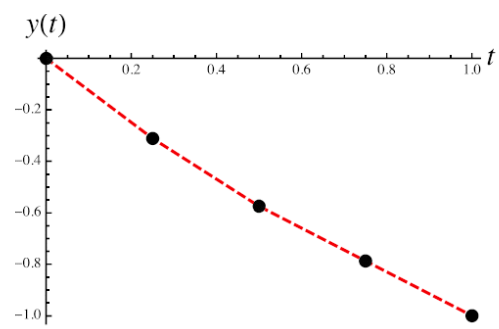


Figure 8.2: Answer for problem 8.8.4.1

8.8.4.3. (a) $y_{i-1} + (-2+h^2)y_i + y_{i+1} = ih^3$, for $i = 1, 2, 3$

(b) The approximate solution is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{238}\begin{bmatrix} 239 \\ 686 \\ 1021 \end{bmatrix} \approx \begin{bmatrix} 1.00420168 \\ 2.88235294 \\ 4.28991597 \end{bmatrix}$. The piecewise linear approximate solution is the red, dashed graph in the figure.

(c) The exact solution of $(\star)$ is $y(t) = x - \cos x + \frac{3+\cos 2}{\sin 2} \cdot \sin x$, whose graph is given in the figure. The results from part (a) look very good, for example, $y_1 = 1.00420168$ vs. the exact value $y(.5) \approx 0.984749672$, $y_2 = 2.88235294$ vs. the exact value $y(1) \approx 2.85081572$, and $y_3 = 4.28991597$ vs. the exact value $y(1.5) \approx 4.26373753$.

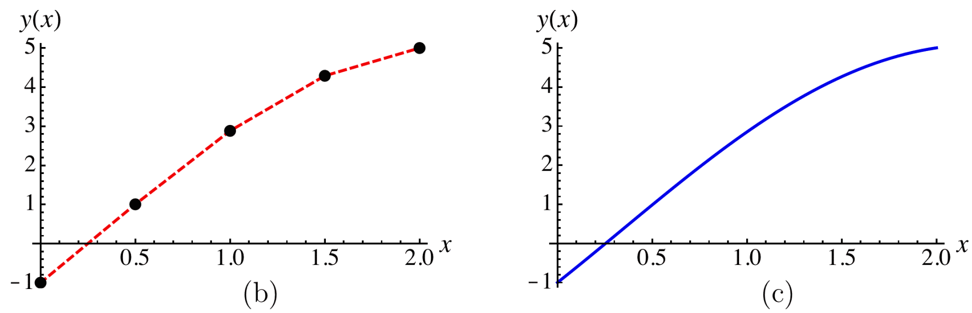


Figure 8.3: Answer for problem 8.8.4.3(b) and (c)

8.8.4.5. (a) $y_{i-1} + (-2 + 2h^2)y_i + y_{i+1} = 0$, for $i = 0, 1, 2, 3$

(b) $\begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \approx \begin{bmatrix} 17.3529170 \\ 19.3097781 \\ 18.8529170 \\ 16.0394412 \\ 11.2210353 \end{bmatrix}$. The fictitious value $y_{-1}$ does not appear in the final conclusion: The piecewise linear approximate solution is the red, dashed graph in the figure.

(c) the exact solution of is $y(x) = \left(5 - \frac{3}{\sqrt{2}} \sin \sqrt{2}\right) \cdot \frac{\cos(\sqrt{2}\, x)}{\cos \sqrt{2}} + \frac{3}{\sqrt{2}} \cdot \sin(\sqrt{2}\, x)$, whose graph is given in the figure. The results from part (a) look good, for example, $y_0 = 19.3097781$ vs. the exact value $y(0) \approx 18.6261587$, $y_1 = 18.8529170$ vs. the exact value $y(.25) \approx 18.2085721$, $y_2 = 16.0394413$ vs. the exact value $y(.5) \approx 15.5385246$, and $y_3 = 11.2210353$ vs. the exact value $y(.75) \approx 10.94630976$.

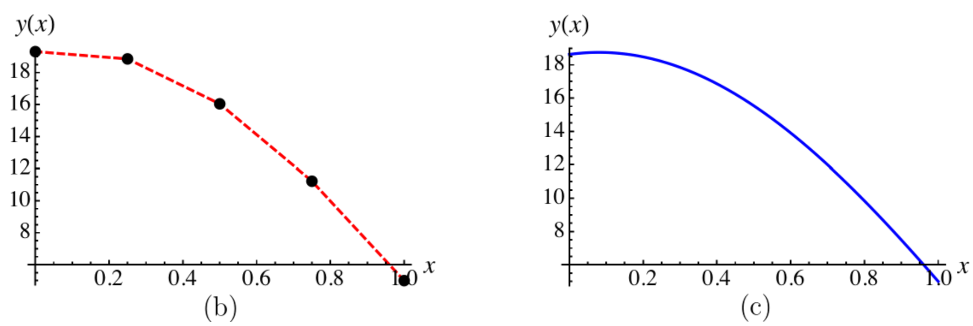


Figure 8.4: Answer for problem 8.8.4.5(b) and (c)

8.8.4.7. (a) For $i = 0, 1, 2, 3$, $y_{i-1} - 2y_i + y_{i+1} = -h^2 x_i = -ih^3$, as well as the BC $y_{-1} + (h-1)y_0 = h$.

(b) The approximate solution is $\begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \approx \begin{bmatrix} .68359375 \\ .57812500 \\ .47265625 \\ .35156250 \\ .19921875 \end{bmatrix}$. The piecewise linear approximate solution is the red, dashed graph in the figure.

(c) The exact solution of is $y(x) = -\frac{5}{12}\,x + \frac{7}{12} - \frac{1}{6}\,x^3$. The solid curve in the figure is the exact solution. The thick dashed curve is the solution using finite differences and agrees well with the exact solution.

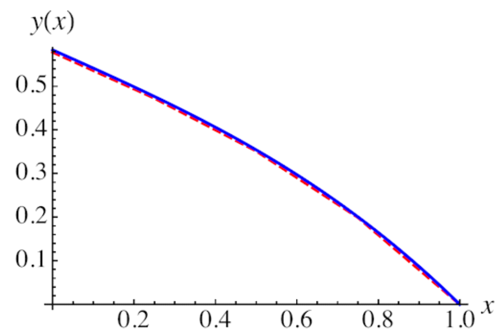


Figure 8.5: Answer for problem 8.8.4.7

8.8.4.9. Using the central difference approximation for the first derivative term, we found approximate eigenvalues

$$\lambda_1 \approx 37.607284106899115, \quad \lambda_2 \approx 138.80108530222495, \quad \lambda_3 \approx 363.84163059087587$$

**Section 8.9.6**

8.9.6.1. (a) the boundary conditions require two equations

$$(1) \quad 0 = y(0) = y_2(x_0) = \frac{1}{6}\,(z_{-1} + 4z_0 + z_1)$$

and

$$(2) \quad -1 = y(1) = y_2(x_2) = \frac{1}{6}\,(z_1 + 4z_2 + z_3)\,.$$

The replacement equations (3) – (5) for the ODE are, respectively, for $j = 0, 1, 2$,

$$(3) - (5) \quad 4\,(z_{j-1} - 2z_j + z_{j+1}) + 2\sqrt{2}\cos\left(\frac{2\pi j \cdot \frac{1}{2}}{3}\right) \cdot \frac{1}{6}\,(z_{j-1} + 4z_j + z_{j+1}) = 0.$$

(b) The approximate solution of the system is $\begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \approx \begin{bmatrix} 0.564829708 \\ 0 \\ -0.564829708 \\ -0.941074435 \\ -1.670872552 \end{bmatrix}$.

The solid curve in Figure is an approximate solution produced by Mathematica. The thick dashed curve is the solution using cubic B-splines. The fact that the latter solution is twice continuously differentiable is apparent and agrees pretty well with Mathematica's approximate solution.

(c) There is no exact solution of this non-constant coefficients linear second order ODE. But we did use the Mathematica command *NDSolve* for the ODE-BVP to find a more accurate approximate solution shown as a solid blue curve in the figure. Mathematica's more accurate approximate solution agrees well with the coarse approximation we found using splines.

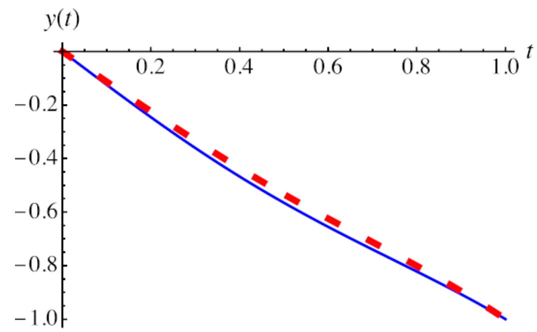


Figure 8.6: Answer for problem 8.9.6.1

8.9.6.3. (a) the boundary conditions require two equations

$$(1) \quad -1 = y(0) = y_4(x_0) = \frac{1}{6}\left(z_{-1} + 4z_0 + z_1\right)$$

and

$$(2) \quad 3 = y(1) = y_2(x_4) = \frac{1}{6}\left(z_3 + 4z_4 + z_5\right).$$

The replacement equations (3) – (7) for the ODE are, respectively, for $j = 0, 1, ..., 4$,

$$(3)-(7) \quad 4\left(z_{j-1} - 2z_j + z_{j+1}\right) + \frac{1}{6}\left(z_{j-1} + 4z_j + z_{j+1}\right) = x_j.$$

(b) The approximate solution of the system is

$$\begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \approx \begin{bmatrix} -1.719908174 \\ -1.041666667 \\ 0.113425159 \\ 0.962038387 \\ 2.046612720 \\ 3 \\ 3.953387280 \end{bmatrix}$$

The approximate solution of the ODE-BVP is.

(c) The exact solution of the ODE-BVP is $y(t) = x - \cos x + \frac{3+\cos 2}{\sin 2} \cdot \sin x$. The solid, blue curve in the figure is the exact solution, as plotted by Mathematica. The thick dashed curve is the solution using cubic B-splines. The fact that the latter solution is twice continuously differentiable is apparent and agrees with well with the exact solution.

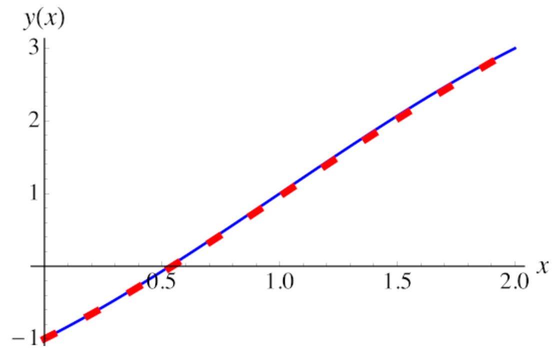


Figure 8.7: Answer for problem 8.9.6.3

8.9.6.5. (a) the boundary conditions require two equations

$$(1) \quad 1 = y(0) - y'(0) = \frac{1}{6}\left(z_{-1} + 4z_0 + z_1\right) - (2h)^{-1}\left(-z_{-1} + z_1\right)$$

and

$$(2) \quad 0 = y(1) = y_4(x_4) = \frac{1}{6}\left(z_3 + 4z_4 + z_5\right).$$

The replacement equations (3) – (7) for the ODE are, respectively, for $j = 0, 1, ..., 4$,

$$(3) - (7) \quad 16\left(z_{j-1} - 2z_j + z_{j+1}\right) = -x_j.$$

(b) The approximate solution is
$$\begin{bmatrix} z_{-1} \\ z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} \approx \begin{bmatrix} 0.6875 \\ 0.583333333 \\ 0.479166667 \\ 0.359375 \\ 0.208333333 \\ 0.0104166667 \\ -0.25 \end{bmatrix}.$$

(c) The exact solution of the ODE-BVP is $y(x) = -\frac{5}{12}x + \frac{7}{12} - \frac{1}{6}x^3$. The solid, blue curve in the figure is the exact solution, as plotted by Mathematica. The thick dashed curve is the solution using cubic B-splines. The fact that the latter solution is twice continuously differentiable is apparent and agrees pretty well with the exact solution.

8.9.6.9. Hint: Begin by choosing any $x$ in the interval $[x_0, x_N]$. Then there is an integer $j$ with $x_{j-1} \le x \le x_j$ and $1 \le j \le N$. Note that for this value of $j$, (8.84) gives

$$L_N(x) = y_{j-1} + \frac{y_j - y_{j-1}}{x_j - x_{j-1}} \cdot (x - x_{j-1}).$$

8.9.6.11. In the figure, (a) shows the curve and the control points; (b) shows that and the control polygon, too.

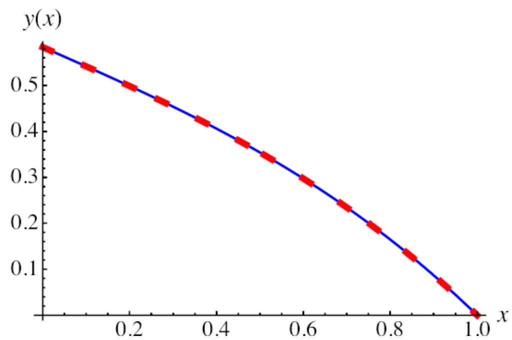


Figure 8.8: Answer for problem 8.9.6.5

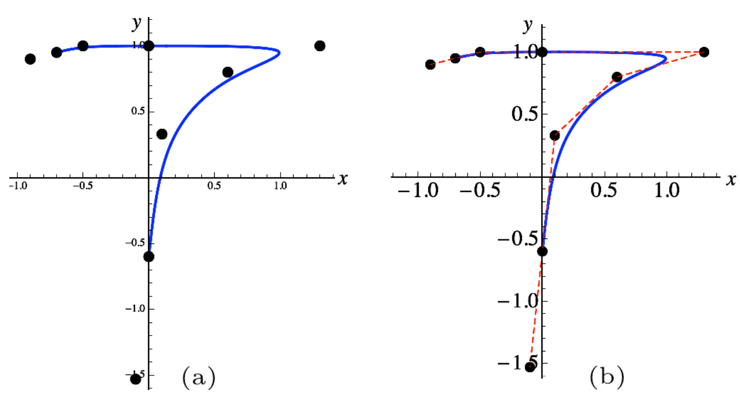


Figure 8.9: Answer for problem 8.9.6.11

# Chapter 9

# Fourier Series

**Section 9.1**

9.1.8.1. $f(x) \doteq f_s(x) = \displaystyle\sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \sin\left(\frac{n\pi}{3}\right) \sin\left(\frac{n\pi x}{3}\right)$

9.1.8.3. $x \doteq f_s(x) = \displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$

9.1.8.5. see figure

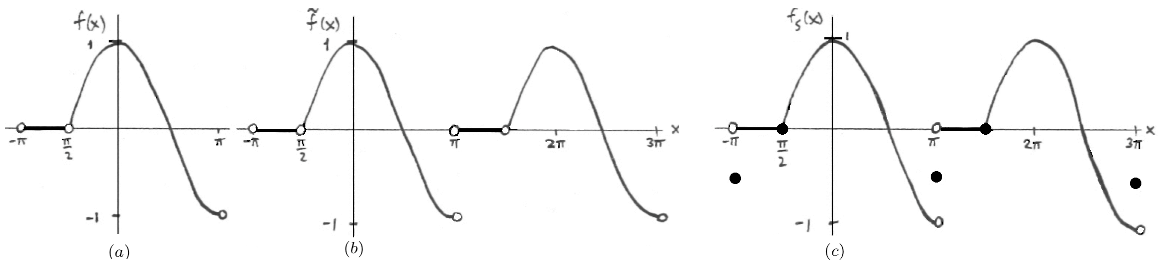


Figure 9.1: Problem 9.1.8.5

9.1.8.7. see figure

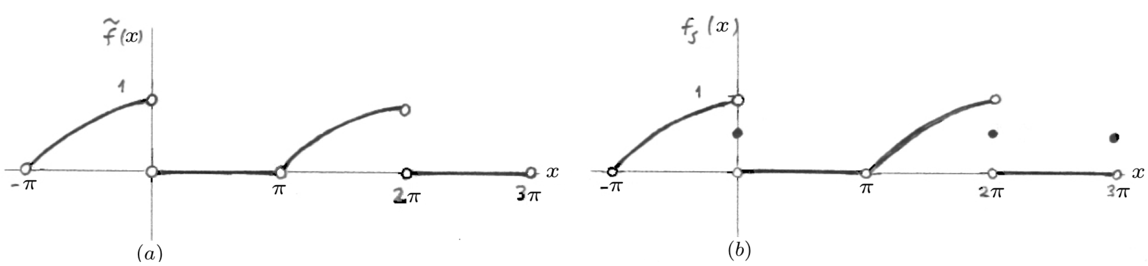


Figure 9.2: Problem 9.1.8.7

9.1.8.9. $|x| \doteq f_s(x) = \dfrac{L}{2} - \dfrac{4L}{\pi^2} \displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\left(\frac{(2k-1)\pi x}{L}\right)$

9.1.8.11. $f(x) \doteq f_s(x) = 1 + \frac{1}{8}\sin x - \frac{3\sqrt{3}}{8}\cos x + \frac{1}{8}\sin 3x - \frac{\sqrt{3}}{8}\cos 3x$

9.1.8.13. $x\,\cos x \doteq f_s(x) = -\frac{1}{2}\,\sin x + \sum_{n=2}^{\infty} \frac{2n}{n^2-1}(-1)^n \sin nx$

9.1.8.15. (a) $f(x) \doteq f_s(x) = \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n} + \frac{2}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)\right)\sin nx$, (b) see figure,

(c) $g(x) = -\frac{1}{4}\sin 4x + \left(\frac{1}{5} + \frac{2}{25\pi}\right)\sin 5x$

(d) see figure: $g(x)$ is the oscillating function; $f(x)$ is the original, sigmoidal function.

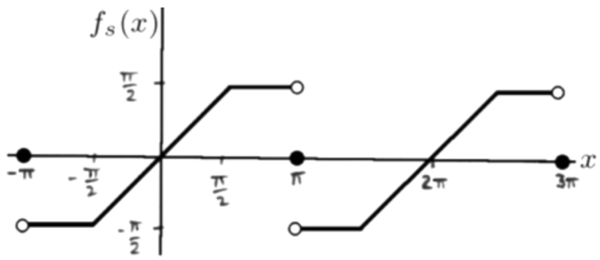


Figure 9.3: Problem 9.1.8.15(b)

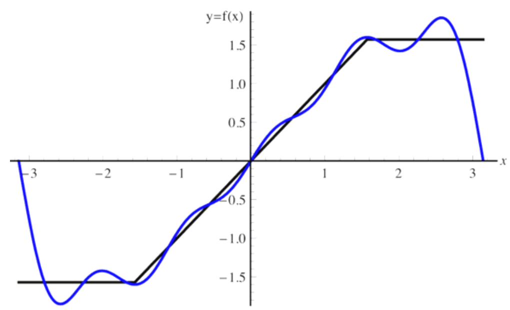


Figure 9.4: Problem 9.1.8.15(d)

9.1.8.17. (a) $f(x) \doteq f_s(x) = \frac{e^L - e^{-L}}{L}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+\left(\frac{n\pi}{L}\right)^2}\left(\cos\left(\frac{n\pi x}{L}\right) + \frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\right)\right)$, (b) see figure

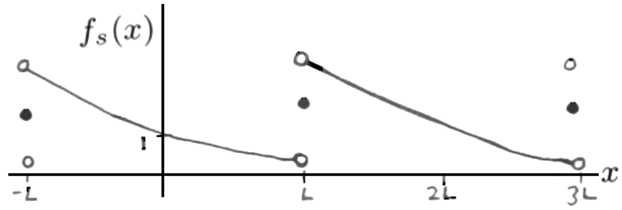


Figure 9.5: Problem 9.1.8.17(b)

9.1.8.19. (a) $f(x) \doteq f_s(x) = -\frac{\pi}{8} - \frac{3}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos nx + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2k-1)^2}\sin\big((2k-1)x\big)$, (b) see the figure.

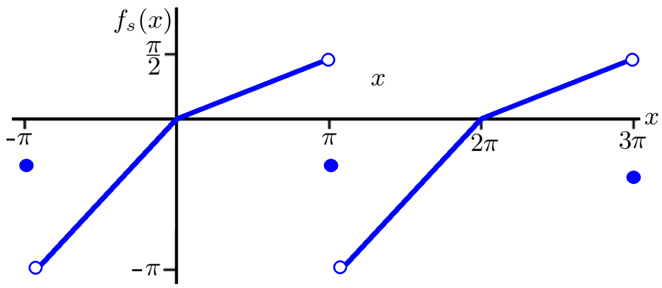


Figure 9.6: Problem 9.1.8.19(a)

9.1.8.21. (a) $f(x) \sim f_s(x) = -\frac{1}{4} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( (-1)^{n+1} + \cos\left(\frac{n\pi}{2}\right) \right) \cos\left(\frac{n\pi x}{2}\right)$

$$+ \sum_{n=1}^{\infty} \left( \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right) \sin\left(\frac{n\pi x}{2}\right)$$

(b) see the figure.

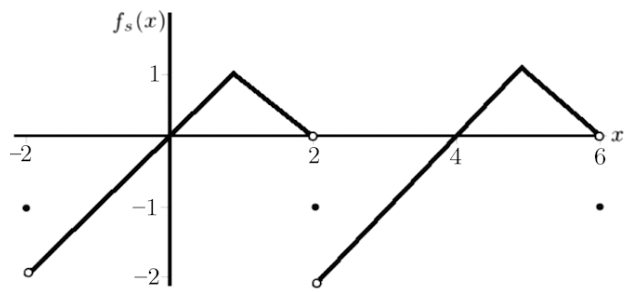


Figure 9.7: Problem 9.1.8.21(b)

9.1.8.23. HInts: First, rewrite $b_n = \frac{1}{L} \left( \int_{-L}^{0} f(x) \sin\left(\frac{n\pi x}{L}\right) dx + \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right)$; next, In the first integral make the substitution $x = -y$.

**Section 9.2**

9.2.4.1. (a) $f_{sin}(x) = -\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{4 + (2k-1)^2}{(2k-1)(-4 + (2k-1)^2)} \sin\big((2k-1)x\big)$,

(b) $f_{cos}(x) = \frac{1}{2} - \cos 2x$ is its own Fourier cosine series, (c) see the figures

9.2.4.3. (a) $f(x) \doteq f_{sin}(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \Big( 2 + (-1)^n - 3\cos\frac{n\pi}{3} \Big) \sin\frac{n\pi x}{3}$,

(b) $f(x) \doteq f_s(x) = \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{n\pi x}{3}\right)$, (c) See figures

9.2.4.5. $f_{sin}(x) = \frac{L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( -n\pi \cos\frac{n\pi}{2} + 6\sin\frac{n\pi}{2} \right) \sin\frac{n\pi x}{L}$, and see the figure.

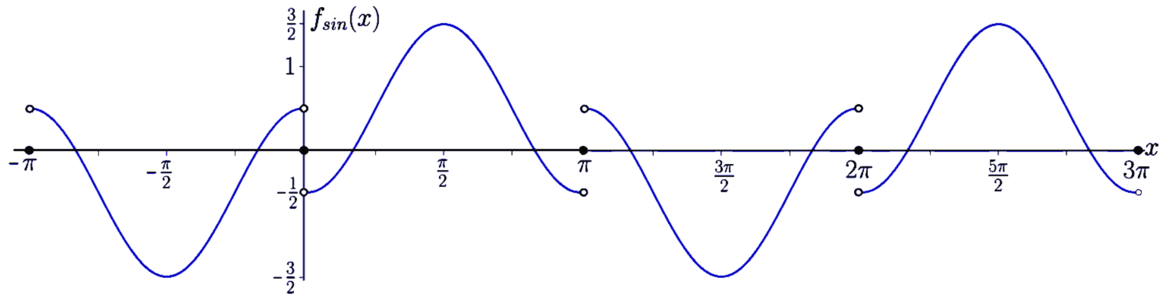


Figure 9.8: Problem 9.2.4.1

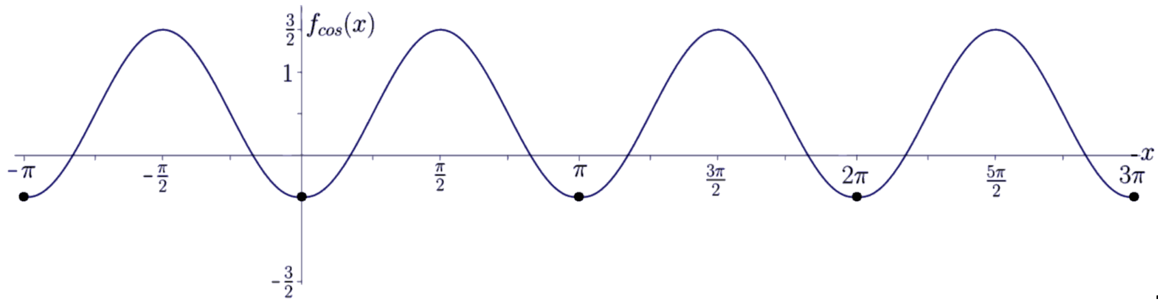


Figure 9.9: Problem 9.2.4.1

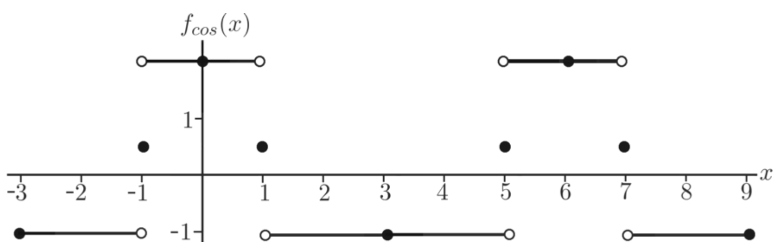


Figure 9.10: Problem 9.2.4.3(c)

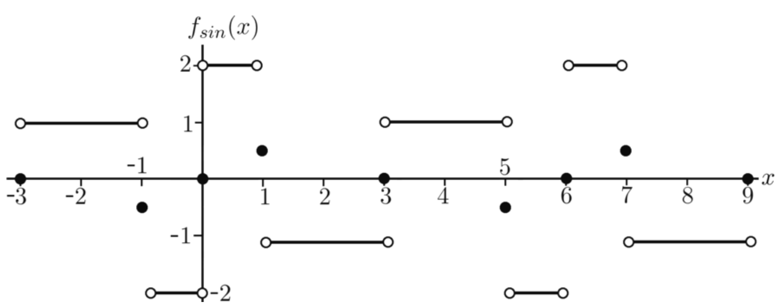


Figure 9.11: Problem 9.2.4.3(c)

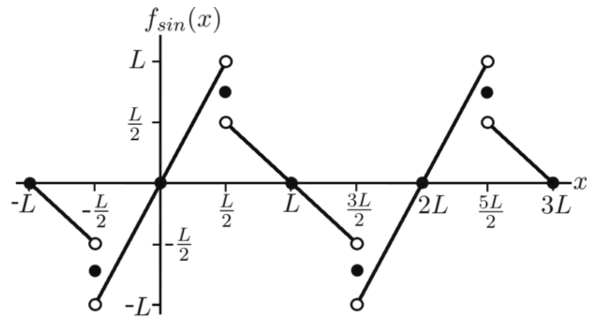


Figure 9.12: Problem 9.2.4.5

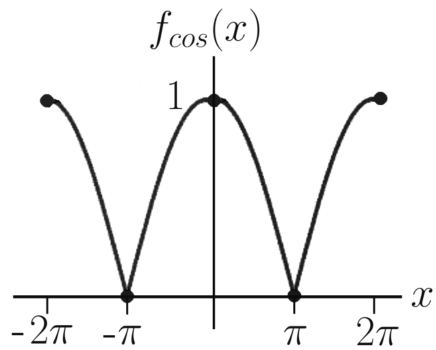


Figure 9.13: Problem 9.2.4.7

9.2.4.7. $f(x) \doteq f_{cos}(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{k=1}^{\infty} \dfrac{1}{4k^2-1}\cos 2kx$, and see the figure.

9.2.4.9. $f(x) \doteq f_{cos}(x) = \dfrac{1}{2} + \dfrac{4}{\pi^2}\displaystyle\sum_{k=1}^{\infty} \dfrac{1}{(2k-1)^2}\cos\big(2(2k-1)\pi x\big)$, and see the figure.

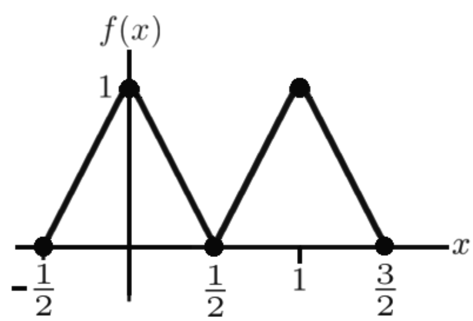


Figure 9.14: Problem 9.2.4.9

9.2.4.11. ODE solution:

$$y(t) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)(9+(2k-1)^2)}\Big((2k-1)e^{-3t} - (2k-1)\cos\big((2k-1)t\big) + 3\sin\big((2k-1)t\big)\Big)$$

The figures show graphs of the approximation of the solution $y(t)$ by a finite sum

$y_N(t) \triangleq \dfrac{4}{\pi}\displaystyle\sum_{k=1}^{N} \dfrac{1}{(2k-1)(4+(2k-1)^2)}\Big((2k-1)e^{-3t} - (2k-1)\cos\big((2k-1)t\big) + 3\sin\big((2k-1)t\big)\Big)$ for $N = 2$ and

$N = 8$. In each picture the approximation is drawn as a dashed curve and the exact solution is drawn as a solid curve.

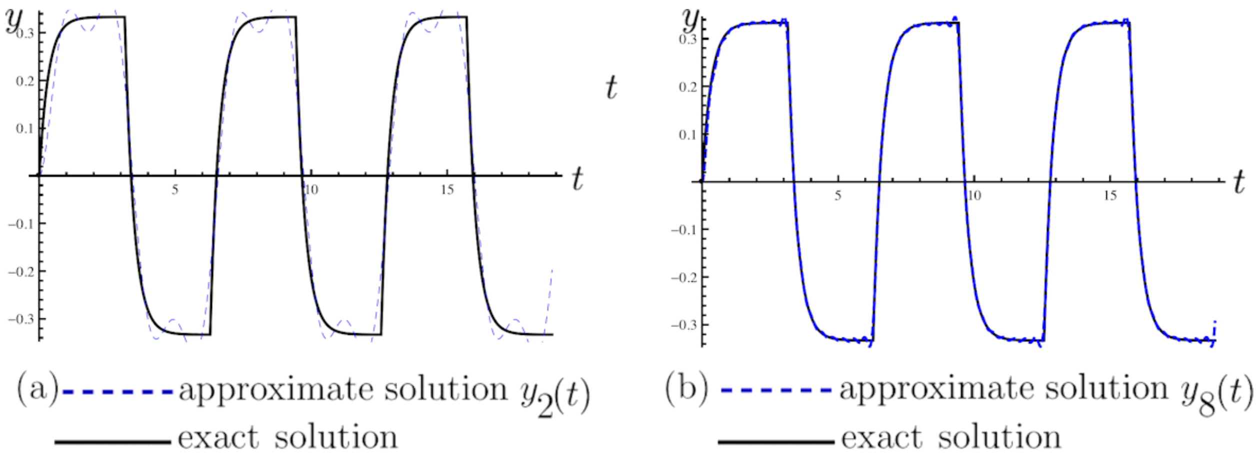


Figure 9.15: Problem 9.2.4.11

9.2.4.13. ODE solution:

$$y(t) = \frac{1}{\sqrt{2}}\,\sin(\sqrt{2}\,t) + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}(2k-1)\big(-(2k-1)^2+2\big)}\Big(-(2k-1)\sin(\sqrt{2}\,t) + \sqrt{2}\,\sin\big((2k-1)t\big)\Big)$$

The figure show graphs of the approximation of the solution $y(t)$ by a finite sum

$y_N(t) \triangleq \dfrac{4}{\pi}\displaystyle\sum_{k=1}^{N}\frac{1}{(2k-1)(4+(2k-1)^2)}\Big((2k-1)e^{-3t} - (2k-1)\cos\big((2k-1)t\big) + 3\sin\big((2k-1)t\big)\Big)$ for $N = 2$ and $N = 8$. In each picture the approximation is drawn as a dashed curve and the exact solution is drawn as a solid curve.

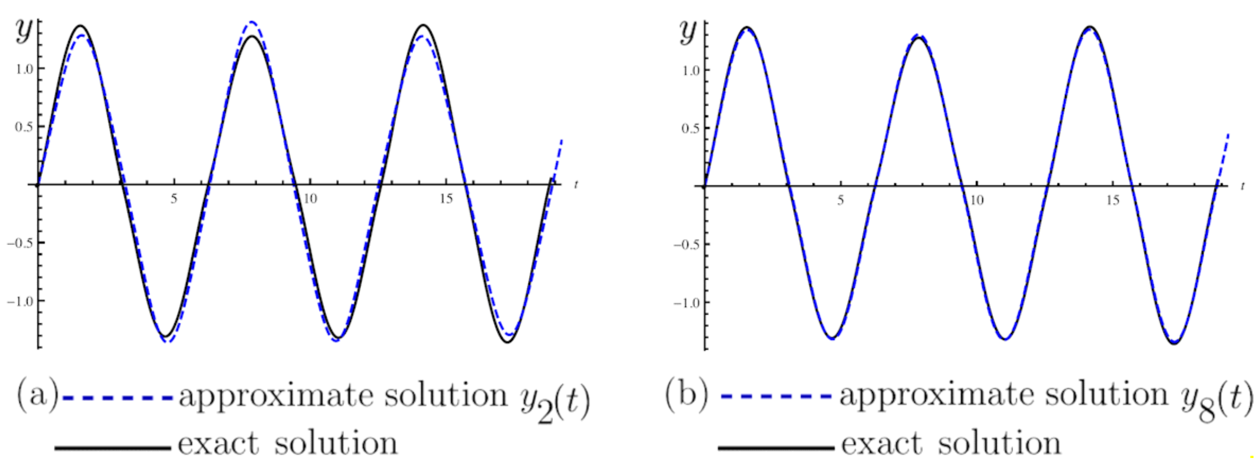


Figure 9.16: Problem 9.2.4.13

**Section 9.3**

9.3.3.1. the only *eigenvalues* are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, ...,$ and corresponding *eigenfunctions* $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, ...$ .

9.3.3.3. the only *eigenvalues* are *eigenvalues* $\lambda_n = \left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2$, $n = 1, 2, ...,$ and corresponding *eigenfunctions* $X_n(x) = \cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$, $n = 1, 2, ...$

9.3.3.5. Hint: Plug solutions $X = c_1 \cos\omega x + c_2 \sin\omega x$ into the BCs (9.21)....

9.3.3.7. The eigenvalues are (1) $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, ...,$ with corresponding *eigenfunctions*
$X_n(x) = e^{-x}\left(\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\right)$, and (2) $\lambda_0 = -1$, with corresponding eigenfunction $X_0(x) \equiv 1$

9.3.3.9. $f(x) \doteq -\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n - 3}\cos\left((n - \frac{1}{2})x\right)$

9.3.3.11. eigenvalues $\mu_n = \frac{1}{L}\left(\frac{\pi}{4} + n\pi\right)$, $n = 0, 1, 2, ...$ .

**Section 9.4**

9.4.3.1. $f(x) \doteq \frac{1}{4}\left(e^{-i3x} + e^{-i3x}\right) + \frac{i2}{\pi}\sum_{k=-\infty}^{\infty} \frac{k}{4k^2 - 9}\, e^{i2kx}$

9.4.3.3. the complex Fourier series is the function itself, $f(x) = \frac{1}{4}e^{-i5x} + \frac{3}{2}\, e^{-i2x} + 1 - \frac{3}{2}\, e^{i2x} + \frac{1}{4}e^{-i5x}$

9.4.3.5. $\mathcal{F}[\, f(t)\,] = \frac{i}{\omega\sqrt{2\pi}}\left(e^{-i3\omega} - e^{-i\omega}\right)$

9.4.3.7. $\mathcal{F}[\, f(t)\,] = \frac{\omega_0}{\sqrt{2\pi}\left(\alpha^2 + \omega_0^2 - \omega^2 + i2\alpha\omega\right)}$

9.4.3.9. $\mathcal{F}[\, e^{-at}\, Step(t)] = \frac{1}{\sqrt{2\pi}}\,\frac{1}{a + i\omega}$

9.4.3.11. $\mathcal{F}^{-1}\left[\frac{1 - e^{-i5\pi\omega}}{4 - \omega^2}\right] = \sqrt{\frac{\pi}{2}} \cdot \left\{\begin{array}{ll} \sin 2x, & 0 \le x \le 5\pi \\ 0, & \text{all other } x \end{array}\right\}$

9.4.3.13. $\mathcal{F}^{-1}\left[\,\frac{\sin b\omega}{\omega}\,\right] = \sqrt{\frac{\pi}{2}} \cdot \left\{\begin{array}{ll} 1, & -b < t < b \\ 0, & |t| > b \end{array}\right\}$

9.4.3.15. $\mathcal{F}[\, g(t)\,] = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\omega}{\omega} \cdot \frac{1}{1 - e^{-1+i2\omega}}$

9.4.3.17. (a) $\mathcal{F}[f(x)\cos kx](\omega)=\frac{1}{2}\left(F(\omega-k)+F(\omega+k)\right)$, (b) $\mathcal{F}[f(x)\sin kx](\omega)=\frac{1}{i2}\left(F(\omega-k)-F(\omega+k)\right)$

9.4.3.19. $\mathcal{F}\left[\frac{\cos(bt)-1}{bt}\right](\omega)=i\sqrt{\frac{\pi}{2}}\cdot\left\{\begin{array}{cl} -1, & -b<\omega<0\\ 1, & 0<\omega<b\\ 0, & |\omega|>b \end{array}\right\}$

**Section 9.5**

9.5.3.1. $\mathbf{F}=DFT[\,\mathbf{f}\,]=\frac{i\sqrt{N}}{2}\left(-\mathbf{d}^{(2)}+\mathbf{d}^{(N-2)}\right)=\frac{i\sqrt{N}}{2}\begin{bmatrix}0\\0\\-1\\0\\ \vdots\\0\\1\\0\end{bmatrix}$

9.5.3.3. $\mathbf{F}=DFT[\,\mathbf{f}\,]=\sqrt{N}\mathbf{d}^{(N-k_0)}=\sqrt{N}\begin{bmatrix}0\\0\\0\\1\\0\\ \vdots\\0\end{bmatrix}\begin{array}{l}\\ \\ \\ \leftarrow (N-k_0)\,place\\ \\ \\ \\ \end{array}$

9.5.3.5. (a) $DFT[\,\mathbf{f}\,]=\begin{bmatrix}1.2 & +i0\\ -0.65 & +i0\\ -0.85 & +i0\\ -0.65 & -i0\end{bmatrix}$,

(b) $DFT[\,\mathbf{f}\,]\approx\begin{bmatrix}2.1 & +i0\\ -0.51 & +i0.20\\ -0.60 & +i0.49\\ -0.41 & +i0.20\\ -0.46 & +i0\\ -0.41 & -i0.20\\ -0.60 & -i0.49\\ -0.51 & -i0.20\end{bmatrix}$, after rounding off to two significant digits.

9.5.3.7. (a) $DFT[\,\mathbf{f}\,]=\begin{bmatrix}1+i0\\ 1-i2\\ 1+i0\\ 1+i2\end{bmatrix}$,

(b) $DFT[\,\mathbf{f}\,]=\begin{bmatrix}1.8 & +i0\\ 0.96 & -i3.2\\ 0.71 & -i0.35\\ 0.46 & -i0.34\\ -0.35 & +i0\\ 0.46 & +i0.34\\ 0.71 & +i0.35\\ 0.96 & +i3.2\end{bmatrix}$, after rounding off to two significant digits.

9.5.3.9. $DFT[\mathbf{f}] = \begin{bmatrix} 0.528 & -i0.0960 \\ 0.416 & +i0.288 \\ 0.192 & +i1.056 \\ 0.864 & -i1.248 \end{bmatrix}$, after rounding off to two significant digits.

9.5.3.11. Hints: use the change of index of summation $m = N-\ell$, that is, $\ell = N-m$, and the "wrap around" convention that $F_{-\ell} \triangleq F_{N-\ell}$ if $\ell = 0, ..., N$.

**Section 9.6**

9.6.4.1. Hints: First multiply the ODE by the eigenfunction $X(x)$ and then integrate over the interval $a < x < b$ and use integration by parts.

9.6.4.3. (a) the only eigenvalues are $\lambda_n = \left(\dfrac{n\pi}{\ln(b/a)}\right)^2$,

(b) the orthogonality relation is that, for $n \neq m$, $\displaystyle\int_a^b R_n(r)R_m(r)\,\frac{1}{r}\,dr = 0$

9.6.4.5. (a) the characteristic equation is $0 = (1-3\lambda)\sin\left(\dfrac{\pi\sqrt{\lambda}}{2}\right) + \sqrt{\lambda}\cos\left(\dfrac{\pi\sqrt{\lambda}}{2}\right)$

(b) the orthogonality relation is that $0 = \displaystyle\int_{\pi/4}^{3\pi/4} X_n(x)X_m(x)\,dx$ for $n \neq m$

9.6.4.7. (a) characteristic equation $0 = 1 - \cosh(\sqrt{\lambda}\,L)\cos(\sqrt{\lambda}\,L)$; there are infinitely many eigenvalues $\lambda_n = -\omega_n^4$, where $\omega_n = \frac{\theta_n}{L} \to \infty$, as $n \to \infty$. Because $\cosh(\theta) \to \infty$, as $\theta \to \infty$, the roots $\theta_n \sim \left(n - \frac{1}{2}\right)\pi$ as $n \to \infty$.

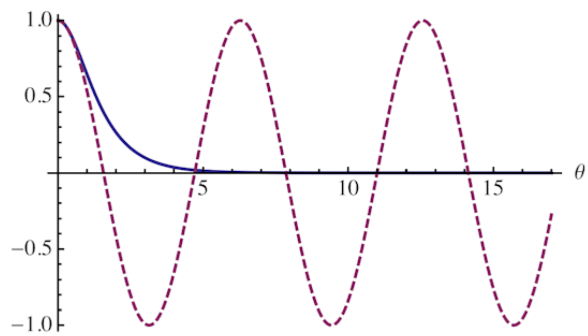


Figure 9.17: Problem 9.6.4.7

(b) the orthogonality relation is $0 = \int_0^L X_n(x)X_m(x)\,dx$, for $n \neq m$

9.6.4.9. The graph of $g(\omega) \triangleq \sqrt{\dfrac{2}{3}} + \dfrac{\tan(\sqrt{2}\omega)}{\tan(\sqrt{3}\omega)}$, where $\omega \triangleq \sqrt{\lambda}$, shown below, *appears* to give infinitely many eigenvalues $\lambda_n = \omega_n^2$, where $\omega_n \sim n$ as $n \to \infty$.

9.6.4.11. If the first column of the adjugate matrix is nonzero it gives eigenfunctions
$X_n(x) = \big(\gamma_0 \sin(\omega_n b) + \gamma_1\,\omega_n \cos(\omega_n b)\big)\cos(\omega_n x) + \big(-\gamma_0 \cos(\omega_n b) + \gamma_1\,\omega_n \sin(\omega_n b)\big)\sin(\omega_n x)$.

If the second column of the adjugate is nonzero it gives eigenfunctions
$X_n(x) = \big(-\epsilon_0 \sin(\omega_n a) + \epsilon_1\,\omega_n \cos(\omega_n a)\big)\cos(\omega_n x) + \big(\epsilon_0 \cos(\omega_n a) + \epsilon_1\,\omega_n \sin(\omega_n a)\big)\sin(\omega_n x)$.

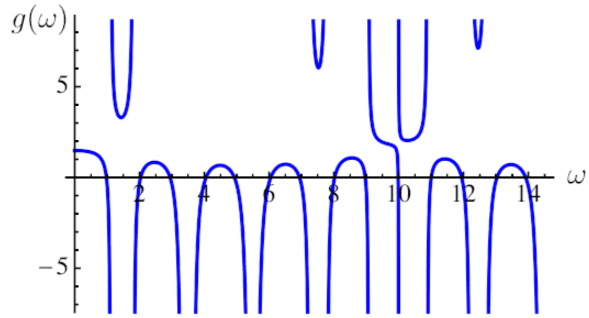


Figure 9.18: Problem 9.6.4.9

Note that at least one of the columns of the adjugate matrix must be a nonzero vector in $\mathbb{R}^2$, because...

**Section 9.7**

9.7.1.1. Hint: Begin by calculating the left hand side, that is, by substituting $\beta X(x)$ into the Rayleigh quotient, as given in (9.83) in Section 9.7:

$$\mathcal{R}_{SL}[\,\beta X(x)\,] \triangleq \frac{-\big[\,p(x)\beta X(x)\beta X'(x)\,\big]_a^b + \displaystyle\int_a^b \big(p(x)\big(\beta X'(x)\big)^2 - q(x)\big(\beta X(x)\big)^2\big)dx}{\displaystyle\int_a^b s(x)\big(\beta X(x)\big)^2 dx} = ...$$

9.7.1.3. An estimate for the minimum eigenvalue $\lambda_1$ would be the minimum of $F(\mu) \triangleq \dfrac{h(1+2\mu L+\mu^2L^2)+\mu^2 L}{L(1+\mu L+\frac{1}{3}\mu^2L^2)}$ over $-\infty<\mu<\infty$, for a given $h$ and $L$.

9.7.1.5. Ex: $\lambda_1 \approx \dfrac{2\pi^2}{L^3}$

9.7.1.7. (a) $\mathcal{R}_{SL}(X) = \dfrac{\displaystyle\int_0^L AC(X'(x))^2dx}{\displaystyle\int_0^L x^2(X(x))^2dx}$

(b) Hint: Use the fact that $X(x)$ satisfies the ODE to calculate $X''(0^+) = \lim_{x\to0^+} X''(x) = ... = 0$.

(c) Hints: Use the fact that $X(x)$ satisfies the ODE to calculate $X''(L^-) = \lim_{x\to L^-} X''(x) = ...$, and explain why there exists a finite number $\alpha$ such that $\alpha = X(L^-) \triangleq \lim_{x\to L^-} X(x)$.

(d) Hint: Take the derivative with respect to $x$ of both sides of the ODE.

(e) Ex: $q(x) = 2-5\left(\dfrac{x}{L}\right)^4 + 3\left(\dfrac{x}{L}\right)^5$ gives estimate for minimum eigenvalue $P_1^2 = \lambda_1 \approx \sqrt{\dfrac{858}{53}}\cdot\dfrac{\sqrt{AC}}{L^2}$

**Section 9.8**

9.8.4.1. Hint: Use work similar to that which established Theorem 9.17, using orthogonality of $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ for positive integers $m \neq n$ to calculate $\int_0^L |f(x)|^2 dx = \langle f(x),\ f(x)\rangle = ...$

9.8.4.3. $Power = L\left(\dfrac{1}{2} + \dfrac{1}{1-e^{-2}} + \dfrac{1}{1-e^{-4}}\right)$

9.8.4.5. <u>Ex</u>.1: Using a square wave function $f(x) = \left\{ \begin{array}{cc} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{array} \right\}$, get $1 = \dfrac{8}{\pi^2}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + ...\right)$, hence $\dfrac{\pi^2}{8} = \displaystyle\sum_{k=1}^{\infty} \dfrac{1}{(2k-1)^2} = \dfrac{1}{1} + \dfrac{1}{9} + \dfrac{1}{25} + \dfrac{1}{49} + ...$

9.8.4.7. Hint: Use convergence of $\dfrac{\pi a_0^2}{2} + \pi\displaystyle\sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right)$ to help explain why $\langle f, \dfrac{1}{\sqrt{\pi}}\sin(nx)\rangle \to 0 = \langle f, 0\rangle$ as $n \to \infty$

# Chapter 10

# Partial Differential Equations Models

**Section 10.1.3**

10.1.3.1. $\dfrac{\partial e}{\partial t}(x,y,t) = -\dfrac{\partial q_x}{\partial x}(x,y,t) - \dfrac{\partial q_y}{\partial y}(x,y,t) + Q(x,y,t)$ is the two space dimensional version of the heat balance equation. Hints: Rewrite $-\oiint_{\mathcal{S}} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS$ as a sum of four double integrals in rectangular coordinates. Integrals $\int_\alpha^\beta \ldots dz = (\beta-\alpha)\cdot(\ldots)$ by using the lack of dependence on $z$. Use results similar to line 4 of page 838.

10.1.3.3. $\dfrac{\partial e}{\partial t}(x,t) = -\dfrac{\partial q_x}{\partial x}(x,t) + \dfrac{2}{R}k(x)\big(u(x,t)-u_0(x,t)\big) + Q(x,t)$ is the one space dimensional version of a heat balance equation modified to include heat loss through the lateral side of the rod by Newton's Law of Cooling. Hints: Rewrite $-\oiint_{\mathcal{S}} \mathbf{q}(\mathbf{r},t) \bullet \widehat{\mathbf{n}}\, dS$ as a sum of three double integrals in an alternative cylindrical coordinates. Integrals $\int_0^R \ldots dr = R\cdot(\ldots)$ by using the lack of dependence on r. Use results similar to line 4 of page 838.

**Section 10.2.4**

10.2.4.1. $u(r) = -\dfrac{1}{4}\left(r^2-a^2\right) + \dfrac{b^2}{2}\ln\left(\dfrac{r}{a}\right) + T_0,\ a<r<b$

10.2.4.3. $u(x) = \bar{T} + (T_0-\bar{T})\cosh(\sqrt{\eta}\,x) + \dfrac{T_1-\bar{T}-(T_0-T)\cosh(\sqrt{\eta}\,L)}{\sinh(\sqrt{\eta}\,L)}\sinh(\sqrt{\eta}\,x)$, for $0<x<L$

10.2.4.5. (a) There is exactly one equilibrium solution when $L \neq \frac{1}{2}\cdot\left(n-\frac{1}{2}\right)\pi$ for *all* integers $n$.
(b) There are infinitely many equilibrium solutions when $L = \frac{1}{2}\cdot\left(n-\frac{1}{2}\right)\pi$ for *any* integer $n$.
(c) There is no no value of $L$ for which there is no equilibrium solution.

10.2.4.7. $u(x) = T_1 + \dfrac{L^2}{\pi^2\kappa}\left(1+\cos\left(\dfrac{\pi x}{L}\right)\right)$

10.2.4.9. The material is homogeneous and has thermal conductivity, specific heat, and mass density all equal to one, in appropriate units. $\mathcal{V} =$ is the lower half of the ball of radius $a$ whose center is at the origin. The problem is circularly symmetric around the $z-$axis. The material is absorbing heat at a rate $Q_0$. The material is insulated on the spherical surface $\rho = a$ for $z \leq 0$. On the $z=0$ disk surface the solid is losing heat at a rate equal to the difference between its temperature and the medium's constant temperature of $20^\circ$, all in appropriate units.The initial temperature of the material is uniformly $40^\circ$.

10.2.4.11. Hints: Explain why the heat flux out of the left end at $x = 0$ is $\widehat{\mathbf{n}} \bullet \mathbf{q}(0,t) = \kappa(0)\frac{\partial u}{\partial x}(0,t)$ and the heat flux out of the right end at $x = L$ is $\widehat{\mathbf{n}} \bullet \mathbf{q}(L,t) = -\kappa(L)\frac{\partial u}{\partial x}(L,t)$.

10.2.4.13. $T = u(b) = \dfrac{\beta}{2\pi\kappa} \ln\left(\dfrac{a}{b}\right)$

10.2.4.15. $u(0^-,t) = v(0^+,t) = w(0^+,t) \quad \text{and} \quad \kappa_1 \dfrac{\partial u}{\partial x}(0^-,t) = \kappa_2 \dfrac{\partial v}{\partial y}(0^+,t) + \kappa_3 \dfrac{\partial w}{\partial x}(0^+,t)$

10.2.4.17. $\displaystyle\int_0^a \frac{\kappa_1}{r} \cdot \frac{\partial u_1}{\partial \theta}\left(r, \frac{\pi}{4}\,{}^+\right) dr = \int_0^a \frac{\kappa_2}{r} \cdot \frac{\partial u_2}{\partial \theta}\left(r, \frac{\pi}{4}\,{}^-\right) dr$

10.2.4.19. $Q_0 = \frac{\beta}{AL}$

10.2.4.21. (b) We will not be able to do this part until Section 11.1.

10.2.4.23. $T = u(b) = 20 - \dfrac{\beta}{4\pi\kappa}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$

**Section 10.3.4**

10.3.4.1. Hint: Integrate Laplace's equation over the interval $[0, L]$...

10.3.4.3. $V(r) = \dfrac{1}{\ln(2.5/1.3)}\left(V_1 \ln\left(\dfrac{r}{1.3}\right) + V_0 \ln\left(\dfrac{2.5}{r}\right)\right)$

10.3.4.5. $k = \dfrac{H^2}{2L}$

10.3.4.7. $0 = a\displaystyle\int_{-\pi}^{\pi} f(\theta)\,d\theta - \int_{-L}^{L} g(x)\,dx$

10.3.4.9. (a) Only for $a = \frac{5}{2}$ is there an equilibrium temperature distribution.
(b) If $a = \frac{5}{2}$, the equilibrium temperature distribution is $u(r) = r^2 + c_1$, where $c_1$ is an arbitrary constant.

10.3.4.11. (a) The solvability condition is $u'(L) = 0$.
(b) rate of heat flow out of the right end is $\mathbf{q}(L) \bullet \widehat{\mathbf{n}} = (-\kappa\boldsymbol{\nabla} u) \bullet (+\hat{\boldsymbol{\imath}}) = -\kappa\frac{du}{dx}(L) = 0$, that is, is zero when the temperature is at equilibrium.

10.3.4.13. $0 = \displaystyle\oiint_{\mathcal{S}} g(\mathbf{r})\,dS$

10.3.4.15. Hints: Begin by using the Calculus I chain rule for the substitution $r = \dfrac{1}{p}$ to get

$$\frac{\partial U}{\partial p}(p,\theta) = \frac{\partial}{\partial p}\Big[\, u\Big(\frac{1}{p},\theta\Big)\Big] = \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) \cdot \frac{\partial r}{\partial p} = \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) \cdot \frac{\partial}{\partial p}\Big[\frac{1}{p}\Big] = -\frac{1}{p^2} \cdot \frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big).$$

Using the product rule,

$$\frac{\partial^2 U}{\partial p^2}(p,\theta) = \frac{\partial}{\partial p}\Big[\frac{\partial U}{\partial p}\Big] = \frac{\partial}{\partial p}\Big[-\frac{1}{p^2}\cdot\frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big)\Big] = \frac{2}{p^3}\cdot\frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big) - \frac{1}{p^2}\cdot\frac{\partial}{\partial p}\Big[\frac{\partial u}{\partial r}\Big(\frac{1}{p},\theta\Big)\Big]...$$

**Section 10.4.6**

10.4.6.1. $\varrho(x)\,\dfrac{\partial^2 y(x,t)}{\partial t^2} = \dfrac{\partial}{\partial x}\left[T(x)\,\dfrac{\partial y(x,t)}{\partial x}\right]$

10.4.6.3. $\varrho \approx 9.87\times 10^{-4}\,kg/m$, to three significant digits

10.4.6.5. $\mathcal{O} = E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

10.4.6.7. Hints: Use (10.75) in Section 10.4, and then Hooke's law, and then (10.74).

10.4.6.9. Hints: Begin with

$$\frac{E}{2(1+\nu)} = \frac{1}{2(1+\nu)}\cdot E = \frac{1}{2(1+\nu)}\cdot\frac{\mu\cdot(3\lambda+2\mu)}{\lambda+\mu} = \frac{1}{2(1+\frac{\lambda}{2(\lambda+\mu)})}\cdot\frac{\mu\cdot(3\lambda+2\mu)}{(\lambda+\mu)} = ...$$

and then $\dfrac{\nu\, E}{(1+\nu)(1-2\nu)} = \dfrac{E}{2(1+\nu)}\cdot\dfrac{2\nu}{(1-2\nu)} = \mu\cdot\dfrac{2\nu}{(1-2\nu)} = ...$

10.4.6.11. Note that the errata page has corrected the formula in line 3 on page 802.

**Section 10.5.5**

10.5.5.1. (a) $u(x,t) = \dfrac{1}{2}\left\{\begin{array}{cc} 0, & x < -2+20t \\ x-20t+2, & -2+20t \le x < 20t \\ 0, & 20t \le x \end{array}\right\} + \dfrac{1}{2}\left\{\begin{array}{cc} 0, & x < -2-20t \\ x+20t+2, & -2-20t \le x < -20t \\ 0, & -20t \le x \end{array}\right\}$

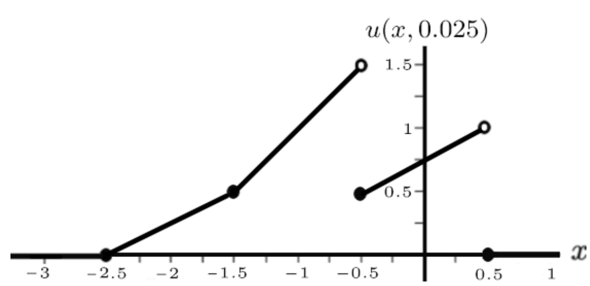


Figure 10.1: Problem 10.5.1: $u(x, 0.025)$

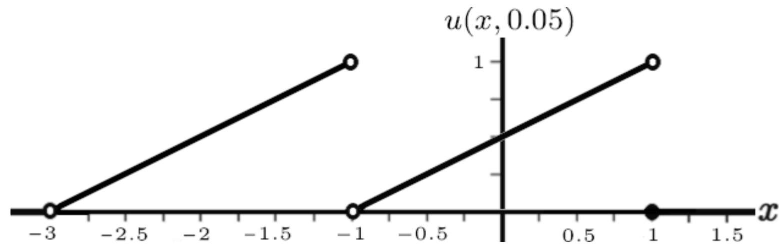


Figure 10.2: Problem 10.5.1: $u(x, 0.05)$

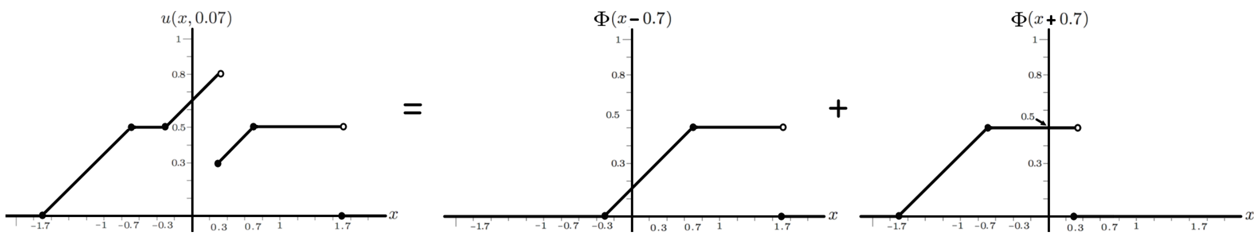


Figure 10.3: Problem 10.5.3: $u(x, 0.07)$

10.5.5.3.

10.5.5.5. (a) $$u(x, 0.025) = \begin{cases} 0, & x < -0.5 \\ 0.5x + 0.25, & -0.5 \le x < 0.5 \\ 0.5, & 0.5 \le x < 1.5 \\ 1.25 - 0.5x, & 1.5 \le x < 2.5 \\ 0, & x \ge 2.5 \end{cases}$$

(b) If $0 < t < \frac{1}{20}$ then $$u(x,t) = \begin{cases} 0, & x < -20t \\ \frac{1}{2}(x + 20t), & -20t \le x < 20t \\ 20t, & 20t \le x < 2 - 20t \\ \frac{1}{2}(2 - x + 20t), & 2 - 20t \le x < 2 + 20t \\ 0, & x \ge 2 + 20t \end{cases}.$$

If $t \ge \frac{1}{20}$, $$u(x,t) = \begin{cases} 0, & x < -20t \\ \frac{1}{2}(x + 20t), & -20t \le x < 2 - 20t \\ 1, & 2 - 20t \le x < 20t \\ \frac{1}{2}(2 - x + 20t), & 20t \le x < 2 + 20t \\ 0, & x \ge 2 + 20t \end{cases}.$$

10.5.5.7. (a) $u(x, 0.025) = \left\{ \begin{array}{ll} 0, & x < 1.5 \\ 0.5x - 0.75, & 1.5 \leq x < 2.5 \\ 1.75 - 0.5x, & 2.5 \leq x < 3.5 \\ 0, & x \geq 2.5 \end{array} \right\}$

(b) If $0 < t < \frac{1}{40}$ then

$$(*) \quad u(x,t) = \left\{ \begin{array}{ll} 0, & x < 2 - 20t \\ \frac{1}{2}(x + 20t - 2), & 2 - 20t \leq x < 2 + 20t \\ 20t, & 2 + 20t \leq x < 3 - 20t \\ \frac{1}{2}(3 - x + 20t), & 3 - 20t \leq x < 3 + 20t \\ 0, & x \geq 3 + 20t \end{array} \right\}.$$

If $t \geq \frac{1}{40}$,

$$u(x,t) = \left\{ \begin{array}{ll} 0, & x < 2 - 20t \\ \frac{1}{2}(x + 20t - 2), & 2 - 20t \leq x < 3 - 20t \\ 1, & 3 - 20t \leq x < 2 + 20t \\ \frac{1}{2}(3 - x + 20t), & 2 + 20t \leq x < 3 + 20t \\ 0, & x \geq 3 + 20t \end{array} \right\}.$$

**Section 10.6.1**

10.6.1.1. Hints: Explain why

$$\overline{PE} = \frac{1}{2L/c} \int_0^{2L/c} PE(t)dt = \frac{c}{2L} \int_0^{2L/c} \left( \frac{T_0 L}{4} \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 \left( a_n \cos\left( \frac{n\pi ct}{L} \right) + b_n \sin\left( \frac{n\pi ct}{L} \right) \right)^2 \right) dt$$

$$= ... = \frac{T_0 \pi^2 L}{8} \sum_{n=1}^{\infty} n^2 \left( a_n^2 + b_n^2 \right) .$$

and

$$KE(t) \triangleq \int_0^L \frac{1}{2} \varrho_0 \left( \frac{\partial u}{\partial t} \right)^2 dx = \frac{\varrho_0 L}{4} \sum_{n=1}^{\infty} \left( \frac{n\pi c}{L} \right)^2 \left( -a_n \sin\left( \frac{n\pi ct}{L} \right) + b_n \cos\left( \frac{n\pi ct}{L} \right) \right)^2 .$$

$$= ... = \frac{\varrho_0 c^2 \pi^2 L}{8} \sum_{n=1}^{\infty} n^2 \left( a_n^2 + b_n^2 \right) .$$

10.6.1.3. Hints: Do calculations that explain why it will be enough to explain why the sum

$$S = \varrho_0 c^2 \left( -a_n \sin\left( \frac{n\pi ct}{L} \right) + b_n \cos\left( \frac{n\pi ct}{L} \right) \right)^2 + T_0 \left( a_n \cos\left( \frac{n\pi ct}{L} \right) + b_n \sin\left( \frac{n\pi ct}{L} \right) \right)^2$$

is constant in time.

# Chapter 11

# Separation of Variables for PDEs

**Section 11.1.4**

11.1.4.1. $T(x,t) = e^{-\alpha t}\sin x - \frac{1}{2}e^{-9\alpha t}\sin 3x$

11.1.4.3. $(x,t) = x + \frac{\pi}{2} - \frac{4}{\pi}\sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos\big((2k-1)x\big)\, e^{-\alpha(2k-1)^2 t}$

11.1.4.5. $T(x,t) = \frac{5}{2} - \dfrac{20}{\pi^2}\sum\limits_{k=1}^{\infty} \dfrac{1}{(2k-1)^2}\cos\left(\dfrac{(2k-1)\pi x}{5}\right) e^{-1.15\times10^{-4}\left(\frac{(2k-1)\pi}{5}\right)^2 t}$

11.1.4.7. $T(x,t) = \frac{1}{2\alpha}(x^2-4x) + \frac{8}{\pi^3\alpha}\sum\limits_{n=1}^{\infty} \frac{1}{\left(n-\frac{1}{2}\right)^3}\sin\left(\frac{\left(n-\frac{1}{2}\right)\pi x}{2}\right) e^{-\alpha\left(\frac{(n-\frac{1}{2})\pi}{2}\right)^2 t}$

11.1.4.9. $T(x,t) = 10x + \sum\limits_{n=1}^{\infty}\left(\frac{80(-1)^n L}{\pi^2(n-\frac{1}{2})^2} + \frac{320\left(\sqrt{2}(-1)^n + \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)}{\pi(4n^2-4n-15)}\right)\sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) e^{-\alpha\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2 t}$

11.1.4.11. $T(x,t) = \cos(x) - \frac{\pi}{2} + \sum\limits_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}\cos((2k-1)x)e^{-(2k-1)^2 t}$

11.1.4.13. $T(x,t) = \sum\limits_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin nx\,dx\right)(t+1)^{-\alpha n^2}\sin nx$

11.1.4.15. $T(x,t) = \frac{4}{\pi}\sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)\left(\left(\frac{(2k-1)\pi}{L}\right)^2 - 1\right)}\left(e^{-t} - e^{-\left(\frac{(2k-1)\pi}{L}\right)^2 t}\right)\sin\left(\frac{(2k-1)\pi x}{L}\right)$

11.1.4.17. $T(x,t) = \frac{4}{\pi}\sum\limits_{k=1}^{\infty} \frac{1}{2k-1}\sin\left(\frac{(2k-1)\pi x}{L}\right)\int_0^t e^{-\alpha\left(\frac{(2k-1)\pi}{L}\right)^2 (t-s)}\, g(s)ds$

11.1.4.19. (a) The eigenvalues/eigenfunctions are $(i)$ $\lambda = \frac{2k\pi}{2L},\ X_{2k}(x) = \sin\left(\frac{2k\pi x}{2L}\right)$ and $(ii)$ $\lambda = \frac{(2k-1)\pi}{2L},\ X_{2k-1}(x) = \cos\left(\frac{(2k-1)\pi x}{2L}\right)$.

(b) $T(x,t) = \frac{400}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \cos\Big(\frac{(2k-1)\pi x}{2L}\Big) e^{-\alpha\left(\frac{(2k-1)\pi}{2L}\right)^2 t} + \frac{400}{\pi} \sum_{\ell=1}^{\infty} \frac{1}{(2\ell-1)} \sin\Big(\frac{(2\ell-1)\pi x}{L}\Big) e^{-\alpha\left(\frac{(2\ell-1)\pi}{L}\right)^2 t}$

11.1.4.21. $T(x,t) = v(x) + w(x,t) = v(x) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(\beta - \alpha\left(\frac{n\pi}{L}\right)^2\right)t}$, where the $b_n$'s are arbitrary constants that depend on the initial condition.

In order for all solutions to go to zero as $t \to \infty$, it is necessary and sufficient that $\beta < \alpha\left(\frac{\pi}{L}\right)^2$. The time constant is $\tau = \frac{1}{\alpha\left(\frac{\pi}{L}\right)^2 - \beta}$.

**Section 11.2.1**

11.2.1.1. $u(x,t) = \frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{n\pi ct}{3}\right) \sin\left(\frac{n\pi x}{3}\right)$

11.2.1.3. $u(x,t) = \cos\left(\frac{\pi ct}{2L}\right) \sin\left(\frac{\pi x}{2L}\right) - \frac{1}{3} \cos\left(\frac{5\pi ct}{2L}\right) \sin\left(\frac{5\pi x}{2L}\right)$

11.2.1.5. $u(x,t) = \dfrac{4}{\pi} \cos\left(\dfrac{\pi ct}{L}\right) \sin\left(\dfrac{\pi x}{L}\right) - \dfrac{8}{\pi} \sum_{k=1}^{\infty} \dfrac{k}{4k^2-1} \cos\left(\dfrac{2k\pi ct}{L}\right) \sin\left(\dfrac{2k\pi x}{L}\right)$

$$+\frac{4}{\pi} \sum_{\ell=2}^{\infty} \frac{1}{2\ell-1} \cos\left(\frac{(2\ell-1)\pi ct}{L}\right) \sin\left(\frac{(2\ell-1)\pi x}{L}\right)$$

11.2.1.7. $u(x,t) = \frac{4L}{\pi^2 c} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2-4} \sin\left(\frac{(2k-1)\pi ct}{L}\right) \sin\left(\frac{(2k-1)\pi x}{L}\right)$

11.2.1.9. $u(x,t) = \delta(t) + \big(-\delta(t) + \epsilon(t)\big) \frac{x}{L} +$

$$+\frac{2}{L} \sum_{n=1}^{\infty} \left( \left( \int_0^L f(x) \sin nx\, dx \right) \cos\left(\frac{2n\pi t}{L}\right) + \left( \int_0^L g(x) \sin nx\, dx \right) \sin\left(\frac{2n\pi t}{L}\right) + \phi_n(t) \right) \sin\left(\frac{n\pi x}{L}\right)$$

$$-\frac{4}{L} \sum_{n=1}^{\infty} \left( \int_0^t \sin\left(\frac{2n\pi}{L}(t-s)\right) \big(-\ddot{\delta}(s) + (-1)^n \ddot{\epsilon}(s)\big) ds \right) \sin\left(\frac{n\pi x}{L}\right)$$

11.2.1.11. $u(x,t) = e^{-t} \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$, where $a_n, b_n$ are arbitrary constants

11.2.1.13. $a_n = 0$ for all $n$ of the form $n = 3k-1$ or $n = 3k-2$ for some integer $k \geq 1$

11.2.1.15. (a) $\lambda = 0$ is not an eigenvalue

(b) $\lambda > 0$ cannot be an eigenvalue because, if it were, $0 = \int_0^L X(x) X''''(x)\, dx + \int_0^L \lambda \big(X(x)\big)^2 dx \int_0^L \big(X''(x)\big)^2 dx + \int_0^L \lambda \big(X(x)\big)^2 dx$ ...

(c) There is an eigenfunction $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ only corresponding to eigenvalues $\lambda_n = -\left(\frac{n\pi}{L}\right)^4$.

11.2.1.17. $y(x,t)$ has steady state oscillations in time, $t$

**Section 11.3.3**

11.3.3.1. $T(x,y) = \dfrac{40}{\pi} \sum\limits_{k=1}^{\infty} \dfrac{\cosh\big((2k-1)x\big)}{(2k-1)^2 \sinh(2(2k-1)\pi)} \sin\big((2k-1)y\big)$

11.3.3.3. $T(x,y) = \frac{80}{\pi} \sum\limits_{k=1}^{\infty} \left( \frac{5}{2k-1} \cosh\Big(\frac{(2k-1)y}{2}\Big) - \frac{1}{(2k-1)^2} \sinh\Big(\frac{(2k-1)y}{2}\Big)\right) \sin\left(\frac{(2k-1)x}{2}\right)$.

11.3.3.5.
$$T(x,y) = \sum_{n=3}^{10} \frac{\alpha_n}{\sinh\left(\frac{2n-1}{2}\right)} \sinh\left(\frac{(2n-1)y}{2}\right) \sin\left(\frac{(2n-1)x}{2}\right) + \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\sinh\left(\frac{2n-1}{2}\right)} \sinh\left(\frac{(2n-1)(1-y)}{2}\right) \sin\left(\frac{(2n-1)x}{2}\right)$$

11.3.3.7. One method gives solution $T(x,y) = 20 + u(x,y) = 20 - \frac{80}{\pi} \sum\limits_{k=1}^{\infty} \frac{\sinh\left(\frac{(2k-1)\pi(a-x)}{b}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi a}{b}\right)} \sin\left(\frac{(2k-1)\pi y}{b}\right)$
.

Another method gives solution $T(x,y) = T_1(x,y) + T_2(x,y)$

$$= \frac{80}{\pi} \sum_{k=1}^{\infty} \left( \frac{\sinh\left(\frac{(2k-1)\pi x}{b}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi a}{b}\right)} \sin\left(\frac{(2k-1)\pi y}{b}\right) + \frac{\sinh\left(\frac{(2k-1)\pi(b-y)}{a}\right) + \sinh\left(\frac{(2k-1)\pi y}{a}\right)}{(2k-1)\sinh\left(\frac{(2k-1)\pi b}{a}\right)} \sin\left(\frac{(2k-1)\pi x}{a}\right)\right)$$

11.3.3.9. $T(x,y) = \dfrac{ay}{4b} - \dfrac{2a}{\pi^2} \sum\limits_{\ell=1}^{\infty} \dfrac{1}{(2\ell-1)^2 \sinh\left(\frac{2(2\ell-1)\pi b}{a\sqrt{3}}\right)} \sinh\left(\dfrac{2(2\ell-1)\pi}{a\sqrt{3}}\, y\right) \cos\left(\dfrac{2(2\ell-1)\pi x}{a}\right)$

11.3.3.11. $T(x,y) = 1 - \frac{\cosh(2\sqrt{2}\,y)}{3\cosh(2\sqrt{2})} \cos 2x + \frac{\cosh(5\sqrt{2}\,y)}{5\cosh(5\sqrt{2})} \cos 5x$

11.3.3.13. (a) The solution of the problem can be written in the form

$$T(x,y) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \left( \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx \right) \sin\left(\frac{n\pi x}{a}\right).$$

(b) $T(x,y) = \int_0^a G(x,y,z)\, f(z)\, dz$, where $G(x,y,z) \triangleq \frac{2}{a} \sum\limits_{n=1}^{\infty} \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi z}{a}\right)$. The latter is called a "Green's function" or "propagator."

**Section 11.4.6**

11.4.6.1. eigenvalues are $\lambda_{m,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$, $m = 0, 1, ...; n = 1, 2, ...$ with corresponding eigenfunctions $\phi_{m,n}(x,y) = \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$

11.4.6.3. $T(x,y,t) = -\frac{16}{\pi} \sin\left(\frac{x}{2}\right) \sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)((2k-1)^2-4)} e^{-\alpha\left(\left(\frac{\pi}{2}\right)^2 + \left(\frac{(2k-1)\pi}{2}\right)^2\right)t} \sin\left(\frac{(2k-1)\pi y}{2}\right)$

11.4.6.5. $T(x,y) = -\dfrac{16}{\kappa\,\pi^2} \displaystyle\sum_{\substack{k=1\\ \ell=1}}^{\infty} \frac{1}{(2k-1)(2\ell-1)\left(\left(\frac{(2k-1)\pi}{a}\right)^2+\left(\frac{(2\ell-1)\pi}{b}\right)^2\right)} \sin\left(\tfrac{(2k-1)\pi x}{a}\right)\sin\left(\tfrac{(2\ell-1)\pi y}{b}\right)$

11.4.6.7. $T(x,y,z) = \dfrac{2}{bc} \displaystyle\sum_{m=1}^{\infty} \frac{1}{\sinh\left(\frac{m\pi a}{b}\right)} \left(\int_0^b \int_0^c f(y,z)\sin\left(\frac{m\pi y}{b}\right)dz\,dy\right)\sinh\left(\frac{m\pi x}{b}\right)\sin\left(\frac{m\pi y}{b}\right)$

$+\frac{4}{bc} \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\sinh\left(\sqrt{\lambda_{m,n}}\,a\right)} \left(\int_0^b\int_0^c f(y,z)\sin\left(\tfrac{m\pi y}{b}\right)\cos\left(\tfrac{n\pi z}{c}\right)dzdy\right)\sinh\left(\sqrt{\lambda_{m,n}}\,x\right)\sin\left(\tfrac{m\pi y}{b}\right)\cos\left(\tfrac{n\pi z}{c}\right),$

where $\lambda_{m,n} = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2$.

11.4.6.9. $u(x,y) = v(y) + w(x,y) = \dfrac{17}{8} + \dfrac{\pi}{4}\,y - \dfrac{1}{2}\,y^2 - \dfrac{3}{8}\,\cos 4y$

$$+\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{8(2k-1)^2\sinh\big(2(2k-1)\pi^2\big)} \left(\pi\cos\left(\frac{(2k-1)\pi}{2}\right) + \frac{2(-1)^{k-1}(4+33(2k-1)^2-7(2k-1)^4}{(2k-1)(-4+(2k-1)^2)}\right)\cdot$$

$$\cdot\Big(\sinh\big(2(2k-1)\pi(\pi-x)\big) + \sinh(2(2k-1)\pi x)\Big)\sin(2(2k-1)y)$$

**Section 11.5.3**

11.5.3.1. $u(r,\theta) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\left(\frac{r}{a}\right)^{2k-1}\cos\big((2k-1)\theta\big)$

11.5.3.3. $V(r,\theta) = \frac{1}{\ln(b/a)}\left(V_a\ln b - V_b\ln a + (V_b - V_a)\ln r\right)$, which can be rewritten as

$$V(r,\theta) = \frac{1}{\ln(b/a)}\left(V_a(\ln b - \ln r) + V_b(\ln r - \ln a)\right) = \frac{1}{\ln(b/a)}\left(V_a\ln\left(\frac{b}{r}\right) + V_b\ln\left(\frac{r}{a}\right)\right)$$

11.5.3.5. (a) eigenvalues are $\lambda_n = \omega_n^2 = n^2$, with corresponding eigenfunctions $\Theta_{2k}(\theta) = \sin 2k\theta$ and $\Theta_{2k-1}(\theta) = \cos\big((2k-1)\theta\big)$; product solutions are $u_n(r,\theta) = R_n(r)\Theta_n(\theta)$, where $r(rR_n')' - n^2R = 0$. ...

(b) $u(r,\theta) = \displaystyle\sum_{k=1}^{\infty}\left(\frac{2}{2k-1}(-1)^{k-1} - \frac{4}{\pi(2k-1)^2}\right)\cdot\left(\frac{r}{a}\right)^{2k-1}\cdot\cos\big((2k-1)\theta\big)$

11.5.3.7. $T(r,z) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{M_n\sinh\left(\frac{\gamma_{n,1}L}{a}\right)} J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)\cdot$

$$\cdot\left(\left(\int_0^a f(r)\,J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)r\,dr\right)\sinh\left(\frac{\gamma_{n,1}}{a}(L-z)\right) + \left(\int_0^a g(r)\,J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)r\,dr\right)\sinh\left(\frac{\gamma_{n,1}}{a}\,z\right)\right),$$

where $M_n \triangleq \displaystyle\int_0^a \left(J_0\left(\frac{\gamma_{n,1}\,r}{a}\right)\right)^2 r\,dr$.

11.5.3.9. (a) $0 = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left[r^2\,\dfrac{\partial T}{\partial r}\right] + \dfrac{1}{r}\dfrac{\partial^2 T}{\partial\theta^2}$

(b) $T(r,\theta) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\,d\theta +$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\left(\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta\right)\cos n\theta + \left(\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta\right)\sin n\theta\right)\left(\frac{r}{a}\right)^{-\frac{1}{2}+\sqrt{n^2+\frac{1}{4}}}$$

11.5.3.11. $T(r,z) = \displaystyle\sum_{n=1}^{\infty} \frac{200 J_1(\gamma_{n,0})}{\gamma_{n,0}} \cdot J_0\Big(\frac{\gamma_{n,0}\, r}{2}\Big) \cdot \frac{\sinh\big(\frac{\gamma_{n,0}}{2}(4-z)\big)}{\sinh\big(\frac{4\gamma_{n,0}}{2}\big)}$

11.5.3.13. $u(r,\theta) = \dfrac{c_0 \ln(r/a) + \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\, d\theta}{2}$

$$-\sum_{n=1}^{\infty} \frac{b^n}{\pi(b^n - a^n)} \cdot \left(\left(\frac{r}{b}\right)^n - \left(\frac{a}{r}\right)^n\right) \left(\left(\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta\right)\cos n\theta + \left(\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta\right)\sin n\theta\right),$$

where $c_0$ is an arbitrary constant

**Section 11.6.3**

11.6.3.1. $T(r,\theta,z) =$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{2\sinh(\sqrt{\lambda_{m,n}}(H-z))}{\pi \sinh(\sqrt{\lambda_{m,n}}\, H)} \cdot \frac{\displaystyle\int_0^{\pi}\int_0^{a} f(r,\theta)\sin(n\theta) J_n(\sqrt{\lambda_{m,n}}\ r) r\, dr d\theta}{\displaystyle\int_0^{a} (J_n(\sqrt{\lambda_{m,n}}\ r))^2 r\, dr} J_n(\sqrt{\lambda_{m,n}}\ r)\sin(n\theta)$$

11.6.3.3. $T(r,\theta,t) \;=\; \displaystyle\sum_{m=1}^{\infty} \frac{a_{m,0}}{2} R_{m,0}(r) e^{-\alpha\lambda_{m,0}t} \;+\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{m,n}(r)\left(a_{m,n}\cos n\theta + b_{m,n}\sin n\theta\right) e^{-\alpha\lambda_{m,n}t}$,

where

$$a_{m,0} = \frac{1}{\pi N_{m,0}} \int_{-\pi}^{\pi}\int_0^a f(r,\theta) R_{m,0}(r)\, r\, dr\, d\theta$$
$$a_{m,n} = \frac{1}{\pi N_{m,n}} \int_{-\pi}^{\pi}\int_0^a f(r,\theta) R_{m,n}(r)\cos(n\theta) r\, dr\, d\theta$$

and

$$b_{m,n} = \frac{1}{\pi N_{m,n}} \int_{-\pi}^{\pi}\int_0^a f(r,\theta) R_{m,n}(r)\sin(n\theta) r\, dr\, d\theta,$$

where, for $n \ge 0$, $m \ge 1$,

$$N_{m,n} = \int_0^a \Big(R_{m,n}(r)\Big)^2 r\, dr \triangleq \int_0^a \Big(-Y_n(\sqrt{\lambda_{m,n}}\ a) J_n(\sqrt{\lambda_{m,n}}\ r) + J_n(\sqrt{\lambda_{m,n}}\ a) Y_n(\sqrt{\lambda_{m,n}}\ r)\Big)^2 r\, dr$$

11.6.3.5. (a) $\left\{\begin{array}{l} 0 = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left[r\dfrac{\partial T}{\partial r}\right] + \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial\theta^2} + \dfrac{\partial^2 T}{\partial z^2},\ 0<\theta<\pi,\ 0\le r<a,\ 0<z<H \\ T(a,\theta,z) = 0,\ 0<\theta\le\pi,\ 0<z<H \\ \dfrac{\partial T}{\partial\theta}(r,0,z) = \dfrac{\partial T}{\partial\theta}(r,\pi,z) = 0,\ 0\le r<a,\ 0<z<H \\ \dfrac{\partial T}{\partial z}(r,\theta,0) = 0,\ T(r,\theta,H) = f(r,\theta),\ 0<\theta<\pi,\ 0\le r<a \end{array}\right\}.$

(b) $T(r,\theta,z) = \displaystyle\sum_{m=1}^{\infty} \left(\frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_0(\sqrt{\lambda_{m,0}}\, r) r dr\, d\theta}{2\pi \displaystyle\int_0^a (J_0(\sqrt{\lambda_{m,0}}\, r))^2 r\, dr}\right) \frac{\cosh(\sqrt{\lambda_{m,0}}\, z)}{\cosh(\sqrt{\lambda_{m,0}}\, H)} J_0(\sqrt{\lambda_{m,0}}\, r) +$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(\frac{\displaystyle\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\cos n\theta J_n(\sqrt{\lambda_{m,n}}\, r) r\, dr\, d\theta}{\pi\displaystyle\int_0^a (J_n(\sqrt{\lambda_{m,n}}\, r))^2 r\, dr}\right) \frac{\cosh(\sqrt{\lambda_{m,n}}\, z)}{\cosh(\sqrt{\lambda_{m,n}}\, H)} J_n(\sqrt{\lambda_{m,n}}\, r)\cos n\theta$$

11.6.3.7. $u(r,\theta,t) = \dfrac{1}{c} \sum_{m=1}^{\infty} \dfrac{1}{N_{m,2}\sqrt{\lambda_{m,2}}} \left( \int_0^a \left(1 - \dfrac{r}{a}\right) J_2(\sqrt{\lambda_{m,2}}\, r)\, r\, dr \right) J_2(\sqrt{\lambda_{m,2}}\, r) \cos(2\theta) \sin(\sqrt{\lambda_{m,2}}\, ct)$

11.6.3.9. (a) Hint: Use the sine addition formula, $\sin(\alpha \pm \beta) = \sin\alpha\, \cos\beta \pm \cos\alpha\, \sin\beta$ to get $\sin\left(\frac{\pi}{2} + \varphi\right) \equiv \sin\left(\frac{\pi}{2} - \varphi\right)$ .

(b) First, explain why a function $h(\phi)$ being odd about $\phi = \frac{\pi}{2}$, implies $\displaystyle\int_0^\pi h(\phi)\, d\phi = 0$. After that, use the cosine addition formula, $\cos(\alpha \pm \beta) = \cos\alpha\, \cos\beta \mp \sin\alpha\, \sin\beta$, to get $\cos\left(\frac{\pi}{2} + \varphi\right) \equiv -\cos\left(\frac{\pi}{2} - \varphi\right)$. So, $\cos(\phi)$ is odd about $\phi = \frac{\pi}{2}$. Also, get that for all odd $n$, $a_n = 0$.

11.6.3.11. $V(\rho,\phi) = \dfrac{10}{3} - \dfrac{5}{27}\rho^2(3\cos^2\phi - 1)$

# Chapter 12

# Numerical Methods II

**Section 12.1.5**

12.1.5.1. (a) Mathematica says that the eigenvalues of $A$ are $\lambda \approx 0.877830, -0.701611, 0.0881093$, all of which have magnitude less than one.

(b) This discretization of the PDE does not give a decay rate that agrees with the decay rate of solutions of the PDE.

12.1.5.3. (a) Mathematica says that the eigenvalues of $A$ are $\lambda \approx 1., -0.823781, 0.544055, -0.367836$. Three of the eigenvalues have magnitude less than one, which corresponds to those solutions of the PDE with corresponding *homogeneous* BCs that are transient in time. The eigenvalue $\lambda = 1$ corresponds to a particular solution $T_P(x,t)$ of the form $T_P(x,t) = xf(t) + (\pi - x)g(t) + w(x,t)$ that satisfies the BCs $\frac{\partial T}{\partial x}(0,t) = \phi(t), \frac{\partial T}{\partial x}(\pi,t) = \psi(t),\ t > 0$ and has $w(x,t) \to 0$ as $t \to \infty$.

(b) This discretization of the PDE does not give a decay rate that agrees with the decay rate of solutions of the PDE.

12.1.5.5. Yes, there are some non-zero initial temperature distributions, $\mathbf{T}^0$, for which $||\mathbf{T}^m|| \to 0,$ as $m \to \infty$.

Ex: For Example 12.2, which has $\triangle x = \frac{\pi}{6}$, let $\mathbf{T}^0 = \begin{bmatrix} T_1^0 \\ T_2^0 \\ T_3^0 \\ T_4^0 \\ T_5^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

**Section 12.2.2**

12.2.2.1. (a) Hints: Use the assumption $\epsilon > 0$, along with $-1 < \cos \frac{n\pi}{N} < 1$ for $n = 1, ..., N-1$, to conclude that (1) $1 + 2\epsilon > 1 - \epsilon\left(-1 + \cos \frac{n\pi}{N}\right)$ and (2) $1 - \epsilon\left(-1 + \cos \frac{n\pi}{N}\right) > 1$. Similarly analyze the numerator of $Q_n$. Eventually, conclude that $Q_n < 1$.

To explain why $Q_n > -1$, suppose instead $Q_n \le -1$, that is, $\frac{1+\epsilon\left(-1+\cos \frac{n\pi}{N}\right)}{1-\epsilon\left(-1+\cos \frac{n\pi}{N}\right)} \le -1$, and eventually get a contradiction.

(b) Because the amplification factor $Q_n$ has magnitude less than one, the Crank-Nicholson method is stable for all $\epsilon > 0$.

12.2.2.3. Hints: If $\epsilon < 0$, define $\beta = -\epsilon > 0$. Then $Q_n = ... = \frac{1-\beta\left(-1+\cos \frac{n\pi}{N}\right)}{1+\beta\left(-1+\cos \frac{n\pi}{N}\right)} \triangleq \frac{1}{R_n}$. Eventually, conclude that $-1 < R_n < 1$, for $n = 1, ..., N-1$.

## Section 12.3.2

Throughout Section 12.3.2, define $u_{ij} \triangleq u(x_i, y_j)$, where $x_i \triangleq i \,\triangle x$, $y_j \triangleq j \,\triangle y$.

12.3.2.1. Approximate solution is
$$\begin{array}{cccc} & 0 & 0 & \\ 0 & \frac{1}{8} & \frac{3}{8} & 1 \\ 0 & \frac{1}{8} & \frac{3}{8} & 1 \\ & 0 & 0 & \end{array}$$
, that is, $u(\frac{1}{3}, \frac{1}{3}) = u(\frac{1}{3}, \frac{2}{3}) \approx \frac{1}{8}$, $u(\frac{2}{3}, \frac{1}{3}) = u(\frac{2}{3}, \frac{2}{3}) \approx \frac{3}{8}$

12.3.2.3. Approximate solution is
$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11}{256} & \frac{14}{256} & \frac{11}{256} & 0 \\ 0 & \frac{14}{256} & \frac{126}{256} & \frac{14}{256} & 0 \\ 0 & \frac{11}{256} & \frac{14}{256} & \frac{11}{256} & 0 \\ 0 & 0 & 0 & 0 & \end{array}$$
, that is,

$u(\frac{1}{4}, \frac{1}{4}) = u(\frac{1}{4}, \frac{3}{4}) = u(\frac{3}{4}, \frac{3}{4}) = u(\frac{3}{4}, \frac{1}{4}) \approx \frac{11}{256}$, $u(\frac{1}{4}, \frac{2}{4}) = u(\frac{2}{4}, \frac{3}{4}) = u(\frac{2}{4}, \frac{1}{4}) = u(\frac{3}{4}, \frac{2}{4}) \approx \frac{14}{256}$, $u(\frac{2}{4}, \frac{2}{4}) \approx \frac{18}{256}$

12.3.2.5. Method I: Using the forward difference approximation of the BC at $y = \pi$, get an approximate solution
$$\begin{array}{cccc} & 5.752814251 & 5.752814251 & \\ 4 & 4.705616700 & 4.705616700 & 4 \\ 4 & 3.267413137 & 3.267413137 & 4 \\ & 0 & 0 & \end{array}$$
, that is,

$u(\frac{\pi}{3}, \frac{\pi}{3}) = u(\frac{2\pi}{3}, \frac{\pi}{3}) \approx 3.267413147$, $u(\frac{\pi}{3}, \frac{2\pi}{3}) = u(\frac{2\pi}{3}, \frac{2\pi}{3}) \approx 4.705616700$, $u(\frac{\pi}{3}, \pi) = u(\frac{2\pi}{3}, \pi) \approx 5.752814251$

Answer Method II: Using central difference approximation of the derivative at $y = \pi$, approximate solution is
$$\begin{array}{cccc} 4 & 5.461173567 & 5.461173567 & 4 \\ 4 & 4.596251443 & 4.596251443 & 4 \\ 4 & 3.230958051 & 3.230958051 & 4 \\ & 0 & 0 & \end{array}$$
, that is,

$u(\frac{\pi}{3}, \frac{\pi}{3}) = u(\frac{2\pi}{3}, \frac{\pi}{3}) \approx 3.230958051$, $u(\frac{\pi}{3}, \frac{2\pi}{3}) = u(\frac{2\pi}{3}, \frac{2\pi}{3}) \approx 4.596251443$, $u(\frac{\pi}{3}, \pi) = u(\frac{2\pi}{3}, \pi) \approx 5.461173567$

12.3.2.7. Using a entral difference of BC at $y=0$ and a central difference approximation of BC at $y=\pi$ gives approximate solution
$$\begin{array}{cccc} 0 & 1.614258153 & 1.614258153 & 0 \\ 0 & 1.209543477 & 1.209543477 & 0 \\ 0 & 2.014372277 & 2.014372277 & 0 \\ 0 & 4.833573356 & 4.833573356 & 0 \end{array}$$
, that is,
$u(\frac{\pi}{3},0)=u(\frac{2\pi}{3},0)\approx 4.833573356$, $u(\frac{\pi}{3},\frac{\pi}{3})=u(\frac{2\pi}{3},\frac{\pi}{3})\approx 2.014372277$,
$u(\frac{\pi}{3},\frac{2\pi}{3})=u(\frac{2\pi}{3},\frac{2\pi}{3})\approx 1.209543477$, $u(\frac{\pi}{3},\pi)=u(\frac{2\pi}{3},\pi)\approx 1.614258153$.

**Section 12.4.3**

12.4.3.3. Hint: First, get $Q=1-\epsilon+\epsilon e^{-i\alpha}=1-\epsilon+\epsilon\cos\alpha-i\epsilon\sin\alpha$

12.4.3.5. (a) there is numerical stability for $c<0$, (b) $c>0$ and $|\epsilon|<1$ implies numerical instability Strangely, if $c>0$ and $\epsilon>1$ then there is numerical stability.

**Section 12.5**

12.5.4.1. Using the same three functions $\phi_j(x)$ as in Example 12.7, we get approximate solution
$$y(x)\approx 0.187969\sin(\pi x)-0.00711276\sin(2\pi x)+0.00704504\sin(3\pi x).$$

12.5.4.3. Using $\phi_1(x)=\cos\left(\frac{\pi x}{2}\right)$, $\phi_2(x)=\cos\left(\frac{3\pi x}{2}\right)$, $\phi_3(x)=\cos\left(\frac{5\pi x}{2}\right)$, get approximate solution
$$y(x)\approx 0.625314\cos\left(\frac{\pi x}{2}\right)-0.0357342\cos\left(\frac{3\pi x}{2}\right)+0.00777959\cos\left(\frac{5\pi x}{2}\right).$$

12.5.4.5. Using
$$\phi_1(x,y)=\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right),\quad \phi_2(x,y)=\cos\left(\frac{3\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right),\quad \phi_3(x,y)=x^2y^2(1-x)(1-y),$$
the approximate solution is
$$u(x,y)=0.35531161853\cos\left(\tfrac{\pi x}{2}\right)\cos\left(\tfrac{\pi y}{2}\right)-0.013588155903\cos\left(\tfrac{3\pi x}{2}\right)\cos\left(\tfrac{\pi y}{2}\right)$$
$$+1.9155528075x^2y^2(1-x)(1-y).$$

# Chapter 13

# Optimization

**Section 13.1.3**

13.1.3.1. the best piece of paper has length $4(1+\sqrt{5})$ in and width $2(1+\sqrt{5})$ in

13.1.3.3. the strongest beam has height $\frac{\sqrt{2}}{\sqrt{3}}m$ and width $\frac{1}{\sqrt{3}}$

13.1.3.5. the hospital should expect the greatest influx of patients after 3 days and about 11 hours

13.1.3.7. the sides of the best triangle have lengths $20\sqrt{5}$, $20\sqrt{5}$, and $20\sqrt{10}$, in m

13.1.3.9. Hints: (a) To explain why $f'' \ge 0$ on $[\,a,b\,]$ implies $f(x)$ is convex on $[\,a,b\,]$, pick any $x<y$ in $[\,a,b\,]$ and any $\lambda$ with $0<\lambda<1$ and define $z \triangleq \lambda x+(1-\lambda)y$. Use the Mean Value Theorem of Calculus I on each of the intervals $[x,z]$ and $[z,y]$... .

13.1.3.11. Hint: Begin by considering the case when there exist distinct $x,y$ in $S$. For any $\lambda$ with $0<\lambda<1$, $f(x)$ being convex helps explain why $f(\lambda x+(1-\lambda)y) \le M$...

13.1.3.13. Partial hint: Explain why

$$f''(x) = \lim_{h\to 0} \frac{f(x+h+h)-f(x+h)-f(x+h)+f(x)}{h\cdot h} = \lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$$

and then use it

**Section 13.2.3**

13.2.3.1. (a) all of the critical points are $(0,0)$, $(-3,\pm\sqrt{6})$,
(b) local minimum at $(0,0)$, saddle points at $(-3,\pm\sqrt{6})$

13.2.3.3. (a) $(x,y)=(0,0),(0,2),(-1,1),(1,1)$,
(b) saddle points at $(-1,1)$, $(1,1)$; local maximum at $(0,0)$; local minimum at $(0,2)$

13.2.3.5. the absolute max. value $2+\sqrt{2}$ is achieved at $(x,y)=(\sqrt{2},0)$

13.2.3.7. absolute maximum of $f$ is 8, achieved at $(x,y)=(\pm 2,0)$; absolute minimum of $f$ is 0, achieved at $(x,y)=(0,0)$

13.2.3.9. Further hints: Use the quadratic formula to explain why the eigenvalues are real. After that, either use the fact that $|A|$ is the product of the two eigenvalues or use the quadratic formula further.

13.2.3.11. Hint: Begin by calculating that $f(\mathbf{x}) = (A\mathbf{x}-\mathbf{b})^T W^T (A\mathbf{x}-\mathbf{b}) = ... = \mathbf{x}^T A^T W^T A\mathbf{x} - 2\mathbf{b}^T W^T A\mathbf{x} + \mathbf{b}^T W^T \mathbf{b}$ and use the results of problem 6.7.6.33 and problem 13.2.3.17.

13.2.3.13. Begin by noting that Example 13.5 in Section 13.2 concluded that $(x, y) = (\frac{3}{4}, -\frac{3}{16})$ is the point on the curve $y = x^2 - x$ closest to the curve $Y = \frac{1}{2}X - 6$. Example 13.5 in Section 13.2 gave $2(Y - y) = 2\mu = 2\big(-2(X - x)\big)$, hence ...

13.2.3.15. Hints: First use $\boldsymbol{\nabla} f = \mathbf{0}$ and then use logical analysis in solving that system of three equations for $h, w, \ell$.

13.2.3.17. Hint: $\mathbf{c}^T\mathbf{x} = c_1x_1 + ... + c_nx_n$

**Section 13.3.3**

13.3.3.1. there is exactly one basic feasible solution: $\mathbf{x} = [\, x_1 \,\vdots\, ... \,\vdots\, x_4 \,]^T = [\, 0 \,\vdots\, 7 \,\vdots\, 0 \,\vdots\, 39 \,]^T$

13.3.3.3. there is no basic feasible solution

13.3.3.5. (a) $\left\{ \begin{array}{rrrrcr} x_1 & +x_2 & +x_3 & & = & 2 \\ -2x_1 & -x_2 & & +x_4 & = & -3 \end{array} \right\}$ is the LP in standard form.

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

13.3.3.7. (a)

$$\left\{ \begin{array}{lllll} \text{Minimize} & -15.55x_1 - 14.66x_2 - 7.23x_3 & & & \\ & & & & \\ \text{Subject to} & -216x_1 - 404x_2 \quad - 363x_3 + x_4 & & & = -300 \\ & -42.8x_1 \; - 6.5x_2 \quad - 4.6x_3 & + x_5 & & = -10 \\ & -2.212x_1 - 3.329x_2 - 0.996x_3 & & + x_6 & = -2.5 \\ & x_1 + \quad x_2 + \quad x_3 & & & = 1 \\ & x_1, ..., x_6 \geq 0 & & & \end{array} \right\}$$

(b) Ex. $[\, x_1 \; ... \; x_6 \,]^T = [\, 0.5 \quad 0.5 \quad 0 \quad 10 \quad 14.65 \quad 0.2705 \,]^T$ is a basic feasible solution of the LP problem in standard form

**Section 13.4**

13.4.2.1. $(x_1, x_2) = (0, 1.5)$ gives the minimum value of 1.5

13.4.2.3. $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 25, 15, 15)$ gives the minimum value of 50; $(x_1, x_2, x_3, x_4, x_5) = (\frac{50}{3}, 0, 0, \frac{20}{3}, \frac{20}{3})$ also gives the minimum value of 50

13.4.2.5. $(x_1, x_2, x_3) = (20, 10, 10)$ gives the minimum value of 80

13.4.2.7. $(x_1, x_2, x_3) = (\frac{37}{2}, \frac{5}{2}, 10)$ gives the minimum value of 40

13.4.2.9. maximum compressive force $\lambda \cdot P = 3\frac{M_{P\ell}}{\ell}$, achieved when $(M_1, M_2, M_3) = M_{P\ell} \cdot (0.25, 1, 1)$

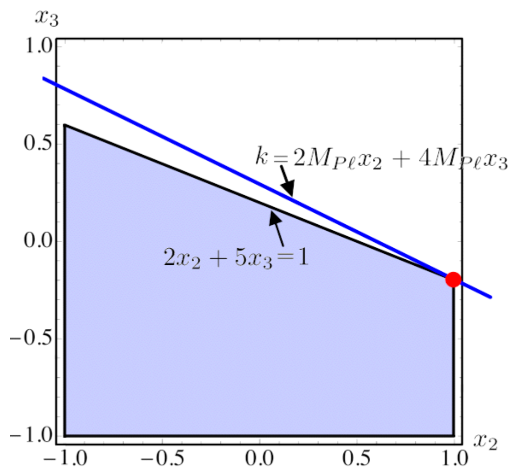


Figure 13.1: Answer key for problem 13.4.2.11

13.4.2.11. maximum compressive force $\lambda = 1.2\frac{M_{P\ell}}{\ell}$, achieved when $(M_1, M_2, M_3) = M_{P\ell} \cdot (1, 1, -0.2)$

13.4.2.13. In terms of the original nutrition problem, we have that the maximum, 15.15 g protein content of 100 g of mixture, is when we use 55.32 g of wheat bran, 44.68 g of oat flour, and 0 g of rice flour.

**Section 13.5**

13.5.3.1. The closest approach of the two regions is where $(x, y) = (\frac{3}{2}, -\frac{3}{4})$, $(X, Y) = (\frac{35}{8}, -\frac{29}{8})$, and the minimum distance is $\frac{23}{4\sqrt{2}}$.

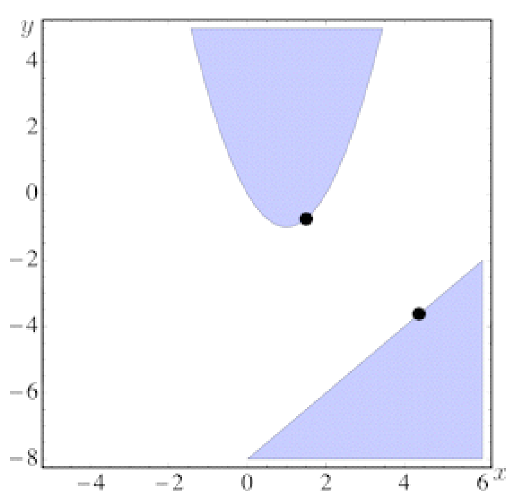


Figure 13.2: Answer key for problem 13.5.3.1

13.5.3.3. minimum value of the objective function with these constraints is $f(\frac{4}{5}, \frac{12}{5}) = \ln \frac{5^5}{4^2 12^3}$

13.5.3.5. Hint: $\lambda_1\mathbf{x}_1 + ... + \lambda_p\mathbf{x}_p = \mu_{p-1}\big(\mu_{p-1}^{-1}(\lambda_1\mathbf{x}_1 + ... + \lambda_{p-1}\mathbf{x}_{p-1})\big) + \lambda_p\mathbf{x}_p = \mu_{p-1}\mathbf{z} + \lambda_p\mathbf{x}_p$, where we define $\mu_{p-1} = \lambda_1 + ... + \lambda_{p-1}$ and $(\star)$ $\mathbf{z} \triangleq \mu_{p-1}^{-1}(\lambda_1\mathbf{x}_1 + ... + \lambda_{p-1}\mathbf{x}_{p-1})$. Because $C$ is convex and $1 = \lambda_1 + ... + \lambda_p = \mu_{p-1} + \lambda_p$, we see that $\lambda_1\mathbf{x}_1 + ... + \lambda_p\mathbf{x}_p$ will be in $C$ as long as we can explain why $\mathbf{z}$, which is a *certain* linear combination of

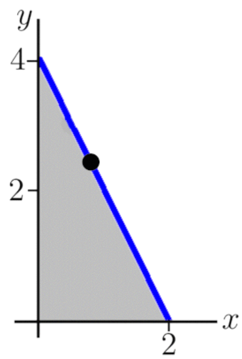


Figure 13.3: Answer Key for problem 13.5.3.3

$\mathbf{x}_1, ..., \mathbf{x}_{p-1}$, is in $C$. Continue in this way, by showing that $\mathbf{z}$ is in $C$ as long as some certain linear combination of $\mathbf{x}_1, ..., \mathbf{x}_{p-2}$ is in $C$, etc. Eventually this will reduce to showing why a certain linear combination of $\mathbf{x}_1$ and $\mathbf{x}_{p-2}$) is in $C$.

13.5.3.7. Hint: Begin by supposing that $S$ contains points $\mathbf{x}$ and $\mathbf{y}$.

13.5.3.9. Hint: Choose any $\mathbf{x}, \mathbf{x}$ in $\mathbb{R}^n$ and any $\lambda$ with $0 < \lambda < 1$. Then

$$\big(\mathbf{g}(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})\big) = f\big(A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + \mathbf{b}\big)$$

$$= f\big(\lambda A\mathbf{x} + (1-\lambda)A\mathbf{y} + \lambda\mathbf{b} + (1-\lambda)\mathbf{b}\big) = ... \,.$$

13.5.3.11. Hint: Pick any $x, y$ that are both in both of $C_1$ and $C_2$.

**Section 13.6**

13.6.3.1. Hint: Generalize the work for the maximization problem in system (13.52).

13.6.3.3. middle eigenvalue $\lambda_2 \approx -1$

# Chapter 14

# Calculus of variations

**Section 14.1.2**

14.1.2.1. $y(x) \approx 0.0130545x^2(\pi - x) + 0.219384 \cos \dfrac{x}{2}$

14.1.2.3. Ex. 1: $u(r) \approx 0.08333333442105048(1-r) + 0.19444444183629322r(1-r)$

Ex. 2: $u(r) \approx 0.2777777777871004(1-r) - 0.19444444445493192(1-r)^2$

Ex. 3: $u(r) \approx -0.16338185122419183(1-r) + 0.24393320411062294 \cos\left(\dfrac{\pi r}{2}\right)$

**Section 14.2.5**

14.2.5.1. $\big(p(x)y'(x)\big) = f(x),\ 0 < x < L$

14.2.5.3. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -f(x,y),\ (x,y) \text{ in } \mathcal{D}$

14.2.5.5. $y(x) \approx 0.2662994102\, J_0(x) - 2.469811754\, Y_0(x)$

14.2.5.7. $\delta J = \displaystyle\int_a^b \left( \frac{\partial F}{\partial y} - \frac{d}{dx}\left[ \frac{\partial F}{\partial y'} \right] \right) \delta y\, dx + \int_a^b \left( \frac{\partial F}{\partial v} - \frac{d}{dx}\left[ \frac{\partial F}{\partial v'} \right] \right) \delta v\, dx$

14.2.5.11. Necessarily, at $\big(x, \mathbf{y}(x), \mathbf{y}'(x)\big) = \big(x, \mathbf{y}_0(x), \mathbf{y}_0'(x)\big)$, we have the Euler-Lagrange equations, for $k = 1, ..., n$,

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left[ \frac{\partial F}{\partial y_k'} \right] \equiv 0,\ a < x < b.$$

**Section 14.3.2**

14.3.2.1. $\left\{ \begin{array}{c} u'' + (q(x) + \lambda)u = 0,\ 0 < x < L \\ u(0) = u'(L) = 0 \end{array} \right\}$

14.3.2.3. $\left\{ \begin{array}{c} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + (f(x,y) + \lambda)u = 0, (x,y) \text{ in } \mathcal{D} \\ \\ u = g(x,y) \text{ on } \partial\mathcal{D} \end{array} \right\}$

**Section 14.4.4**

14.4.4.1. $$\left\{\begin{array}{ll}\text{Minimize} & \displaystyle\iint_{\mathcal{D}} \left( \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right) dA \\ \text{Subject to} & \displaystyle\iint_{\mathcal{D}} |u(x,y)|^2 \, dx\, dy = 1 \text{ and} \\ & \displaystyle\frac{\partial u}{\partial n} \equiv 0 \text{ on } \partial\mathcal{D} \end{array}\right\}.$$

14.4.4.3. $$\left\{\begin{array}{ll}\text{Minimize} & \displaystyle\iint_{\mathcal{D}} \left( \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - f(x,y)u^2 \right) dA \\ \text{Subject to} & \displaystyle\iint_{\mathcal{D}} \sigma(x,y)|u(x,y)|^2 \, dx\, dy = 1 \\ & \text{and } u \equiv 0 \text{ on } \partial\mathcal{D} \end{array}\right\}$$

14.4.4.5. $$\left\{\begin{array}{c} X''(x) + \big(\lambda + q(x)\big)X(x) = 0,\ 0 < x < L, \\ X(0) = X(L) = 0 \end{array}\right\}$$

**Section 14.5.4**

14.5.4.1. $y(x) \approx 0.127332639564\, T_1(x) + 0.178564177107\, T_2(x) + 0.141182350726\, T_3(x)$

14.5.4.3. $y(x) \approx -0.707679286807 C_{-1}(x) + 0.0445065089233 C_0(x) + 0.529653251113 C_1(x)$

$$+0.697245134775 C_2(x) + 0.529653251113 C_3(x)$$

14.5.4.5. (a) Hint: $F(x,y,y') \triangleq \frac{1}{2}(y')^2 + \frac{1}{2}y^2 - \frac{1}{4}y^4 - xy$,
(b) $y(x) \approx 0.035220059648\, T_1(x) + 0.056871998478\, T_2(x) + 0.050527720768\, T_3(x)$

# Chapter 15

# Functions of a Complex Variable

**Section 15.1.4**

15.1.4.1. Hint: Begin with $LHS = \overline{wz} = \overline{(x+iy)(u+iv)} = \overline{(xu-yv)+i(xv+yu)} = ....$

15.1.4.3. Hint: Let $w = z_1\overline{z}_2$ and calculate $\overline{w} = \overline{z_1\overline{z}_2} = ...$

15.1.4.5. Hints: First, note that $z=0$ cannot be a solution. Next, if $z \neq 0$ solves the equation, then multiply both sides by $z$...

15.1.4.7. (a) $4e^{i\pi/3} = 2 + i2\sqrt{3}$, (b) $e^{-i\,2\arctan(4/3)} = -\frac{7}{25} + i\,\frac{24}{25}$, (c) $\sqrt{2}e^{i11\pi/12} = \frac{-1-\sqrt{3}}{4} + i\,\frac{-1+\sqrt{3}}{4}$

15.1.4.9. (a) Hint: Begin with $\sin 3\theta = \mathcal{Im}\left(e^{i3\theta}\right) = \mathcal{Im}\left((\cos\theta + i\sin\theta)^3\right) = ...$
(b) Hint: Begin with $\cos 4\theta = \mathcal{Re}\left(e^{i4\theta}\right) = \mathcal{Re}\left((\cos\theta + i\sin\theta)^4\right) = ...$

15.1.4.11. Hint: Explain why the angle between the vector $\overline{Oz_1}$ and $\overline{Oz_0}$ is $2\pi/3$, etc.; The length of each of the sides is $\sqrt{3}\,\rho^{1/3}$.

15.1.4.13. (a) $2e^{i\pi/4} = \sqrt{2} + i\sqrt{2}$ and $2e^{i5\pi/4} = -\sqrt{2} - i\sqrt{2}$, (b) $2e^{-i\pi/3} = 1 - i\sqrt{3}$ and $2e^{i2\pi/3} = -1 + i\sqrt{3}$.
(c) $e^{i0} = 3 + i0$, $3e^{i2\pi/3} = -\frac{3}{2} + i\,\frac{3\sqrt{3}}{2}$, and $3e^{i4\pi/3} = -\frac{3}{2} - i\,\frac{3\sqrt{3}}{2}$.
(d) $\sqrt{2}e^{-i\pi/4} = 1 - i$, $\sqrt{2}e^{i5\pi/12} = \sqrt{2}\cos\left(\frac{5\pi}{12}\right) + i\sqrt{2}\sin\left(\frac{5\pi}{12}\right)$, and $\sqrt{2}e^{-i11\pi/12} = \sqrt{2}\cos\left(\frac{11\pi}{12}\right) - i\sqrt{2}\sin\left(\frac{11\pi}{12}\right)$.
(e) $2e^{i2\pi/9} = 2\cos\left(\frac{2\pi}{9}\right) + i2\sin\left(\frac{2\pi}{9}\right)$, $2e^{i8\pi/9} = 2\cos\left(\frac{8\pi}{9}\right) + 2\sin\left(\frac{8\pi}{9}\right)$, and $2e^{-i4\pi/9} = 2\cos\left(\frac{4\pi}{9}\right) - 2\sin\left(\frac{4\pi}{9}\right)$.
(f) $3e^{i\pi/4} = \dfrac{3}{\sqrt{2}} + i\,\dfrac{3}{\sqrt{2}}$, $3e^{i3\pi/4} = -\dfrac{3}{\sqrt{2}} + i\,\dfrac{3}{\sqrt{2}}$, $3e^{-i3\pi/4} = -\dfrac{3}{\sqrt{2}} - i\,\dfrac{3}{\sqrt{2}}$, and $3e^{-i\pi/4} = \dfrac{3}{\sqrt{2}} - i\,\dfrac{3}{\sqrt{2}}$.

15.1.4.15. $2^{1/3}\,e^{-i2\pi/9}$, $2^{1/3}\,e^{i4\pi/9}$, and $2^{1/3}\,e^{-i8\pi/9}$.

15.1.4.17. (a) $z = -1 \pm \sqrt{2}$, (b) $\mp\frac{\sqrt{3}}{6} + i\,\frac{1}{2}$

15.1.4.19. (a) image of $A$ under the inversion mapping is $f(A) = \left\{w : |w| = \dfrac{1}{3}\right\}$
(b) the image of $\widehat{A} \triangleq \{z : |z-1| = 1,\ z \neq 0\}$ under the inversion mapping is $f(\widehat{A}) = L = \left\{w = \frac{1}{2} + iv : -\infty < v < \infty\right\}$, a vertical line in the $w-$plane
(c) the image of $\widehat{A} \triangleq \{z : |z+2| = 2,\ z \neq 0\}$ under the inversion mapping is $f(\widehat{A}) = L = \left\{w = -\frac{1}{4} + iv : -\infty < v < \infty\right\}$, a vertical line in the $w-$plane
(d) the image of $\widehat{A} \triangleq \{z : |2z+i| = 1,\ z \neq 0\}$ under the inversion mapping is $f(\widehat{A}) = L = \{w = u + i : -\infty < u < \infty\}$, a vertical line in the $w-$plane. Also, see the picture.

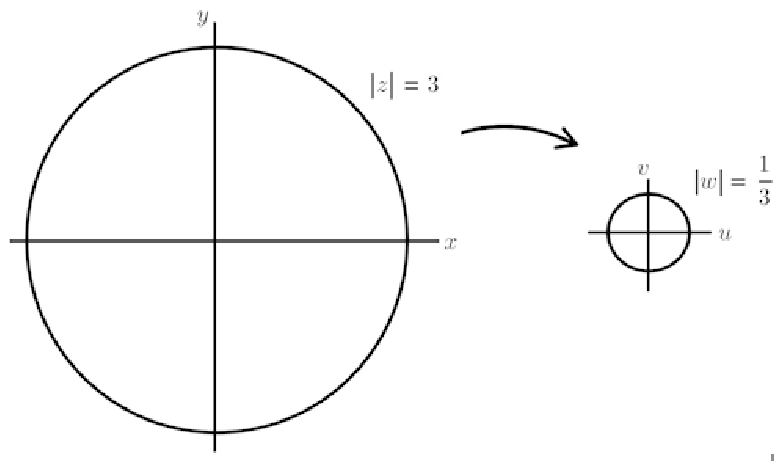


Figure 15.1: Transformation in 15.1.4.19(a)

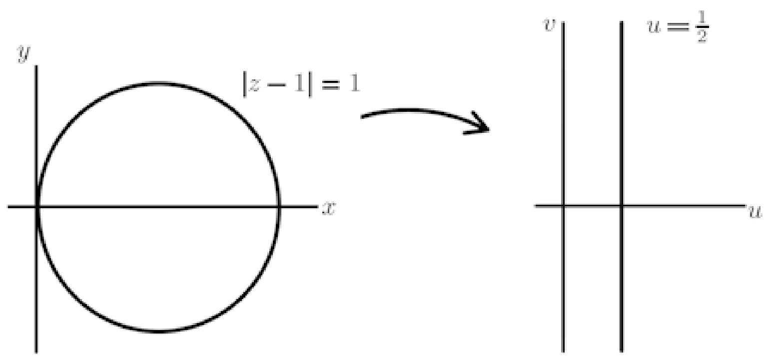


Figure 15.2: Transformation in 15.1.4.19(b)

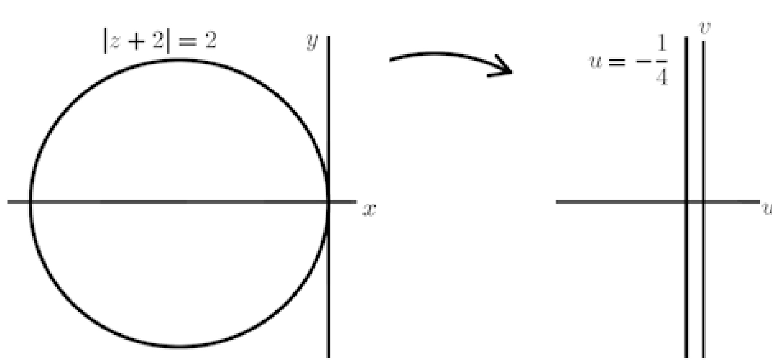


Figure 15.3: Transformation in 15.1.4.19(c)

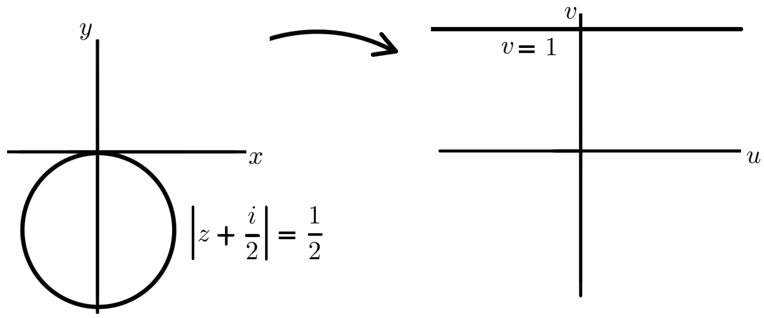


Figure 15.4: Transformation in 15.1.4.19(d)

15.1.4.21. (a) the image of $A$ under the mapping $f(z)$ is $f(A) = \{w : |w + 3| = 1\}$, the circle of radius 1 and center at $-3$ in the $w-$plane

(b) the image of $A$ under the mapping $f(z)$ is the horizontal line $f(A) = \{w : \mathcal{I}m(w) = 2\} = \{u + i2 : -\infty < u < \infty\}$

(c) the image of $A$ under the mapping $f(z)$ is the horizontal line $f(A) = \{w : \mathcal{R}e(w) = -2\} = \{-2 + iv : -\infty < v < \infty\}$

(d) the image of $A$ under the mapping $f(z)$ is the line $f(A) = \{w = u + iv : v = u + 3\} = \{u + i(u + 3) : -\infty < u < \infty\}$

15.1.4.23. (a) Hint: Calculate $LHS = |a + Re^{i\theta}| = |a + R\cos\theta + iR\sin\theta| = \sqrt{(a + R\cos\theta)^2 + (R\sin\theta)^2}$ and $RHS = |a + Re^{-i\theta}| = |a + R\cos\theta - iR\sin\theta| = ...$

(b) Hint: Calculate $RHS = |ae^{i\theta} + R| = |e^{i\theta}(a + Re^{-i\theta})| = ...$

**Section 15.2.5**

15.2.5.1. (a) $-6$, (b) 6, (c) $-1$

15.2.5.3. (a) $\widetilde{f}(z) \triangleq \left\{ \begin{array}{cc} \dfrac{3(z^2-1)}{z+1}, & z \neq -1 \\ \\ -6, & z = -1 \end{array} \right\}$, (b) $\widetilde{g}(z) \triangleq \left\{ \begin{array}{cc} \dfrac{2(z^3+1)}{z+1}, & z \neq -1 \\ \\ 6, & z = -1 \end{array} \right\}$

15.2.5.5. (a) $f'(z) = i2z + 2$ exists everywhere, (b) $g'(z) = 4\pi(2z - i)$ exists everywhere

(c) $h'(z) = 5z^4 + 9z^2 + 8z + 2$ exists everywhere, (d) $k'(z) = \frac{-i3}{(z-i)^2}$ exists everywhere except at $z = i$

(e) $\ell'(z) = \frac{-2(z-i)}{(z^2-i2z-4)^2}$ exists everywhere except at $z = i \pm \sqrt{3}$

(f) $m'(z) = \frac{-\pi^2 z(3z-2)}{z^4(z-1)^2}$ exists everywhere except at $z = 0$ and except at $z = 1$.

15.2.5.7. (a) everywhere, (b) nowhere, (c) $f'(z) = i - i2z$

15.2.5.9. (a) the imaginary axis, (b) everywhere except the imaginary axis, (c) $f'(z) = i(1 + z - \bar{z})$ for $z$ on the imaginary axis

15.2.5.11. (a) at the origin, (b) everywhere except the origin, (c) $f'(0) = 0$

15.2.5.13. Hint: Begin by explaining why $\dfrac{\partial u}{\partial y} \equiv -\dfrac{\partial v}{\partial x}$, but $\dfrac{\partial u}{\partial x} \equiv \dfrac{\partial v}{\partial y}$ only at the points on the circle $x^2+y^2 = 1$.

15.2.5.15. Hint: Begin by explaining why $\left\{\mathbf{q}_1, \widehat{\mathbf{T}}\right\}$ is an o.n. basis for $\mathbb{R}^2$

**Section 15.3.3**

15.3.3.1. $f(z)$ satisfies the Cauchy-Riemann equations at all $z$, so $f(z)$ is an entire function. In fact, $f'(z) = 1+i$.

15.3.3.3. $f(z)$ is analytic and has $f'(z) = -\frac{1}{(z-i)^2}$, everywhere except at $z = i$.

15.3.3.5. $v(x,y) = y - y^2 + x^2 + c$, where $c$ =arb. const.

15.3.3.7. $f(z) = z + iz^2, \quad f'(z) = 1 + 2iz$

15.3.3.9. $f(z) = 2 - z^2 + z^3, \quad f'(z) = -2z + 3z^2$

15.3.3.11. $v(x,y) = \frac{x}{x^2+y^2} + c$, for any real constant $c$, on the domain $\mathcal{D} = \{z = x + iy : \, x^2 + y^2 > 0\}$

15.3.3.13. $V(r,\theta) = v_\infty \left(r - \frac{a^2}{r}\right) \sin\theta + c$, for any real constant $c$

15.3.3.15. $v(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2} + c$, for any real constant $c$

15.3.3.17. $V(r,\theta) \triangleq -r^n \cos(n\theta)$

15.3.3.19. (a) (a) $f(z) = e^{ax-by}\cos(bx+ay) + ie^{ax-by}\sin(bx+ay), \quad$ (b) everywhere

15.3.3.21. Hint: define $w(x,y) = -u(x,y)$

15.3.3.23. The solution curves, that is, the streamlines, are $y^2 - x^2 = c$ =constant.

**Section 15.4.6**

15.4.6.1. $z = \frac{(-1+i)(1\pm\sqrt{7})}{2}$

15.4.6.3. (a) $arg(-1+i) = \left\{\frac{3\pi}{4} + 2\pi k : \, k \text{ is any integer}\right\}$ and $Arg(-1+i) = \frac{3\pi}{4}$
(b) $arg(-\sqrt{3}-i) = \left\{-\frac{5\pi}{6} + 2\pi k : k \text{ is any integer}\right\}$ and $Arg(-\sqrt{3}-i) = -\frac{5\pi}{6}$
(c) $arg(2e^{i\,\frac{5\pi}{3}}) = \left\{\frac{5\pi}{3} + 2\pi k : k \text{ is any integer}\right\}$ and $Arg\big(2e^{i\,\frac{5\pi}{3}}\big) = -\frac{\pi}{3}$
(d) $arg(\frac{-\sqrt{3}-i}{-1+i}) = \left\{\frac{5\pi}{12} + 2\pi k : k \text{ is any integer}\right\}$ and $Arg\Big(\dfrac{-\sqrt{3}-i}{-1+i}\Big) = \dfrac{5\pi}{12}$

15.4.6.5. (a) False; counterexample given by $z = 2e^{i3\pi/4}$
(b) True; Hint: for all $z \neq 0$, $Log(\sqrt{z}) = ... = Log\Big(|z|^{1/2}e^{iArg(z)/2}\Big)$

15.4.6.7. (a) $z$ for which $-\pi < \mathcal{I}m(z) \leq \pi, \quad$ (b) all $z \neq 0$

15.4.6.9. $z = -\frac{1}{2}\ln 2 + i\left(\frac{3\pi}{4} + 2n\pi\right)$, for all integers $n$.

15.4.6.11. <u>Ex</u>. $z = e^{i5\pi/6}$ for $n = 2$

15.4.6.13. $\left\{ e^{-\frac{\pi}{2} - 2\pi k} \cdot e^{i \ln 2} : k \text{ is any integer} \right\}$

15.4.6.15. $\sqrt[3]{2}\, e^{i5\pi/18}$

15.4.6.17.differentiable everywhere except the ray $Arg(z) = \pi$, also known as the non-positive real axis

15.4.6.19. Hint: Explain why $|e^w| = ... = e^u$

15.4.6.21. Hint: Explain why $z^\alpha \triangleq e^{\alpha log(z)} = ... = \left\{ e^{\alpha \left( \ln 2 + iArg(z) \right)} \right\}$

15.4.6.25. Hints: Do calculations such as

$$\frac{\partial v}{\partial x}(x, 0) = \lim_{h \to 0} \frac{v(x+h, 0) - v(x, 0)}{h} = \lim_{h \to 0} \frac{\cos^{-1}\left( \frac{x+h}{\sqrt{(x+h)^2 + 0^2}} \right) - 0}{h} = ...$$

and

$$-\lim_{y \to 0^+} \frac{1}{\sqrt{1 - \left( \frac{x}{\sqrt{x^2+y^2}} \right)^2}} \cdot \frac{\partial}{\partial y} \left( \frac{x}{\sqrt{x^2+y^2}} \right) = -\lim_{y \to 0^+} \frac{1}{\sqrt{\frac{y^2}{x^2+y^2}}} \cdot \frac{-xy}{(x^2+y^2)^{3/2}} = ...$$

**Section 15.5.1**

15.5.1.1. $\frac{1}{\sqrt{2}} \left( \cosh(1) + i \sinh(1) \right)$

15.5.1.3. $\{ 2k\pi + i \ln(2 + \sqrt{5}) : \text{ any integer } k \}$

15.5.1.5. $\left\{ 2k\pi + i \ln\left( 3 + \sqrt{10} \right) : \text{ any integer } k \right\} \cup \left\{ (2\ell - 1)\pi + i \ln\left( -3 + \sqrt{10} \right) : \text{ any integer } \ell \right\}$

15.5.1.7. $\{ n\pi + i\,(-1)^{n+1} \frac{\pi}{2} : \text{ any integer } n \}$

15.5.1.9. $\left\{ \left( n - \frac{1}{2} \right) \pi + i\, 3\, (-1)^n : \text{ any integer } n \right\}$

15.5.1.11. $\{ -\frac{\pi}{4} + n\pi + i\, 0 : \text{ any integer } n \}$

15.5.1.13. Hint: $\sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \frac{e^{iz_1} - e^{-iz_1}}{i2} \cdot \frac{e^{iz_2} + e^{-iz_2}}{2} + \frac{e^{iz_1} + e^{-iz_1}}{2} \cdot \frac{e^{iz_2} - e^{-iz_2}}{i2} = ...$

15.5.1.15. Hint: Analyze the solutions of the system of equations $\left\{ \begin{array}{lll} (1) & b = & \cos x \cosh y \\ (2) & 0 = & -\sin x \sinh y \end{array} \right\}$.

15.5.1.17. Hint: Analyze the solutions of the system of equations $\left\{ \begin{array}{lll} (1) & b = & \cosh x \cos y \\ (2) & 0 = & \sinh x \sin y \end{array} \right\}$

**Section 15.6.4**

15.6.4.1. (a) $f(z) = 3z^{-1} + \sum_{j=0}^{\infty} -(2^{j+1}) \cdot z^j$, for $0 < |z| < 2$, (b) $f(z) = 5z^{-1} + \sum_{j=2}^{\infty} 2^{j-1} \cdot z^{-j}$, for $2 < |z|$

15.6.4.3. (a) $f(z) = \sum_{k=1}^{\infty} z^{-k} + \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{j+1}} \cdot z^j$, for $1 < |z| < 2$

(b) $f(z) = -\sum_{j=0}^{\infty} \left( -1 + \frac{(-1)^j}{2^{j+1}} \right) z^j$, for $0 < |z| < 1$

15.6.4.5. Ex. 1: $f(z) = -\frac{1}{z} + \sum_{j=0}^{\infty} z^j$ converges for $0 < |z| < 1$

Ex. 2: $f(z) = \sum_{\ell=2}^{\infty} z^{-\ell}$ converges for $1 < |z| < \infty$

15.6.4.7. $f(z) = \frac{1}{2} \sum_{j=0}^{\infty} \left( -\frac{z}{2} \right)^j - \frac{3}{2z} \sum_{j=0}^{\infty} \left( -\frac{1}{2z} \right)^j$ converges for $\frac{1}{2} < |z| < 2$

15.6.4.9. $f(z) = \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \cdot z^{-j} + (1+z)$ converges for $0 < |z| < \infty$

15.6.4.11. Hint: Use the Binomial Theorem, stating that ... $(z_1 + z_2)^j = \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!} z_1^{\ell} z_2^{j-\ell}$, and use the convolution formula for the product of two series of complex numbers, which implies that, $\left( \sum_{\ell=0}^{\infty} a_\ell \right) \left( \sum_{n=0}^{\infty} b_n \right) = \sum_{j=0}^{\infty} \left( \sum_{\ell=0}^{j} a_\ell b_{j-\ell} \right)$

**Section 15.7.2**

15.7.2.1. Ex: $f(z) = \frac{1}{(z^2+1)(z+1)^3}$

15.7.2.3. two

15.7.2.5. $z_k = i\big(2k+1\big)\pi$ is a simple pole for all integers k

15.7.2.7. Ex: $f(z) = \frac{z^2+1}{(z+1)^3}$

15.7.2.9. Hint: Begin with $f(z) = \frac{h(z)}{(z-z_0)^m}$ for some function $h$ that is analytic at $z_0$ and satisfies $h(z_0) \neq 0$.

15.7.2.11. Hint: Begin with $h(z) = (z-z_0)^m k(z)$, where $k(z_0) \neq 0$ and $k(z)$ is analytic at $z_0$.

15.7.2.13. Hints: Begin with $g(z) = k(z)(z-z_0)^m$ and $h(z) = \ell(z)(z-z_0)^n$, where $k(z)$ and $\ell(z)$ that are analytic and non-zero at $z_0$.

15.7.2.15. Hints: Begin with $h(z) = \ell(z)(z-z_0)^n$, where $\ell(z)$ is analytic and non-zero at $z_0$, and $g(z) = (z-z_0)^{-m} k(z)$, where $k(z)$ that is analytic and non-zero at $z_0$.

**Section 15.8.3**

15.8.3.1. $i18\pi$

15.8.3.3. $-2\pi i$

15.8.3.5. $-i\,\pi$

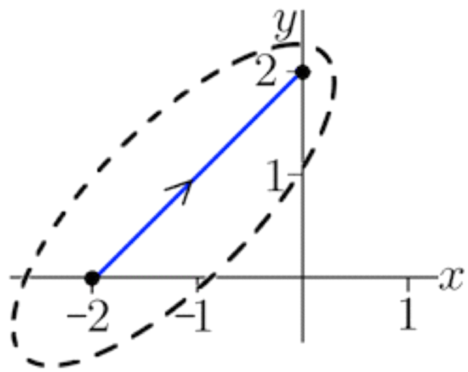


Figure 15.5: Answer key for problem 15.8.3.5

**Section 15.9.4**

15.9.4.1. The singularities of $f$ are at $z = 0, -1, 2$ and the corresponding residues are $\mathsf{Res}[\,f;\ 0\,] = -\frac{1}{2}$, $\mathsf{Res}[\,f;\ -1\,] = \frac{1}{3}$, $\mathsf{Res}[\,f;\ 2\,] = \frac{1}{6}$.

15.9.4.3. In $D_2(\frac{\pi}{2})$, the only singularities of $f$ are at $z = 0, \pi$ and the corresponding residues are $\mathsf{Res}[\,f;\ 0\,] = 0, \mathsf{Res}[\,f;\ \pi\,] = -\frac{1}{\pi}$.

15.9.4.5. $2\pi i\big(\cos(1) - \sin(1)\big)$

15.9.4.7. 0

15.9.4.9. $\frac{i\,\pi}{3}$

15.9.4.11. $-2\pi^2 i$

15.9.4.13. $2\pi i\,\cosh(\omega\xi)$

**Section 15.10.5**

15.10.5.1. $\pi\sqrt{2}$

15.10.5.3. $\dfrac{\pi}{\sqrt{2}}$

15.10.5.5. $\dfrac{\pi}{4}$

15.10.5.7. $\sqrt{\dfrac{2}{3}} \cdot \pi$

15.10.5.9. $\frac{\pi}{\sqrt{2}} e^{-\omega/\sqrt{2}} \cos\left(\frac{\omega}{\sqrt{2}} - \frac{\pi}{4}\right)$

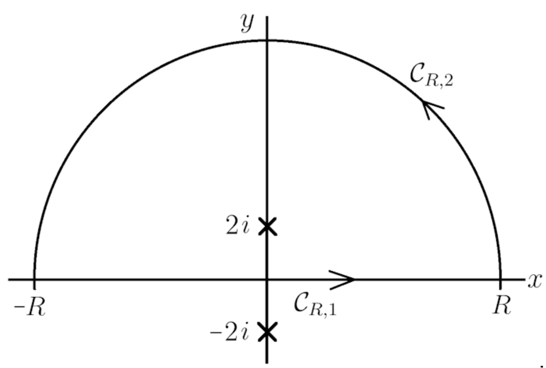


Figure 15.6: Complex integration to evaluate a real integral

# Chapter 16

# Conformal Mapping

**Section 16.1.5**

16.1.5.1. Hint: Note that $M(z) = \dfrac{az+b}{d}$ is not constant if, and only if, $a \neq 0$.

16.1.5.3. $\{z : \mathcal{I}m(z) \neq 0\} \cup \{z : z \text{ is real and } -3 < z < 3\}$

16.1.5.5. all $z$ except $z = 0 + ik\pi$, where $k$ is any integer

**Section 16.2.5**

16.2.5.1. $M_+(z) = \dfrac{4(z-4)}{4z-1}$ or $M_-(z) = \dfrac{4z-1}{4(z-4)}$

16.2.5.3. $M_+(z) = \dfrac{2z-9-3\sqrt{5}}{2z-9+3\sqrt{5}}$ or $M_-(z) = \dfrac{2z-9+3\sqrt{5}}{2z-9-3\sqrt{5}}$

16.2.5.5. $M_+(z) = \dfrac{2z-11-3\sqrt{5}}{2z-11+3\sqrt{5}}$ or $M_-(z) = \dfrac{2z-11+3\sqrt{5}}{2z-11-3\sqrt{5}}$

16.2.5.7. $M_+(z) = \dfrac{4z+i}{4(z+i4)}$ or $M_-(z) = \dfrac{4(z+i4)}{4z+i}$

16.2.5.9. $M(z) = 2z + i$

16.2.5.11. $M(z) = \dfrac{2z - i2}{(-1+i)(z+1)}$

16.2.5.13. $M(z) = \dfrac{i2z}{z-2}$

16.2.5.15. $M(z) = \dfrac{z-2}{z+2}$

16.2.5.17. $M(z) = \dfrac{-4}{z-2}$

16.2.5.19. Hint: Start with $\left|M(e^{i\theta}\right|^2 = \left|\dfrac{e^{i\theta}-\alpha}{e^{i\theta}-\frac{1}{\alpha}}\right|^2 = \left|\dfrac{(\cos\theta-\alpha)+i\sin\theta}{(\cos\theta-\frac{1}{\alpha})+i\sin\theta}\right|^2$ .

16.2.5.21. (a) Hint: Begin by writing $M(\delta+i\gamma)-M(\delta-i\gamma)=\dfrac{\delta+i\gamma-\alpha}{\delta+i\gamma-\beta}-\dfrac{\delta-i\gamma-\alpha}{\delta-i\gamma-\beta}$.

16.2.5.23. Hint: Begin by observing that $w$ is in $f(\mathcal{C})$ if, and only if, $|w-w_0|=\left|\dfrac{1}{z_0+Re^{i\theta}}-\dfrac{\overline{z_0}}{|z_0|^2-R^2}\right|$.

16.2.5.25. Hints: Define $g(\varphi)\triangleq\dfrac{\sin\varphi}{1+\cos\varphi}$ and use L'Hôpital's Rule.

**Section 16.3.5**

16.3.5.1. Hint: Use the Students' Solutions Manual for problem 16.2.5.1.

Final conclusion: $\phi=\phi(x,y)=2-\dfrac{1}{2\ln 2}\ln\left(\dfrac{16(4x^2+4y^2-17x+4)^2+(60y)^2}{\big((4x-1)^2+16y^2\big)^2}\right)$

16.3.5.3. Hint: Use the Students' Solutions Manual for problem 16.2.5.3.

Final conclusion: $\phi=\phi(x,y)=\left(\ln\left(\dfrac{3-\sqrt{5}}{2}\right)\right)^{-1}\cdot\left(\ln(9-4\sqrt{5})-\ln\left(\dfrac{16\big(x^2+y^2-9x+9\big)^2+720y^2}{\big((2x-9+3\sqrt{5})^2+4y^2\big)^2}\right)\right)$

16.3.5.5. $\phi(x,y)\equiv 1$

16.3.5.7.
$$\phi(x,y)=1+\frac{2}{\pi}\cos^{-1}\left(\frac{R^2-(x^2+y^2)}{\sqrt{\big(2Ry+\frac{1}{\sqrt{3}}\left((x+R)^2+y^2\right)\big)^2+\big(R^2-(x^2+y^2)\big)^2}}\right)$$
$$-\frac{2}{\pi}\cos^{-1}\left(\frac{R^2-(x^2+y^2)}{\sqrt{\big(2Ry-\frac{1}{\sqrt{3}}\left((x+R)^2+y^2\right)\big)^2+\big(R^2-(x^2+y^2)\big)^2}}\right).$$

16.3.5.9. Hints for part (d): Solve $z=J(\zeta)\triangleq\dfrac{1}{2}\left(\zeta+\dfrac{a^2}{\zeta}\right)$ to get $\zeta=\zeta_\pm\triangleq z\pm\sqrt{z^2-a^2}$, and note that $J(\zeta)$ has an interesting property: $J(\zeta)=J\left(\dfrac{a}{\zeta}\right)$, for all $\zeta\neq 0$.

16.3.5.11. Hint: For small $\varepsilon$, Taylor's series gives the approximation $\sqrt{1+2\varepsilon\alpha+\varepsilon^2\beta}\approx 1+\alpha$.

16.3.5.13. $H\approx 0.07$, $T\approx 0.0909327$, lift is $F_y\approx 2.14\pi\varrho_0 v_\infty^2\sin(\alpha+0.14)$

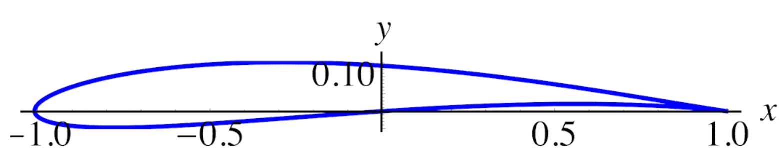


Figure 16.1: Answer key for problem 16.3.5.13

# Chapter 17

# Integral Transform Methods

**Section 17.1.2**

17.1.2.1. $\sqrt{\dfrac{2}{\pi}} \cdot \dfrac{1}{1+\omega^2}$ agrees with table entry F.6

17.1.2.3. $\sqrt{\dfrac{\pi}{2}} \cdot \Big(-i\, sgn(\omega) e^{-|\omega|}\Big)$. By the way, Mathematica writes the final conclusion as

$$\mathcal{F}\left[\frac{x}{1+x^2}\right](\omega) = i\sqrt{\frac{\pi}{2}}\,\left(e^{\omega}\,Step(-\omega) - e^{-\omega}\,Step(\omega)\right).$$

17.1.2.5. $\sqrt{\dfrac{2}{\pi}} \cdot \dfrac{1}{1+t^2}$

17.1.2.7. $\dfrac{1}{\sqrt{2\beta}} \cdot e^{-x^2/(4\beta)}$

17.1.2.9. $-i\sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x)\,\sin(\omega x)\,dx$

17.1.2.11. Hint: Use $e^{-i\omega x} = \big(\cos(\omega x) - i\sin(\omega x)\big)$ and Theorem 9.2 in Section 9.1.

**Section 17.2.2**

17.2.2.1. $u(x,t) = \dfrac{\sqrt{\pi}}{2\sqrt{2\alpha t}}\left(2\mathrm{erf}\Big(\dfrac{x}{2\sqrt{\alpha t}}\Big) - \mathrm{erf}\Big(\dfrac{x+\pi}{2\sqrt{\alpha t}}\Big) - \mathrm{erf}\Big(\dfrac{x-\pi}{2\sqrt{\alpha t}}\Big)\right)$

17.2.2.3. $u(x,y,t) = \dfrac{1}{4\pi\alpha t} \cdot \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\big((x-\xi)^2+(y-\eta)^2\big)/(4\alpha t)}\, u(\xi,\eta,0)\,d\xi\,d\eta$

17.2.2.5. For $x>0$ and $0<y<H$, $\quad u(x,y) = \frac{\sqrt{\pi/2}}{2H} \cdot \sin\left(\frac{\pi}{H}(H-y)\right)\cdot$

$$\cdot\int_0^\infty \left(\frac{1}{\cosh\left(\frac{\pi(x-\xi)}{H}\right)+\cos\left(\frac{\pi}{H}(H-y)\right)} + \frac{1}{\cosh\left(\frac{\pi(x+\xi)}{H}\right)+\cos\left(\frac{\pi}{H}(H-y)\right)}\right) f(\xi)\,d\xi\,.$$

17.2.2.7. Use evenness to turn this problem into a Fourier inversion problem, and then use an entry from Table 17.1 and Theorem 17.3(a) in Section 17.1.

17.2.2.11. Hint: Use the result of Example 17.3 in Section 17.1 with $\beta = \dfrac{1}{4\alpha t}$.

**Section 17.3.2**

17.3.2.1. Hints: <u>Ex</u>. Bromwich contour $\mathcal{C}:\ s = 0 + iy,\ -\infty < y < \infty$; each of poles $s = -1 \pm i2$ gives a residue

17.3.2.3. $u(x,t) = \dfrac{\sin\left(\frac{2(L-x)}{c}\right)}{\sin\left(\frac{2L}{c}\right)} \cdot \sin 2t \ + \ \dfrac{4c}{L}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{4 - \left(\frac{n\pi c}{L}\right)^2} \cdot \sin\left(\dfrac{n\pi(L-x)}{L}\right) \sin\left(\dfrac{n\pi ct}{L}\right)$

17.3.2.5. $u(x,t) = \dfrac{\sin\left(\frac{x}{c}\right)}{\sin\left(\frac{L}{c}\right)} \cdot \cos t \quad + \quad \dfrac{2c^2}{L}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{1 - \left(\frac{(n-\frac{1}{2})\pi c}{L}\right)^2} \cdot \sin\left(\dfrac{(n-\frac{1}{2})\pi x}{L}\right) \cdot \cos\left(\dfrac{(n-\frac{1}{2})\pi ct}{L}\right).$

17.3.2.7. $$u(x,t) = -\frac{1}{\pi}\left((\pi - x)\sin(t)\cos(\pi - x) + \frac{1}{2}\, t\cos(t)\sin(\pi - x) - \frac{1}{2}\,\sin(t)\sin(\pi - x)\right)$$
$$+ \ \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{1-n^2} \cdot \sin\big(n(\pi - x)\big) \cdot \sin(nt).$$

17.3.2.11. The Errata webpage adds that we should assume ICs $u(x,0) = v(x,0) = 0$ for $0 < x < \infty$. Hints: the PDE system becomes
$$\left\{ \begin{array}{l} s\,U(x,s) = \alpha\, \dfrac{\partial^2 U}{\partial x^2} \\[2ex] s\,V(x,s) = \alpha\, \dfrac{\partial^2 V}{\partial x^2} + \beta\, s\, U(x,s) \end{array} \right\},$$
and then get
$$V(x,s) = \left(\frac{\sqrt{\alpha}}{s\sqrt{s}} \ + \frac{\beta}{2s}\right) \cdot e^{-(x/\sqrt{\alpha})\sqrt{s}} \ + \frac{\beta}{2\sqrt{\alpha s}}\, x\, e^{-(x/\sqrt{\alpha})\sqrt{s}}.$$
With $k = (x/\sqrt{\alpha})$, use entries L2.6, L2.7, and L2.8 of Table 17.2 at the end of Section 17.3. In summary, the solution of the original system of PDEs is
$$\left\{ \begin{array}{l} u(x,t) = \operatorname{erfc}\left(\dfrac{x}{2\sqrt{\alpha t}}\right) \\[2ex] v(x,t) = \left(\dfrac{\beta}{2} - x\right)\operatorname{erfc}\left(\dfrac{x}{2\sqrt{\alpha t}}\right) + \left(2\sqrt{\dfrac{\alpha t}{\pi}} + \dfrac{\beta x}{2\sqrt{\pi \alpha t}}\right) e^{-x^2/(4\alpha t)} \end{array} \right\}.$$

**Section 17.4.1**

17.4.1.1. $\sqrt{r}g'(r)$, $\sqrt{r}g(r)$, $\sqrt{r}f'(r)$, and $\sqrt{r}f(r)$ are bounded as $r \to \infty$, where $g(r) = \Delta f(r) = \dfrac{1}{r}\dfrac{d}{dr}\left[ r\dfrac{df}{dr}(r)\right]$, and also that $\lim_{r\to 0^+} |g'(r)| < \infty$, $\lim_{r\to 0^+} |g(r)| < \infty$, $\lim_{r\to 0^+} |f'(r)| < \infty$, and $\lim_{r\to 0^+} |f(r)| < \infty$.

17.4.1.3. $c(r,z) = -\dfrac{c_0 z}{\pi r^{3/2}}\displaystyle\int_0^a \dfrac{1}{\xi^{3/2}}\ Q'_{-\frac{1}{2}}\left(\dfrac{z^2 + r^2 + \xi^2}{2\xi r}\right) d\xi$

17.4.1.5. $w(r,t) = \dfrac{1}{\beta t} \cdot \displaystyle\int_0^{\infty} J_0\left(\dfrac{r\xi}{2\beta t}\right) \cdot \left(\sin\left(\dfrac{r^2 + \xi^2}{4\beta t}\right) f(\xi) \ + \ \cos\left(\dfrac{r^2+\xi^2}{4\beta t}\right) h(\xi)\right) \xi\, d\xi,$

where $f(r) = w(r,0)$ and $h(r) \triangleq \mathcal{H}_0^{-1}\left[\dfrac{1}{\beta k^2}\,\mathcal{H}_0\left[\dfrac{\partial w}{\partial t}(r,0)\right]\right]$

# Chapter 18

# Nonlinear Ordinary Differential Equations

**Section 18.1.5**

18.1.5.1. origin is a saddle point; curved solutions approach the line through $[1 \quad \sqrt{3}]^T$ as $t \to \infty$ and approach the line through $[-\sqrt{3} \quad 1]^T$ as $t \to -\infty$

18.1.5.3. origin is a stable spiral point because eigenvalues are $-1 \pm i2$

18.1.5.5. I'm sorry, this is the same problem as 18.1.5.3.

18.1.5.7. general solution is $\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$, where $c_1, c_2$ =arb. consts. A phase plane picture is in the figure.

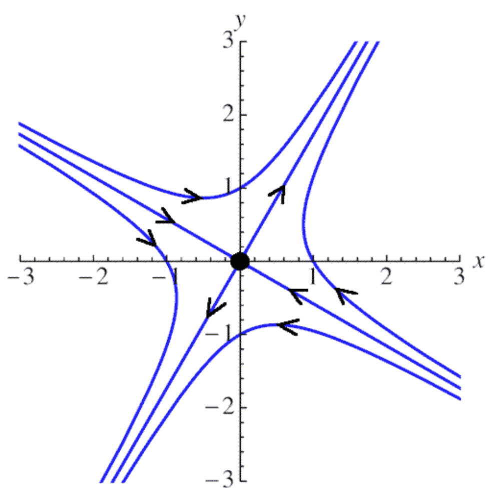


Figure 18.1: Answer key for problem 18.1.5.7

18.1.5.9. (a) Hint: Use uniqueness of solutions of an ODE system [unfortunately, not seen until Theorem 18.21 in Section 18.7 but analogous to Theorem 3.6 in Section 3.2]

(b) $x(t-2) \triangleq 1 - \left(\dfrac{1-e^{t-2}}{1+e^{t-2}}\right)^2$

**Section 18.2.3**

18.2.3.1. (a) $(x, y) = (0, 0)$, (b) $(x, y) = (0, 0)$ is asymptotically stable

18.2.3.3. (a) $(x_1, x_2) = (0, 0)$ and all points on the circle $x_1^2 + x_2^2 = 1$

(b) $(x_1, x_2) = (0, 0)$ is asymptotically stable; all equilibria on the circle $x_1^2 + x_2^2 = 1$ are unstable.

18.2.3.5. (a) $(x, y) = (0, 0)$ and $(2, 4)$

(b) $(x, y) = (0, 0)$ is an unstable equilibrium point; $(x, y) = (2, 4)$ is an unstable equilibrium point.

18.2.3.7. (a) $(x, y) = (n\pi, 0)$, where $n$ is any integer

(b) $(x, y) = (2k\pi, 0)$ are unstable equilibria; $(x, y) = \big((2\ell - 1)\pi, 0\big)$ are asymptotically stable equilibria

18.2.3.9. (a) $(x, y) = (0, 0)$, $(0, 4)$, and $\left(-\dfrac{1}{8}, 5\right)$

(b) $(x, y) = (0, 0)$ is unstable; $(x, y) = (0, 4)$ is asymptotically stable; $(x, y) = \left(-\dfrac{1}{8}, 5\right)$ is unstable

18.2.3.11. (a) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2b \end{bmatrix} \mathbf{x}$; (b) the only equilibrium point is at $(x, y) = (0, 0)$; $(\theta, \dot{\theta}) = (0, 0)$ is asymptotically stable if $b > 0$. If $b = 0$, we cannot use linearization to decide if the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$ is stable or unstable.

18.2.3.13. $(0, 0, 0)$ is asymptotically stable if $\beta < 0$; $(0, 0, 0)$ is stable but not asymptotically stable if $\beta = 0$; $(0, 0, 0)$ is unstable if $\beta > 0$

**Section 18.3.3**

18.3.3.1. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ -\frac{1}{3} c_1^2 e^{2t} + c_2 e^{-t} \end{bmatrix}$, where $c_1$ and $c_2$ are arbitrary constants. The phase plane picture is shown in the figure.

18.3.3.3. Theorem 18.6 in Section 18.3 guarantees that the original, nonlinear system $(\star)$ has a $2\pi-$periodic solution $\mathbf{x}(t)$ that satisfies

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} e^{-2t} + 3e^{2t} & -e^{-2t} + e^{2t} \\ -3e^{-2t} + 3e^{2t} & 3e^{-2t} + e^{2t} \end{bmatrix} \left( \mathbf{x}_0 + \frac{1}{4} \int_0^t \begin{bmatrix} (3e^{-2s} + e^{2s}) x_1^3(s) + (e^{-2s} - e^{2s}) \sin s \\ (3e^{-2s} - 3e^{2s}) x_1^3(s) + (e^{-2s} + 3e^{2s}) \sin s \end{bmatrix} ds \right).$$

**Section 18.4.5**

18.4.5.1. (a) $V(x, y) = x^2 + 3y^2$ is positive definite with respect to $(0, 0)$, (b) $V(x, y) = \sin^2 x + 3y^2$ is positive definite with respect to $(\pi, 0)$, (c) $V(x, y) = x^3 + (y - 1)^6$ is indefinite with respect to $(0, 1)$

18.4.5.3. $(x, y) = (0, 0)$ is stable; but, we cannot decide whether or not $(x, y) = (0, 0)$ is asymptotically stable without further information or theory

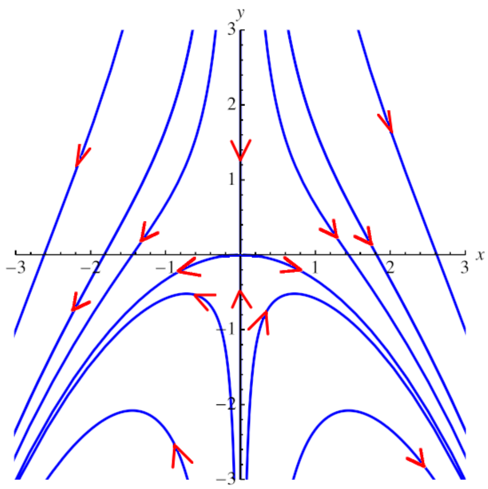


Figure 18.2: Answer key for problem18.3.3.1

18.4.5.5. $\mathbf{x} = \mathbf{0}$ is asymptotically stable

18.4.5.7. for example, $A = 2, B = 3$ gives asymptotic stability of the origin

18.4.5.9. for example, $A = 5$ and $B = 1$ gives asymptotic stability of the origin

18.4.5.11. Using the given (Liapunov) function we see that the origin is unstable. The linearization does not give a conclusion about the stability of the original, nonlinear system of ODEs.

18.4.5.13. $(x, y) = (0, 0)$ is unstable.

18.4.5.15. $V(\mathbf{x}) \triangleq \mathbf{x}^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \mathbf{x}$

18.4.5.17. $V(\mathbf{x}) \triangleq \mathbf{x}^T \begin{bmatrix} \frac{3}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{3}{16} \end{bmatrix} \mathbf{x}$

18.4.5.19. Note the change on the Errata webpage. For $\alpha > 0$, the origin is unstable; for $\alpha < 0$, the origin is asymptotically stable; for $\alpha = 0$, the origin is stable but not asymptotically stable.

18.4.5.21. $H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(k_1+k_2)q_1^2 - k_2 q_1 q_2 + \frac{1}{2}(k_2+k_3)q_2^2 + \frac{1}{2m_1} p_1^2 + \frac{1}{2m_2} p_2^2$ gives Hamiltonian system $\left\{ \begin{array}{c} \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \end{array} \right\}$, where $\mathbf{q} = [\,q_1 \quad q_2\,]^2$, $\mathbf{p} = [\,p_1 \quad p_2\,]^2$, and $(x_1, x_1, \dot{x}_1, \dot{x}_2) = (0, 0, 0, 0)$ is stable in $\mathbb{R}^4$

**Section 18.5.2**

18.5.2.1. $\mathbf{0}$ is asymptotically stable

18.5.2.3. $\mathbf{0}$ is asymptotically stable

18.5.2.5. $\mathcal{S} \triangleq \{(y,v) : T = M\}$, the *set* consisting of *all* of the equilibria, is asymptotically stable for the system of Example 5.8 in Section 5.2.

**Section 18.6.5**

18.6.5.1. $x^2+y^2 \equiv 1$ gives a stable limit cycle and $x^2+y^2 \equiv 4$ gives an unstable limit cycle

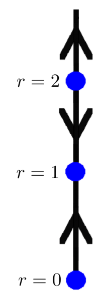


Figure 18.3: Answer key for problem18.6.5.1

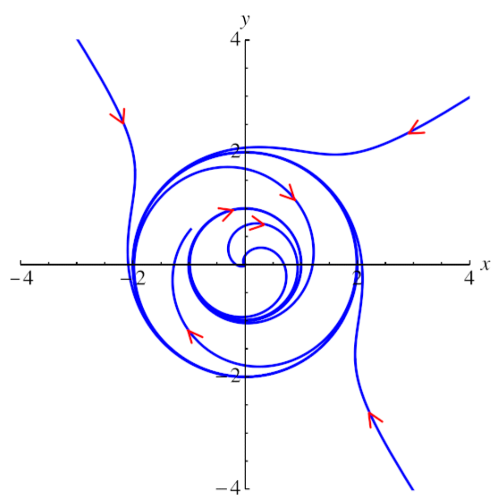


Figure 18.4: Answer key for problem18.6.5.1: Phase plane picture

18.6.5.3. $r = \ln 2$ is a stable limit cycle and there is no other limit cycle

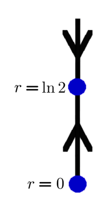


Figure 18.5: Answer key for problem18.6.5.3

18.6.5.7. (b) Hint: Use the Levinson-Smith Theorem, that is, Theorem 18.16 in Section 18.6.

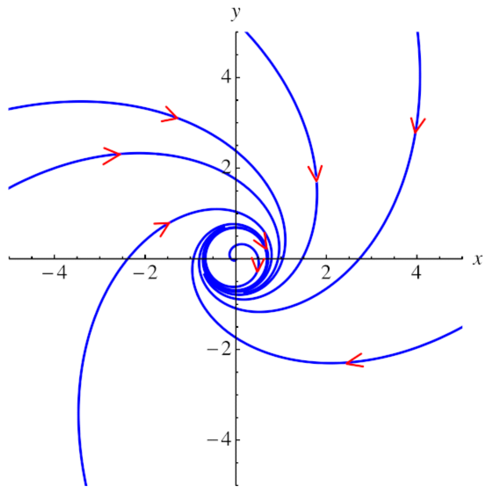


Figure 18.6: Answer key for problem18.6.5.3: Phase plane picture

18.6.5.9. Note that $0.01x$ has been changed to $x$ on the Errata webpage. Use the Levinson-Smith Theorem, that is, Theorem 18.16 in Section 18.6.

18.6.5.13. In fact, there is no limit cycle for any value of $\beta$. But, there is a change in the nature of solutions, that is, a bifurcation, at $\beta = 0$.

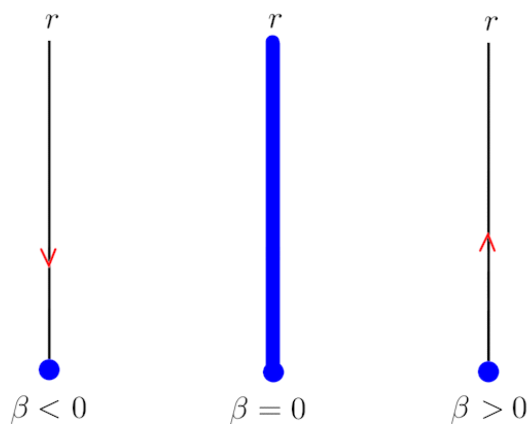


Figure 18.7: Answer key for problem18.6.5.13: Phase line pictures for $\dot{r} = \beta r^3$ as $\lambda$ varies

**Section 18.7.2**

18.7.2.1. $y_1(t) \equiv 1$, $y_2(t) = 1 + (t-2)$, $y_3(t) = 1 + (t-2) + \frac{1}{2}(t-2)^2$, $y_4(t) = 1 + (t-2) + \frac{1}{2!}(t-2)^2 + \frac{1}{3!}(t-2)^3$. Guess $y_k(t) = 1 + (t-2) + \frac{1}{2!}(t-2)^2 + \frac{1}{3!}(t-2)^3 + ... + \frac{1}{(k-1)!}(t-2)^{k-1}$, and then guess $y_\infty(t) = 1 + (t-2) + \frac{1}{2!}(t-2)^2 + ... + \frac{1}{k!}(t-2)^k + ... \equiv e^{t-2} = y(t)$, the exact solution of the IVP.

18.7.2.3. $\mathbf{x}_1(t) \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad \mathbf{x}_2(t) = \begin{bmatrix} 1 \\ -t \end{bmatrix}$, $\quad \mathbf{x}_3(t) = \begin{bmatrix} 1 - \frac{1}{2}t^2 \\ -t \end{bmatrix}$, $\quad \mathbf{x}_4(t) = \begin{bmatrix} 1 - \frac{1}{2}t^2 \\ -t + \frac{1}{3!}t^3 \end{bmatrix}$,

$\mathbf{x}_5(t) = \begin{bmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 \\ \\ -t + \frac{1}{3!}t^3 \end{bmatrix}$. Guess that for $k = odd$, $\quad \mathbf{x}_k(t) = \begin{bmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 + ... + (-1)^{(k-1)/2}\frac{1}{(k-1)!}t^{k-1} \\ \\ -t + \frac{1}{3!}t^3 + ... + (-1)^{(k-1)/2}\frac{1}{(k-2)!}t^{k-2} \end{bmatrix}$, and for $k = even$, $\mathbf{x}_k(t) = \begin{bmatrix} 1 - \frac{1}{2}\,t^2 + \frac{1}{4!}\,t^4 + ... + (-1)^{(k-2)/2}\,\frac{1}{(k-1)!}\,t^{k-2} \\ \\ -t + \frac{1}{3!}\,t^3 + ... + (-1)^{k/2}\,\frac{1}{(k-1)!}\,t^{k-1} \end{bmatrix}$. This suggests

$\mathbf{x}_\infty(t) = \begin{bmatrix} 1 - \frac{1}{2}\,t^2 + \frac{1}{4!}\,t^4 + ... + (-1)^{\ell}\,\frac{1}{(2\ell)!}\,t^{2\ell} + ... \\ \\ -t + \frac{1}{3!}\,t^3 + ... + (-1)^{\ell}\,\frac{1}{(2\ell+1)!}\,t^{2\ell+1} + ... \end{bmatrix}$, which equals $\mathbf{x}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$, the exact solution of the IVP system.

18.7.2.5. (a) Hint: Use the Fundamental Theorem of Calculus, that is, Theorem 7.4(b) in Section 7.1

(b) Hint: Take the Take the derivative with respect to $t$ of both sides and use the Fundamental Theorem of Calculus, that is, Theorem 7.4(a) in Section 7.1.

18.7.2.7. $\mathbf{x}(t; t_1, \mathbf{x}_1) = e^{(t-t_1)A}\mathbf{x}_1, \quad Y(t) \triangleq \dfrac{\partial \mathbf{x}}{\partial \mathbf{x}_1}(t; t_1, \mathbf{x}_1) = e^{(t-t_1)A}$

18.7.2.9. Note the changes on the Errata webpage.

(a) $\mathbf{x}(t;\lambda) = \begin{bmatrix} a\cos(\sqrt{\lambda}\,t) + b\lambda^{-1/2}\sin(\sqrt{\lambda}\,t) \\ \\ -a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) + b\cos(\sqrt{\lambda}\,t) \end{bmatrix}$, where $a, b =$ constants

(b) $\mathbf{z}(t;\lambda) = \frac{1}{2}\,t \cdot \lambda^{-1} \cdot \begin{bmatrix} -a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) + b\cos(\sqrt{\lambda}\,t) \\ \\ -a\lambda\cos(\sqrt{\lambda}\,t) - b\lambda^{1/2}\sin(\sqrt{\lambda}\,t) \end{bmatrix} - \frac{1}{2}\,\lambda^{-3/2} \cdot \begin{bmatrix} b\sin(\sqrt{\lambda}\,t) \\ \\ a\lambda\sin(\sqrt{\lambda}\,t) \end{bmatrix}$

(c) $\dot{\mathbf{z}}(t;\lambda) = A\mathbf{z}(t;\lambda) + \frac{1}{2}\,\lambda^{-1} \cdot \begin{bmatrix} -a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) + b\lambda^{-1/2}\sin(\sqrt{\lambda}\,t) \\ \\ a\lambda^{1/2}\sin(\sqrt{\lambda}\,t) - 2a\lambda\cos(\sqrt{\lambda}\,t) - b\lambda^{1/2}\sin(\sqrt{\lambda}\,t) \end{bmatrix}$

18.7.2.11. (a) $\delta = \frac{\pi}{2}$, (b) Picard's Theorem 3.6 in Section 3.2 guarantees the existence and uniqueness of a solution on *some* open time interval, (c) $\bar{\alpha}$ is any positive number strictly less than $\frac{1}{2}$, which is much less than $\delta$

**Section 18.9**

18.9.3.1. $\lambda = a + b\,e^{-\,r\lambda/2} + c\,e^{-\,r\lambda}$

18.9.3.3. $\det(\lambda I - A - e^{-\,r\lambda}B) = 0$