Part X

Appendices

Introduction

This is a very long (over 200 pages) set of 7 appendices. Their purpose is to spice the conceptual coverage of the text with a little technical material for those readers and instructors who are interested in such material.

Appendix A discusses the common units used in physics and various conversion factors connecting them. Appendix B introduces radian as a measure of angles and covers the connection between angle, distance, and size. Appendix C is on vectors, their algebraic properties, and their components.

Appendix D is a large collection of examples with slightly more emphasis on numerical and algebraic calculation than is used in the book. Every example has a reference in the book and the textbook page on which it is referred to appears in the margin of that example. The placement of the numerical examples (and the Math Notes in Appendix E) on the CD makes it convenient for the reader to have both the book and the PDF file open at the same time and refer to either as necessary. With the exception of the last part of Example D.7.3 where a little trigonometry is used, no example uses mathematics beyond high school algebra.

Appendix E is composed mostly of algebraic derivations of equations mentioned (or occasionally used) in the text. Like Numerical Examples, every math note has a marginal note indicating where in the textbook it is referred to. With the exception of the last part of Math Note E.27.1 where a little trigonometry is used, one paragraph of Math Notes E.37.2, E.37.4, E.38.1, and Math Notes E.37.5, E.37.6, and E.39.2 where a little calculus is used, all math notes use mathematics at the level of high school algebra.

Appendix F is a short discussion of the technical aspects of spacetime geometry, including the rules governing its application. Finally Appendix G is a large set of numerical exercises which could be assigned as homework, or worked out in class by students in groups of 3 or 4 nearest neighbors, while the instructor walks around in the class answering questions.

Appendix A

Units in Physics

Physics studies the smallest particles and the largest galaxies in the universe. Being the most quantitative of all sciences, it makes measurements profusely and in domains of various sizes. The convenience of using different measuring apparatuses in different domains makes the variety of units unavoidable. You cannot measure the inside of an atom with the same yardstick that you measure the periphery of a galaxy.

Such a span of size demands a power-of-ten representation of numbers. Some of these powers have names, which usually come as a prefix to the unit used. Table A.1 contains the most common prefixes in use.

Prefix	Definition	Symbol	Power of ten
giga-	One billion	G	10^9
mega-	One million	M	10^6
kilo-	One thousand	k	10^3
centi-	One hundredth	c	10^{-2}
milli-	One thousandth	m	10^{-3}
micro-	One millionth	μ	10^{-6}
nano-	One billionth	n	10^{-9}

Table A.1: Most commonly used power-of-ten prefixes.

The units of length, time, and mass are the *fundamental units*, because other physical quantities can be expressed in terms of them. There are two systems of measurements in use: *Systèm International* (SI) and *United States Customary System* (USCS) (formerly known as the *British Imperial System*). The latter, which is used only in the US, measures length in foot, weight in pound, and time in seconds. The former, also called the international system or the metric system, measures length in meters, time in seconds, and mass in kilogram. Table A.2 shows some common quantities in SI units and the symbols used for them.

Meter was originally taken to be 1/20,000,000 of a meridian of Earth. Once determined, an equal length was marked off on a bar of platinum-iridium alloy. This bar is now kept in the International Bureau of Weights and Measures in France. **Second** used to be defined as 1/86400 of a mean solar day. However, as the Earth is slowing down in its motion around the Sun, such a definition was not accurate enough. So, in 1956, it was agreed to define a second in terms of the mean solar day of the year 1900. In 1964, second was defined to be the duration of 9,192,631,770 periods of the electromagnetic wave corresponding to the transition between two hyperfine levels of the ground state of the cesium-133 atom. In 1997 it was added that the cesium atom is to be assumed at rest and at 0 K. With second

Quantity	Unit	Symbol
Length	meter	m
Time	second	s
Mass	kilogram	kg
Force	newton	N
Charge	coulomb	C
Energy	joule	J
Temperature	kelvin	K

Table A.2: SI units.

so defined, the 17th Conférence générale des poids et mesures (CGPM) in 1983 adopted the definition of meter as the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second. The speed of light is now defined to be *exactly* 299,792,458 m/s. The reason for this exactness is that very accurate distance measurements are much harder to ascertain than very accurate time measurements. **Kilogram** is defined as the mass of one liter (0.001 m^3) of water at 4° Celsius.

Although no other country uses the USCS, for the benefit of our American readers, we provide the factors for converting the USCS units to SI units in Tables A.3 and A.4 .

USCS	mile	foot	inch	gallon	quart	fluid ounce
	5280 ft	12 in		4 qt	2 pt	
SI	1609 m	0.3048 m	0.0254 m	0.003785 m^3	0.000946 m^3	
	1.609 km	30.48 cm	2.54 cm	3.785 lit	0.946 lit	29.57 cm^3

Table A.3: Conversion of USCS to SI: length and volume.

USCS	pound	ounce	Btu
	16 oz		777.6 ft lb
SI	453.6 g	28.35 g	1054.4 J

Table A.4: Conversion of USCS to SI: mass and energy.

Information about physical constants such as the universal gravitational constant, speed of light, Planck constant, and a host of other numbers are available on the internet. Some helpful sites are given below.

http://physics.nist.gov/cuu/Constants/index.html
http://pdg.lbl.gov/
http://pdg.lbl.gov/2009/reviews/rpp2009-rev-phys-constants.pdf
http://www.physlink.com/Reference/PhysicalConstants.cfm
http://hyperphysics.phy-astr.gsu.edu/hbase/tables/funcon.html

Appendix B

Radian, Distance, and Size

In this appendix, we summarize the relation between angles and distances, because the distance and/or size of celestial objects are calculated using angles.

B.1 Radian

A convenient unit to use for such calculations is **radian**, which we describe now.

> **Box B.1.1.** *Draw any circle with center at the vertex of the angle. Measure length of arc of circle subtended by the angle. Divide this length by the radius of the circle. This ratio is the size of the angle in* ***radians****.*

Because the arc length of the circle varies in proportion to the length of its radius, the size of the angle in radians is independent of the circle used (see Figure B.1(a)). Radian is usually abbreviated as *rad*. It turns out that

$$1 \text{ rad} = 57.3^\circ$$

For detail see the example that follows.

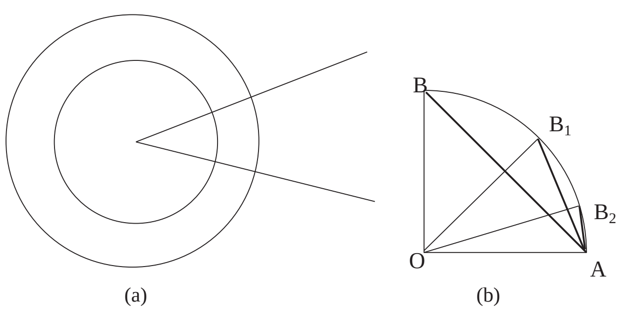


Figure B.1: (a) The ratio of the bigger arc subtended by the angle to the bigger radius is the same as the ratio of the smaller arc to the smaller radius. (b) As the angle gets smaller the length of the line segment and the arc length become almost equal. While the arc AB and the line segment $\overline{AB}$ are substantially different, the arc AB_2 and the line segment $\overline{AB_2}$ are indistinguishable in the drawing.

Angle in deg	Angle in radian	Approximation
0.1	0.001745328	0.001745328
0.5	0.008726639	0.008726611
1	0.017453278	0.017453056
3	0.052359833	0.052353852
5	0.087266389	0.087238701
7	0.122172944	0.122096976
10	0.174532778	0.174311339

Table B.1: The middle column gives the size of the angle in radian. The last column uses the length of the line segment instead of the arc length.

Example B.1.2. In going around the circle, we introduce an angle that is 360°. The arc associated with this angle is simply the circumference of the circle, which is $2\pi r$. Taking the ratio, we obtain

$$360^\circ = \frac{\text{circumference}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi \text{ rad}$$

Therefore,

$$1\text{rad} = \frac{360^\circ}{2\pi} = \frac{360^\circ}{6.2831853} = 57.29578 \text{ degrees}$$

which we usually round off to 57.3°.

Because smaller and smaller arcs look more and more straight, we can replace arcs with line segments without introducing too much error. Figure B.1(b) shows how the line segment and the arc become almost equal when the angle gets smaller and smaller. While the arc AB and the line segment $\overline{AB}$ are substantially different, the arc AB_1 and the line segment $\overline{AB_1}$ are equal to a better approximation, and the arc AB_2 and the line segment $\overline{AB_2}$ are almost indistinguishable in the drawing. Table B.1 compares the size of the angle in radians when the arc length is used with the approximation in which the length of the line segment is used instead. It is clear that, as long as the angle is small enough, the two results are almost identical. Even for the fairly large angle of 10 degrees, there is agreement to three significant figures.

B.2 Using Radian to Find Distances

This tells us something very useful: To find the angle (in radian) subtended by an object at a point O, divide the size of the object by its distance from O,

$$\text{angle in radian} = \frac{\text{size}}{\text{distance}} \tag{B.1}$$

Conversely, and more powerfully

Box B.2.1. *If you know the angle in radians and the distance, you can find the size of the distant object! If you know the angle in radians and the size of the distant object, you can find its distance!*

Let us see how this works.

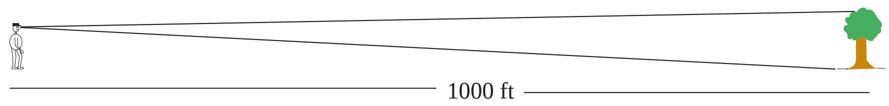


Figure B.2: From the distance and the angle you can find the height of the tree.

Example B.2.2. Suppose that you are standing 1000 ft away from a tree (see Figure B.2). You decide to measure the height of the tree without getting close to it. The first thing you want to do is measure the angular size of the tree. How? Take a sheet of paper and a long straight thin stick. Point it to the top of the tree; mark your line of sight; point the stick to the bottom of the tree; mark your line of sight; measure the angle between these two lines of sight with a protractor. Suppose that this angle is 2 degrees.

To use Equation (B.1), you need to find the angle in radians. Since each radian has 57.3° in it, we have

$$\text{angle in radian} = \frac{2}{57.3} = 0.0349$$

Now use Equation (B.1),

$$0.0349 = \frac{\text{size}}{1000} \quad \Rightarrow \quad \text{size} = 0.0349 \times 1000 = 34.9 \text{ ft}$$

■

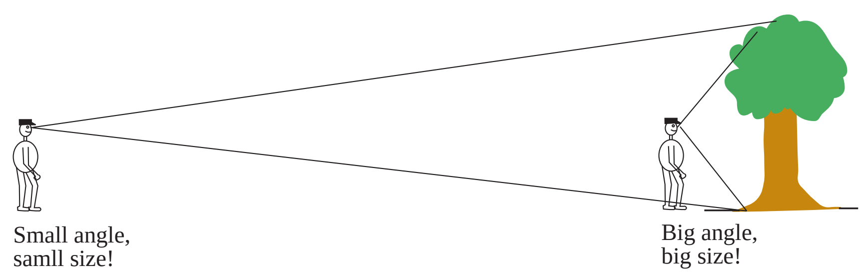


Figure B.3: The angle decreases as you move away from the tree.

Angular extent determines the apparent size of an object. As you move away from an object (or if the object moves away from you) its size appears to decrease, because the angle subtended by it decreases (Figure B.3). How far does it have to move before it appears as a single point? In other words, what is the smallest angle that the human eye can measure? If you draw two dots close together on a piece of paper, and move the paper away, the two dots appear closer and closer together until they become so close that the eye cannot separate them. Dividing the separation between the two dots by their distance from your eyes gives the smallest angle (in radians) your eye can perceive, or your eye's *resolving power*. It turns out that

Box B.2.3. *The resolving power of the eyes of most people is about 2 millidegrees or* 3.5×10^{-5} *radian.*

This means that an object that is so far away that it subtends an angle of 2 millidegrees or less, will appear as a single point regardless of its shape.

Now suppose that the object is very far and moving sideways (transverse motion). If the time is not long enough for the object to move a considerable distance,[1] we will not be able

[1] Enough to make the angle subtended by the transverse distance 2 millidegrees or more.

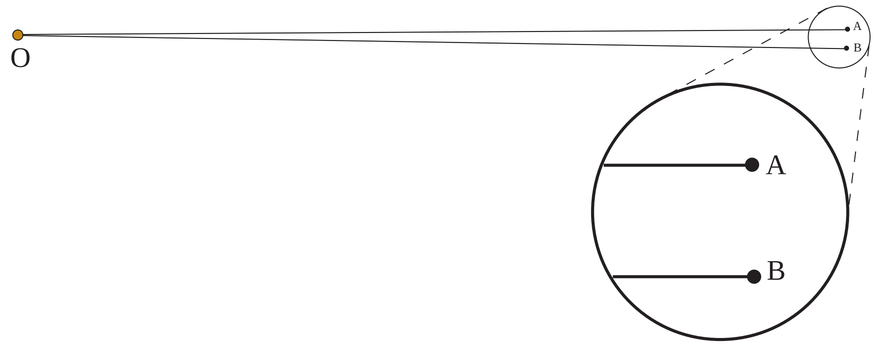


Figure B.4: As long as $\overline{AB}$ is much much smaller than $\overline{OA}$ or $\overline{OB}$, the rays coming from O appear parallel at A or B, or anywhere in between.

to detect the transverse motion. That is why the stars do not seem to be moving. They are at such an enormous distance that the angle subtended by their transverse distance is well below our resolving powers even for time intervals comparable to a human lifetime.

Example B.2.4. A truck 2 m high moves on a straight highway. We want to figure out how far it has to move away from us before we see it as a single point. For this to happen, the angular extension of the truck has to be 2 millidegrees or 3.5×10^{-5} radian. Using Equation (B.1), we write

$$3.5 \times 10^{-5} = \frac{2 \text{ m}}{\text{distance}} \quad \Rightarrow \quad \text{distance} = \frac{2 \text{ m}}{3.5 \times 10^{-5}} = 57,143 \text{ m} = 35.5 \text{ miles}$$

where in the last step we divided by 1,610 to convert meters to miles. ■

Suppose that a (point) light source is located far away sending light rays to an object which is small compared to the distance of the light source. If $\overline{AB}$ of Figure B.4 represents the size of the object, and $\overline{AB}$ is much much smaller than $\overline{OA}$ or $\overline{OB}$, then, as the blow-up of the region shows, the two rays $\overline{OA}$ and $\overline{OB}$ appear parallel. For example, suppose A is the north pole of Earth, and B its south pole. Then the two rays $\overline{OA}$ and $\overline{OB}$ from the Sun (represented by the point O) will appear parallel. In fact, any two Sun rays reaching any two points on Earth will be parallel, because the angle subtended by those two rays will be even smaller than the angle $\angle AOB$. Thus

> **Box B.2.5.** *The light rays coming from the Sun or any other star are all parallel as they reach the Earth.*

Appendix C

Vectors

In this appendix we introduce some simple properties of coordinate systems an vectors. The coordinate systems we introduce include nonperpendicular systems because of their application in relativity theory.

C.1 Coordinate Systems

A point P in space is an intrinsic entity independent of any "observer." However, in all applications, one chooses a **coordinate system** and assigns three numbers (x, y, z) to the point, its **coordinates**, as in Figure C.1(a). However, almost always we confine ourselves to a plane, the xy-plane. Then a point has only two coordinates (x, y) as in Figure C.1(b). How are these coordinates determined? There are two ways to get the coordinates of a point P:

- From P drop a perpendicular to each axis and measure the distance from the origin to the foot of the perpendicular.
- From P draw a parallel line to each axis and measure the distance from the origin to the intersection of the axis and the parallel line.

The two procedures above are, of course, equivalent. But there are situations where they give different results.

In some cases, for instance in the special theory of relativity, it is necessary to consider nonperpendicular axes. How does one determine the coordinates of a point P in such cases? If one used the "perpendicular" procedure, one would have a situation shown in Figure C.1(c), where a point on the y-axis would have a nonzero x-coordinate. Of course, we don't want this. So we have to use the "parallel" procedure as shown in Figure C.1(d).

C.2 Vectors

In all subsequent discussions it is helpful to go back to the most illustrative prototype of vectors, i.e., the position vector.

Many physical quantities have not only a numerical value, indicating their strength or **magnitude**, but also a *direction.* Such quantities are collectively called **vectors**. Practically every operation defined for ordinary numbers can also be defined for vectors: We can add vectors, subtract them, multiply them, differentiate and integrate them (vector calculus), etc. In fact, because of their additional property of direction, vectors have a much richer mathematics than ordinary numbers.

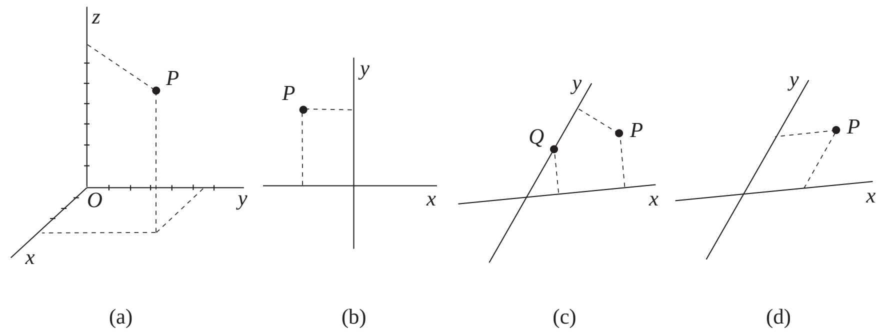


Figure C.1: (a) The point P has coordinates $(4, 5.5, 7)$ in the coordinate system O. (b) We usually deal with points in a plane, which require only two numbers as their coordinates. (c) For a nonperpendicular system coordinates cannot be determined by perpendicular drops. (d) Coordinates of a point are determined by lines parallel to the axes.

We do not intend to carry out a thorough investigation of vectors. Instead, we shall summarize some of their most basic properties needed in our future discussions. It is helpful to picture the **displacement** vector as a prototype of all vectors. There are many operations we can perform on vectors, some of which we introduce below.

Parallel transportation: A vector is not affected if you move it *parallel to itself* and do not stretch or shrink it. This process is called *parallel transportation.*

Length or magnitude: To every vector $\mathbf{v}$ is associated [1] a positive number denoted by v, or, sometimes, $|\mathbf{v}|$ and called the *length* or *magnitude* of $\mathbf{v}$. It is simply the length of the line segment representing the vector.

Equality of vectors: It should be clear that two vectors are equal if and only if they have the same length and they are parallel to one another. In other words, a vector is determined uniquely by its length and its direction.

Multiplication by a number: Given a vector $\mathbf{v}$ and a number t, we define the vector $t\mathbf{v}$ to have a *length* $|t|$ times[2] the length of $\mathbf{v}$ and a direction which is the same as $\mathbf{v}$ if t is positive, and opposite to it if t is negative (Figure C.2).

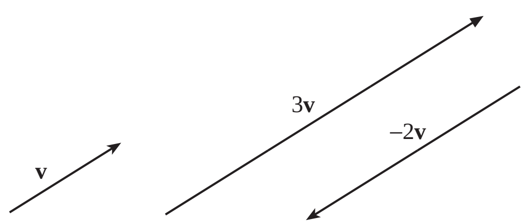


Figure C.2: Product of a number and a vector.

Addition of vectors: This operation is a generalization of the corresponding operation for displacement. Suppose an object in motion starts from the point A_1, goes to point A_2, then from A_2 moves on to A_3 (Figure C.3). The total displacement is clearly $\mathbf{r}_{13}$. But this can also be thought of as the sum of the displacements $\mathbf{r}_{12}$ and $\mathbf{r}_{23}$. Thus one writes

$$\mathbf{r}_{13} = \mathbf{r}_{12} + \mathbf{r}_{23}$$

[1] It is customary to denote vectors by **boldface** type, and we adhere to this notation throughout the book.

[2] Two vertical lines flanking a number indicate the *absolute value* of that number.

Generalizing this for arbitrary vectors, we define vector addition as follows.

> **Box C.2.1.** *Given vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, to find $\mathbf{v}_1 + \mathbf{v}_2$, parallel-transport $\mathbf{v}_2$ so that its tail coincides with the tip of $\mathbf{v}_1$, then draw the directed line segment from the tail of $\mathbf{v}_1$ to the tip of $\mathbf{v}_2$.*

Figure C.4(b) shows how addition is done. It also shows that the operation of addition is commutative: $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$. Obviously, if there are more than two vectors, we can add them by finding the sum of the first two, then adding the third to this sum and so on. This amounts to putting the tail of one at the head of another and continuing until all vectors are drawn. The vector obtained by connecting the tail of the first to the head of the last is the sum of all vectors. The sum thus obtained will not depend on the order in which the vectors are added to one another, because at each step the order of addition is irrelevant.

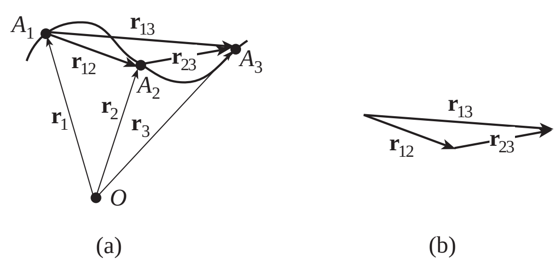


Figure C.3: (a) Position vectors and displacements. (b) The displacement r_{13} is the sum of the displacements r_{12} and r_{23}.

Subtraction: To subtract $\mathbf{v}_2$ from $\mathbf{v}_1$ simply add $-\mathbf{v}_2$ to $\mathbf{v}_1$ as shown in Figure C.4(c). Note that $\mathbf{v}_1 - \mathbf{v}_2 = -(\mathbf{v}_2 - \mathbf{v}_1)$. Equivalently, the difference between any two vectors is determined as follows: Draw the two vectors from a common point. This may involve transporting the vectors parallel to themselves. Now draw the directed line segment from the tip of the "initial vector" (i.e., the vector being subtracted) to the tip of the final vector. In the case of displacement vector, this procedure seems "natural." For other vectors it may not appear as intuitive, but we note that the difference so defined has the property that when it is added to the "initial" vector, it gives the final vector. Thus, if we add the vector labeled $\mathbf{v}_1 - \mathbf{v}_2$ in the second diagram of Figure C.4(c) to $\mathbf{v}_2$, we get $\mathbf{v}_1$, as we should. For any mathematical quantity, this is the property we expect of the difference.

Decomposition: Given a vector $\mathbf{v}$, we can write it as the sum of a pair of vectors. The pair is not unique in the sense that there are many (in fact, infinitely many) different pairs whose sum equals the given vector. We say that $\mathbf{v}$ is **decomposed** into the two vectors, and that each vector in a pair is a **component** of $\mathbf{v}$. Figure C.5(a) shows a vector $\mathbf{v}$ being decomposed into three different pairs. Figure C.5(b) shows the same vector being decomposed into its most commonly used horizontal and vertical components. The horizontal and vertical components could be thought of as lying along the axes of a coordinate system. In that case we speak of the x- and y-components of a vector $\mathbf{v}$, and we write v_x and v_y instead of $\mathbf{v}_{\text{hor}}$ and $\mathbf{v}_{\text{ver}}$. v_x and v_y are simply the projections of $\mathbf{v}$ on the two axes. The length of $\mathbf{v}$ can be obtained from its components by using the Pythagorean theorem: $|\mathbf{v}|^2 = v_x^2 + v_y^2$ or $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$.

Given a vector and a set of axes, there is a unique way that it can be decomposed along those axes. Stated differently, the components of a vector along the axes of a given

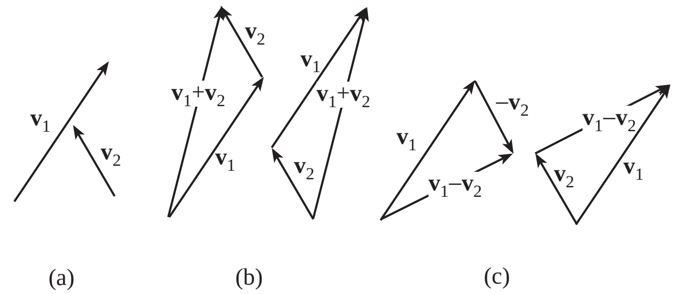


Figure C.4: (a) Two arbitrary vectors. (b) To add, draw one vector from the tip of the other; then the tail of the first to the tip of the second. (c) To subtract, either add the negative of the vector, or draw both vectors from the same point; then connect the tip of the vector being subtracted to the tip of the other vector.

coordinate system determine the vector uniquely. Thus, a vector is determined uniquely either by its magnitude and direction, or by its components along the axes of a coordinate system. Since a projection can never be larger than the vector itself, the maximum value a component of $\mathbf{v}$ can assume is v.

Three-dimensional vectors: In the discussion above, we have implicitly assumed that the vectors are drawn in a plane. With the exception of decomposition, all discussions go through unchanged for vectors in space. When we are dealing with a single vector (or even two vectors), we can do our decomposition in a plane (in the case of two vectors, the plane defined by them). However, in general, we need three axes to decompose (or project) vectors, because physical vectors are generally three-dimensional. Figure C.5(c) shows such a vector with its three components (or projections) along the axes of a (three-dimensional) coordinate system. The magnitude of the vector is obtained by using the (three-dimensional) Pythagorean theorem: $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$.

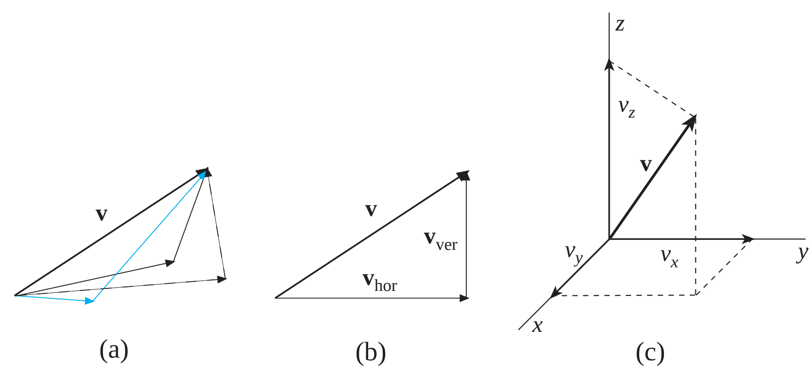


Figure C.5: (a) A vector **v** is decomposed into a pair of vectors in three different ways. (b) The most common decomposition is into vertical and horizontal components. (c) A real physical vector generally has *three* components.

Appendix D

Numerical Examples

D.3 Numerical Examples for Chapter 3

Example of Kepler's 3rd law
(page 42 of the book)

Example D.3.1. The constant of proportionality can be determined by examining the motion of one of the planets, say Earth. The period of revolution T of the Earth around the Sun is approximately one year and a quarter of a day. For later reference, we want to use the scientific units. So, T should be converted into seconds. This is easily done

$$365\tfrac{1}{4} \times 24 \times 3600 = 3.15576 \times 10^7 \text{ seconds}$$

Since the orbit of the Earth is almost circular, the semimajor axis is the same as the (average) radius of the orbit, i.e., the Earth–Sun distance, which happens to be 150 million km, or 1.5×10^{11} m. Substituting these two numbers in Kepler's third law gives

$$\left(3.15576 \times 10^7\right)^2 = k \times \left(1.5 \times 10^{11}\right)^3$$

or, computing the powers on both sides

$$9.9588 \times 10^{14} = 3.375 \times 10^{33} k \quad \text{or} \quad k = \frac{9.96 \times 10^{14}}{3.375 \times 10^{33}} = 2.9508 \times 10^{-19}$$

This k will work only if T is given in seconds and a in meters.

With k at our disposal, we can find the period of other planets if we know their semimajor axes, or vice versa. As an example, consider Mercury, whose orbit is very elliptical. Its closest distance to the Sun is 46 million km, but its farthest distance is 70 million km. This gives a semimajor axis of about 58 million km. Mercury's period can thus be found from

$$T^2 = 2.9508 \times 10^{-19} \times \left(5.8 \times 10^{10}\right)^3 = 5.75728 \times 10^{13}$$

and

$$T = \sqrt{5.756 \times 10^{13}} = 7.588 \times 10^6 \text{ s}$$

Note that we had to convert the orbital radius from km to meter, and that the final answer is in seconds. We can find this period in Earth days. Since there are $24 \times 3600 = 86,400$ seconds in an Earth day, we have

$$T_{\text{Mercury}} = \frac{7.588 \times 10^6}{86,400} = 87.8 \text{ Earth days.}$$

Thus, Mercury's years are less than a quarter of an Earth year.

Until 1962 it was thought that Mercury's "day" was the same length as its "year" so as to keep the same face to the Sun much as Moon does to the Earth. But this was shown to be false in 1965 by doppler radar observations. It is now known that Mercury spins three times in two of its years.

The third law is sometimes written in a more convenient form by using two planets. Let T_1 and T_2 be the periods of two planets and a_1 and a_2 their semi-major axes. Then $T_1^2 = ka_1^3$ and $T_2^2 = ka_2^3$. By taking the ratio of these equations the third law becomes

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^3 \tag{D.1}$$

Because k is absent in this equation, one can measure T's and a's in any convenient units. For instance, T's could be measured in years, or days, or seconds (as long as both are measured in the *same* units), and a's in million kilometers, kilometers, or meters.

As a concrete example, let 1 refer to Mercury and 2 to Earth. Then $a_1 = 58$ million kilometers, $T_2 = 365.25$ days, and $a_2 = 150$ million kilometers. Therefore,

$$\left(\frac{T_1}{365.25}\right)^2 = \left(\frac{58}{150}\right)^3 \quad \text{or} \quad \frac{T_1^2}{365.25^2} = (0.3867)^3 = 0.0578$$

and

$$T_1 = \sqrt{0.0578 \times 365.25^2} = 87.8\text{days}$$

Note that no conversion to seconds and meters was necessary.

D.4 Numerical Examples for Chapter 4

An example of UAM **(page 59 of the book)**

Example D.4.1. A mad driver is moving at the rate of 35 m/s on a residential street! A cat suddenly jumps in front of the car 105 m away. The driver slams on the brakes "immediately" with a reaction time delay of 0.2 second. The brakes cause a deceleration of 6 m/s^2.

Q: Is the cat dead or alive?

A: To answer this question, we will

1. find how far the car travels before the brakes are *actually* applied;
2. plug in all the known quantities in the two kinematics formulas for uniformly accelerated motion, paying attention to the *sign* of the acceleration;
3. determine how long it will take for the car to come to a complete stop;
4. knowing the time of decelerated travel, we will calculate the distance the car travels while decelerating to a complete stop, and finally
5. add the two distances to get the total distance.

1. The reaction time is the time that it takes for the act of seeing the cat in the middle of the street to translate into applying brakes. This delay is due to the physical fact that no signal or information travels with infinite speed. So, for the eye to transmit the sight of the panicked cat to the brain and for the brain to analyze the situation and decide what to do, and then transmit the decision to the muscles of the leg takes some time. In this case, 0.2 second. During this time the car travels uniformly with its initial speed. Thus, the distance in this case is

$$x = vt = 35 \times 0.2 = 7 \text{ m}$$

2. With $v_0 = 35$ and $a = -6$, Equations (4.3) and (4.4) yield

$$v(t) = 35 - 6t$$
$$x(t) = 35t - 3t^2.$$

These two equations give the speed and distance traveled for any given time t *for this particular problem.*

3. For the car to come to a complete stop, v has to be zero. Therefore,

$$0 = 35 - 6t \quad \Rightarrow \quad 6t = 35 \quad \Rightarrow \quad t = \frac{35}{6} = 5.833 \text{ s}$$

4. Substitute this t in the equation for $x(t)$ to find the distance:

$$x = 35 \times 5.833 - 3(5.833)^2 = 102 \text{ m}.$$

5. The total distance is the sum of the two distances found in (1) and (4):

$$x_{\text{tot}} = 7 + 102 = 109 \text{ m}.$$

The cat is dead!

By the way, the reason that the driver is designated as “mad” can be appreciated by converting the speed to mph:

$$35 \text{ m/s} = \frac{35}{0.4472} = 78.26 \text{ mph!}$$

On a residential street?

D.6 Numerical Examples for Chapter 6

Calculating the minuscule star parallax (**page 76 of the book**)

Example D.6.1. To see the enormity of stellar distances and the minuteness of their parallaxes, consider Epsilon Eridani, a star that is a mere 10.8 light years away. Figure D.1 (on the left) shows the Earth E moving around the Sun on a circular orbit. Two locations of Earth six months apart are shown as E_1 and E_2. Two position vectors from Earth to a star S are also shown. Note that $\overline{E_1E_2} = \overline{S_1S_2}$, and that the Earth–Sun distance is 150 million km. Thus,

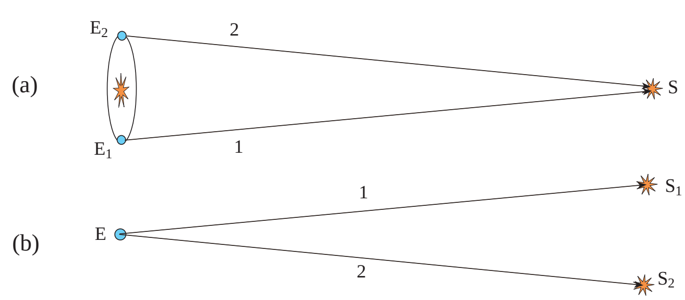


Figure D.1: (a) Earth moves around the Sun from its first position E_1 to its second position E_2. (b) The star appears to have moved from S_1 to S_2, describing a parallax. The Earth–Sun distance is vastly exaggerated to show the parallax. Note that $\overline{E_1E_2} = \overline{S_1S_2}$.

$$\overline{E_1E_2} = 2 \times 150,000,000 = 300,000,000 = 3 \times 10^8 \text{ km}$$

On the other hand, noting that a light year is 9.45×10^{12} km, you find the distance between Earth and Epsilon Eridani to be

$$10.8 \times 9.45 \times 10^{12} = 1.02 \times 10^{14} \text{ km}$$

Thus, the maximum possible parallax—corresponding to the largest displacement of Earth around the Sun—in radian (see Appendix B) is

$$\text{parallax in radian} = \frac{\overline{E_1 E_2}}{\text{distance of Epsilon Eridani}} = \frac{3 \times 10^8}{1.02 \times 10^{14}} = 2.94 \times 10^{-6}$$

At 57.3 degrees per radian, this is only 0.000169 degree. A minute amount indeed! If a close star such as Epsilon Eridani has such a small maximum parallax, the other stars which are considerably farther than Epsilon Eridani—some thousands of light years away—must have even smaller parallaxes. No wonder there was so much resistance in accepting the heliocentric theory of the solar system in ancient times!

Example D.6.2. A car moves east from A to B for one hour at 60 mph, and north from B to C for $1\frac{1}{2}$ hours at 40 mph (see Figure D.2).

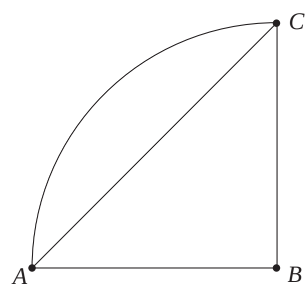


Figure D.2: A car moves from A to C through B and returns to A along the quarter-circle.

Illustration of speed and velocity
(page 78 of the book)

Q: What is the distance the car travels and its average speed?
A: The distance is the length of $\overline{AB}$ plus the length of $\overline{BC}$. But

$$\overline{AB} = 1 \times 60 = 60 \text{ miles} \quad \text{and} \quad \overline{BC} = 1.5 \times 40 = 60 \text{ miles}.$$

So, the total distance is 120 miles. The average speed is

$$v_{\text{avg}} = \frac{\text{distance}}{\Delta t} = \frac{120 \text{ miles}}{2.5 \text{ hours}} = 48 \text{ mph}.$$

Q: What is the car's displacement and its average velocity?
A: The displacement, whose length is $\overline{AC}$, is obtained by the Pythagoras theorem:

$$\overline{AC} = \sqrt{60^2 + 60^2} = \sqrt{7200} = 84.85 \text{ miles} \quad \Rightarrow \quad \Delta\mathbf{r} = 84.85 \text{ miles, northeast}.$$

From this we calculate the average velocity:

$$\mathbf{v}_{\text{avg}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{84.85 \text{ miles, northeast}}{2.5 \text{ hours}} = 33.9 \text{ mph, northeast},$$

Now suppose that the car comes back from C to A in $1\frac{1}{2}$ hours along the quarter-circle shown.

Q: What are the average velocity and average speed?

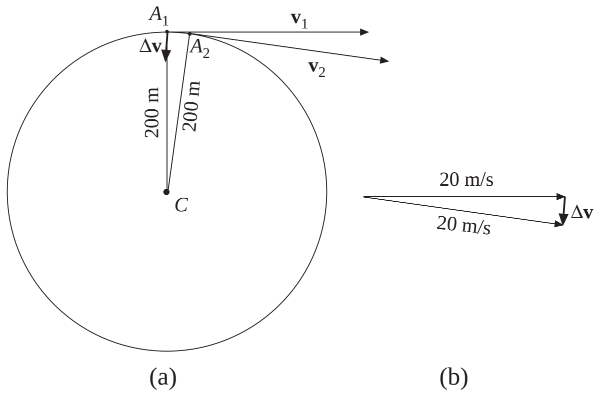


Figure D.3: (a) As the car moves from A_1 to A_2, its velocity changes slightly. (b) The change in velocity $\Delta\mathbf{v}$.

A: The average velocity is easily obtained,

$$\mathbf{v}_{\text{avg}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{84.85 \text{ miles, southwest}}{1.5 \text{ hours}} = 56.57 \text{ mph, southwest.}$$

To find the average speed, we need to calculate the distance, which is the length of a quarter-circle of radius 60 miles. This length is one fourth of the circumference of the (full) circle:

$$\text{circumference} = 2\pi r = 2 \times 3.14159 \times 60 = 377 \text{ miles} \quad \Rightarrow \quad \text{distance} = \frac{377}{4} = 94.25.$$

Thus,

$$v_{\text{avg}} = \frac{\text{distance}}{\Delta t} = \frac{94.25 \text{ miles}}{1.5 \text{ hours}} = 62.83 \text{ mph.}$$

Acceleration of a car on a circle
(page 82 of the book)

Example D.6.3. A car is moving on a curved road in the shape of an arc of a circle of radius 200 m. Suppose that the car is moving eastward momentarily with a constant speed of 20 m/s, as shown in Figure D.3(a) on page 17.

Q: What is the magnitude and direction of the centripetal acceleration of the car at that moment?

A: The car is initially at point A_1. To find the *instantaneous* acceleration, we have to know the change in the velocity a short while, say a second, later. In that period, the car has moved to A_2, which is 20 m away from A_1 on the circle. The angle A_1CA_2 is $20/200 = 0.1$ radian (see Appendix B). This is the same as the angle between the velocities $\mathbf{v}_1$ at A_1 and $\mathbf{v}_2$ at A_2 as shown in Figure D.3(b). Since the angle is small (about 5.7°), we can approximate $\Delta\mathbf{v}$ by the arc of a circle whose radius is the length of the velocity vector. Then the definition of *radian* yields

$$\text{angle in radian} = \frac{\text{arc length}}{\text{radius}} \quad \text{or} \quad 0.1 = \frac{\text{arc length}}{20 \text{ m/s}} \quad \Rightarrow \quad \text{arc length} = 2 \text{ m/s}$$

Therefore, $\Delta\mathbf{v}$ = arc length = 2 m/s *southward*. The specification of the direction is important, because $\Delta\mathbf{v}$ is a vector. Of course, $\Delta\mathbf{v}$ does not point southward *exactly* (it is slightly west of south), but for smaller and smaller angles it gets closer and closer to southward. Since the change in the velocity took place in 1 s, the acceleration is 2 m/s^2. This turns out to be identical to the exact result obtained by using Equation (6.1).

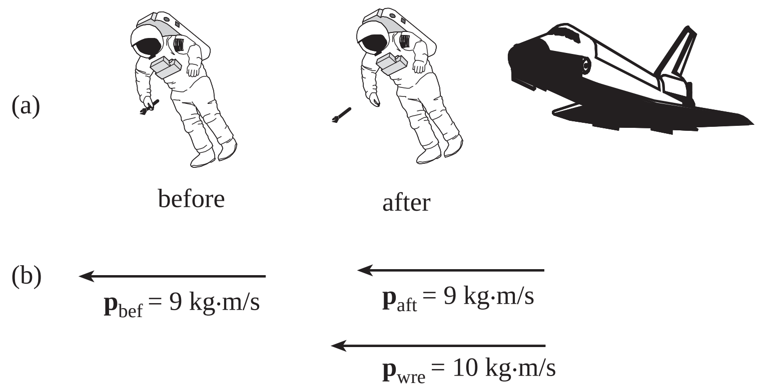


Figure D.4: (a) The stranded astronaut with his wrench before and after the throw. The wrench moves away from the astronaut and the spaceship. (b) The total momentum before and after the throw *are equal*. For the astronaut to be able to get back to the ship, the wrench must be thrown *away* from the spaceship with a momentum *larger* than the total momentum.

D.7 Numerical Examples for Chapter 7

Astronaut lost in space
(page 92 of the book)

Example D.7.1. An astronaut with a total mass (astronaut plus all his equipment) of 90 kg is detached from his spaceship and moves with a speed of 0.1 m/s away from it (to the left) at a distance of 20 m [Figure D.4(a)]. The commander tells the astronaut to throw the 0.5-kg wrench he is holding as hard as he can. The astronaut follows the order, throwing the wrench at a speed of 20 m/s relative to the spaceship. Can the astronaut make it back to the spaceship? Consider the astronaut plus the wrench as a system.

Q: What is the total momentum of this system before the throwing of the wrench?

A: Use the defining equation (7.1) and get

$$\mathbf{p}_{\text{bef}} = m\mathbf{v} = 90 \times 0.1 = 9 \text{ kg·m/s, to the left.}$$

This momentum is shown as an arrow in Figure D.4(b).

The momentum of this isolated system remains the same after the astronaut throws the wrench. It should be clear that he should throw the wrench to the left if he is to have a chance of returning to the spaceship to the right.

Q: What is the momentum of the wrench?

A: That also is given by (7.1):

$$\mathbf{p}_{\text{wre}} = m_{\text{wre}}\mathbf{v}_{\text{wre}} = 0.5 \times 20 = 10 \text{ kg·m/s, to the left.}$$

This momentum is also shown as an arrow in Figure D.4(b).

Q: What is the momentum of the astronaut?

A: The total momentum *after the throw* is the momentum of the wrench plus the momentum of the astronaut. And this total momentum must be 9 kg·m/s to the left. Since the wrench has a momentum of 10 kg·m/s to the left, the astronaut must have a momentum of 1 kg·m/s to the *right*.

Q: What is the speed of the astronaut?

A: Well, since $\mathbf{p}_{\text{ast}} = m_{\text{ast}}\mathbf{v}_{\text{ast}}$, and since $\mathbf{p}_{\text{ast}}$ is 1 kg·m/s to the right, and m_{ast} is 89.5 kg (total mass minus the mass of the wrench), then

$$1 = 89.5\mathbf{v}_{\text{ast}} \quad \text{or} \quad \mathbf{v}_{\text{ast}} = \frac{1}{89.5} = 0.0112 \text{ m/s, to the right.}$$

Q: How long does it take the astronaut to reach the spaceship?

A: Since the distance is 20 m and the speed is 0.0112 m/s, we have

$$x = vt \quad \Rightarrow \quad 20 = 0.0112t \quad \Rightarrow \quad t = \frac{20}{0.0112} = 1790 \text{ s}$$

or about half an hour.

If the astronaut had thrown the wrench with a speed too much slower than 20 m/s, say 15 m/s, then $\mathbf{p}_{\text{wre}}$ in Figure D.4(b) would have been $0.5 \times 15 = 7.5$ kg·m/s to the left. Since the total momentum is 9 kg·m/s to the left, the astronaut would have had to have a momentum of 1.5 kg·m/s *to the left*, and away from the spaceship.

The same kind of analysis can be used to show that under no circumstances should the wrench be thrown *toward* the spaceship.

Finding speed of a bullet **(page 92 of the book)**

Example D.7.2. A 50-gram bullet is fired on a 2-kg wooden block resting on a smooth surface. The bullet penetrates into the block and the whole system is seen to move with a speed of 3 m/s. What was the initial speed of the bullet?

The momentum after collision is

$$p_{\text{after}} = (2 + 0.05) \times 3 = 2.05 \times 3 = 6.15 \text{ kg·m/s}$$

Here, we converted grams (mass of the bullet) to kg. Since the momentum of an isolated system (here, the bullet-block system) does not change, and since the block was initially stationary, the initial momentum of the bullet must have been 6.15 kg·m/s. Therefore,

$$6.15 = 0.05v \quad \text{or} \quad v = \frac{6.15}{0.05} = 123 \text{ m/s.}$$

Motion of a projectile **(page 94 of the book)**

Example D.7.3. Decomposing both $\mathbf{F}_{\text{net}}$ and $\mathbf{a}$ along the horizontal and vertical directions, and (for convenience) suppressing the subscript "net", you obtain the following two equations:

$$\begin{aligned} F_{\text{hor}} &= ma_{\text{hor}} \\ F_{\text{vert}} &= ma_{\text{vert}} \end{aligned} \tag{D.2}$$

The motion of a projectile has two components: a horizontal component that has no force, and thus no acceleration, and a vertical component that has the gravitational force, leading to the gravitational acceleration of 9.8 m/s^2 downward. Therefore the two equations in (D.2) give $a_{\text{hor}} = 0$ and, assuming that up is the positive direction, $a_{\text{vert}} = -9.8$ m/s^2.

Let x denote the horizontal distance and y the vertical distance. Now suppose that we throw a projectile with a horizontal speed of 3 m/s, and an initial vertical speed of 4 m/s. The x motion is uniform, and the y motion is uniformly accelerated. Using Equations (4.1) and (4.4), we obtain

$$\begin{aligned} x &= 3t \\ y &= 4t - 4.9t^2 \end{aligned} \tag{D.3}$$

The first equation gives $t = x/3$, which, when plugged in the second equation, yields

$$y = \tfrac{4}{3}x - 4.9\left(\frac{x}{3}\right)^2 = -0.544x^2 + 1.333x \tag{D.4}$$

which is the equation of a parabola.

An interesting property of (D.4) is obtained when you write it as

$$y = x(-0.544x + 1.333)$$

because it now clearly shows that $y = 0$ (i.e., the projectile is at the ground) when $x = 0$ and $x = 1.333/0.544 = 2.448$ m. The first x corresponds to the firing of the projectile, the second to its landing. So, 2.448 m is the *range* of the projectile.

It is instructive to consider a general projectile motion in which the horizontal speed is v_{0x}, the vertical speed is v_{0y}, and the vertical acceleration is denoted by g. Then, Equation (D.3) becomes

$$\begin{aligned} x &= v_{0x}t \\ y &= v_{0y}t - \tfrac{1}{2}gt^2 \end{aligned} \tag{D.5}$$

The first equation gives $t = x/v_{0x}$, which, when plugged in the second equation, yields

$$y = \frac{v_{0y}}{v_{0x}}x - \tfrac{1}{2}g\left(\frac{x}{v_{0x}}\right)^2 = -\frac{g}{2v_{0x}^2}x^2 + \frac{v_{0y}}{v_{0x}}x \tag{D.6}$$

which is again the equation of a (general) parabola. The range R can be obtained as before:

$$y = x\left(\frac{v_{0y}}{v_{0x}} - \frac{g}{2v_{0x}^2}x\right) \quad \Rightarrow \quad R = \frac{\dfrac{v_{0y}}{v_{0x}}}{\dfrac{g}{2v_{0x}^2}} = \frac{2v_{0x}v_{0y}}{g} \tag{D.7}$$

Familiarity with trigonometry is needed for this paragraph!

Readers familiar with trigonometry note that, if v_0 is the initial velocity and θ is the angle of the projectile, then $v_{0x} = v_0 \cos\theta$ and $v_{0y} = v_0 \sin\theta$, and Equation (D.7) gives

$$R = \frac{2v_{0x}v_{0y}}{g} = \frac{2(v_0\cos\theta)(v_0\sin\theta)}{g} = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g} \tag{D.8}$$

This is an interesting result. It says that for any given initial velocity, the maximum range is obtained when $\sin(2\theta) = 1$, or $\theta = 45°$.

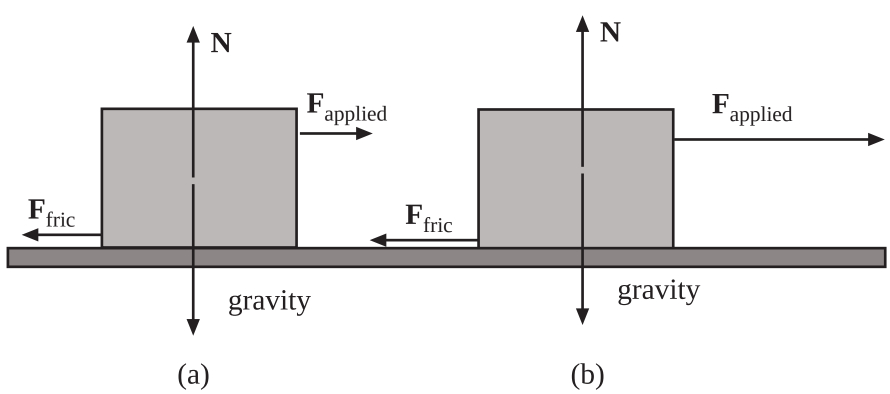


Figure D.5: (a) When the applied force is smaller than the maximum frictional force, the latter adjusts itself to the former. (b) Frictional force can be measured with this simple experiment.

Finding force of friction (**page 95 of the book**)

Example D.7.4. Consider a box whose mass is 3 kg. It is observed that when a force of 10 N is applied to it, the box acquires an acceleration of 2 m/s^2.

Q1: What is the net force on the block?

A1: From the information given and the second law, we obtain[1]

$$F_{\text{net}} = ma = 2 \times 3 = 6 \text{ N}$$

[1] In most examples in this book, we need not worry about the most general *direction* of vectors, because all vectors of a given problem are usually along the same line. The present example is an illustration, and that is why I am abandoning the boldface (vector) notation.

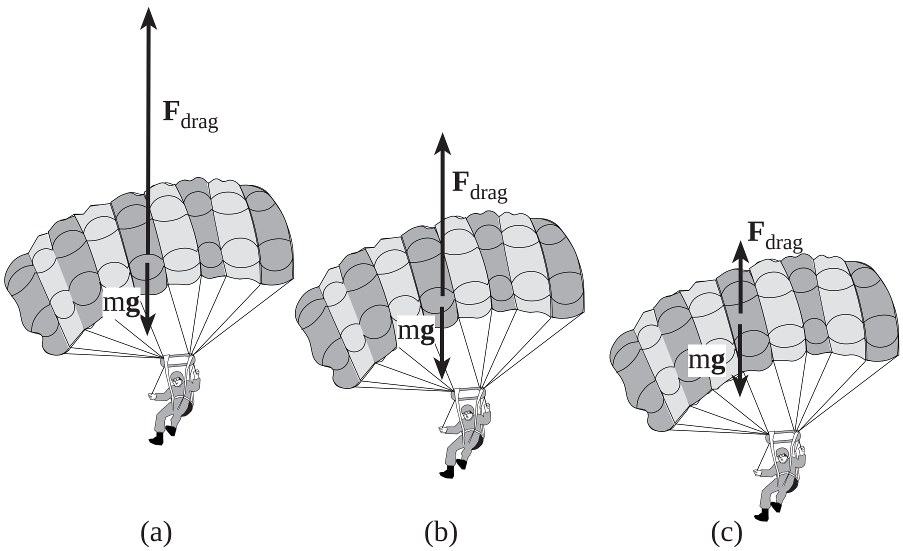


Figure D.6: The praying parachutist! (a) As soon as he opens the parachute—after falling freely for a while—the deceleration is large. (b) After a while the speed (and with it the air drag) decreases slowing down the process of slowing down! (c) The terminal speed is reached when the force of air drag balanced the weight.

Q2: What is the frictional force on the block?

A2: Since the applied force and the frictional force point in opposite directions, and since the net force is the vector (in this case, algebraic) sum of all vectors, we have

$$F_{\text{net}} = F_{\text{applied}} - F_{\text{fric}}$$

Substituting the numbers yields

$$6 = 10 - F_{\text{fric}} \quad \Rightarrow \quad F_{\text{fric}} = 10 - 6 = 4 \text{ N}.$$

Example D.7.5. Consider the motion of a parachutist as shown in Figure D.6. Suppose that the force of air drag can be written as $F_{\text{drag}} = 200v$ where v is the speed of the parachutist, that the mass of the parachutist plus the parachute and everything else in motion is 100 kg, and that the parachutist does not open the parachute for a while. He falls down freely until his speed reaches 20 m/s. Now he opens the parachute. We want to analyze his (vertical) motion.

Motion of a parachutist (**page 96 of the book**)

There are two forces acting on the parachutist, the weight and the drag force. The weight is

$$\mathbf{w} = m\mathbf{g} = 100 \times 9.8 = 980 \text{ N downward}$$

The drag force—when the speed is 20 m/s—is

$$F_{\text{drag}} = 200v = 200 \times 20 = 4000 \text{ N upward}$$

because *the drag force always opposes the motion.* These forces are shown in Figure D.6(a). The net force is therefore $\mathbf{F}_{\text{net}} = 4000 - 980 = 3020$ N upward, and the second law of motion gives the acceleration:

$$\mathbf{F}_{\text{net}} = m\mathbf{a} \quad \Rightarrow \quad 3020 \text{ N upward} = 100\mathbf{a} \quad \text{or} \quad \mathbf{a} = \frac{3020}{100} = 30.2 \text{ m/s}^2 \text{ upward.}$$

The acceleration is opposite to the direction of motion. Therefore, the system slows down.

A little later we find the system moving at the rate of 10 m/s. The air drag is reduced to $\mathbf{F}_{\rm drag} = 200 \times 10 = 2000$ N upward, and since the weight has not changed, the net force is now [see Figure D.6(b)]

$$\mathbf{F}_{\rm net} = 2000 - 980 = 1020 \text{ N upward.}$$

Once again, the second law of motion gives the acceleration:

$$\mathbf{F}_{\rm net} = m\mathbf{a} \quad \Rightarrow \quad 1020 \text{ N upward} = 100\mathbf{a} \quad \text{or} \quad \mathbf{a} = \frac{1020}{100} = 10.2 \text{ m/s}^2 \text{ upward.}$$

The acceleration is still opposite to the direction of motion. Therefore, the system keeps slowing down.

The parachutist keeps falling—and slowing—down. But the process of slowing down cannot go on forever. In fact, once the speed is so small that the drag force exactly balances the weight, the process stops. Why? Because if the speed gets smaller, the drag force decreases; the weight overpowers the drag force; the net force will be pointing downward; and the system starts to accelerates. But as soon as it accelerates, the speed increases, the drag force increases until it is equal to the weight again. The speed at which the drag force and the weight are equal [see Figure D.6(c)] is called the **terminal speed** or **terminal velocity**. It is obtained by equating the weight and the general form of the drag force:

$$980 = 200v \quad \Rightarrow \quad v = \frac{980}{200} = 4.9 \text{ m/s.}$$

Once this speed is reached, the parachutist will neither speed up nor slow down. Parachutes are designed in such a way that the terminal speed is low enough to be safe for landing.

Motion of an elevator
(page 97 of the book)

Example D.7.6. Consider the downward motion of a 50-kg person standing on a scale in an elevator as shown in Figure D.7. Let us first concentrate on the motion at the very beginning, i.e., when the elevator starts moving down with an acceleration of 4 m/s^2. The second law tells us that the net force on the person is

$$\mathbf{F}_{\rm net} = m\mathbf{a} = 50 \times 4 = 200 \text{ N downward.}$$

because the acceleration is downward. This net force is the result of two forces acting on the person, her weight acting downward and the force of the scale acting upward. The weight is

$$\mathbf{w} = m\mathbf{g} = 50 \times 9.8 = 490 \text{ N downward.}$$

Therefore, the force of scale on the person $\mathbf{F}_{\rm sp}$ must be pointing up and have the magnitude

$$F_{\rm sp} = 490 - 200 = 290 \text{ N upward.}$$

All these forces are shown in Figure D.7(a).

Now if the scale is exerting the force $F_{\rm sp}$ on the person upward, *by the third law of motion*, the person must exert a force $F_{\rm ps} = -F_{\rm sp}$ on the scale downward. The reading on the scale is precisely the force exerted on it. So, the scale reads the "weight" of the person to be 290 N. It is everybody's common experience that as the elevator starts to come down, one feels slightly lifted up.

After a while the elevator will reach a *constant* speed, at which point the acceleration becomes zero, the net force becomes zero, and therefore, the two forces on the person balance each other. So, $F_{\rm sp} = 490$ N pointing up. Therefore, the person must be pushing down on the scale by the same force and the scale shows the true weight of the person [see Figure D.7(b)].

Now the elevator approaches the bottom and starts slowing down with a deceleration of 2 m/s^2. The net force on the person is now

$$\mathbf{F}_{\rm net} = m\mathbf{a} = 50 \times 2 = 100 \text{ N upward,}$$

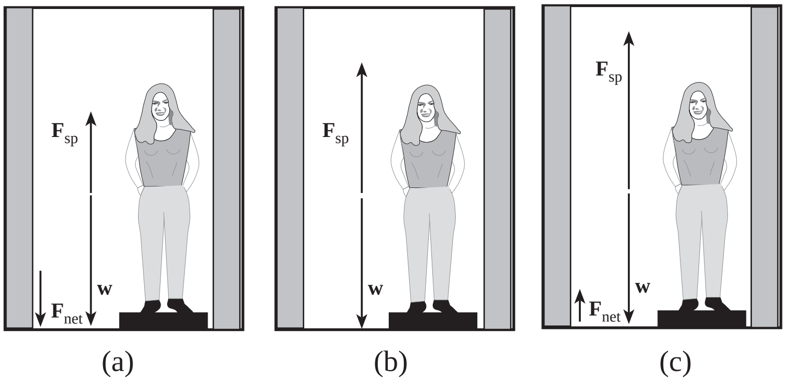


Figure D.7: The person on a scale in an elevator. (a) The elevator starts accelerating downward. (b) The elevator has cruised to a constant speed. (c) The elevator is decelerating to stop.

because the acceleration—opposing the downward motion—is upward. This means that $\mathbf{F}_{\text{sp}}$ must overpower the weight by 100 N. Therefore, $\mathbf{F}_{\text{sp}}$ must be 590 N pointing up. By the third law, the person must be pushing down on the scale by the same force. So, the scale reads the "weight" of the person to be 590 N. It is again common experience that as the elevator starts to stop on its way down, one feels slightly pushed against the floor [see Figure D.7(c)].

Let us go back to the beginning and suppose that, as the elevator starts descending from the top, its acceleration is 9.8 m/s^2. The net force on the person is

$$\mathbf{F}_{\text{net}} = m\mathbf{a} = 50 \times 9.8 = 490 \text{ N downward.}$$

Since her weight is 490 N, the other force must be zero. The scale is exerting no force on the person. The third law prohibits the person from exerting any force on the scale. The reading on the scale is therefore zero! She is weightless!

The reader notes that in the last case the elevator is in free fall, i.e., the supporting cables are not holding it back. Since all objects fall at the same rate,[2] the elevator and all its occupants fall at the same rate, so there is no reason for the person to "speed up" to the bottom of the elevator. She will float in midair!

Example D.7.7. The car of the roller coaster of Figure D.8 is upside *down* but not *falling* down. How can that be? Let us analyze its motion. Suppose that the mass of the car plus its passenger is 150 kg and it is moving with a speed of 10 m/s on a circle of radius 5 m. The acceleration of the system is

Looping roller coaster **(page 97 of the book)**

$$a = a_{\text{cent}} = \frac{v^2}{r} = \frac{10^2}{5} = 20 \text{ m/s}^2 \text{ downward,}$$

because a_{cent} always points toward the center and the center is just below the car. Thus, the net force on the car is

$$F_{\text{net}} = ma = 150 \times 20 = 3000 \text{ N downward.}$$

Gravity accounts for

$$w = mg = 150 \times 9.8 = 1470 \text{ N downward}$$

[2]It was Galileo who discovered this phenomenon. That is why the statement "The gravitational acceleration is 9.8 m/s^2." makes sense. It is the same for all objects, heavy, light, big, small, black, or white!

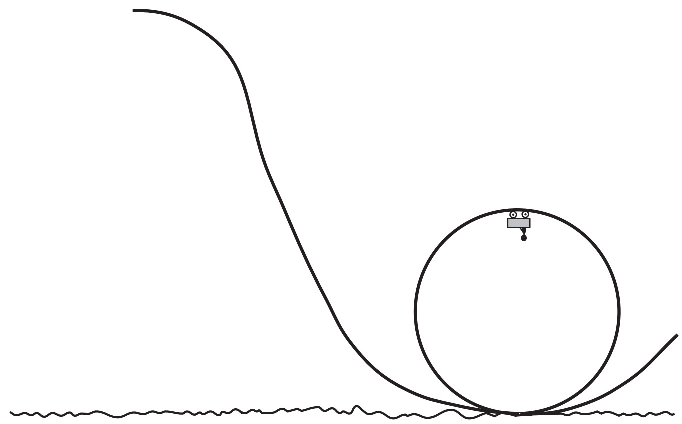


Figure D.8: The roller coaster must have a sufficiently large speed to ensure the presence of a normal force on the cars.

of it. The rest is the force exerted on the car by the track. This force is very important. If the car is to stay on the track, it must push on it. The third law of motion then says that the track must push on the car. So, as long as the force of the track on the car is nonzero, the car will stay up without falling down. In order for this to happen, the car must be moving fast enough.

What is the minimum speed with which the car can circle the loop without falling down? The force of the track on the car must be nonzero, but it could be very small. Let us say that this force is 0.0001 N. Then, at the height of the loop, the total (net) force on the car will be the weight plus this small force, i.e., 1470.0001 N. The second law now gives

$$F_{\text{net}} = ma \quad \Rightarrow \quad 1470.0001 = 150a \quad \text{or} \quad a = \frac{1470.0001}{150} = 9.80000067 \text{ m/s}^2 \text{ downward}$$

But this is the centripetal acceleration which is related to speed. Invoking this relation, we obtain

$$a = \frac{v^2}{r} \quad \Rightarrow \quad 9.80000067 = \frac{v^2}{5} \quad \text{or} \quad v^2 = 9.80000067 \times 5 = 49.0000033$$

and

$$v = \sqrt{49.0000033} = 7.00000024 \text{ m/s}$$

The two digits 24 at the end of this number come about because of the small force of the track on the car. The smaller this force, the smaller the difference between the calculated speed and 7 m/s. In fact, we can set the force of the track on the car equal to zero and obtain what could be called the *critical speed*, v_{crit}. For the car to remain on its track, its speed must *exceed* v_{crit}. This speed had better be independent of the mass of the car (plus the passenger), otherwise heavier (or lighter) passengers may fall, their critical speed being different! **Math Note E.7.1** on **page 79** of *Appendix.pdf* proves this fact.

D.8 Numerical Examples for Chapter 8

Finding speed of a ball fired from a table top **(page 111 of the book)**

Example D.8.1. A ball of mass 100 grams (equal to 0.1 kg) is fired horizontally from a table top 1 m high with a speed of 10 m/s. What is the speed of the ball when it reaches the floor?

Measure heights from the floor; i.e., let the floor be the reference level. Then, initially the total energy is

$$ME = \tfrac{1}{2}mv_i^2 + mgh_i = \tfrac{1}{2}0.1 \times 10^2 + 0.1 \times 9.81 \times 1 = 5.981 \text{ Joules}$$

This initial energy must equal the energy at the end of the ball's flight where the height is zero. Height of zero means zero PE; so, at the end, the energy is all kinetic. Setting the total energy equal to KE at the end, we get

$$5.981 = \tfrac{1}{2}mv_f^2 = 0.05v_f^2 \quad \Rightarrow \quad v_f^2 = \frac{5.981}{0.05} = 119.62$$

or

$$v_f = \sqrt{119.62} = 10.94 \text{ m/s}.$$

Analyzing motion of a roller coaster **(page 111 of the book)**

Example D.8.2. The roller coaster of Figure D.9 has a loop, around which the cars will circle. Assume that $h_0 = 80$ m, $h_1 = 50$ m, and $h_2 = 40$ m. The car and its passenger have a total mass of 100 kg, and start from rest at A. We want to find the speed of the car at points B, C, and D. It is clear that a convenient reference height is the lowest point of the track, i.e., point B.

The total mechanical energy of the system is obtained at point A:

$$ME = KE_A + PE_A = 0 + mgh_0 = 100 \times 9.8 \times 80 = 78,400 \text{ J}$$

Although calculated at the specific point A, this ME will remain the same at any point of the track.

At B, the height is zero, so we can write

$$ME = KE_B + PE_B \quad \text{or} \quad 78,400 = \tfrac{1}{2}mv_B^2 + 0 = \tfrac{1}{2} \times 100v_B^2 = 50v_B^2$$

Therefore

$$v_B^2 = \frac{78,400}{50} = 1568 \quad \text{and} \quad v_B = \sqrt{1568} = 39.6 \text{ m/s}$$

At C, the height is 50 m, so

$$ME = KE_C + PE_C \quad \text{or} \quad 78,400 = \tfrac{1}{2}mv_C^2 + mgh_1 = \tfrac{1}{2} \times 100v_C^2 + 100 \times 9.8 \times 50$$

or $78,400 = 50v_C^2 + 49,000 \quad \Rightarrow \quad 50v_C^2 = 78,400 - 49,000 = 29,400$, giving

$$v_C^2 = \frac{29,400}{50} = 588 \quad \text{and} \quad v_C = \sqrt{588} = 24.25 \text{ m/s}$$

The car has slowed down a bit since B.

At D, the height is 40 m, so we have

$$ME = KE_D + PE_D \quad \text{or} \quad 78,400 = \tfrac{1}{2}mv_D^2 + mgh_2 = \tfrac{1}{2} \times 100v_D^2 + 100 \times 9.8 \times 40$$

or $78,400 = 50v_C^2 + 39,200 \quad \Rightarrow \quad 50v_D^2 = 78,400 - 39,200 = 39,200$, yielding

$$v_D^2 = \frac{39,200}{50} = 784 \quad \text{and} \quad v_D = \sqrt{784} = 28 \text{ m/s}$$

The car has speeded up again.

Q: What is the force exerted by the track on the car at D?

A: The (centripetal) acceleration of the car at D is

$$a = \frac{v^2}{r} = \frac{28^2}{20} = 39.2 \text{ m/s}^2$$

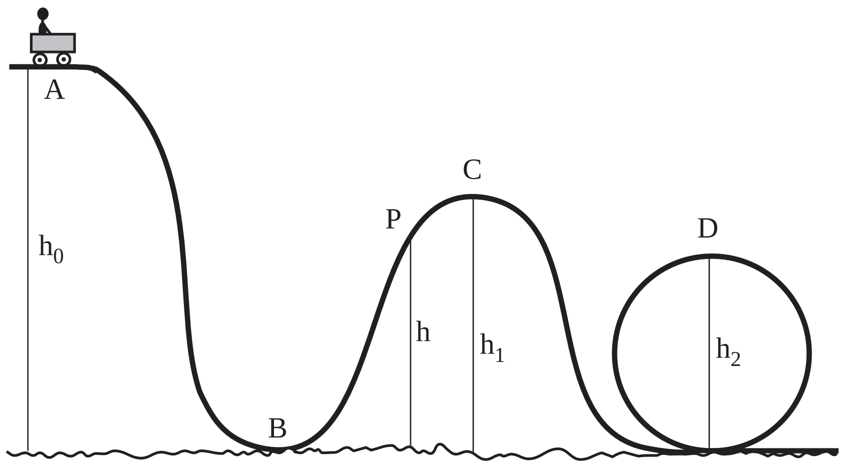


Figure D.9: A car of a roller coaster as it starts to begin its motion.

So, from the second law, the net force is

$$F_{\text{net}} = ma = 100 \times 39.2 = 3,920 \text{ N}$$

The weight can provide only $mg = 100 \times 9.8 = 980$ N of the net force. The rest must be coming from the track. So, the (downward) force of the track on the car is $3,920 - 980 = 2,940$ N. The car pushes back on the track (remember the third law!) and will not fall (see also Example D.7.7).

D.9 Numerical Examples for Chapter 9

Comparing acceleration of an apple and Moon (**page 132 of the book**)

Example D.9.1. An apple—or any other object—falls to the ground with an acceleration of 9.81 m/s^2. The Moon, at a distance of 384,400 km, circles the Earth in 27.322 days,[3] or

$$27.322 \times 24 \times 3600 = 2.36 \times 10^6 \text{ s}$$

and the distance it travels during this time is the circumference of its orbit, i.e.,

$$2\pi r = 2 \times 3.1416 \times 384,400,000 = 2.415 \times 10^9 \text{ m}$$

It follows that the speed of the Moon as it moves around the Earth is

$$v = \frac{\text{distance}}{\text{time}} = \frac{2.415 \times 10^9}{2.36 \times 10^6} = 1023 \text{ m/s}$$

Using this value for the speed in the relation $a = v^2/r$ for the centripetal acceleration, we obtain $a = 0.002725$ m/s^2. The acceleration at the surface of the Earth is $9.81/0.002725 = 3600$ times larger than the acceleration of the Moon. On the other hand, Moon is $384,400/6400 = 60$ times farther than the apple from the center of the Earth (see Figure 9.1). From these two numbers and the assumption that both accelerations are caused by the Earth's gravity, one can conclude that the gravitational acceleration falls off as the *inverse square* of the distance.

Estimating G (**page 133 of the book**)

Example D.9.2. We want to use Equation (9.1) to estimate G. Consider a 1-kg coconut

[3] The actual distance and period of the Moon are slightly different from these numbers. I chose these numbers so that the distance ratio and the acceleration ratio obtained below turn out "nice," and have an "obvious" relation.

falling from a tree. We know the force F on the coconut: it is its weight. Thus,

$$F = \text{weight} = w = mg = 1 \times 9.8 = 9.8 \text{ N}.$$

So far, we know m (it is 1 kg), and F. If we knew r and M, we could determine G. What is r? It is the distance from the coconut to the center of the Earth. But the coconut (even at the top of the tree, say 10 m high) is so close to the surface of the Earth (compared to the radius of the Earth, over 6 million meter) that we can safely say that

$$r = R_\oplus \equiv \text{radius of Earth} = 6.4 \times 10^6 \text{ m}.$$

The last value was not only known to Newton, but also to the Greeks as early as the third century BC (see Section 1.3.2).

M is the most difficult to determine. Nevertheless, we can estimate it by looking at a sample of the stuff, of which the Earth is made. The lightest substance on (and in) Earth is water. What would the mass of the Earth be if it were made up *entirely of water*? The density ρ (mass of one cubic meter) of water is 1000 kg/m^3. If we can determine how many cubic meters there are in the Earth—i.e., if we know the volume of the Earth—we can calculate its mass. The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$. So, if the Earth were made up of only water, its mass would be

$$M = \tfrac{4}{3}\pi R^3 \rho = \tfrac{4}{3}\pi (6.4 \times 10^6)^3 \times 1000 = 1.1 \times 10^{24} \text{ kg}$$

Substituting all the known quantities in Equation (9.1), we obtain

$$9.8 = \frac{G(1)(1.1 \times 10^{24})}{(6.4 \times 10^6)^2} \quad \Rightarrow \quad G = \frac{9.8(6.4 \times 10^6)^2}{1.1 \times 10^{24}} = 3.65 \times 10^{-10}$$

Now let us go to the other extreme and assume that the Earth is entirely made up of a very heavy element, say gold with a density of about 20,000 kg/m^3. In that case,

$$M = \tfrac{4}{3}\pi R^3 \rho = \tfrac{4}{3}\pi (6.4 \times 10^6)^3 \times 20000 = 2.2 \times 10^{25} \text{ kg}$$

and, using the same argument as above,

$$G = \frac{9.8(6.4 \times 10^6)^2}{2.2 \times 10^{25}} = 1.82 \times 10^{-11}$$

We therefore conclude that the actual value of G must be between these two extremes:

$$1.82 \times 10^{-11} < G < 3.65 \times 10^{-10}$$

Finding the speed of a low-altitude satellite **(page 136 of the book)**

Example D.9.3. You can use Equation (9.3) to obtain the speed of a satellite circulating the Earth if you know its altitude. Let's assume that a satellite is launched into a circular orbit at an altitude of 100 km.

Q: What is the speed of the satellite?

A: First note that r in Equation (9.3) is the *distance to the center of the Earth.* Thus to find r you must add the altitude to the radius of the Earth,$R_\oplus$. This gives

$$r = R_\oplus + h = 6,400 \text{ km} + 100 \text{ km} = 6,500 \text{ km} = 6.5 \times 10^6 \text{ m}$$

Substituting in Equation (9.3) now yields

$$v = \sqrt{\frac{GM_\oplus}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{6.5 \times 10^6}} = 7,847 \text{ m/s} = 17,554 \text{ mph}$$

Thus to launch a satellite into a circular orbit at an altitude of 100 km, you must boost it to an orbital speed of about 17,500 mph.

Finding the mass of the Sun
(page 137 of the book)

Example D.9.4. Using Equation (9.4), we can weigh the Sun! All we have to do is plug in the values of T and r for a planet such as the Earth and solve for the mass of the sum, $M_\odot$. For Earth, we have

$$T = 365 \times 24 \times 3600 = 3.15 \times 10^7 \text{ s}, \quad r = 150,000,000 \text{ km} = 1.5 \times 10^{11} \text{ m}$$

Therefore,

$$\left(3.15 \times 10^7\right)^2 = \frac{4\pi^2}{6.67 \times 10^{-11} M_\odot} \left(1.5 \times 10^{11}\right)^3$$

or

$$M_\odot = \frac{4\pi^2 \times \left(1.5 \times 10^{11}\right)^3}{6.67 \times 10^{-11} \times (3.15 \times 10^7)^2} = 2 \times 10^{30} \text{ kg}$$

Thus, the Sun is about 330,000 times heavier than the Earth.

Finding the altitude of a synchronous satellite
(page 137 of the book)

Example D.9.5. We use Equation (9.4) to find the altitude of a synchronous satellite. Here, the period is one day or 86,400 seconds and M is the mass of the Earth, which we found in Example 9.1.5:

$$(86,400)^2 = \frac{4\pi^2}{(6.67 \times 10^{-11})\,(6 \times 10^{24})} r^3$$

This yields

$$r^3 = \frac{(86,400)^2 \left(6.67 \times 10^{-11}\right) \left(6 \times 10^{24}\right)}{4\pi^2} = 7.57 \times 10^{22}$$

or

$$r = \sqrt[3]{7.57 \times 10^{22}} = 4.23 \times 10^7 \text{ m}$$

which is about 6.6 times the radius of the Earth. Note that r is the distance from the satellite to the *center* of the Earth. The altitude is one Earth radius less, or 5.6 Earth radii.

Numerical dark matter
(page 137 of the book)

Example D.9.6. A star has a *visible* mass of 3×10^{30} kg. One of its planets, at a distance of 4×10^{11} m, moves around it every 2.5 years.

Q1: Does the Kepler third law hold for this star-planet system?

A1: The period is $2.5 \times (3.15 \times 10^7) = 7.88 \times 10^7$ seconds. So, the left-hand side of Kepler's third law is $(7.88 \times 10^7)^2$ or 6.2×10^{15}. On the other hand, the right-hand side is

$$\frac{4\pi^2}{GM} r^3 = \frac{4 \times (3.1416)^2}{(6.67 \times 10^{-11})(3 \times 10^{30})} (4 \times 10^{11})^3 = 1.26 \times 10^{16}$$

not equal to the left-hand side!

Q2: What is the best way to reconcile the discrepancy?

A2: Kepler's third law is on such a firm theoretical and observational footing that we have no choice but to accept it. The value of G has been determined to a very high accuracy; so we have no room to fudge it. We are therefore left with r, M, and T. The first and the last quantities can be measured directly and accurately. The period can be measured by charting the planet's position in the field of a super-powerful telescope (such as the Hubble Space Telescope). The distance can be measured accurately by finding the angular separation of the star and the planet once the distance of the star from Earth is known.

The only alternative left for explaining the discrepancy is to assume that the mass of the star in not necessarily its *visible* mass. If we assume the validity of Kepler's third law, and the accuracy of the distance and period, the mass of the star can be determined:

$$T^2 = \frac{4\pi^2}{GM} r^3 \quad \Rightarrow \quad M = \frac{4\pi^2 r^3}{GT^2} = \frac{4 \times (3.1416)^2 \times (4 \times 10^{11})^3}{(6.67 \times 10^{-11})(7.88 \times 10^7)^2}$$

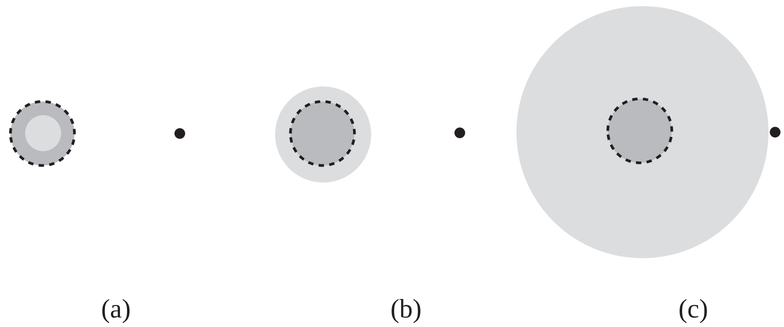


Figure D.10: The distribution of the invisible mass experienced by a planet (the black dot). The dashed circle represents the boundary of the *visible* part of the star. (a) The entire invisible mass could be concentrated inside the visible region of the star. It could also spread out slightly beyond the visible region, as in (b), or all the way up to the planet (c).

or $M = 6.1 \times 10^{30}$ kg. This suggests that more than half the mass of the star is *invisible*!

Q3: How is this extra mass distributed?

A3: Since r is the distance between the planet and the *center* of the star, the extra mass need not be inside the visible region of the star. In fact, it could be distributed in a sphere whose radius can be as large as r. Figure D.10 shows three of the infinitely many possibilities for the distribution of the invisible mass experienced by a planet. The invisible mass could be confined entirely inside the visible region [Figure D.10(a)] within a small sphere as shown or as large a sphere as the visible region itself. It could also spread beyond the visible region; either in a sphere only slightly larger than the visible region [Figure D.10(b)], or in a sphere that has a radius as large as the orbital radius of the planet [Figure D.10(c)]. All these distributions of the invisible mass give rise to the same period (or speed) of the planet, because in all cases the distance of the planet from the center of the star is the same.

Calculating the PE of an apple in two ways (**page 141 of the book**)

Example D.9.7. Take an apple with a mass of 0.1 kg located 50 m above the surface of the Earth. Using the approximate formula first, we have

$$PE = mgh = 0.1 \times 9.81 \times 50 = 49.05 \text{ Joules}$$

This must be interpreted as the *difference* between the potential energy at h and at the surface of the Earth.

Now let us calculate this difference using the exact formula. We have to substitute the most precise values for the parameters of the equation. Using a good table of physical constants, we have (note that capital letters usually denote properties of the large gravitating body)

$$\begin{aligned}(PE)_{\text{surface}} &= -\frac{GM_\oplus m}{R_\oplus} = -\frac{6.673 \times 10^{-11} \times 5.977 \times 10^{24} \times 0.1}{6,371,000} \\ &= -6,260,323.5 \text{ Joules} \\ (PE)_h &= -\frac{GM_\oplus m}{r} = -\frac{6.673 \times 10^{-11} \times 5.977 \times 10^{24} \times 0.1}{6,371,050} \\ &= -6,260,274.4 \text{ Joules}\end{aligned}$$

From these two values we evaluate the difference in potential energy to be

$$-6,260,274.4 - (-6,260,323.5) = 49.1 \text{ Joules}$$

which agrees very well with the approximate value.

What goes up may or may not come down
(page 142 of the book)

Example D.9.8. A 500-kg space probe is launched with the aid of booster rockets. When it reaches the altitude of 600 km, the boosters are detached, at which point the probe has a speed of 12 km/s.

Q1: Will the probe move on forever, as it should?

A1: If the probe is to move forever, its total energy must be positive or zero. Let us check this at the point where the boosters are detached. At that point

$$r = 6,400 + 600 = 7,000 \text{ km} = 7 \times 10^6 \text{ m} \quad \text{and} \quad v = 12,000 \text{ m/s}$$

because altitude plus the radius of the Earth gives the distance from the center of the Earth. It follows from Equation (9.7) that

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 - \frac{GMm}{r} = \tfrac{1}{2}(500)(12,000)^2 - \frac{6.67 \times 10^{-11}(6 \times 10^{24})(500)}{7 \times 10^6} \\ &= 3.6 \times 10^{10} - 2.86 \times 10^{10} = 7.4 \times 10^9 \text{ J} \end{aligned}$$

Since the energy is positive, the probe will move on forever.

We can find out what the speed of the probe will be when it reaches the boundary of the solar system, i.e., Pluto's orbit at a distance of 5.9 billion km. Again we use Equation (9.7) with everything known except v:

$$7.4 \times 10^9 = \tfrac{1}{2}(500)v^2 - \frac{6.67 \times 10^{-11}(6 \times 10^{24})(500)}{5.9 \times 10^{12}}$$

or

$$250v^2 = 7.4 \times 10^9 + 33915 \approx 7.4 \times 10^9 \quad \Rightarrow \quad v = \sqrt{\frac{7.4 \times 10^9}{250}} = 5441 \text{ m/s}$$

This is the speed the probe will have for the rest of its journey. In fact, we saw above that the contribution of the PE to the total energy was only 33,915 J at the orbit of Pluto; further out, this contribution is even less. Therefore, the total energy of 7.4×10^9 J is, to a very good approximation, the KE of the probe for the rest of its journey, and for this KE, the speed is 5441 m/s, as calculated above.

We have to emphasize that the foregoing discussion completely ignores the influence of the Sun and other bodies in the solar system. In fact, at far enough distances, the gravity of the Sun will overpower that of the Earth, and cannot be ignored. In actual launches, such details must be (and are) taken into account using sophisticated mathematical and computational techniques.

As a second example, suppose we throw straight up a 1-kg cannon ball with a speed of 10 km/s.

Q2: Will the ball move on forever or will it eventually come back? If it comes back, what is the maximum height it reaches?

A2: At the surface of the Earth

$$r = 6,400 \text{ km} = 6.4 \times 10^6 \text{ m} \quad \text{and} \quad v = 10,000 \text{ m/s}$$

The total energy of such a ball is therefore

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 - \frac{GMm}{r} = \tfrac{1}{2}(1)(10,000)^2 - \frac{6.67 \times 10^{-11}(6 \times 10^{24})(1)}{6.4 \times 10^6} \\ &= 5 \times 10^7 - 6.25 \times 10^7 = -1.25 \times 10^7 \text{ J} \end{aligned}$$

Since $E < 0$, the ball must eventually stop. To find the height at which it stops, first denote the distance from the center of the Earth to the point at which the ball stops by r_s. At this distance $KE = 0$, and we have

$$-1.25 \times 10^7 = 0 - \frac{6.67 \times 10^{-11}(6 \times 10^{24})(1)}{r_s} \quad \text{or} \quad 1.25 \times 10^7 = \frac{4 \times 10^{14}}{r_s}$$

This gives

$$r_s = \frac{4 \times 10^{14}}{1.25 \times 10^7} = 3.2 \times 10^7 \text{ m}$$

or 32,000 km. The height h is obtained by subtracting the radius of the Earth from this distance:

$$h = 32,000 - 6,400 = 25,600 \text{ km}.$$

Example D.9.9. We can find the numerical value of the Earth's escape velocity by substituting the mass and the radius of the Earth in Equation (9.8):

Calculating escape velocities
(page 143 of the book)

$$v_{\text{esc}} = \sqrt{\frac{2 \times (6.67 \times 10^{-11})\,(6 \times 10^{24})}{6.4 \times 10^6}} = \sqrt{1.25 \times 10^8} = 11183 \text{ m/s}$$

which is approximately 25,000 mph.

Similarly, the escape velocity of the Moon, which has a mass of 7.35×10^{22} kg and a radius of 1740 km, is

$$v_{\text{esc}}^{\text{moon}} = \sqrt{\frac{2 \times (6.67 \times 10^{-11})\,(7.35 \times 10^{22})}{1.74 \times 10^6}} = \sqrt{5.64 \times 10^7} = 2374 \text{ m/s}$$

Example D.9.10. A satellite is circling a planet of mass M at a distance r from the planet's center. From Equation (9.3), the KE of the satellite is

Calculating the binding energy of a satellite
(page 143 of the book)

$$KE = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2 = \frac{GMm}{2r}$$

The potential energy of the satellite is $-GMm/r$. So, its total energy is

$$E = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

showing that the total energy of the satellite is *negative*.

The minimum energy that the satellite needs to "unbind" itself from M is that which makes the total energy zero (in which case the satellite reaches the escape velocity).

Example D.9.11. What should the radius of the Sun be if it is to act as a black hole?

Radius of black holes
(page 144 of the book)

The speed of light is a known physical quantity equal to 300,000 km/s or 3×10^8 m/s. The mass of the Sun was calculated in Example D.9.4 with the result that $M_\odot = 2 \times 10^{30}$ kg. Substituting these and the value of G in Equation (9.9) yields

$$\frac{2 \times 10^{30}}{R} \geq \frac{(3 \times 10^8)^2}{2 \times 6.67 \times 10^{-11}}$$

or

$$R \leq \frac{(2 \times 10^{30})(2 \times 6.67 \times 10^{-11})}{9 \times 10^{16}} = 2964 \text{ m} \approx 3 \text{ km}$$

This is much much smaller than the actual radius of the Sun which is 6.96×10^8 m.

D.10 Numerical Examples for Chapter 10

Example D.10.1. The gravitational force F_{ae} exerted on a 0.5-kg apple by the Earth is

Force of the Sun on apple and Moon
(page 152 of the book)

$$F_{\text{ae}} = \frac{GM_\oplus m}{R_\oplus^2} = \frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 0.5}{(6.4 \times 10^6)^2} = 4.9 \text{ N}$$

What is the gravitational force F_{as} exerted on the same apple by the Sun? The answer to this question is a little more subtle, because the Sun is not pulling just the apple, but the Earth as well! If the Sun were pulling both at exactly the same rate, then no force of the Sun on the apple would be detectable. This situation is not unlike two cars moving on a highway at exactly the same speed in exactly the same direction: no motion of one car is detectable by the driver of the other—no change occurs in the position vector of one car relative to the other.

To calculate the force of the Sun on the apple, we need to find the *difference* in the acceleration of the Earth and the apple by the Sun and multiply the result by the mass of the apple. Suppose that the apple is between the Earth and the Sun, so that the apple is closer to the Sun than (the center of the) Earth by one Earth radius. The acceleration of the apple toward the Sun would be

$$g_{\text{as}} = \frac{GM_{\odot}}{r^2} = \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}}{(1.5 \times 10^{11} - 6.4 \times 10^6)^2} = \frac{1.334 \times 10^{20}}{2.24981 \times 10^{22}} = 0.00592939$$

and the acceleration of the Earth toward the Sun would be

$$g_{\text{es}} = \frac{GM_{\odot}}{r^2} = \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}}{(1.5 \times 10^{11})^2} = \frac{1.334 \times 10^{20}}{2.25 \times 10^{22}} = 0.00592889$$

Thus the Sun pulls the apple away from Earth with a force of

$$F_{\text{as}} = m(g_{\text{as}} - g_{\text{es}}) = 0.5(0.00592939 - 0.00592889) = 2.5 \times 10^{-7} \text{ N},$$

over ten million times smaller than the force of Earth on the apple! If the apple happens to be on the far side of Earth, the answer will be the same as the reader can easily verify (the only difference is that we have a plus sign in the denominator of the first equation of this paragraph).

For the same reason, the pull of the Sun on the Moon is obtained by calculating the difference in the acceleration of the Earth and the Moon by the Sun. When the Moon is located between Earth and Sun, we have

$$g_{\text{ms}} = \frac{GM_{\odot}}{r^2} = \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}}{(1.5 \times 10^{11} - 4 \times 10^8)^2} = \frac{1.334 \times 10^{20}}{2.238 \times 10^{22}} = 0.00596068$$

The acceleration of the Earth toward the Sun is the same as before. Thus the Sun pulls the Moon away from Earth with an acceleration of[4]

$$a_{\text{ms}} = g_{\text{ms}} - g_{\text{es}} = 0.00596068 - 0.00592889 = 3.2 \times 10^{-5} \text{ m/s}^2,$$

The Earth pulls the Moon with an acceleration of 0.0027 m/s^2 (see Example D.9.1), in excess of 80 times larger than the Sun's acceleration of the Moon.

D.11 Numerical Examples for Chapter 11

Finding the number of fringes (**page 92** of *Appendix.pdf*)

Example D.11.1. To find the number of the fringes, refer to Figure D.11 in which is shown two coherent sources of wave S_1 and S_2 and the point E at the edge of the screen.

Q: How many fringes (bright lines or circles) are there on the screen?

A: It should be clear that the answer is in the path difference Δl between $\overline{S_1E}$ and $\overline{S_2E}$, where E is the end "point" (really the edge) of the screen. The number of times the wavelength λ fits in this path difference determines the number of fringes. For example, if $\overline{S_1E} - \overline{S_2E} = 4.5\lambda$, there will be 4 bright spots between C_0 and the (left) end of the screen;

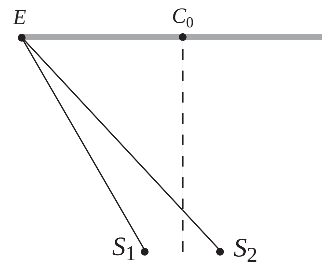


Figure D.11: Two coherent sources of wave S_1 and S_2 produce interference, a pattern of high and low intensity that remain unchanged. Point E is at the edge of the screen.

and another 4 bright spots on the other side of the screen. The edges of the screen coincide with a dark "spot."[5]

If the sources are circular, this path difference creates one central spot C_0 and four circles. (The two points C_1 in Figure 11.7(a) belong to the same circle that cuts through the plane of the figure at those points.) If the sources are rectangular perpendicular to the plane of the figure, the path difference above creates nine lines: C_0 plus four on each side

Changing the path difference $\Delta l = \overline{S_1E} - \overline{S_2E}$ alters the number of fringes. For instance, Δl changes if we move S_1 relative to S_2, or if we move both sources relative to the screen. We can express the number of fringes either as the ratio $\Delta l/\lambda$ or in terms of time and frequency as follows:

$$\text{number of fringes} = \frac{\Delta l}{\lambda} = \frac{c\Delta t}{\lambda} = f\Delta t \tag{D.9}$$

where c is the speed of the wave and Δt is the time that it takes the wave to travel the path difference.

Example D.11.2. Let's calculate the *frequency* of the siren of a police car when it is approaching with a speed of 34 m/s and subsequently receding with the same speed. The original frequency of the siren is assumed to be 450 Hz and the speed of sound is taken to be 340 m/s.

Finding frequency change is a police siren **(page 169 of the book)**

To be able to use Equation (E.10), you need the (original) wavelength of the sound. But this is simply given by

$$\lambda = \frac{c}{f} = \frac{340}{450} = 0.756 \text{ m}.$$

For the case of the approach, $v = -34$ m/s. Thus, the first equation in (E.10) gives

$$\lambda_{\text{det}} = 0.756\left(1 - \frac{34}{340}\right) = 0.756(1 - 0.1) = 0.756 \times 0.9 \approx 0.68 \text{ m}$$

For recession, $v = +34$ m/s and

$$\lambda_{\text{det}} = 0.756\left(1 + \frac{34}{340}\right) = 0.756(1 + 0.1) = 0.756 \times 1.1 \approx 0.83 \text{ m}$$

The corresponding frequencies are obtained from $c = \lambda f$. So, for approach,

$$f_{\text{app}} = \frac{c}{\lambda_{\text{app}}} = \frac{340}{0.68} = 500 \text{ Hz},$$

[4] We could calculate the *force* exerted on the Moon by Sun or Earth, but since acceleration is proportional to this force (the proportionality being the mass of Moon), the acceleration gives exactly the same information.

[5] The path difference is an odd (nine) multiple of a half wavelength.

and for recession,

$$f_{\text{rec}} = \frac{c}{\lambda_{\text{rec}}} = \frac{340}{0.83} \approx 410 \text{ Hz},$$

The difference in frequency (or pitch) is $500 - 410 = 90$ Hz, which is easily detectable by human ear.

Finding frequency change when detector moves **(page 169 of the book)**

Example D.11.3. Let's compare the frequency of sound when the source is approaching the stationary detector with that when the detector is approaching the stationary source. Suppose that instead of 34 m/s, the *source* is moving at half the speed of sound, or 170 m/s while producing a sound whose frequency is 450 Hz (and whose wavelength is, therefore, 0.756 m, as found in Example D.11.2). Then the approach wavelength is [using the first relation in Equation (E.10)]

$$\lambda_{\text{det}} = 0.756\left(1 - \frac{170}{340}\right) = 0.378 \text{ m}$$

with the corresponding frequency of $f_{\text{app}} = 340/0.378 \approx 900$ Hz.

On the other hand, if the *detector* moves with the same speed, the wavelength will be

$$\lambda_{\text{det}} = \frac{0.756}{1 - (-170/340)} = 0.504 \text{ m}$$

with the corresponding frequency of $f_{\text{app}} = 340/0.504 \approx 675$ Hz. Substantially different from 900 Hz!

Example D.11.4. Let us go back to Example D.11.2, but assume that the detector moves—with the same speed of 34 m/s—rather than the source. When the detector is approaching the police car, the wavelength is

$$\lambda_{\text{det}} = \frac{\lambda}{1 - (v/c)} = \frac{0.756}{1 - (-34/340)} = \frac{0.756}{1 + 34/340} = \frac{0.756}{1.1} = 0.687 \text{ m},$$

Doppler effect when detector moves slowly **(page 170 of the book)**

and the frequency is $f_{\text{app}} = 340/0.687 = 495$ Hz, which is only slightly lower than the approach frequency 500 Hz of Example D.11.2. The receding frequency can also be calculated. First we find the corresponding wavelength:

$$\lambda_{\text{det}} = \frac{\lambda}{1 - (v/c)} = \frac{0.756}{1 - (+34/340)} = \frac{0.756}{0.9} = 0.84 \text{ m},$$

and the frequency is $f_{\text{rec}} = 340/0.84 = 408$ Hz, which is also only slightly lower than the receding frequency of Example D.11.2.

We thus see that—even though 34 m/s is 10% the the sound speed, and 10% is a fairly large number—there is very little difference between the Doppler shifts caused by the motion of the source and the motion of the detector. The reader is urged to redo this example and Example D.11.2 using a speed of 3.4 m/s instead of 34 m/s, and see that Equations (E.10) and (E.11) agree even better.

Radar detectors **(page 170 of the book)**

Example D.11.5. A policeman driving his car at 90 mph chases a speeder and sends a radar signal of wavelength 2 cm to the car. The reflected wavelength is 3×10^{-10} m *shorter* than the original signal.

Q1: What is the relative speed of the two cars? What is the speed of the car?. The speed of radar waves is the same as light speed, 300,000 km/s.

A1: Since the wavelength of the reflected signal is shorter, the car is *approaching* the police car. This means that the police car is gaining on the speeder. Thus, we expect the front car's speed to be less than 90 mph. Substituting the (negative) value of $\Delta\lambda$ in Equation (E.10), we obtain

$$-\frac{3\times 10^{-10}}{0.02} = 2\frac{v_{\rm rel}}{3\times 10^8} \Rightarrow v_{\rm rel} = -\frac{(3\times 10^{-10})(3\times 10^8)}{2\times 0.02} = -2.25 \text{ m/s} \approx -5 \text{ mph}$$

Q2: How fast is the speeder going?

A2: The police car is *approaching* the speeder, i.e., it is moving this much *faster* than the other car. So, the speeder's speed is $90 - 5 = 85$ mph.

D.12 Numerical Examples for Chapter 12

Finding electric field of a point charge
(page 178 of the book)

Example D.12.1. Let's calculate the field of a point charge q at a distance r from the charge. Coulomb's law says that the force on a *test charge* q' located at a distance r from q is $F = k_e qq'/r^2$. On the other hand, by the definition of the electric field, $E = F/q'$. Combine these two ideas to obtain

Electric field of a point charge

$$E = \frac{k_e qq'/r^2}{q'} = \frac{k_e q}{r^2}$$

The electric field of other more complicated sources is found by summing this expression (vectorially) over all the point charges comprising the source. Note the similarity between the equation above and the expression for the gravitational field, $g = Gm/r^2$ given in Equation (9.2).

By an argument similar to the one that led to Equation (9.6), you can obtain the mathematical expression for the potential energy of a system of two charges. This leads to

$$PE = \frac{k_e q_1 q_2}{r} \tag{D.10}$$

The negative sign of Equation (9.6)—which comes from the attractiveness of the gravitational force—is absent, because the electric force can be attractive or repulsive, depending on the relative sign of q_1 and q_2. If they are of opposite signs, you automatically get a negative potential as in (9.6). For two charges of the same sign, the potential will be positive, a situation which was absent in gravity.

Example D.12.2. A 50-megawatt generator is feeding electricity to a nearby town using 10 cables, each having a resistance of 100 Ω. Suppose that the emf of the plant is 50,000 volts, and the current at the plant is divided equally among the cables. What is the heat loss in the cables?

Calculating the transmission power loss
(page 187 of the book)

From $P = i\mathcal{E}$ we can find i, because we know both P (it is 50 million Watts) and $\mathcal{E}$ (it is 50,000 volts):

$$50,000,000 = i \times 50000 \quad \Rightarrow \quad i = \frac{50,000,000}{50000} = 1,000 \text{ amps}$$

This is the current at the generator. Each cable gets one tenth of this current or 100 amps. So, the power loss in each cable is

$$P = Ri^2 = 100 \times 100^2 = 1,000,000 \text{ Watts},$$

and the total loss is ten times this or 10 megawatts, or 20% of the plant capacity.

Now suppose that the plant increases its emf to 500,000 volts. What is the heat loss in the cables now? The current is calculated as before

$$50,000,000 = i \times 500000 \quad \Rightarrow \quad i = \frac{50,000,000}{500000} = 100 \text{ amps}$$

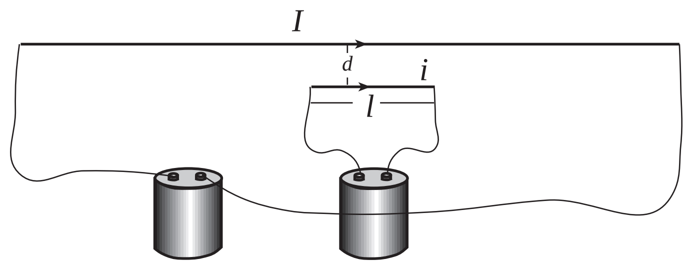


Figure D.12: A long wire carrying a current I exerts a force on a (shorter) wire carrying current i.

and each cable carries one tenth of this or 10 amps. The power loss in each cable is now

$$P = Ri^2 = 100 \times 10^2 = 10{,}000 \text{ Watts},$$

and the total loss is ten times this or 100 kilowatts, or only 0.2% of the plant capacity. You see that increasing the output voltage of the plant dramatically decreases the loss in the transmission lines.

D.13 Numerical Examples for Chapter 13

Calculating the magnetic force between two wires **(page 195 of the book)**

Example D.13.1. Ampère showed that a long wire carrying current I exerts a force F on another wire of length l carrying current i parallel to the long wire (see Figure D.12) given by

$$F = k_m \frac{2Iil}{d} \tag{D.11}$$

where d is the distance between the two wires. The force is attractive if the currents are in the same direction (as in Figure D.12), and repulsive if in opposite directions.

The constant k_m is one of the fundamental constants of electromagnetism, which can be measured experimentally:[6] If the currents in the two wires are one ampere each, the shorter wire is one meter, and the wires are one meter apart, then the force is measured to be 10^{-7} N. Thus, in the system of unit we are using, $k_m = 10^{-7}$.

Although the magnetic force of Equation (D.11) is small when the wires are far apart and the current is small, one sees a dramatic effect when larger currents run through the wires and the distance between them is small. Suppose that each wire carries a current of 20 amps and they are 1 cm apart. Let the length of the shorter wire be still one meter. Then the force between the wires will be

$$F = 10^{-7} \frac{2 \times 20 \times 20 \times 1}{0.01} = 0.4 \text{ N}$$

which is large enough to move the wires.

This force can easily be demonstrated using ordinary wires and batteries; however, the large currents cause a huge production of heat [see Equation (E.17) and the discussion surrounding it] in the wires—even possibly their meltdown.

Finding magnetic field of a current **(page 199 of the book)**

Example D.13.2. In modern notation, the Biot-Savart law can be expressed as

$$B = \frac{2k_m i}{r}, \tag{D.12}$$

[6]In reality, Equation (D.11) *defines* ampere as the unit of current by *assigning* the value 10^{-7} to k_m, but you need not worry about such details.

where $k_m = 10^{-7}$ is the magnetic constant encountered before [Equation (D.11)], i is the current of the long wire in amperes, r is the distance from the wire in meters, and B is the magnetic field of the wire.

A power line 20 meters above the ground is carrying a current of 200 amps. What is the magnetic field at a point on the ground directly beneath the line?

Equation (D.12) gives the answer:

$$B = \frac{2 \times 10^{-7} \times 200}{20} = 2 \times 10^{-6} \text{ tesla} = 0.02 \text{ gauss}$$

which is much weaker than the magnetic field of the Earth. This shows that it takes a gigantic current to produce any appreciable magnetic field.

Finding magnetic field of a solenoid **(page 200 of the book)**

Example D.13.3. The strength of the field inside a solenoid is

$$B = 4\pi k_m n i, \tag{D.13}$$

where n is the number of windings per unit length. For example, if a current of 1 amp runs through a solenoid with 1000 turns per meter, the magnetic field inside will be

$$B = 4 \times 3.1416 \times 10^{-7} \times 1000 \times 1 = 1.26 \times 10^{-3} \text{ tesla} = 12.6 \text{ gauss}.$$

Again this shows that creating a sizable magnetic field requires enormous currents.

Transformer in an electric shaver **(page 204 of the book)**

Example D.13.4. An electric shaver requires 1.2 volts and 1.5 amps to operate. The primary coil of its transformer has 1100 turns. Assume that the shaver is plugged into a 110-volt outlet.

Q: How many turns does its secondary coil have?

A: The first formula in Equation (E.18) gives

$$\frac{110}{1100} = \frac{1.2}{N_2} \quad \text{or} \quad N_2 = 12$$

Q: What is the current in the primary coil?

A: Use the first formula in Equation (E.18) to obtain

$$1100 i_1 = 12 \times 1.5 \quad \Rightarrow \quad i_1 = \frac{18}{1100} = 0.016 \text{ amp}.$$

Q: How much power does the shaver use?

A: The power used by the shaver is the product of its voltage and current:

$$P = 1.2 \times 1.5 = 1.8 \text{ Watts}.$$

Note that we could have used this to find the current in the primary, because the primary and the secondary powers are equal, and the primary voltage is given. Here is the one-liner detail:

$$1.8 = 110 i_1 \quad \Rightarrow \quad i_1 = \frac{1.8}{110} = 0.016 \text{ amp}.$$

D.14 Numerical Examples for Chapter 14

EM wave energies of Sun and wires **(page 218 of the book)**

Example D.14.1. In Chapter 11, we noted that the intensity of a wave is proportional to the square of its amplitude (see Box 11.2.2). The same idea holds true for *plane* electromagnetic waves. A useful quantity related to the energy of EM waves is the **energy flux**. This is the amount of energy that the wave delivers to a unit area (square meter) per second. For example, the energy delivered by sunlight to each square meter of Earth is measured to be about 1400 Joules per second (1400 Watts). This is known as the *solar constant*. The EM energy flux, denoted by ϕ_{em} is related to the fields as follows:

$$\phi_{em} = \frac{c}{4\pi k_e} E^2 = \frac{c}{4\pi k_m} B^2 \tag{D.14}$$

Knowing the solar constant, we can determine the magnetic field of the sunlight using Equation (D.14):

$$1400 = \frac{3 \times 10^8}{4 \times 3.14159 \times 10^{-7}} B^2 \;\Rightarrow\; B^2 = 5.87 \times 10^{-12} \quad \text{or} \quad B = 2.42 \times 10^{-6} \text{ tesla}$$

This is about 0.023 gauss or one twentieth the magnetic field of Earth.

On the other hand, from the magnetic field of a wire carrying AC electricity, we can determine its EM energy flux. Example D.13.2 showed that the magnetic field of a huge current of 100 amps was about 0.01 gauss. The EM energy flux of this field is

$$\phi_{em} = \frac{3 \times 10^8}{4 \times 3.14159 \times 10^{-7}} (0.02 \times 10^{-4})^2 = 955 \text{ Watts/m}^2$$

which is about three fourths the energy flux of the Sun.

From this calculation, it is clear that even living right under a power line, one is exposed to only three fourths the "danger" of Sun bathing. This "danger" is reduced even further, once quantum mechanical effects, the more accurate way of calculating the energy of EM waves, are taken into account. According to the quantum theory, it is the frequency of the EM wave that determines its energy. Sunlight has frequencies in the range of 10^{13}–10^{15} Hz, including the harmful ultraviolet. Power line waves, on the other hand, have frequencies in the range of 50–60 Hz. The energy of such waves are extremely small; so small that biological tissues do not feel their presence.

D.16 Numerical Examples for Chapter 16

Finding probabilities for 4 coins **(page 230 of the book)**

Example D.16.1. Instead of explicitly counting the outcomes as done above for up to three coins, let us analyze the case of four coins in a manner that can lead easily to generalization. First, we note, that there are

$$2 \times 2 \times 2 \times 2 = 2^4 = 16$$

possible outcomes. This can be seen by noting that there are two possibilities for the first coin, two for the second coin, etc., and the total is simply the product of all these possibilities.

The frequencies of various outcomes can be determined as follows. There is only one way that we can get zero H, namely when all the coins turn up tail. There are four ways that we can obtain one H, namely when coin number 1 is H and the rest are T, or coin number 2 is H and the rest are T, etc. The frequency for getting two H's is a little more complicated. We note that for the first H we have 4 possibilities while for the second H only 3 choices are left. Thus, the total number of choices seems to be $4 \times 3 = 12$. However, this is an overcount, because there is no difference between a case where we choose, say coin #2 first and then #4, and the case in which we reverse the order of our choices. Since there are two possibilities for reordering the first and the second choices, we must divide the total number obtained above by 2. Hence, the total number of choices for two H's is $12/2 = 6$. For 3 H's the calculation, at least in the case of 4 coins, becomes simple again because three H's are equivalent to one T, whose frequency is 4. Finally the frequency for four H's is obviously 1.

D.17 Numerical Examples for Chapter 17

Example D.17.1. A mixture of oxygen molecules and helium atoms is at a temperature of 27 $°C$. The oxygen molecule is eight times as massive as the helium atom.

Q1: On average, how much faster are the helium atoms moving than the oxygen molecules?

Finding rms speeds (**page 244 of the book**)

A1: Since the temperatures of both gases are equal, the average KE of their "particles" must be the same by Equation (17.1). Let 1 stand for helium and 2 for oxygen. Then

$$\langle KE_1\rangle = \langle KE_2\rangle, \quad \text{or} \quad \langle \tfrac{1}{2}m_1 v_1^2\rangle = \langle \tfrac{1}{2}m_2 v_2^2\rangle, \quad \text{or} \quad m_1\langle v_1^2\rangle = m_2\langle v_2^2\rangle$$

But we know that $m_2 = 8m_1$. So, $\langle v_1^2\rangle = 8\langle v_2^2\rangle$.

root mean square explained

The square root of the average of the square of a quantity is called its **root mean square**, and subscripted with "rms". The "rms" value is a measure of the "average" of the magnitude of a *vector* quantity such as velocity. In a random situation in which all directions are equally likely, the average *velocity* will be zero, but the average *speed* is, of course, not zero. In our present discussion, the rms value of velocity measures the average speed. And for the mixture above,

$$v_{1\text{rms}} = \sqrt{\langle v_1^2\rangle} = \sqrt{8\langle v_2^2\rangle} = \sqrt{8}\sqrt{\langle v_2^2\rangle} = 2.83 v_{2\text{rms}}$$

Q2: Given that the helium atomic mass is 6.64×10^{-27} kg, what is the average speed of the particles of each gas?

A2: Use Equation (17.1) to find the average speed for *any* particle:

$$\tfrac{1}{2}m\langle v^2\rangle = \tfrac{3}{2}k_B T \quad \text{or} \quad \langle v^2\rangle = \frac{3k_B T}{m} \quad \Rightarrow \quad v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}} \tag{D.15}$$

For helium, this yields

$$v_{1\text{rms}} = \sqrt{\frac{3(1.38 \times 10^{-23})(27 + 273.16)}{6.64 \times 10^{-27}}} = 1368 \ \text{ m/s}$$

where we were careful to convert the Celsius temperature value to Kelvin. Oxygen molecules move 2.83 times slower: $v_{2\text{rms}} = 1368/2.83 = 483$ m/s.

Example D.17.2. An application of Equation (17.2) is to the study of the change in the pressure of tires of a car at extreme seasonal temperatures. When you inflate the tires in the summer at, say 302.6 K—equivalent to 85 °F—the pressure inside will be appropriate for that temperature. In the winter, when the temperature is 266.5 K—equivalent to 20 °F—the pressure of the tires will be lower. In fact it is easy to determine the ratios of the two pressures. Since the volume of the tires changes very little with temperature, we take it to be constant. Also, assuming very little leakage, we take the number of molecules to be the same (this is true if you have not inflated the car in the meantime). Subscripting the winter quantities by w and the summer ones by s, we have

$$P_w V = N k_B T_w, \quad P_s V = N k T_s$$

Dividing both sides of these equations, we obtain

$$\frac{P_w V}{P_s V} = \frac{N k_B T_w}{N k_B T_s} \quad \Rightarrow \quad \frac{P_w}{P_s} = \frac{T_w}{T_s}$$

Substituting the above values for the temperatures, we get

$$\frac{P_w}{P_s} = \frac{266.5}{302.6} = 0.88 = 88\%$$

Thus, there is a 12% drop in the pressure of the tire in the winter.

To increase the pressure, one pumps air in the tire. This increases the number of molecules, and since the volume is assumed constant, the result is an increase in pressure.

Q: How much air do you have to pump into the tires to restore the pressure to it original summer value?

A: In this case the values of pressures are the same; so we write $P_w = P_s \equiv P$. On the other hand the number of molecules in winter is different form that of the summer. So, we write N_w and N_s for these two quantities. We then have

$$\frac{PV}{PV} = \frac{N_w k T_w}{N_s k T_s} \quad \Rightarrow \quad 1 = \frac{N_w T_w}{N_s T_s} \quad \Rightarrow \quad \frac{N_s}{N_w} = \frac{T_w}{T_s} = 0.88$$

or

$$\frac{N_w}{N_s} = \frac{1}{0.88} = 1.136$$

This corresponds to an increase of 13.6% in the amount of air in the tires.

Finding the Avogadro number
(**page 245 of the book**)

Example D.17.3. It is known that one mole of any gas at 1 atmospheric pressure—equal to 1.013×10^5 Pascal—and 0 °C occupies a volume of approximately 22.4 liters or 0.0224 m^3. From this information we can calculate the Avogadro's number. We use $P = 1.013 \times 10^5$, $V = 0.0224$, $T = 273$ K and $k_B = 1.3806 \times 10^{-23}$ in Equation (17.2) to obtain

$$(1.013 \times 10^5)(0.0224) = N(1.3806 \times 10^{-23})(273.15)$$

or

$$N = \frac{(1.013 \times 10^5)(0.0224)}{(1.3806 \times 10^{-23})(273.15)} = 6.023 \times 10^{23}$$

as the Avogadro's number.

Temperature of a reservoir does not change
(**page 248 of the book**)

Example D.17.4. Suppose system A has 100 particles and is at a temperature of 50 degrees. For system B, let the number of particles be 10,000,000 and its temperature 20 degrees. Then Equation (E.39) yields

$$T_f = \frac{100 \times 50 + 10,000,000 \times 20}{10,000,000 + 100} = \frac{200,005,000}{10,000,100} = 20.0003,$$

which is very close to the temperature of the reservoir.

Illustration of the law of increase of entropy
(**page 250 of the book**)

Example D.17.5. Consider two systems A and B initially in contact with two different reservoirs with temperatures—assumed to be equal[7] to the average energy—$T_A = 1/3$ and $T_B = 2/15$. Assume that A has 12 coins and B has 60 coins. Thus, the total energy of A is $12 \times 1/3 = 4$ implying 8 positive and 4 negative coins. Similarly B has a total energy of $60 \times 2/15 = 8$ implying 34 positive and 26 negative coins.

We now separate the two systems from their respective reservoirs and bring them in contact with one another. After a while they will reach thermal equilibrium. The average energy (or temperature) of the whole system will then be

$$T = E_{avg} = \frac{\text{total energy}}{\text{total number of coins}} = \frac{4+8}{12+60} = \frac{1}{6}$$

The final total energy of A is therefore, $12 \times 1/6 = 2$ implying 7 positive and 5 negative coins. Similarly, the final total energy of B is $60 \times 1/6 = 10$ implying 35 positive and 25 negative coins.

We now ask the question: What is the number of possibilities, or *accessible states*, just before the two systems are brought together and what is this number when they reach

[7] We are ignoring the proportionality constant which does not affect the analysis here.

equilibrium? The number of accessible states is simply the number of accessible states of A times that of B:

$$\text{initial \# of accessible states of A} = f_{12}(8) = \frac{12!}{8!4!} = 495$$
$$\text{initial \# of accessible states of B} = f_{60}(34) = \frac{60!}{34!26!} = 6.99 \times 10^{16}$$

Therefore, initially

$$\text{total initial \# of accessible states} = (495)(6.99 \times 10^{16}) = 3.46 \times 10^{19}$$

Similarly, when equilibrium is reached, we have

$$\text{final \# of accessible states of A} = f_{12}(7) = \frac{12!}{7!5!} = 792$$
$$\text{final \# of accessible states of B} = f_{60}(35) = \frac{60!}{35!25!} = 5.19 \times 10^{16}$$

and

$$\text{total final \# of accessible states} = (792)(5.19 \times 10^{16}) = 4.1 \times 10^{19}$$

We see that the total number of accessible states increases when two systems at different temperatures are put in thermal contact with one another. Since entropy is just the (natural) logarithm of the number of accessible states, the entropy increases as well.

D.18 Numerical Examples for Chapter 18

ΔT in MEH measurement
(page 265 of the book)

Example D.18.1. Let us work backwards, and calculate the temperature change of the water when a kilogram falls a distance of 1 meter. The work done by a mass of 1 kg falling 1 meter is 9.8 Joules (the product of the weight, which is 9.8 N and displacement, which is 1 meter). Let's assume that there is 5 kg (about 11 pounds) of water in the bucket. We use Equation (E.48) to calculate the temperature change. $Q = 9.8$ J, $m = 5$ kg, and, using Table 18.1, $c = 4186$. Substituting all this information in Equation (E.48), we get

$$9.8 = 5 \times 4186(T_f - T_i) \quad \Rightarrow \quad T_f - T_i = 9.8/20930 = 0.000468\ ^\circ\text{C}$$

Small numbers with which Joule had to cope.

This clearly shows how small the temperature change is in a typical experiment, and why the academic community of the mid 19th century was reluctant to accept Joule's results.

By increasing the falling mass and the distance it falls, and decreasing the amount of water, one can increase $T_f - T_i$. For example, if the falling mass is 5 kg and the distance is 10 meters, then the amount of work will be

$$5 \times 10 \times 9.8 = 490 \text{ Joules}$$

If the amount of water in the bucket is 3 kg, the temperature difference will be given by

$$490 = 3 \times 4186(T_f - T_i) \quad \Rightarrow \quad T_f - T_i = \frac{490}{12558} = 0.039\ ^\circ\text{C}$$

This is still a very small amount, especially if we recall that the 19th century instruments were not as accurate as today's. Nevertheless, Joule was able to measure the MEH using various methods all of which gave approximately the same number.

Measuring c for copper
(page 266 of the book)

Example D.18.2. A sample of copper weighing 100 grams is placed in boiling water for a long enough time so that it eventually acquires a temperature of 100 °C. When subsequently placed in 200 grams of cold water at 10 °C, it is seen that the temperature of the water eventually rises to 14 °C.

Q: What is the specific heat of copper?

A: Let us label all quantities appearing in Equation (E.48) for the copper by index 1 and those for the water by index 2. Assuming no loss to the environment, we can write $Q_1 + Q_2 = 0$ or $m_1c_1(T_{1f} - T_{1i}) + m_2c_2(T_{2f} - T_{2i}) = 0$. Note that since the quantity of heat is negative for copper and positive for water and the two are equal in magnitude, this equation makes sense. Converting grams to kg and substituting values for the known quantities, we obtain

$$0.1c_1(14 - 100) + 0.2 \times 4186 \times (14 - 10) = 0 \quad \Rightarrow \quad -8.6c_1 + 3348.8 = 0$$

The last equation gives $c_1 = 3348.8/8.6 = 389.4$, which is close to the value given in Table 18.1.

Heat pump vs space heater (**page 271 of the book**)

Example D.18.3. In this example, we compare the energy consumption of an electric space heater and a heat pump. Suppose the the outside temperature is -10 °C, and we want to maintain the inside temperature at 22 °C. The amount of heat lost to the outside is 2000 J per second (the heat power loss is 2000 Watts). To keep the inside temperature constant, we need to replenish this heat loss.

If we want to use an electric heater, we need a 2000-Watt one; and if it runs 10 hours a day, it will use

$$2000 \text{ Watt} \times 10 \text{ hour} = 2 \text{ kiloWatt} \times 10 \text{ hour} = 20 \text{ kWh}.$$

At 15 cents per kWh, this will cost \$3 per day.

Now consider an ideal heat pump, which as an engine, has an efficiency of

$$\epsilon = 1 - \frac{T_c}{T_h} = 1 - \frac{273.16 - 10}{273.16 + 22} = 0.1084.$$

But $\epsilon = W/Q_h$, where W is the work put into the heat pump (i.e., the electricity used), and Q_h is the heat delivered into the house (2000 J per second). Thus, the electric consumption per second is

$$W = \epsilon Q_h = 0.1084 \times 2000 = 216.8 \text{ Watt} = 0.2168 \text{ kWatt}$$

Running this for 10 hours consumes 2.168 kWh of electricity or just a little over 30 cents.

Although the actual numbers vary to some degree, this example illustrates the advantage of using heat pumps over electric heaters. The reason is that while an electric heater replenishes all of the Q_h directly from electricity (so that $W = Q_h$), a heat pump "pumps" some of the required heat—what we denoted as Q_c—from the cold outside into the house (so that $W = Q_h - Q_c$).

D.20 Numerical Examples for Chapter 20

From laboratory physics to the universe (**page 287 of the book**)

Example D.20.1. Sun is not a perfect black body radiator, but its radiation pattern can be approximated by a BBR curve resembling the one shown in Figure 20.2, where the wavelength on the horizontal axis is measured in μm (micrometer, or 10^{-6} m). The peak of the curve can be seen to occur at approximately 0.5 μm. This determination of the peak leads readily to the surface temperature of Sun:

$$T\lambda_{max} = 0.0029 \quad \text{or} \quad 0.5 \times 10^{-6}T = 0.0029 \quad \Rightarrow \quad T = 5800 \text{ °K}$$

Now that we know the temperature, we can find Sun's energy flux using the Stefan-Boltzmann law,

$$J_e = 5.67 \times 10^{-8}T^4 = 5.67 \times 10^{-8} \times (5800)^4 = 6.4 \times 10^7 \text{ Watts/m}^2$$

Imagine a typical table top (about one square meter); imagine spreading 640,000 one-hundred-Watt light bulbs evenly on it. Now *try* to imagine how bright that surface is. Thats how bright the Sun's surface is!

If we know Sun's radius, we can calculate its total power output. Actually, we do know Sun's radius; it is about 700,000 km. A sphere of radius r has an area of $4\pi r^2$. So, the area of the Sun is $4\pi(7\times10^8)^2$ or 6.16×10^{18} m^2. Since each square meter of Sun's surface gives off 6.4×10^7 Watts, the total power output of the Sun is this number time its area:

$$\text{Total power output of Sun} = (6.4\times10^7)\times(6.16\times10^{18}) = 3.94\times10^{26}\ \text{Watts.}$$

Where does this power come from? What fuels this colossal release of energy? The answer to these questions had to wait the discovery of relativity and the nucleus of the atom. It is the mass of the Sun that, in a most dramatic demonstration of the famous $E=mc^2$, turns into energy through a nuclear process known as **thermal fusion**. The mass depletion corresponding to the above power is obtained from $E=mc^2$:

$$3.94\times10^{26} = m\times(3\times10^8)^2 \quad \text{or} \quad m = \frac{3.94\times10^{26}}{9\times10^{16}} = 4.38\times10^9\ \text{kg/s.}$$

Sun loses over 4 million tons of its mass every *second* to shine. So, it will eventually die, but not when it loses *all* its mass. The combination of the nuclear and gravitational processes seal the fate of the Sun far ahead of its complete annihilation. Chapter 39 discusses these processes in some more detail.

Example D.20.2. Equating the photon energy (hf or hc/λ) to the energy required to "unglue" the electron from the metal (this is the work function W) plus the (maximum) kinetic energy of the electron, Einstein obtained

Illustration of photoelectric effect **(page 294 of the book)**

$$\begin{array}{llll} \text{In terms of frequency} & hf = W + KE_{\max} & \text{or} & hf = W + eV_{\text{stop}} \\ \text{In terms of wavelength} & \dfrac{hc}{\lambda} = W + KE_{\max} & \text{or} & \dfrac{hc}{\lambda} = W + eV_{\text{stop}} \end{array} \qquad \text{(D.16)}$$

Zinc has a work function of about 3.5 eV. The EM radiation with the longest wavelength that can release photoelectrons from the surface of a zinc plate is that λ for which $KE_{\max}$ is as small as possible. Thus as long as

$$\frac{hc}{\lambda} > W \quad \text{or} \quad \lambda < \frac{hc}{W}$$

we will have some electrons coming out. Thus, the (borderline) longest wavelength is given by

$$\lambda = \frac{hc}{W} = \frac{(6.626\times10^{-34})(3\times10^8)}{3.5\times(1.6\times10^{-19})} = 3.55\times10^{-7}\ \text{m}$$

or 0.355 μm (micrometer) or 355 nm (nanometer), which is in the ultraviolet region of the EM spectrum. Notice how we were careful to change the unit of W from eV to J in the formula above.

If UV light of wavelength 150 nm strikes the surface of zinc, what is the maximum KE of the electrons released? To answer this question, we use (D.16):

$$\frac{(6.626\times10^{-34})(3\times10^8)}{150\times10^{-9}} = \underbrace{3.5\times(1.6\times10^{-19})}_{\text{conversion of eV to J}} + KE_{\max}$$

This yields $KE_{\max} = 7.65\times10^{-19}$ J or 4.78 eV. If we are interested in the speed of such electrons, we can use the definition of KE and the fact that the mass of an electron is 9.11×10^{-31} kg:

$$KE = \tfrac{1}{2}mv^2 \quad \Rightarrow \quad 7.65\times10^{-19} = \tfrac{1}{2}(9.11\times10^{-31})v^2$$

or

$$v = \sqrt{\frac{2 \times 7.65 \times 10^{-19}}{9.11 \times 10^{-31}}} = 1.3 \times 10^{6} \text{ m/s}$$

D.21 Numerical Examples for Chapter 21

Wavelength of light emitted by hydrogen **(page 305 of the book)**

Example D.21.1. The electron in a hydrogen atom makes a transition from the third to the second orbit.

Q: What is the wavelength of the photon emitted?

A: The initial energy is $E_i = E_3 = -13.6/3^2 = -1.51$ eV. The final energy is $E_f = E_2 = -13.6/2^2 = -3.4$ eV. Thus, the photon energy is

$$E_\gamma = E_i - E_f = -1.51 - (-3.4) = 1.89 \text{ eV}$$

or $1.89 \times (1.6 \times 10^{-19}) = 3 \times 10^{-19}$ J. To find the wavelength, we use the Planck relation $E = hc/\lambda$. This yields

$$3 \times 10^{-19} = \frac{(6.626 \times 10^{-34})(3 \times 10^{8})}{\lambda} \quad \text{or} \quad \lambda = \frac{1.988 \times 10^{-25}}{3 \times 10^{-19}} = 6.6 \times 10^{-7} \text{ m}$$

or 0.66 μm. This is a visible red photon.

D.25 Numerical Examples for Chapter 25

Relatively inertial frames **(page 366 of the book)**

Example D.25.1. In this example we want to show that if an object moves with uniform velocity relative to one observer, then it moves with uniform velocity relative to all other observers moving uniformly relative to the original observer. Refer to Figure D.13 where we have labeled the origin of RF_1 by O, that of RF_2 by B, and the object moving relative to RF_2 by A. We have also assumed, for simplicity, that all points coincide at the initial time, $t = 0$. The snapshots of motion at four subsequent times, separated equally, say by one second, is also shown in the figure. After one second B goes to B_1 and A goes to A_1 so that A has moved a distance of $\overline{B_1A_1}$ in one second. After two seconds, B is found at B_2 and A at A_2. Note that $\overline{B_2A_2} = 2\overline{B_1A_1}$ because A is assumed to be moving uniformly relative to B. Similarly, $\overline{B_3A_3} = 3\overline{B_1A_1}$, etc. It is an easy exercise in geometry to show that A, A_1, A_2, and A_3 all lie along a single straight line and that

$$\overline{OA_1} = \overline{A_1A_2} = \overline{A_2A_3} = \dots$$

so that A moves with constant velocity relative to O.

D.26 Numerical Examples for Chapter 26

Possibility of space travel **(page 383 of the book)**

Example D.26.1. The spaceship Viking is a super fast cosmic explorer that can attain a speed of $0.999c$. This spaceship is charged with exploring the star system Zeta Leporis located at a distance of 70 light years from Earth.[8]

Q: How long does it take Viking to reach (one of the planets of) Zeta Leporis (a) according to the Earth clock, and (b) according to the spaceship clock?

A: For (a), we simply use

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{70 \text{ light years}}{0.999c} = \frac{70 \text{ years} \cdot c}{0.999c} = \frac{70}{0.999} \text{ years} = 70.07 \text{ years}$$

[8] A light year is a *distance* obtained by multiplying a year by the speed of light. Multiplying 3.15×10^{7}—the number of seconds in a year—by 3×10^{8} m/s, the speed of light, one obtains the result that 1 light year $= 9.45 \times 10^{15}$ m. In some calculations it is more convenient to write light year as *year* $\cdot$ *c*.

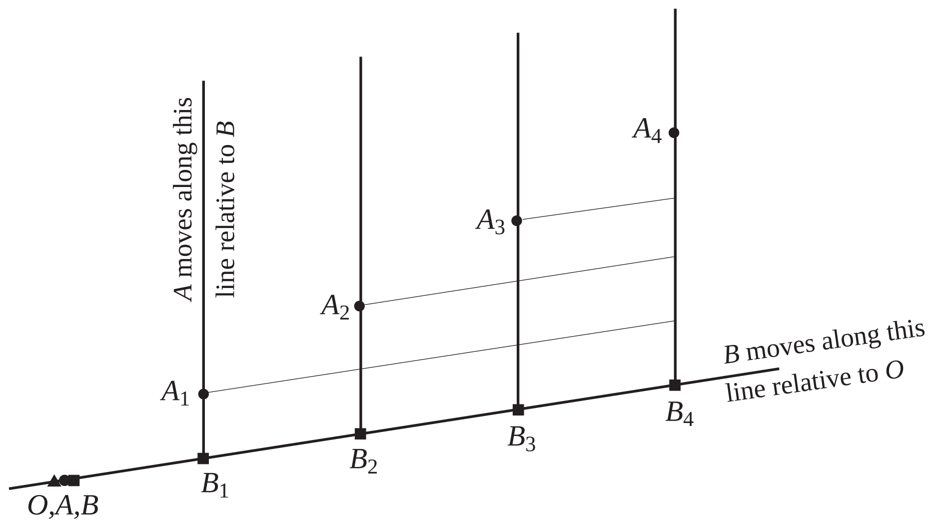


Figure D.13: An RF inertial relative to an inertial RF is inertial.

So, as expected, it takes a little over 70 years for Viking to get to Zeta Leporis, as seen by ground observers.

To obtain (b), we note that the spaceship clock measures a proper time between two events: spaceship leaves the Earth (event E_1), and spaceship lands on Zeta Leporis (event E_2), and the clock is present at both of these. We thus write

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-(v/c)^2}} \quad \Rightarrow \quad 70.07 \text{ years} = \frac{\Delta\tau}{\sqrt{1-(0.999)^2}}$$

or

$$\begin{aligned}\Delta\tau &= 70.7 \text{ years} \times \sqrt{1-(0.999)^2} = 70.7 \text{ years} \times \sqrt{0.001999} \\ &= 70.07 \text{ years} \times 0.0447 = 3.13 \text{ years}\end{aligned}$$

So, it takes a little over 3 years for the crew of Viking to reach Zeta Leporis!

Example D.26.2. The spaceship Enterprise is charged with an exploratory mission that takes it to a planet in a star system far away. The captain of the ship, who has just had a baby, is 30 years old when Enterprise takes off with a speed of $0.98c$. It takes the crew of the spaceship 5 years to get to their destination. They spend 6 months on the planet and then head back home with a speed of $285,000$ km/s.

Father younger than daughter
(page 383 of the book)

The journey can be naturally divided into three parts: moving at $0.98c$ toward the planet; landing and staying on the planet on which their speed is zero (or very small); heading back home with a speed of $285,000$ km/s. When using Equation (26.1), you have to apply it *separately* to the three portions of the journey.

Q1: How old is the baby when her father lands on the distant planet?

A1: The proper time interval—measured by the crew—between take off from Earth and landing on the planet is 5 years. The time interval between the same two events as measured by Earth people is

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}} = \frac{5}{\sqrt{1-(0.98)^2}} = 25.13 \text{ years}$$

Q2: How far is the planet from Earth?

A2: It takes 25.13 years for the ship to get there while moving at 98% light speed. Therefore, is

$$\text{distance} = \text{speed} \times \text{time} = 0.98c \times 25.13 \text{ years} = 24.63\, c \cdot \text{yr} = 24.63 \text{ ly}$$

Q3: According to the "baby" how long does it take the captain to travel from the planet back to Earth?

A3: Now we have the distance and the speed (285,000 km/s is $0.95c$); therefore, we can find the time interval:

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{24.63\ \cancel{l}\text{y}}{0.95\cancel{c}} = 25.93 \text{ years}$$

Q4: How long does it take the captain to travel from the planet back to Earth?

A4: The crew measures the proper time again. We have the coordinate time Δt and we want to find $\Delta\tau$:

$$\Delta\tau = \Delta t\sqrt{1-(v/c)^2} = 25.93\sqrt{1-(0.95)^2} = 8.1 \text{ years}$$

Q5: How old is the "baby" when her father arrives back on Earth?

A5: We simply add the time intervals, keeping in mind that the 6 months that the crew spends on the planet is the same for Earth observers. This is because the spaceship is not moving relative to Earth,[9] i.e., it belongs to Earth's RF. Thus,

$$\text{age of the "baby"} = 25.13 + 0.5 + 25.93 = 51.56 \text{ years}$$

Q6: How old is the captain when he arrives back on Earth?

A6: Again, we just add the time intervals to his initial age:

$$\text{age of the captain} = 30 + 5 + 0.5 + 8.1 = 43.6 \text{ years}$$

The father is almost 8 years younger than his daughter!

"Explaining" why space travel is possible
(page 384 of the book)

Example D.26.3. Going back to Example D.26.2, we can now see why Enterprise can cover the Earth-planet distance in 5 years. On their way to their destination, the Enterprise passengers see this distance in motion, and they measure it to be

$$L = L_0\sqrt{1-(v/c)^2} = 24.63\sqrt{1-(0.98)^2} = 4.9 \text{ ly}$$

Since they are moving with a speed of $0.98c$, their travel time is

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{4.9\ \cancel{l}\text{y}}{0.98\cancel{c}} = 5 \text{ years}$$

as given in the statement of Example D.26.2.

On their way back, they measure the same distance as

$$L = L_0\sqrt{1-(v/c)^2} = 24.63\sqrt{1-(0.95)^2} = 7.69 \text{ ly}$$

and with a speed of $0.95c$, their travel time is

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{7.69\ \cancel{l}\text{y}}{0.95\cancel{c}} = 8.1 \text{ years}$$

which is what we found in Example D.26.2 using time-dilation formula.

[9] We ignore the small speed the planet may have relative to Earth.

Example D.26.4. As a concrete example, let us apply Equation (E.72) of Math Note E.26.5 to the calculation of the shrinkage of a length. In the hope that we will be able to see the relativistic effect, we take the fastest vehicle of Table 26.1 and build the longest satellite possible. Let's assume that the length is of the order of a football field, about 100 m. Then from the first equation in (E.72) and the fifth column of the table, we obtain

$$L_0 - L = \tfrac{1}{2}(v/c)^2 L_0 = 3.5 \times 10^{-10} \times 100 = 3.5 \times 10^{-8} \text{ m}$$

which is the size of a molecule!

A jet plane flies for 10 hours according to a clock on the ground. Using Table 26.1, estimate the amount by which the plane's clock goes slower.

Smallness of length contraction and time dilation
(page 387 of the book)

The difference between the time intervals Δt and $\Delta\tau$ is found by using the second formula in (E.72):

$$\Delta t - \Delta\tau = \tfrac{1}{2}(v/c)^2 \Delta t = (4 \times 10^{-13}) \times (10 \times 3600) = 2.88 \times 10^{-8} \text{ sec}$$

Although this is too small a time interval to measure by ordinary clocks, it is easily measurable by atomic clocks. In fact, a test very similar to this was done in 1975 and the result agreed perfectly with the prediction of the STR.

Example D.26.5. The crew of Apollo 23 goes to the Moon with a speed of 15 km/s. It spends 10 hours exploring the Moon, and comes back with the same speed. The captain of the spaceship has just had a baby when he leaves on the mission. The whole trip takes 24 hours for the crew.

To feel the enormity of Apollo's speed, estimate its travel time from New York to Los Angeles (a distance of about 5000 km):

Time dilation for a Moon trip
(page 387 of the book)

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{5000}{15} = 333.33 \text{ s} = 5.55 \text{ min}$$

An extraordinarily fast vehicle by any human standard!

Q1: How many hours does the captain spend on the way from Earth to Moon?

A1: The round-trip travel time is $24 - 10 = 14$ hours. Since the speed is the same outbound and inbound, the time for each leg of the trip is half this amount, i.e., 7 hours or $7 \times 3600 = 25,200$ s.

Q2: How many more seconds does it take the baby for her father to reach the Moon?

A2: Here we are seeking the difference between the proper and the coordinate time. Since the speed is small compared to light speed (the ratio v/c is 15/300,000 or 0.00005), we can use the second equation in (E.72). Since the right-hand side of this equation is very small, the difference between Δt and $\Delta\tau$ is very small, i.e., $\Delta t \approx \Delta\tau$. Thus, we use $\Delta\tau$ instead of Δt on the right-hand side, because we know $\Delta\tau$ (it is 7 hours or 25,200 s):

$$\Delta t - \Delta\tau \approx \tfrac{1}{2}(v/c)^2 \Delta\tau = \tfrac{1}{2}(0.00005)^2 \times 25,200 = 0.0000315 \text{ s}$$

So, it will take the captain 25,200 s to get to the Moon, during which time the baby has aged 25200.0000315 s.

Q3: How many more seconds has the baby aged than her father when Apollo 23 returns?

A3: The time difference develops only when the spaceship is in motion.[10] Therefore, the aging difference is twice the difference for each leg of the trip, or $2 \times 0.0000315 = 0.000063$ s.

Q4: How far is Moon from Earth according to the Earth observers?

A4: Since we know the speed and the time, we can find the distance by multiplying:

$$\text{distance} = \text{speed} \times \text{time} = 15 \times 25,200.0000315 = 378000.00045 \text{ km}$$

[10]Strictly speaking, this is not true! While in ultrarelativistic cases we could ignore the motion of the planet, here the speed of the Moon is comparatively not as small as the speed of the planet relative to the spaceship. Nevertheless, it is small enough that we can ignore it.

Q5: What is the Earth-Moon distance according to the crew?
A5: We use the same formula:

$$\text{distance} = \text{speed} \times \text{time} = 15 \times 25,200 = 378000 \text{ km}$$

The Earth-Moon distance has shrunk by only 0.00045 km or 0.45 m (less than a foot and a half) for the crew!

D.27 Numerical Examples for Chapter 27

Spacetime distance of zero
(page 398 of the book)

Example D.27.1. Observer O spots a light beam at x_1 at time t_1 (event E_1). A little later he finds the beam at x_2 at time t_2 (event E_2).

Q: What is the spacetime distance for this light beam?

A: Since light travels from x_1 to x_2 with speed c, we have

$$x_2 - x_1 = c(t_2 - t_1) \quad \text{or} \quad \Delta x = c\Delta t$$

Therefore,

$$\Delta s = \sqrt{c^2(\Delta t)^2 - (\Delta x)^2} = \sqrt{c^2(\Delta t)^2 - (c\Delta t)^2} = 0$$

which holds for any light signal, as the two events above are quite general.

Another observer O' sees these two events as (x'_1, ct'_1) and (x'_2, ct'_2). The invariance of the spacetime distance tells us that for her Δs is also zero. Therefore,

$$0 = \Delta s = \sqrt{c^2(\Delta t')^2 - (\Delta x')^2} \quad \Rightarrow \quad c(\Delta t') = \Delta x' \quad \text{or} \quad \frac{\Delta x'}{\Delta t'} = c$$

i.e., she measures the speed of the light signal to be c, consistent with the second principle of relativity.

Quantitative analysis of tunnel and train
(page 400 of the book)

Example D.27.2. The time of the occurrence of event E_2 (or E_3) according to Karl is the segment $\overline{E_1E_3}$ (see Figure D.14). Furthermore, rule 2 of Box F.0.2 gives $\overline{E_2E_3}/\overline{E_1E_3} = \beta$. But $\overline{E_2E_3}$ is just $\overline{AB} = 500$ m. Thus, $\overline{E_1E_3} = 500/0.75 = 667$ m. This is the projection of $\overline{E_1E_2}$—a segment on Emmy's time axis—onto Karl's time axis; so by rule 4 of Box F.0.3, $\overline{E_1E_2} = \overline{E_1E_3}/\gamma = 667/1.51 = 441$ m. This is the time of occurrence of the coincidence of B and D (times the speed of light) according to Emmy.

What is the time of occurrence of the coincidence of A and C (times the speed of light) according to Emmy? It is the line segment $\overline{DG}$, which is the projection of $\overline{E_1E_3}$ onto the ct-axis. Again, by rule 4 of Box F.0.3,

$$\overline{DG} = \gamma\overline{E_1E_3} = 1.51 \times 667 = 1007 \text{ m.}$$

Therefore, E_3 occurs $1007 - 441 = 566$ m or $\frac{566}{3\times10^8} = 1.9 \times 10^{-6}$ s later than E_2 according to Emmy.

The stretch factor is $1.51\sqrt{1 + 0.75^2} = 1.89$. This means that the ticks used on Emmy's axes should be 1.89 times farther apart than those used for Karl. The stretched units (of 100 m each) used in Figure D.14 to measure the length of the train takes this fact into account.

Trying to stop Bruno's execution
(page 402 of the book)

Example D.27.3. To stop the execution, we need to find an observer for whom "now" (17 February 2003) and Bruno's execution are simultaneous, i.e., they both lie on the x-axis of the observer. We look among our Galactic Explorers and find observer O 500 light years away, far enough that with the proper speed will have an x-axis that passes through the event E, Bruno's death [see Figure D.15(a)].

Q1: What speed should O have? In which direction?

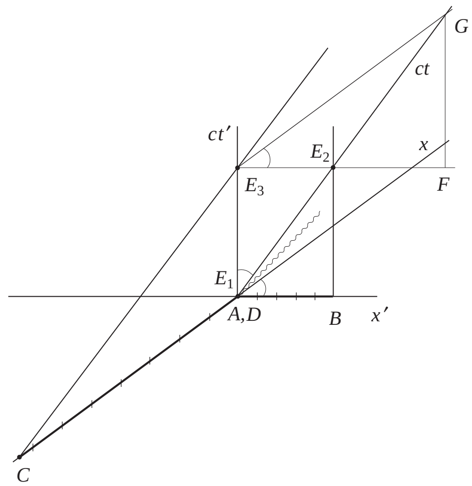


Figure D.14: The spacetime diagram of the train and the tunnel.

A1: Figure D.15(a) shows that O should be moving *away from* us, because the ct-axis makes an acute angle with the x'-axis. The same figure shows that $\beta = \overline{EO'}/\overline{O'O}$ or $\beta = c \times 403$ years/(500 ly) $= 0.806$; so O must be moving with 80.6% the speed of light away from us.

Q2: How far away is Bruno's execution taking place from O?

A2: By rule 4 of Box F.0.3, $\overline{O'O} = \gamma\overline{EO}$. But $\gamma = 1/\sqrt{1-0.806^2} = 1.69$. Therefore, $\overline{EO} = \overline{O'O}/\gamma = 500/1.69 = 296$ light years, too far away to prevent the execution.

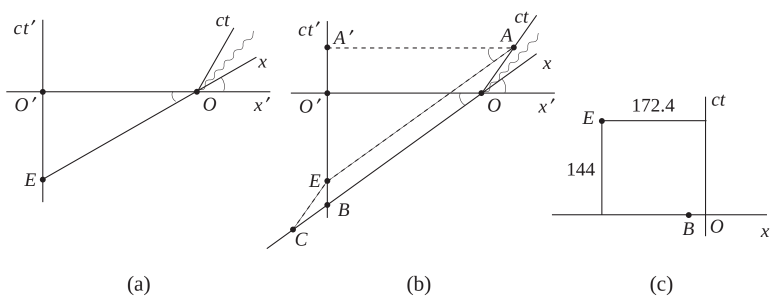


Figure D.15: (a) Spacetime diagram for rescuing Bruno by finding an observer whose present time is Bruno's execution. (b) The coordinates of E in the nonperpendicular coordinate system O. (c) The events B and E as seen by O.

It appears that γ determines how close we can get to the execution; the larger the γ, the closer we might get. Since the enormity of γ is determined by β, we try to choose β closer to 1. So we call on another observer who is 404 light years away. The required β—that which ensures that the x-axis of the observer passes through B—is $\beta = 403/404 = 0.99752$, giving $\gamma = 14.22$. In this case, $\overline{EO} = \overline{O'O}/\gamma = 404/14.22 = 28.4$ light years, still too far away to prevent the execution.

Q3: Is there any way that we can get to E?

A3: To answer this question, we write $\overline{EO}$ in terms of $\overline{EO'}$. We do this by simply noting that $\overline{EO} = \overline{O'O}/\gamma$ and $\overline{O'O} = \overline{EO'}/\beta$. Then,

$$\overline{EO} = \frac{\overline{O'O}}{\gamma} = \frac{\overline{EO'}/\beta}{\gamma} = \frac{\overline{EO'}}{\beta\gamma}$$

and thus, for $\overline{EO}$ to be very small, γ has to be very large; and unless we go *at the speed of light*, we can never shrink $\overline{EO}$ to zero! Although we *can* find observers for whom Bruno's death occurs *NOW*, we can never find an observer who is present *at the location* of the execution.

Since we can "go back" in time, why not go further back, to a time *before* the event of interest happened and "wait" for the event? Let's try this and go back 10 years prior to Bruno's execution, to event B of Figure D.15(b). In this case, the x-axis is the line OB, with the corresponding ct-axis drawn at equal angle from the light world line, as done before. Assume that O is 414 light years away. This gives

$$\beta = \frac{\overline{BO'}}{\overline{O'O}} = \frac{413}{414} = 0.99758 \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - 0.99758^2}} = 14.4$$

where β was found using rule 2 of Box F.0.2. To see if E can be prevented, we need to know where the location of E is in the spacetime plane of O.

Q4: When is E happening according to O?

A4: Draw a line from E parallel to the x-axis to cut the ct-axis at A. Then $\overline{OA}$ is the time coordinate of E in O. By rule 4 of Box F.0.3,

$$\overline{OA} = \gamma\overline{BE} = 14.4 \times 10 = 144 \text{ ly}$$

So the execution is happening 144 years from now according to O.

Q5: What is the x-coordinate of E according to O?

A5: Draw a line from E parallel to the ct-axis to cut the x-axis at C. Then $-\overline{OC}$ is the x-coordinate of E in O. But $\overline{OC} = \overline{EA}$. To find $\overline{EA}$, draw a line from A parallel to the x'-axis to cut the ct'-axis at A'. Then the length of $\overline{EA}$ *as measured by an ordinary ruler* by O' is given by the Pythagoras theorem:

$$\overline{EA} = \sqrt{(\overline{EA'})^2 + (\overline{A'A})^2} = \sqrt{(\overline{EA'})^2 + (\overline{EA'}/\beta)^2} = \frac{\overline{EA'}}{\beta}\sqrt{1+\beta^2}$$

where the second equality follows from rule 2 of Box F.0.2. By rule 5 of Box F.0.3, the *real length* of $\overline{EA}$ as measured by O is obtained by dividing $\overline{EA}$ by the stretch factor $\gamma\sqrt{1+\beta^2}$. Therefore, denoting the x-coordinate of E by x_E, we get

$$x_E = -\frac{\overline{EA}}{\gamma\sqrt{1+\beta^2}} = -\frac{\overline{EA'}}{\gamma\beta} \tag{D.17}$$

The only thing that is left now is to find $\overline{EA'}$, which is the sum of $\overline{EO'}$ and $\overline{O'A'}$. The first one is given, and the second one can be obtained from $\overline{OA}$ by another application of the rule 4 of Box F.0.3:

$$\overline{O'A'} = \gamma\overline{OA} = 14.4 \times 144 = 2074 \text{ ly}$$

Equation (D.17) now yields

$$x_E = -\frac{403 + 2074}{(14.4)(0.99758)} = -172.4 \text{ ly}$$

Thus, according to O, the execution will happen 144 years from now at a distance of 172.4 ly (in the negative x-direction). Any probe sent from O must have a fractional speed

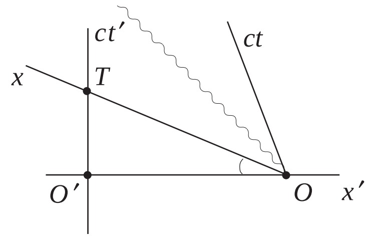


Figure D.16: Spacetime diagram for "traveling" to the future. T stands for tomorrow. OT must be the x-axis, because T has to happen NOW for O.

of 172.4/144; i.e., it must travel 1.2 times faster than light! Even a laser pulse sent with the purpose of stunning the executioner will not be able to make it in time!

For the sake of completeness, we also calculate the coordinates of B relative to O. Clearly, B has zero time coordinate. Its x-coordinate x_B is obtained by rule 4 of Box F.0.3: $\overline{OO'}$ is 414 ly and is γ times $\overline{OB}$. Therefore,

$$x_B = -\overline{OB} = -\frac{\overline{OO'}}{\gamma} = -\frac{414}{14.4} = -28.75 \text{ ly}$$

Figure D.15(c) shows the two events B and E in the coordinate system O.

Time traveling to future **(page 402 of the book)**

Example D.27.4. So far, we have been trying to travel to the past, without success. Would we have a better luck with "future" travel? Tomorrow is "only one day away;" it is probably not too much to ask the theory of relativity to help us get there. Utilizing the experience we gained in the case of Bruno's death, we find observer O who is only 24.5 light hours away.[11]

Q1: What speed should O have? In which direction?

A1: Figure D.16 shows that O should be moving *toward* us (because the ct-axis, the worldline of O, is bending towards the ct'-axis), and that $\beta = \overline{TO'}/\overline{O'O} = c \times$ 24 hours/(24.5 light hours) $= 0.9796$; so O must be moving with 97.96% the speed of light toward us.

Q2: How far away is "tomorrow" taking place from O?

A2: First we calculate γ: $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-0.9796^2} = 4.975$. By rule 4 of Box F.0.3, $\overline{O'O} = \gamma\overline{TO}$. Therefore, $\overline{TO} = \overline{O'O}/\gamma = 24.5/4.975 = 4.9$ light hours, or $4.9 \times 3600 \times (3 \times 10^8) = 5.3 \times 10^{12}$ m, or 3.3 billion miles! Again, "tomorrow" is too far away.

Finding time interval for one observer when simultaneous for another **(page 403 of the book)**

Example D.27.5. This example calculates the time difference between the explosion of the two firecrackers on the train as measured by Karl when Emmy sees them simultaneously (see Figure 25.6 of Chapter 25). Let Karl be O' and Emmy O, and assume that right is positive. Thus, Emmy is moving in the positive direction implying a positive β. Suppose E_1 is the explosion of A and E_2 the explosion of B. Let the origin of Emmy's coordinate system be where she is located, i.e., the middle of the train, whose length is L. Also let the time of explosion be when Emmy's clock starts ticking, i.e., the explosions occur at time zero. Under these assumptions, E_1 and E_2 have respective coordinates $(-L/2, 0)$ and $(+L/2, 0)$ according to Emmy.

If we take Δx to be $x_2 - x_1$ then $\Delta x = +L/2 - (-L/2) = L$ and $\Delta t = t_2 - t_1 = 0 - 0 = 0$. The second of the Δ-equations of Box F.4.1 yields

$$c\Delta t' = t'_2 - t'_1 = \gamma\beta\Delta x = \gamma\beta L$$

[11]A light hour is the distance that light travels in one hour. For comparison, Saturn is about 1.25 light hours away from Sun.

For example, if the train moves at half the speed of light and the car is 30 m long, then $\beta = 0.5$, $\gamma = 1/\sqrt{1-(0.5)^2} = 1.155$, and

$$c\Delta t' = 1.155 \times 0.5 \times 30 = 17.32 \text{ m}$$

or

$$\Delta t' = t_2' - t_1' = \frac{17.32}{3 \times 10^8} = 5.8 \times 10^{-8} \text{ s},$$

i.e., $t_2' = t_1' + 5.8 \times 10^{-8}$ s. This shows that t_2', the time of the occurrence of B according to Karl is larger than t_1', the time of the occurrence of A. Thus, Karl sees A before B, as explained in Chapter 25.

Time dilation and length contraction from Lorentz transformation **(page 403 of the book)**

Example D.27.6. This example shows that the Lorentz transformation implies time dilation and length contraction. If $\Delta x = 0$, then Δt is the proper time between the two events. The second of the Δ-equations of Box F.4.1 gives

$$c\Delta t' = \gamma(0 + c\Delta t) \quad \text{or} \quad \Delta t' = \frac{\Delta t}{\sqrt{1-(v/c)^2}}$$

which is the relation between proper and coordinate time as given in Equation (26.1).

For length contraction, first we have to determine how to measure the length of an object that is moving. To measure the length of such an object, we have to spot the two ends of the object *at the same time*, i.e., we can call $\Delta x'$ "the length of the object" only if $\Delta t' = 0$. This happens only if $\beta\Delta x + c\Delta t = 0$ or $c\Delta t = -\beta\Delta x$. Substituting this in the first Δ-equation of Box F.4.1 yields

$$\Delta x' = \gamma(\Delta x - \beta^2\Delta x) = \frac{1}{\sqrt{1-\beta^2}}(1-\beta^2)\Delta x = \sqrt{1-\beta^2}\,\Delta x$$

which is the length contraction formula once we identify β as v/c. We could have obtained the same formula if we had started with the Lorentz transformation having the O' coordinates on the right-hand side.

Lorentz transformation and time travel **(page 403 of the book)**

Example D.27.7. The year is 2163, and the American delegation to the Intergalactic Space Federation, stationed on Earth, is submitting a proposal to use the laws of relativity to stop the assassination of President Kennedy. The ISF accepts the proposal and starts to seek reference frames in which the assassination takes place NOW (in 2163). It finds the Spaceship Diracus 210 light years (ly) away, which is just passing one of the ISF outposts there.

Q1: How fast is Diracus moving relative to Earth (or the outpost)?

A1: Let the RF of Diracus be O and the Earth's RF be O'. The two events of interest are E_1, the assassination of President Kennedy (on Earth) in *Earth year* 1963, and E_2, the passage of Diracus by the outpost in Earth year 2163. These events are shown in Figure D.17 in the Earth reference frame O'. E_1 has coordinates $(0, -200 \text{ ly})$ because we are assuming that NOW is the year 2163.[12] Similarly, E_2 has coordinates $(210 \text{ ly}, 0)$.

$\Delta x' \equiv x_2' - x_1'$ is positive because $x_2' = 210$ ly and $x_1' = 0$; $c\Delta t'$ is positive because $t_2' = 0$ and $t_1' = -200$ years.

In the language of the Δ-equations of Box F.4.1, $\Delta x' = 210$ ly, $c\Delta t' = 200$ ly, and since we want the two events to be simultaneous on Diracus, $\Delta t = 0$. Thus, the two Δ-equations in Box F.4.1 reduce to

$$210 \text{ ly} = \gamma\Delta x, \quad 200 \text{ ly} = \gamma\beta\Delta x = \beta(\gamma\Delta x) \tag{D.18}$$

To find the relative speed, substitute the left-hand side of the first equation in the right-hand side of the second:

$$200 \text{ ly} = \beta(210 \text{ ly}) \quad \text{or} \quad \frac{200}{210} = \beta \quad \text{or} \quad \beta = 0.9524$$

[12] Recall that $c \times$ year is the same as light year. So the second coordinate of E_1 is -200 ly.

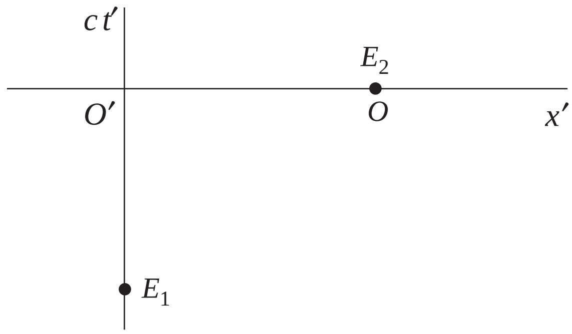


Figure D.17: The events E_1 and E_2 in the Earth RF. Note that the time of the origin (and thus the time of all the events on the x'-axis) is NOW which is the year 2163.

Thus, Diracus must be moving at slightly over 95% the speed of light. Since β is positive, Diracus must be moving *away* from Earth.

Q2: According to the Diracus crew, how far away is the assassination taking place?

A2: Now that we have the relative speed, we can find the distance Δx between the two events for the Diracus crew by using either of the two equations in (D.18). But we need γ first: $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-(0.9524)^2} = 3.28$. Using the first equation in (D.18) yields

$$\Delta x = \frac{210 \text{ ly}}{\gamma} = \frac{210}{3.28} = 64 \text{ ly}$$

So, although the Diracus crew knows that Kennedy's assassination is taking place now, it can do nothing about it, because it is taking place 64 light years away. Kennedy cannot be saved!

Going further back in time

Having gained experience from the failure of their attempt in the previous example, the ISF looks for another RF for which *1961 is NOW*. That way, argues ISF's project director, the crew will have two extra years to "prepare" for the event. Their plan is to aim at the building from which the shooting takes place. The Spaceship Diracus II, 205 ly away, seems to be a good candidate. The commander of Diracus II consults the chief physicist of the mission, and asks her the following questions. Let's see if we can answer them.

Q1: How fast should Diracus II be moving relative to Earth (or the outpost)?

A1: As in the previous case, let the RF of Diracus II be O and the Earth's RF be O'. The two events of interest are E_1, "the building—in Earth year 1961—in which the shooting will take place" and E_2 the "passage of Diracus II by the outpost in Earth year 2163." These events are shown in Figure D.18(a) in the Earth reference frame O'. E_1 has coordinates $(0, -202 \text{ ly})$ because again we are assuming that NOW is the year 2163. Similarly, E_2 has coordinates $(205 \text{ ly}, 0)$.

Because there are three events to deal with, it is helpful to label the Δ quantities. For example, use Δx_{21} to denote $x_2 - x_1$, and $c\Delta t_{31}$ to denote $ct_3 - ct_1$, etc. Then, we have $\Delta x'_{21} = 205$ ly, $c\Delta t'_{21} = 202$ ly, and since we want the two events to be simultaneous on Diracus II, $\Delta t_{21} = 0$. Thus, the two Δ-equations of Box F.4.1 reduce to

$$205 \text{ ly} = \gamma \Delta x_{21}, \quad 202 \text{ ly} = \gamma\beta\Delta x_{21} = \beta(\gamma\Delta x_{21}) \tag{D.19}$$

Substitute the left-hand side of the first equation on the right-hand side of the second equation to obtain

$$202 = \beta(205) \quad \text{or} \quad \frac{202}{205} = \beta \quad \text{or} \quad \beta = 0.9854$$

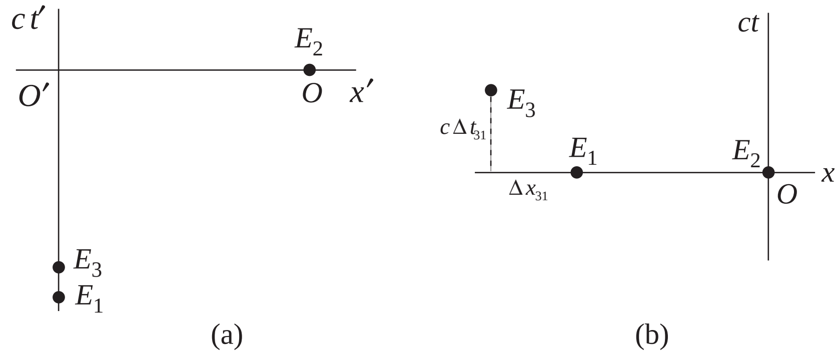


Figure D.18: (a) The events E_1, E_2, and E_3 in the Earth RF. Note that the time of the origin (and thus the time of all the events on the x'-axis) is NOW which is the year 2163. (b) The same three events as seen by the crew of Diracus II.

Since β is positive, Diracus II must be moving *away* from Earth.

Q2: According to the Diracus II crew, how far away is the building in which the assassin will be hiding?

A2: We can find the spatial distance Δx_{21} between E_1 and E_2 for the Diracus II crew by using either of the two equations in (D.19), for which we have to calculate γ first: $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-(0.9854)^2} = 5.87$. Using the first equation in (D.19) yields

$$\Delta x_{21} = x_2 - x_1 = \frac{205 \text{ ly}}{\gamma} = \frac{205}{5.87} = 34.9 \text{ ly}$$

Since $x_2 = 0$ (origin of Diracus II), $x_1 = -34.9$ ly. Figure D.18(b) shows these two events in the RF of Diracus II.

The chief physicist reports these results to the commander. The commander, knowing that nothing can move as fast as light, asks him to look into the possibility of sending a powerful laser beam to stun the assassin exactly at the time of the shooting. The chief physicist starts to calculate. Let's see if we can anticipate his results.

Q3: According to the Diracus II RF, what is the time and space difference between the two events "assassination building in Earth year 1961" and "assassination building in Earth year 1963?" Denote the latter event by E_3 and designate their space and time difference with the subscript 31.

A3: The (inverse) Lorentz transformation—i.e., the transformation with the *primed quantities* on the right-hand side—gives the answer. Since O' (the Earth RF) is moving *in the negative* direction of O (the Diracus II RF), we must use $-\beta$ in the formulas. Now, the space difference between the two events is zero according to Earth RF, because they both occur in the same building. The Earth time interval is 2 years. It follows that

$$\begin{aligned}\Delta x_{31} &= \gamma(\Delta x'_{31} - \beta c\Delta t'_{31}) = 5.87(0 - 0.9854c \times 2) = -11.57 \text{ ly}\\ c\Delta t_{31} &= \gamma(-\beta\Delta x'_{31} + c\Delta t'_{31}) = 5.87(0 + c \times 2) = 11.74 \text{ ly} \quad \text{or} \quad \Delta t_{31} = 11.74 \text{ yrs}\end{aligned}$$

Q4: According to the Diracus II RF, what are the coordinates of the event E_3, "assassination building in Earth year 1963?"

A4: Recall that $\Delta x_{31} = x_3 - x_1$, and we have already calculated x_1 to be -34.9 ly. Therefore,

$$-11.57 = x_3 - (-34.9) = x_3 + 34.9, \quad \text{or} \quad x_3 = -11.57 - 34.9 = -46.47 \text{ ly}.$$

Similarly, $c\Delta t_{31} = ct_3 - ct_1$, or $11.74 = ct_3 - 0$ ($ct_1 = 0$ because E_1 is happening NOW for Diracus II). Thus, $ct_3 = 11.74$ ly. All these events are shown in Figure D.18(b) in the RF of Diracus II. You may ask why we did not use the first set of equations in Box F.4.1. The answer is that those equations require the two coordinate systems to have the *same origin*; O and O' do not.

Q5: How fast should the laser beam be moving to be present in the assassination building in time?

A5: The beam must cover a distance of 46.47 ly in 11.74 years. Therefore, $\beta = \Delta x / c\Delta t = 46.47/11.74 = 3.96$, i.e., it must move at about four times the speed of light! So, although the crew of Diracus II knows that Kennedy's assassination *will* take place 11.74 years from now, they can do nothing about it, because to get to the event light speed must be surpassed, violating relativity. Once again, Kennedy cannot be saved!

The discussion above illustrates the implausibility of traveling back in time, but does not *prove* the impossibility of such a travel. A simple diagrammatic version of such a proof is given on page 402. A more tedious algebraic proof can be found in **Math Note E.27.12** on **page 127** of *Appendix.pdf*.

Example D.27.8. First let us note that each tick of Emmy's clock is 2×10^{-8} s according to Emmy, and

$$\Delta t = \frac{2 \times 10^{-8}}{\sqrt{1 - (0.866)^2}} = 4 \times 10^{-8} \text{ s}$$

according to Earth. Next, let's place Karl's clock so that the emitter is at the origin, and the mirror is 3 m away from the origin. Let E_1 be the emission of light and E_2 its reflection. Karl measures Δx_{21} to be 3 m and Δt_{21} to be the time it takes light to go from one end of the clock to the other, i.e., $3/3 \times 10^8$ or 10^{-8} s. It follows that $c\Delta t_{21}$ is also 3 m. What is $c\Delta t'_{21}$, the time interval between the same two events according to Earth? With $\beta = 0.866$, we get $\gamma = 1/\sqrt{1 - 0.866^2} = 2$, and the second Δ-equation in Box F.4.1 yields

MM clock orientation and its time measurement **(page 403 of the book)**

$$c\Delta t'_{21} = \gamma(\beta \Delta x_{21} + c\Delta t_{21}) = 2(0.866 \times 3 + 3) = 11.196 \text{ m}$$

giving $\Delta t'_{21} = 11.196/3 \times 10^8 = 3.732 \times 10^{-8}$ s. So, the Earth people measure the trip of the light signal from the bottom of Karl's clock to its top to be only slightly less than an entire tick of Emmy's clock. Is the second half of the light trip as long as the first? That would be disastrous, because it would imply that according to Karl's clock, the Earth people must age almost twenty years,[13] rather than ten years, as Emmy's clock suggests! Before jumping to any conclusion, let's calculate the second half of the light signal's trip.

Denote the event of the arrival of signal back to the bottom of the clock as E_3. Then using the same equation as above (with due consideration to signs),[14] we get

$$c\Delta t'_{32} = \gamma(\beta \Delta x_{32} + c\Delta t_{32}) = 2[0.866 \times (-3) + 3] = 0.804 \text{ m}$$

yielding $\Delta t'_{32} = 0.804/3 \times 10^8 = 0.268 \times 10^{-8}$ s. Adding the two flight times, we get

$$3.732 \times 10^{-8} + 0.268 \times 10^{-8} = 4 \times 10^{-8} \text{ s}$$

Exactly the same as the Earth measurement of the tick of Emmy's clock!

Was the above agreement of clocks a luck of numbers? Did we choose the length and the speed so cleverly as to make the two clocks agree? Proving that the coincidence was not the result of numerical tricks is not that hard. Just keep the symbols in the formulas rather than numbers. Let L be the length of the clock according to Emmy and Karl. Then for both, a tick is $2L/c$, and Emmy's tick becomes $\Delta t = \gamma(2L/c)$ for Earth observers. How does Karl's tick appear to Earth observers?

[13] More than ten years not less, as we had suspected!
[14] We are employing the convention that $\Delta x_{32} = x_3 - x_2$.

The flight time from the bottom of Karl's clock to its top is $\Delta t_{21} = L/c$, or $c\Delta t_{21} = L$; and the space interval is $\Delta x_{21} = +L$ (the plus sign indicates that the top is to the right of the bottom). The same flight time as measured by Earth is

$$c\Delta t'_{21} = \gamma(\beta \Delta x_{21} + c\Delta t_{21}) = \gamma(\beta L + L) = \gamma L(1+\beta)$$

The flight time from the top of Karl's clock back to its bottom is $\Delta t_{32} = L/c$, or $c\Delta t_{32} = L$; and the space interval is $\Delta x_{32} = -L$ (the minus sign indicates that the bottom is to the left of the top). The same flight time as measured by Earth is

$$c\Delta t'_{32} = \gamma(\beta \Delta x_{32} + c\Delta t_{32}) = \gamma[\beta(-L) + L] = \gamma L(1-\beta)$$

A complete tick is of duration $\Delta t = \Delta t'_{21} + \Delta t'_{32}$. Thus,

$$\Delta t = \frac{\gamma L(1+\beta)}{c} + \frac{\gamma L(1-\beta)}{c} = \frac{\gamma L(1+\beta) + \gamma L(1-\beta)}{c} = \frac{2\gamma L}{c}$$

identical to Δt as obtained from Emmy's clock.

Lorentz transformation and addition of velocities **(page 405 of the book)**

Example D.27.9. Emmy (observer O) is riding on a supertrain moving at $0.9c$. She fires a bullet from a supergun in the forward direction. Call the "firing of the bullet" event $E_1 = (x_1, ct_1)$. One microsecond later the bullet is found 250 m away from the gun. We note (or *should* note) immediately that the "finding of the bullet" is a natural second event $E_2 = (x_2, ct_2)$ and that

$$\Delta x = x_2 - x_1 = 250 \text{ m}, \quad c\Delta t = c(t_2 - t_1) = 3 \times 10^8 (10^{-6}) = 300 \text{ m}$$

all according to Emmy. We can find the speed of the bullet relative to Emmy, because we are given both the distance (Δx) and the travel time (Δt). Calling this v_b we get $v_\mathrm{b} = \Delta x/\Delta t = 250/10^{-6} = 2.5 \times 10^8$ m/s or $0.83c$.

Karl, standing on the platform, looks at the same two events and measures his own $\Delta x'$ and $\Delta t'$. These values are given by the Δ-equations of Box F.4.1, where $\beta = v/c$ and v is the speed of the *train* relative to the *platform*. In this case, $\beta = 0.9$ and $\gamma = 1/\sqrt{1-(0.9)^2} = 2.294$. Therefore,

$$\begin{aligned} \Delta x' &= \gamma(\Delta x + \beta c\Delta t) = 2.294(250 + 0.9 \times 300) = 1192.9 \text{ m} \\ c\Delta t' &= \gamma(\beta \Delta x + c\Delta t) = 2.294(0.9 \times 250 + 300) = 1204.4 \text{ m} \end{aligned}$$

We can now find the speed of the bullet relative to Karl. Calling this v'_b we get $v'_\mathrm{b}/c = \Delta x'/(c\Delta t') = 1192.9/1204.4 = 0.9905$, which is less than the speed of light, as it should be.

Box D.27.10. *Do not confuse the speed of an object in a reference frame with the speed of* ***that reference frame*** *relative to* ***another reference frame****. The speed of the bullet calculated by Emmy and Karl in Example D.27.9 above has nothing to do with the speed used in the Lorentz transformations.*

Example D.27.11. Karl gets on a spaceship that travels to a planet of a star system 12 ly away on a world line drawn with thick lines in Figure D.19 as seen by observer O, Emmy. All units are in light years, and for easier reading most of the calibration of the ct-axis is made on the worldline parallel to it.

Q1: What is the speed of the spaceship between E_1 and E_2? Between E_2 and E_3? Between E_3 and E_4?

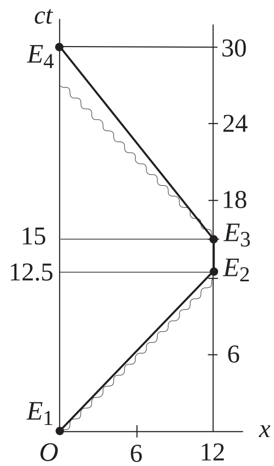


Figure D.19: Karl moves on the heavy worldline while Emmy remains on Earth with ct-axis being her worldline.

A1: Recall from Box F.0.2 that β is the slope of the angle between axes of the two RFs. Since the worldline represents the time axis, β is the slope of $\overline{E_1E_2}$ relative to the ct-axis, or more intuitively, just distance divided by time:

$$\beta = \frac{\Delta x}{c\Delta t} = \frac{12 \text{ ly}}{12.5 \text{ ly}} = 0.96 \quad \text{or} \quad v = 0.96c$$

Similarly $\beta = 0$ for the speed between E_2 and E_3, and

$$\beta = \frac{\Delta x}{c\Delta t} = \frac{12 \text{ ly}}{15 \text{ ly}} = 0.8 \quad \text{or} \quad v = 0.8c$$

for the speed between E_3 and E_4.

Q2: How long is the time interval between take-off from Earth (E_1) and landing on the planet (E_2) according to Emmy?

A2: The vertical axis is Emmy's ct-axis. The (c times) time interval is shown to be 12.5 ly. Thus, $c\Delta t = 12.5$ ly; canceling the "c" and the "l" from both sides, we get $\Delta t = 12.5$ y.

Q3: How long is the time interval between landing (E_2) and departure (E_3) from the planet according to Emmy?

A3: E_3 is at 15 ly mark. Therefore, $c\Delta t = 15 - 12.5 = 2.5$ ly, or $\Delta t = 2.5$ y.

Q4: How long is the time interval between departure (E_3) and landing on Earth (E_4) according to Emmy?

A4: E_4 is at 30 ly mark. Therefore, $c\Delta t = 30 - 15 = 15$ ly, or $\Delta t = 15$ y.

Q5: How long does the entire trip take according to Emmy?

A5: The time interval between E_1 and E_4 is 30 ly. Therefore, $c\Delta t = 30$ ly, or $\Delta t = 30$ y.

Q6: What is Δs_{21}, the spacetime interval for the two events E_1 and E_2?

A6: With E_1 and E_2 having coordinates $(0, 0)$ and $(12, 12.5)$, respectively, we get

$$\Delta s_{21} = \sqrt{(c\Delta t_{21})^2 - (\Delta x_{21})^2} = \sqrt{12.5^2 - 12^2} = 3.5 \text{ ly}$$

Q7: What is Δs_{32}, the spacetime interval for the two events E_2 and E_3?

A7: E_2 has coordinates $(12, 12.5)$ and E_3 has coordinates $(12, 15)$. Therefore,

$$\Delta s_{32} = \sqrt{(c\Delta t_{32})^2 - (\Delta x_{32})^2} = \sqrt{2.5^2 - 0^2} = 2.5 \text{ ly}$$

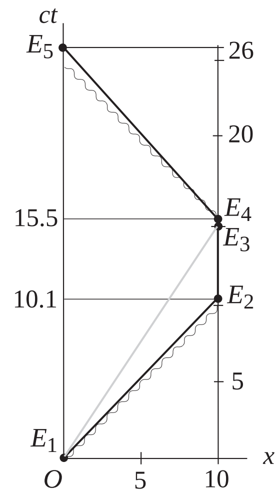


Figure D.20: Emmy moves on the heavy black worldline. Karl moves on the grey worldline first and then joins Emmy to return home. Pat remains on Earth.

Q8: What is Δs_{43}, the spacetime interval for the two events E_3 and E_4?
A8: As in the two previous cases, we have

$$\Delta s_{43} = \sqrt{(c\Delta t_{43})^2 - (\Delta x_{43})^2} = \sqrt{15^2 - 12^2} = 9 \text{ ly}$$

Q9: What is Δs for the entire trip? How long did this trip take according to Karl?
A9: Adding the three spacetime intervals, we obtain

$$\Delta s = \Delta s_{21} + \Delta s_{32} + \Delta s_{43} = 3.5 + 2.5 + 9 = 15 \text{ ly}$$

Since the broken worldline is that of Karl and Δs is the spacetime interval (or length) of this worldline, Δs is related to Karl's proper time via $\Delta s = c\Delta\tau$. Thus, $c\Delta\tau = 15$ ly and $\Delta\tau = 15$ y.

Q10: Who measures the proper time interval between E_1 and E_4, Karl or Emmy (or both)?

A10: Since they are both present at the two events, they both measure the proper time

Example D.27.12. Karl, Emmy, and Pat are newly born triplets. Karl and Emmy are put on two different spaceships that travel to a planet of a star system 10 ly away. Emmy lands on the planet 10.1 years later as seen by observer O, Pat. She waits 4.9 years until Karl, who is traveling slower, lands on the same planet (see Figure D.20). After six months they both return home on the same spaceship and land on Earth 26 years after their departure. All times and distances are given according to the Earth observers, and all units shown in the figure are in light years, and for easier reading most of the calibration of the ct-axis is made on the worldline parallel to it.

Q1: What is the speed of Emmy's spaceship on her journey to the planet?
A1: Fractional speed is distance divided by (c times) time:

$$\beta = \frac{\Delta x}{c\Delta t} = \frac{10 \text{ ly}}{10.1 \text{ ly}} = 0.99 \quad \text{or} \quad v = 0.99c$$

Q2: What is the speed of Karl's spaceship on his journey to the planet?
A2: The calculation is similar to above:

$$\beta = \frac{\Delta x}{c\Delta t} = \frac{10 \text{ ly}}{15 \text{ ly}} = 0.667 \quad \text{or} \quad v = 0.667c$$

Q3: How old is Emmy when she meets Karl? How old is Karl?

A3: Emmy's age is the spacetime length of her worldline up to her meeting with Karl, the broken line $E_1E_2E_3$ (divided by c):

$$\Delta s_{21} + \Delta s_{32} = \sqrt{(10.1)^2 - (10)^2} + \sqrt{(15 - 10.1)^2 - (10 - 10)^2} = 6.3 \text{ ly}$$

Thus, Emmy is 6.3 years old. Karl's age is the spacetime length of $\overline{E_1E_3}$, which we expect to be longer than the broken line $E_1E_2E_3$, because it is a side of the triangle $E_1E_2E_3$:

$$\Delta s_{31} = \sqrt{15^2 - 10^2} = 11.2 \text{ ly}$$

making Karl 11.2 years old.

Q4: How old is Emmy when she lands back on Earth? How old is Karl? How old is Pat?

A4: To find Emmy's and Karl's age when they land on Earth we have to add to their age at event E_3 the spacetime length of the broken line $E_3E_4E_5$ (divided by c). This length is

$$\Delta s_{43} + \Delta s_{54} = \sqrt{(15.5 - 15)^2 - (0)^2} + \sqrt{(26 - 15.5)^2 - (10)^2} = 0.5 + 3.2 = 3.7 \text{ ly}$$

Thus, Emmy is $6.3 + 3.7 = 10$ and Karl is $11.2 + 3.7 = 14.9$ years old when they land on Earth. Pat, having been left behind, has the longest proper time and is 26 years old when all three meet at event E_5.

D.28 Numerical Examples for Chapter 28

Example D.28.1. Emmy is on a train moving at $0.9c$ in the positive direction of Karl's axis. She sees a bullet dashing by with a speed of $0.95c$ in the forward direction.

Q1: What are the components of the bullet's spacetime velocity according to Emmy?

A1: Equation (28.1) gives these components

$$\begin{aligned} u_{bx} &= \frac{v_b}{\sqrt{1 - (v_b/c)^2}} = \frac{0.95c}{\sqrt{1 - (0.95)^2}} = 3.042435c \\ u_{bt} &= \frac{c}{\sqrt{1 - (v_b/c)^2}} = \frac{c}{\sqrt{1 - (0.95)^2}} = 3.202563c \end{aligned}$$

These two components satisfy

$$u_{bt}^2 - u_{bx}^2 = (3.202563c)^2 - (3.042435c)^2 = 0.99999904c^2$$

verifying the last equation of (28.1).

Q2: What are the components of the bullet's spacetime velocity according to Karl?

A2: Equation (28.2) gives these components in terms of Emmy's. In that equation, γ is given in terms of the *relative speed of the two observers*:

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}} = \frac{1}{\sqrt{1 - (0.9)^2}} = 2.294157$$

$$\begin{aligned} u'_{bx} &= \gamma(u_{bx} + \beta u_{bt}) = 2.294157(3.042435c + 0.9 \times 3.202563c) = 13.592288c \\ u'_{bt} &= \gamma(u_{bx} + \beta u_{bt}) = 2.294157(0.9 \times 3.042435c + 3.202563c) = 13.629024c \end{aligned}$$

These two components satisfy

$$u'^{\,2}_{bt} - u'^{\,2}_{bx} = (13.629024c)^2 - (13.592288c)^2 = 0.99998908c^2$$

again verifying the last equation of (28.1), and illustrating that $u_{bt}^2 - u_{bx}^2$ is frame-independent.

Energy and getting to light speed (page 418 of the book)

Example D.28.2. The impossibility of attaining light speed for massive objects is demonstrated by the difficulty of speeding up when the object is already moving close to light speed. Suppose that a small vehicle of mass 1000 kg is moving at $0.9999c$ and we want to speed it up to $0.99999c$.

Q: How much energy do we need?

A: According to Equation (28.4), the initial energy of the vehicle is (we ignore the subscript b)

$$E_i = \frac{mc^2}{\sqrt{1-\beta^2}} = \frac{1000 \times (3 \times 10^8)^2}{\sqrt{1-0.9999^2}} = 6.36 \times 10^{21}\ \text{J}$$

and the final energy is

$$E_f = \frac{1000 \times (3 \times 10^8)^2}{\sqrt{1-0.99999^2}} = 2.01 \times 10^{22}\ \text{J}$$

So, the energy needed is the difference between these two energies or 1.38×10^{22} J.

Now suppose that we have already reached a speed of $0.999999c$ and we want to speed it up to $0.9999999c$. Then the energy needed will be the difference between

$$E_i = \frac{1000 \times (3 \times 10^8)^2}{\sqrt{1-0.999999^2}} = 6.36 \times 10^{22}\ \text{J}$$

and

$$E_f = \frac{1000 \times (3 \times 10^8)^2}{\sqrt{1-0.9999999^2}} = 2.01 \times 10^{23}\ \text{J}$$

or 1.38×10^{23} J, ten times larger than the previous energy difference. Thus, the closer you get to the speed of light, the harder it gets to increase your speed.

It is instructive to compare these answers with the (incorrect) classical results. In the first case the vehicle has an initial energy of

$$E_i = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(1000)(0.9999 \times 3 \times 10^8)^2 = 4.4991 \times 10^{19}\ \text{J}$$

and a final energy of

$$E_f = \tfrac{1}{2}(1000)(0.99999 \times 3 \times 10^8)^2 = 4.49991 \times 10^{19}\ \text{J}$$

Thus the energy needed is 8.1×10^{15} J. In the second case, the energy needed in the difference between

$$E_i = \tfrac{1}{2}(1000)(0.999999 \times 3 \times 10^8)^2 = 4.499991 \times 10^{19}\ \text{J}$$

and

$$E_f = \tfrac{1}{2}(1000)(0.9999999 \times 3 \times 10^8)^2 = 4.4999991 \times 10^{19}\ \text{J}$$

or 8.1×10^{13} J, which is only one per cent of the previous energy difference. Thus, from a classical point of view, the closer you get to light speed, the easier it gets to speed up. In fact, not only can you reach light speed, but also surpass it. This conclusion is of course completely wrong as the classical physics fails drastically when objects move with speed comparable to light speed.

Numerical example of relativistic Doppler formula (page 419 of the book)

Example D.28.3. Since the speed of a photon is the same for all observers, one might suspect that its momentum should also be the same. Math Note E.28.4 investigates this suspicion and, as a by-product, derives the **relativistic Doppler formula**:

$$\lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda \tag{D.20}$$

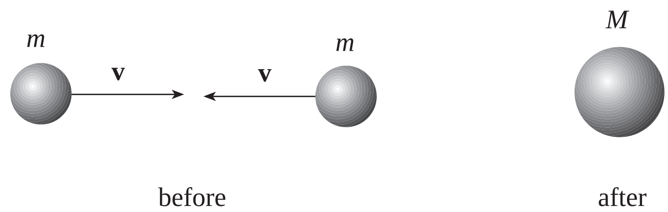


Figure D.21: The relativistic conservation of momentum in this collision implies the nonconservation of mass.

where β is positive for relative recession and negative for relative approach.

Suppose in the observation of a galaxy we find that in the spectral lines of a certain element the green light, whose wavelength is 0.55 μm, has shifted to 1.455 μm. We wish to calculate the speed with which this galaxy is moving away from us. If we were to use the classical result of Equation (E.10), we would get

$$\lambda_{\text{det}} = \lambda\left(1+\frac{v}{c}\right), \quad \text{or} \quad 1.455 = 0.55\left(1+\frac{v}{c}\right) \quad \Rightarrow \quad 1+\frac{v}{c} = \frac{1.455}{0.55} = 2.645$$

which gives $v/c = 1.645$, a nonsensical result! Relativistic Doppler formula, on the other hand, gives

$$\lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda \quad \text{or} \quad 1.455 = \sqrt{\frac{1+\beta}{1-\beta}}\,0.55 \quad \Rightarrow \quad \sqrt{\frac{1+\beta}{1-\beta}} = \frac{1.455}{0.55} = 2.645$$

Squaring both sides gives

$$\frac{1+\beta}{1-\beta} = 6.996 \quad \text{or} \quad 1+\beta = 6.996(1-\beta) = 6.996 - 6.996\beta$$

which is equivalent to $7.996\beta = 5.996$ or $\beta = 0.75$. So, the galaxy is moving away from us at 75% the speed of light, a possible result.

Although the nonrelativistic Doppler formula gives nonsensical results (as illustrated in this example), when the relative speed of the source and the detector is much smaller than the light speed, Equation (D.20) reduces to the nonrelativistic formula as shown in **Math Note E.28.5** on **page 133** of *Appendix.pdf*. Since β is the *relative* motion of the source and the detector, the same Math Note also proves that in relativity theory an absolute frame, such as the one mentioned in Section 11.4, is meaningless.

Example D.28.4. Consider the collision situation depicted in Figure D.21: Two equal masses are moving in opposite directions with equal speed. They collide and coalesce to form a single mass M. Both classically and relativistically the two momenta are equal but opposite in direction, giving a total momentum of zero. Therefore, after the collision the momentum of M is zero.

Example of nonconservation of mass **(page 422 of the book)**

Q: What is the mass M in terms of the initial mass m?

A: Classically, the mass is conserved; so $M = 2m$. Relativistically, it is the energy that is conserved. The total energy before the collision is

$$E_{\text{tot}} = E_1 + E_2 = \gamma mc^2 + \gamma mc^2 = 2\gamma mc^2$$

After collision, M is at rest, but its relativistic energy is *not* zero; it is Mc^2. Equating these two quantities, we get

$$Mc^2 = 2\gamma mc^2 \quad \text{or} \quad M = 2m\gamma = \frac{2m}{\sqrt{1-(v/c)^2}}$$

Therefore, mass is not conserved in relativistic collisions. For v much smaller than light speed, the denominator is almost 1 and $M \approx 2m$, regaining the classical conservation of mass.

As a numerical example, let two 1-kg masses approach each other with a speed of $0.9c$, then the mass formed at the end of their collision is

$$M = \frac{2m}{\sqrt{1-(v/c)^2}} = \frac{2}{\sqrt{1-(0.9)^2}} = 4.588 \text{ kg}$$

So 2 kg of mass has turned into 4.588 kg. Where has the extra 2.588 kg come from? It is the kinetic energy of the colliding particles that has transformed into mass. To see this, note that each particle has a KE of

$$KE = \frac{mc^2}{\sqrt{1-(v/c)^2}} - mc^2 = \frac{9 \times 10^{16}}{\sqrt{1-(0.9)^2}} - 9 \times 10^{16} = 1.165 \times 10^{17} \text{ J}$$

and the two of them carry twice this much KE or 2.33×10^{17} J. On the other hand, the energy "hidden" in the extra mass of 2.588 kg is

$$2.588 \times (3 \times 10^8)^2 = 2.33 \times 10^{17} \text{ J};$$

equal to the KE of the two particles! This process of the transformation of KE into mass is the underlying principle of particle accelerators (or atom smashers).

D.29 Numerical Examples for Chapter 29

Example D.29.1. A green light of wavelength 0.5 μm is sent down from the top of Mount Everest to its foot. The altitude of Mount Everest is 8848 m, and the gravitational acceleration (field) is 9.8 m/s^2. (Neglect the slight variation of the gravitational acceleration due to the height of the mountain.)

Minuteness of gravitational Doppler shifts **(page 433 of the book)**

Q: By how much does the wavelength change when it reaches the foot?

A: With $h_{\text{det}} = 0$ and $h_{\text{src}} = 8848$ m, Equation (E.116) gives

$$\frac{\lambda_{\text{det}} - \lambda_{\text{src}}}{\lambda_{\text{src}}} = \frac{g(h_{\text{det}} - h_{\text{src}})}{c^2} = \frac{9.8 \times (0 - 8848)}{(3 \times 10^8)^2} = -9.6 \times 10^{-13}$$

implying that

$$\lambda_{\text{det}} - \lambda_{\text{src}} = -9.6 \times 10^{-13} \lambda_{\text{src}} = -9.6 \times 10^{-13} \times 0.5 = -4.8 \times 10^{-13} \ \mu\text{m}$$

a very minuscule change! The negative sign indicates that λ_{det} is shorter than λ_{src}, as expected.

Numerical example of gravitational time dilation **(page 433 of the book)**

Example D.29.2. Two identical clocks are synchronized at sea level. One of them is then moved to the top of Mount Everest (8848 m high). A year later they are brought together for comparison. Neglect the slight variation of the gravitational acceleration (field) due to the height of the mountain, and take g to be 9.8 m/s^2.

Q1: Which clock runs faster? By how much?

A1: The difference between the heights (8848 m) is much smaller than the Earth radius. Therefore, we can use the first equation in E.117. Let h_1 be the height of the clock at sea level and h_2 the height of the clock at the top of the mountain. Since gravity points from top to bottom, h_2 is larger than h_1. So t_2 will be larger than t_1, and the mountain clock runs faster. To find by how much, just plug the numbers in Equation (E.117):

$$\frac{t_2 - t_1}{t} = \frac{g(h_2 - h_1)}{c^2} = \frac{9.8 \times (8848)}{(3 \times 10^8)^2} = 9.6 \times 10^{-13} \quad \Rightarrow \quad t_2 - t_1 = 9.6 \times 10^{-13} t$$

Substitute 3.15×10^7 s for t (remember that there are 3.15×10^7 seconds in a year) to obtain $t_2 - t_1 = 3 \times 10^{-5}$ s.

What if we had chosen h_1 to be the clock at the mountain top? Would we have concluded that the clock at the sea level runs faster? Let's see! In the new situation, Equation (E.117) gives $h_2 - h_1 = -8848$ m and $t_2 - t_1 = -3 \times 10^{-5}$ s. However, t_2 is the time measured by the *clock at height* h_2, i.e., the clock at the sea level. Since the difference is negative, this clock is running *slower*, same as before.

For a more pronounced difference in the running of clocks, let us take one of the clocks to a satellite circling the Earth at an altitude of 1.86×10^7 m.

Q2: By how much does the satellite clock run faster in one day?

A2: In this case, the height is so large that we have to use the second equation in E.117. Let Φ_2 be the potential of the satellite clock and Φ_1 that of the Earth clock. Then, adding the Earth radius to the height of the satellite to get r_2, we obtain

$$\begin{aligned}\Phi_2 - \Phi_1 &= -\frac{GM}{r_2} - \left(-\frac{GM}{r_1}\right) \\ &= -\frac{(6.67 \times 10^{-11}) \times (6 \times 10^{24})}{1.86 \times 10^7 + 6.4 \times 10^6} + \frac{(6.67 \times 10^{-11}) \times (6 \times 10^{24})}{6.4 \times 10^6} \\ &= 4.65 \times 10^7\end{aligned}$$

This shows that Φ_2 is larger than Φ_1, and therefore, t_2 is larger than t_1, i.e., the satellite clock runs faster. By how much? Equation (E.117) gives the answer:

$$\frac{t_2 - t_1}{t} = \frac{\Phi_2 - \Phi_1}{c^2} = \frac{4.65 \times 10^7}{(3 \times 10^8)^2} = 5.1 \times 10^{-10} \quad \Rightarrow \quad t_2 - t_1 = 5.1 \times 10^{-10} t$$

Substitute $24 \times 3600 = 86,400$ s for t to obtain $\Delta t = 4.4 \times 10^{-5}$ s. Thus, the satellite clock runs ahead of the Earth clock in one day more than the Mount Everest clock does in one year.

Example D.29.3. The red light emitted by a hydrogen atom has a wavelength of $0.657\ \mu$m. This same light is received from a galaxy and is observed that its wavelength has increased by $0.0335\ \mu$m.

Doppler red shift
(page 445 of the book)

Q1: How fast is the galaxy receding from us?

A1: The received wavelength is 0.657+0.0335=0.6905 μm. So the ratio $\lambda'/\lambda = 0.6905/0.657 = 1.051$. According to Equation (E.91), this ratio is simply $\sqrt{\dfrac{1+\beta}{1-\beta}}$. Therefore,

$$\frac{1+\beta}{1-\beta} = 1.051^2 = 1.105 \quad \Rightarrow \quad 1 + \beta = 1.105 - 1.105\beta$$

This yields $\beta = +0.05$, where the positive sign is indicative of the recession of the galaxy. The actual speed is $0.05 \times 3 \times 10^8 = 1.5 \times 10^7$ m/s.

Q2: How far is the galaxy from us if the Hubble constant is 21 km/s per Mly?

A2: In units of km/s, the speed is 15,000. Put this number on the left-hand side of Equation (29.2) and use $H = 21$ km/s per Mly to obtain

$$15,000 = 21d \quad \text{or} \quad d = 714 \text{ Mly}.$$

Example D.29.4. Consider the Milky Way and another distant galaxy separated by a distance d. How long did it take for the galaxy to reach this distance starting at the same point as the Milky Way? If the galaxy is moving (and has been moving all this time) with speed v, the time t is simply $t = d/v$. With v given by Hubble law (29.2), this t becomes

Calculating age of the universe
(page 446 of the book)

$$t = \frac{d}{v} = \frac{d}{Hd} = \frac{1}{H} \tag{D.21}$$

Note that t is independent of the distance: all galaxies have taken this much time to reach their present position starting at the same point as the Milky Way. It follows that Equation (D.21) gives the time in the past at which *all* galaxies were on top of each other. If we calculate H in the scientific units, Equation (D.21) gives t in seconds. Now H has been given in km/s per Mly or (km/s)/(Mly). We have to convert this unit into scientific units. A light year (ly) is the distance light travels in one year. With 3.15×10^7 seconds in a year, light, moving at the speed of 3×10^8 m/s, covers a distance of 9.45×10^{15} m. It follows that

$$1(\text{km/s})/(\text{Mly}) = \frac{\text{km/s}}{\text{million ly}} = \frac{1000 \text{ m/s}}{10^6 \times (9.45 \times 10^{15}) \text{ m}} = 1.06 \times 10^{-19} \text{ s}^{-1}$$

For H equal to 21 km/s per Mly, Equation (D.21) gives

$$t = \frac{1}{21 \times (1.06 \times 10^{-19})} \text{ s} = 4.5 \times 10^{17} \text{ s} \quad \text{or} \quad \frac{4.5 \times 10^{17}}{3.15 \times 10^7} = 1.4 \times 10^{10} \text{ years},$$

and for H equal to 23 km/s per Mly, (D.21) yields

$$t = \frac{1}{23 \times (1.06 \times 10^{-19})} \text{ s} = 4.1 \times 10^{17} \text{ s}, \quad \text{or} \quad \frac{4.1 \times 10^{17}}{3.15 \times 10^7} = 1.3 \times 10^{10} \text{ years}$$

D.31 Numerical Examples for Chapter 31

Deriving mass excess formula
(page 460 of the book)

Example D.31.1. By definition, $\Delta = Mc^2 - (Z+N)e_u$, where e_u is the energy equivalent of a unified atomic mass unit. Using Equation (31.1), we can rewrite this as

$$\begin{aligned}\Delta &= [Zm_p + Nm_n - (Z+N)m_b]c^2 - (Z+N)e_u \\ &= Zm_pc^2 + Nm_nc^2 - (Z+N)e_b - Ze_u - Ne_u\end{aligned}$$

or

$$\Delta = Z(m_pc^2 - e_u) + N(m_nc^2 - e_u) - (Z+N)e_b$$

But $m_pc^2 = 938.27$ MeV, $m_nc^2 = 939.57$ MeV, and $e_u = 931.49$ MeV. Therefore,

$$\begin{aligned}\Delta &= Z(938.27 - 931.49) + N(939.57 - 931.49) - (Z+N)e_b \\ &= 6.78Z + 8.08N - (Z+N)e_b \end{aligned} \tag{D.22}$$

Mass excess for iron
(page 460 of the book)

Example D.31.2. Iron nucleus, consisting of 26 protons and 30 neutrons has a binding energy per nucleon of 8.55 MeV.

Q1: What is the mass of the iron nucleus in atomic mass units?

A1: We use Equation (31.1) with $Z = 26$, $N = 30$, and $e_b = 8.55$ MeV. First we convert e_b to the equivalent mass m_b in atomic mass units:

$$m_b = \frac{8.55}{931.5} = 0.009179 \text{ u}$$

Next we substitute this and the masses of the proton and neutron (in u) in Equation (31.1):

$$M = 26 \times 1.007276 + 30 \times 1.008665 - (26+30)0.009179 = 55.935 \text{ u}$$

Q2: What is the mass excess of the iron nucleus in MeV?

A2: From Equation (D.22), we get

$$(26+30) \times 8.55 = 6.78 \times 26 + 8.08 \times 30 - \Delta \quad \text{or} \quad 478.8 = 418.68 - \Delta$$

This gives $\Delta = -60.12$ MeV.

Example D.31.3. A sample of uranium ore has 10^{15} radioactive radium nuclei $^{226}_{88}\text{Ra}$ with a half-life of 1,600 years.

Q: How many radium nuclei are left after 50 years?

A: Here $N_0 = 10^{15}$, $t_{\text{half}} = 1,600$ years, and $t = 50$ years. Therefore, Equation (31.2) yields

$$N(50 \text{ years}) = \frac{10^{15}}{2^{50/1600}} = \frac{10^{15}}{2^{0.03125}} = \frac{10^{15}}{1.0219} = 9.786 \times 10^{14}$$

So, 97.86% of the sample remains, and only 2.14% of it decays.

Radium and oxygen radioactivity **(page 463 of the book)**

A sample has 10^{15} radioactive oxygen nuclei $^{19}_{8}\text{O}$ with a half-life of 27 seconds.

Q: How many nuclei are left after 10 minutes?

A: Here $N_0 = 10^{15}$, $t_{\text{half}} = 27$ s, and $t = 600$ s. Therefore, Equation (31.2) yields

$$N(10 \text{ minutes}) = \frac{10^{15}}{2^{600/27}} = \frac{10^{15}}{2^{22.22}} = \frac{10^{15}}{4.89 \times 10^6} = 2.04 \times 10^8$$

So, only 0.00002% of the sample remains, and 99.99998% of it decays.

Estimating the age of a Dead Sea Scroll **(page 463 of the book)**

Example D.31.4. The oldest "Messianic" scrolls had, in 1991, a $^{14}\text{C}/^{12}\text{C}$ ratio of approximately 9.91×10^{-13}.

Q: How long ago was that scroll written?

A: Since the initial $^{14}\text{C}/^{12}\text{C}$ ratio in a fresh scroll manufactured from live organic material is 1.3×10^{-12}, the ratio $N(t)/N_0$ for the scroll in question is $9.91 \times 10^{-13}/1.3 \times 10^{-12}$ or 0.7625. Therefore, t (the time in the past when radioactivity of ^{14}C started) is the solution of

$$0.7625 = \frac{1}{2^{t/5730}} \quad \text{or} \quad 2^{t/5730} = \frac{1}{0.7625} = 1.311$$

By trial and error, we can find that $2^{0.3907}$ is approximately 1.311. So

$$\frac{t}{5730} \approx 0.3907 \quad \Rightarrow \quad t \approx 2239$$

Thus the scroll must have been written 2239 years prior to 1991 or about 248 BC.

Estimating age of Earth **(page 464 of the book)**

Example D.31.5. A typical natural radioactivity leading to a *stable* nucleus occurs in several steps, at each of which other radioactive elements may be produced. However, since these intermediate elements have short half-lives (compared to the parent nucleus), they soon disappear and can be ignored. So assume that we start with N_0 parent nuclei. After a time t, we have $N_p = N_0/2^{t/t_{\text{half}}}$ parent nuclei and $N_d = N_0 - N_p$ daughter nuclei. The ratio of daughter nuclei to parent nuclei is

$$r = \frac{N_d}{N_p} = \frac{N_0 - N_p}{N_p} = \frac{N_0 - N_0/2^{t/t_{\text{half}}}}{N_0/2^{t/t_{\text{half}}}} = \frac{1 - 1/2^{t/t_{\text{half}}}}{1/2^{t/t_{\text{half}}}} = 2^{t/t_{\text{half}}} - 1$$

or

$$2^{t/t_{\text{half}}} = 1 + r \quad \text{or} \quad (t/t_{\text{half}}) \ln(2) = \ln(1 + r)$$

which gives

$$t = \frac{t_{\text{half}}}{\ln(2)} \ln(1 + r) \tag{D.23}$$

A chemical analysis of an appropriate rock sample can determine r, and consequently t, the age of the Earth.

The final stable by-product of ^{238}U decay is ^{206}Pb. In a certain rock sample, there are as many ^{238}U (parent) atoms as there are ^{206}Pb (daughter) atoms, i.e., $N_d = N_p$. This yields $r = 1$ and $t = t_{\text{half}}$, which for ^{238}U is 4.5 billion years.

Example D.31.6. In any of the fusion reactions, if a particle appears on the right of one equation and the left of another, it cancels from the overall outcome. The overall outcome is what is left on the left hand turning into what is left on the right hand side.

To see the net outcome of the first proton-proton cycle, write the three reactions one under the next, and cancel the particles that show up on both sides:

$$\begin{aligned} p + p &\to {}^2_1\cancel{\mathrm{D}} + \mathrm{e}^+ + \nu \\ p + {}^2_1\cancel{\mathrm{D}} &\to {}^3_2\cancel{\mathrm{He}} + \gamma \\ p + {}^3_2\cancel{\mathrm{He}} &\to {}^4_2\mathrm{He} + \mathrm{e}^+ + \nu \end{aligned}$$

Net result of the p-p cycle (page 467 of the book)

This shows that the ingoing particles are 4 protons and the final product is a helium, two positrons, two neutrinos, and a gamma particle.

For the second proton-proton cycle, we have

$$\begin{aligned} p + p &\to {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu \\ p + {}^2_1\mathrm{D} &\to {}^3_2\mathrm{He} + \gamma \\ {}^3_2\mathrm{He} + {}^3_2\mathrm{He} &\to {}^4_2\mathrm{He} + p + p \end{aligned}$$

The last reaction has two ${}^3_2\mathrm{He}$ while there is only one ${}^3_2\mathrm{He}$ in the previous reaction. So, we multiply the second reaction by 2 (this means that two middle reactions should take place before the last reaction can occur). However, this will introduce two ${}^2_1\mathrm{D}$ on the left of the middle reaction while the first reaction has only one ${}^2_1\mathrm{D}$ on the right. So, we need to multiply that reaction by two as well. Thus, we have

$$\begin{aligned} 2p + 2p &\to 2{}^2_1\mathrm{D} + 2\mathrm{e}^+ + 2\nu \\ 2p + 2{}^2_1\mathrm{D} &\to 2{}^3_2\mathrm{He} + 2\gamma \\ 2{}^3_2\mathrm{He} &\to {}^4_2\mathrm{He} + 2p \end{aligned}$$

Canceling particles, we end up with

$$\begin{aligned} 2p + 2p &\to 2{}^2_1\cancel{\mathrm{D}} + 2\mathrm{e}^+ + 2\nu \\ 2\cancel{p} + 2{}^2_1\cancel{\mathrm{D}} &\to 2{}^3_2\cancel{\mathrm{He}} + 2\gamma \\ 2{}^3_2\cancel{\mathrm{He}} &\to {}^4_2\mathrm{He} + 2\cancel{p} \end{aligned}$$

indicating that the ingoing particles are 4 protons and the final product is a helium, two positrons, two neutrinos, and two gamma particle.

D.34 Numerical Examples for Chapter 34

Numerics of proton decay experiment (page 533 of the book)

Example D.34.1. We first note that if x is small compared to 1, then for any positive a, a^x is approximately equal to $1 + x \ln a$; and the approximation gets better and better as x gets smaller and smaller. Since t is incredibly smaller than t_{half} (which for proton decay is 10^{31}), the approximation works extremely well in Equation (31.2). So, substitute $t = 1$, $N(t) = N_0 - 100$, and $t_{\mathrm{half}} = 10^{31}$ in Equation (31.2), and use this approximation to get

$$N_0 - 100 = \frac{N_0}{1 + (1/10^{31}) \ln 2} \quad \text{or} \quad (N_0 - 100)(1 + 10^{-31} \ln 2) = N_0$$

Multiplying out yields

$$N_0 + 10^{-31}(\ln 2) N_0 - 100 - 10^{-29} \ln 2 = N_0 \quad \text{or} \quad 10^{-31}(\ln 2) N_0 = 100$$

This give $N_0 = 10^{33}/\ln 2$ or $N_0 = 1.44 \times 10^{33}$.

Next we calculate the number of protons in a ton of water. A mole of water (equal to 18 grams, 2 grams of which is hydrogen and 16 grams oxygen) contains 6.02×10^{23} molecules of water (see Section 17.1 for the definition of a mole and the Avogadro number). Since each molecule contains 10 protons (2 coming from the two H's and 8 from the O), there are 6.02×10^{24} protons in each mole of water. Since a kilogram is 1000 grams or $1000/18 = 55.56$ moles, there are $55.56 \times 6.02 \times 10^{24} = 3.3 \times 10^{26}$ protons in a kilogram of water or 3.3×10^{29} protons in a ton of water. So 1000 tons of water contains 3.3×10^{32} protons, which is 23% of N_0 calculated above, which should yield 23% of 100, or 23 proton decays per year. So, as a rule, we expect 23 proton decays per year from each 1000 tons of water.

D.37 Numerical Examples for Chapter 37

Estimating critical density of the universe **(page 570 of the book)**

Example D.37.1. The two extreme values of H—in the unit appropriate for formulas—are $21 \times (1.06 \times 10^{-19})$ or 2.23×10^{-18} s^{-1} and $23 \times (1.06 \times 10^{-19})$ or 2.44×10^{-18} s^{-1}. Therefore the lower limit for the critical density is

$$\rho_c = \frac{3 \times (2.23 \times 10^{-18})^2}{8\pi \times (6.67 \times 10^{-11})} = 8.9 \times 10^{-27} \text{ kg/m}^3$$

and the upper limit is

$$\rho_c = \frac{3 \times (2.44 \times 10^{-18})^2}{8\pi \times (6.67 \times 10^{-11})} = 1.157 \times 10^{-26} \text{ kg/m}^3$$

Now suppose that the radiation contribution to the critical density can be neglected, so that ρ_c is composed entirely of matter.[15] The ordinary matter of the present universe consists of atoms, and the mass of an atom is concentrated in its nucleus, which is composed of nucleons. The mass of a nucleon is about 1.67×10^{-27} kg, and ρ_c can be expressed as a nucleon **number density**. This number density is between

$$\frac{8.9 \times 10^{-27}}{1.67 \times 10^{-27}} = 5.3 \text{ nucleons/m}^3$$

and

$$\frac{1.157 \times 10^{-26}}{1.67 \times 10^{-27}} = 6.9 \text{ nucleons/m}^3$$

These are both far more than the observed nucleon number density.

Universe when it was 1% its current size **(page 572 of the book)**

Example D.37.2. Use the first row of Table E.3 to trace the development of some properties of the universe. For example, you can find out how long after the big bang the universe was one percent of its present size. The first equation in the first row of Table E.3 gives the answer if you know the present matter density ρ_{m0} of the universe. Astronomical observations put ρ_{m0} between 2.4×10^{-27} and 2.7×10^{-27} kg/m^3. Taking the intermediate value of 2.5×10^{-27} for ρ_{m0}, the first equality of the first row of Table E.3 gives

$$0.01 = \left[6\pi(6.67 \times 10^{-11})(2.5 \times 10^{-27})\right]^{1/3} t^{2/3} = 1.465 \times 10^{-12} t^{2/3} \quad \text{or} \quad t^{2/3} = 6.8 \times 10^9$$

which yields $t = (6.8 \times 10^9)^{3/2} = 5.6 \times 10^{14}$ s, or 18 million years. This calculation is valid if the universe was matter-dominated at 18 million years after the big bang. As we shall see later, this was indeed the case.

Had you used the *second* equality of the first equation of Table E.3, you would have obtained a different answer, namely, 13.7 million years instead of 18 million years. This has to do with the uncertainty in the actual value of ρ_{m0}, to which we shall return below.

[15] As we shall see later, this is indeed the case, and the contribution of the radiation component to ρ_c is less than 0.1 per cent of the matter contribution.

What about the density of the universe? Since the density falls as inverse cube of the scale of the universe, we expect the density to have been 100^3 or a million times denser than today.

Finally, the Hubble parameter had a value of

$$H = \frac{2}{3t} = \frac{2}{3 \times (5.6 \times 10^{14})} = 1.2 \times 10^{-15}\ \mathrm{s}^{-1}$$

In a more familiar unit,

$$H = \frac{1.2 \times 10^{-15}}{1.06 \times 10^{-19}} = 11,200\ \mathrm{km/s\ per\ Mly}.$$

This is approximately 600 times larger than the present value of the Hubble parameter, indicating the faster rate of expansion of the universe when it was much younger.

Horizon expansion
(page 574 of the book)

Example D.37.3. When the universe is one hour old (at which time radiation is dominant), the horizon radius is approximately 0.000228 ly (twice the distance that light travels in one hour), while the scale of the universe is (taking a scale of 500 Mly for the present universe) approximately 5.4 ly,[16] and the ratio of the horizon *volume* to the scale volume is

$$\left(\frac{0.000228}{5.4}\right)^3 = 7.5 \times 10^{-14}$$

When the universe is one week old, the horizon radius is approximately 0.038 ly (twice the distance that light travels in one week), while the scale of the universe is approximately 70 ly, and the ratio of the horizon volume to the scale volume is

$$\left(\frac{0.038}{70}\right)^3 = 1.6 \times 10^{-10}$$

over 2000 times larger than the ratio at 1 hour.

Finally, when the universe is one year old, the horizon radius is 2 ly while the scale of the universe is approximately 500 ly, and the ratio of the horizon volume to the scale volume is

$$\left(\frac{2}{500}\right)^3 = 6.4 \times 10^{-8}$$

about 400 times larger than the ratio at 1 week and over 800,000 times larger than the ratio at 1 hour. It is clear that the horizon covers a larger and larger fraction of the universe as time passes, and that at the earliest times the horizon was flat.

Size of early universe as a function of time
(page 578 of the book)

Example D.37.4. During the times of interest in that example, the universe was dominated by radiation. Therefore, you can use the first equation of the second row of Table E.3, with

$$\rho_{\gamma 0} = 8.36 \times 10^{-33} T_0^4 = 8.36 \times 10^{-33}(2.725)^4 = 4.61 \times 10^{-31}\ \mathrm{kg/m^3}$$

to obtain

$$\frac{R(t)}{R_0} = \left(\frac{32\pi(6.67 \times 10^{-11})(4.61 \times 10^{-31})}{3}\right)^{1/4} t^{1/2} = 1.79 \times 10^{-10}\sqrt{t}$$

Assuming that $R_0 = 500$ Mly, this gives

$$R(t) = 500,000,000 \times 1.79 \times 10^{-10}\sqrt{t} = 0.08958\sqrt{t}\ \mathrm{ly} \tag{D.24}$$

[16] We shall see later in Example D.37.4 how to estimate the size of the universe as a function of time.

For example, one hour after the big bang, $t = 3600$ s, and

$$R(t) = 0.08958\sqrt{3600} = 5.375 \text{ ly}$$

while the horizon radius r_h increases as $2ct$:

$$r_h = 2 \times (3 \times 10^8) \times (3600) = 2.16 \times 10^{12} \text{ m} = \frac{2.16 \times 10^{12}}{9.45 \times 10^{15}} = 0.0002286 \text{ ly}$$

One week (or $7 \times 24 \times 3600 = 604,800$ s) after the big bang,

$$R(t) = 0.08958\sqrt{604,800} = 69.66 \text{ ly}$$

while the horizon distance becomes

$$r_h = 2 \times (3 \times 10^8) \times (604,800) = 3.63 \times 10^{14} \text{ m} = \frac{3.63 \times 10^{14}}{9.45 \times 10^{15}} = 0.0384 \text{ ly}$$

And finally, one year (or 3.15×10^7 s) after the big bang,

$$R(t) = 0.08958\sqrt{3.15 \times 10^7} = 502.8 \text{ ly}$$

while the horizon distance is two light years.

D.38 Numerical Examples for Chapter 38

Example D.38.1. Use Equation (38.1) to estimate the time passed since the big bang when the second epoch starts. To find α, you need the particles present in the universe. There are three charged leptons, three neutrinos and five quarks, each contributing a factor of 7/8 because they are fermions, and a factor of 2 because they all have antiparticles. The neutrinos also need a factor of $\frac{1}{2}$ because they have only one spin orientation. There are also a *total of* 8 gluons and antigluons (so no need for a factor of 2), and one photon. Therefore,

Age of universe at start of 2nd epoch **(page 587 of the book)**

$$\alpha = \underbrace{\frac{7}{8} \times 2 \times (3 + \tfrac{3}{2} + 5)}_{\text{leptons and quarks}} + 8 + 1 = 26.75$$

and

$$t = \frac{2.3 \times 10^{20}}{\sqrt{\alpha}\, T^2} = \frac{2.3 \times 10^{20}}{\sqrt{26.75}\, (10^{14})^2} = 4.4 \times 10^{-9} \text{ s} = 4.4 \text{ nanoseconds.}$$

Example D.38.2. The major contributors to the density of the universe after μ^+-μ^- annihilation are the electron, three neutrino, and photon species.[17] Therefore,

Density in third epoch **(page 588 of the book)**

$$\alpha = \tfrac{7}{8} \times 2 + \underbrace{\tfrac{7}{8} \times 2 \times 3 \times \tfrac{1}{2}}_{\nu \text{ contribution}} + 1 = 5.375$$

Notice the introduction of $\frac{1}{2}$ for the neutrino contribution. This is because neutrinos have only one spin orientation (helicity). Use this value of α and $T = 10^{11}$ K in the fourth row of Table E.3 to obtain

$$\rho = 5.375 \times (8.36 \times 10^{-33}) \times (10^{11})^4 = 4.5 \times 10^{12} \text{ kg/m}^3$$

A grain of sand is about a millimeter on each side. So, its volume is $0.001^3 = 10^{-9}$ m^3. If this grain of sand were made of the material of this epoch, it would weigh

$$4.5 \times 10^{12} \text{ kg/m}^3 \times 10^{-9} = 4500 \text{ kg}$$

or 4.5 metric tons!

[17] The word "species" is used when we want to group each particle with its antiparticle: electron "species" consists of both e^- and e^+.

Inequality of protons and neutrons **(page 588 of the book)**

Example D.38.3. We can understand the inequality of proton and neutron numbers from the Boltzmann factor discussed in Section 17.2.2. Think of protons and neutrons as two "states" of a nucleon. A proton is a nucleon with a smaller mass (or energy $E_p = m_p c^2$) of 938.27 MeV, and neutron a nucleon with a larger mass (or energy $E_n = m_n c^2$) of 939.57 MeV. The probability of a nucleon being a proton is proportional to e^{-E_p/k_BT}, and the probability of a nucleon being a neutron is proportional to e^{-E_n/k_BT}. Thus the ratio of the two probabilities is

$$\frac{P(E_n)}{P(E_p)} = \frac{e^{-E_n/k_BT}}{e^{-E_p/k_BT}} = e^{-\Delta E/k_BT}, \qquad \Delta E \equiv E_n - E_p$$

Substitute for ΔE and k_B to get

$$\frac{\Delta E}{k_B} = \frac{\overbrace{(1.3\times 10^6)(1.6\times 10^{-19})}^{\Delta E \text{ in Joules}}}{1.38\times 10^{-23}} = 1.5\times 10^{10}$$

and

$$\frac{P(E_n)}{P(E_p)} = e^{-1.5\times 10^{10}/T} \tag{D.25}$$

If T is much larger than 1.5×10^{10} K, the exponent of Equation (D.25) is almost zero, the exponential is 1, and the two probabilities are equal. So p and n populations are almost equal. This is what we expect at the earlier epochs. As T drops, the negative exponent increases in magnitude, leading to a small ratio. For example, at $T = 1.5\times 10^{10}$ K, the exponent becomes -1, $P(E_n)/P(E_p) = 0.368$, and with $P(E_n) + P(E_p) = 1$, we get $P(E_n) = 0.269$. So, neutrons constitute about 27% and protons 73%. A detailed calculation of n-p abundance is much more complicated than the one presented here, although the Boltzmann factor plays a significant role in that calculation.

Age of universe at 5th epoch **(page 589 of the book)**

Example D.38.4. To find the age of the universe at the beginning of the fifth epoch, we have to know what α is. The constituents of the universe are radiation, neutrinos, and matter. The contribution of matter to the density is very small as shown below. Therefore, α is the sum of the contributions from radiation (which is 1) and neutrinos. Recall that after their decoupling, neutrinos' temperature T_ν fell below the radiation temperature T_γ in such a way that $T_\nu = \sqrt[3]{\frac{4}{11}}T_\gamma$. So, in finding α, we need to take this into account. Denoting the neutrino's contribution to α by α_ν, we have

$$\alpha_\nu = \tfrac{7}{8}\times 3\times 2\times \tfrac{1}{2}\times (\sqrt[3]{4/11})^4 = 0.681$$

where the first factor arises from the fact that neutrinos are fermions; the second factor because there are 3 neutrino species; the third factor because they have antiparticles; the fourth factor because they have only one helicity; and the last factor is due to the temperature difference between the radiation and neutrino gases. Adding this to the radiation contribution yields $\alpha = 1.681$. Therefore, Equation (38.1) with $T = 10^9$ K gives

$$t = \frac{2.3\times 10^{20}}{\sqrt{1.681}\,(10^9)^2} = 177.4 \text{ s.}$$

The matter density is negligible for the following reason. A particle is considered relativistic if its thermal energy $2.7k_BT$ is much larger than its rest energy mc^2. At a temperature of 10^9 K, the thermal energy is 3.73×10^{-14} J. Thus, only particles whose rest energies are much smaller than this can be considered relativistic. The rest energy of an electron is about 8.2×10^{-14} J, which is over twice the thermal energy above. Therefore, the electrons are nonrelativistic, and the protons and neutrons more so. It follows that the matter density is just the mass of a nucleon (electrons are negligibly less massive than protons and neutrons) times the nucleonic number density n_b. But n_b is $1/(1.6 \times 10^9)$ the photon number density, and the latter is

$$n_\gamma = 2.7 \times 10^7 T_\gamma^3 = 2.7 \times 10^7 (10^9)^3 = 2.7 \times 10^{34} \text{ photons/m}^3$$

giving a value of $2.7 \times 10^{34}/(1.6 \times 10^9) = 1.69 \times 10^{25}$ for n_b. With the mass of each nucleon being 1.67×10^{-27} kg, we obtain

$$\rho_m = 1.67 \times 10^{-27} n_b = (1.67 \times 10^{-27}) \times (1.69 \times 10^{25}) = 0.028 \text{ kg/m}^3$$

The radiation density can be calculated from the fourth row of Table E.3:

$$\rho_\gamma = 8.36 \times 10^{-33} T^4 = 8.36 \times 10^{-33} (10^9)^4 = 8360 \text{ kg/m}^3$$

The neutrino density is

$$\rho_\nu = \alpha_\nu \rho_\gamma = 0.681 \times 8360 = 5690 \text{ kg/m}^3$$

Therefore, the universe is radiation- and neutrino-dominated, and matter contributes next to nothing to its density. Dark matter has not been taken into account, not because it does not contribute to the matter density, but because we don't know much about it. Nevertheless, even *with* dark matter, radiation and neutrino dominate the density since dark matter raises the matter contribution by at most a factor of ten.

Neutron decay in 5th epoch
(page 589 of the book)

Example D.38.5. The neutron decay, like all other decays, obeys the exponential law of Equation (31.2). If you start with N_0 neutrons, t seconds later you will have $N(t)$ neutrons, where

$$N(t) = \frac{N_0}{2^{t/t_{\text{half}}}}$$

and t_{half} is the half-life of neutron.

The helium formation starts when the temperature reaches 950 million Kelvin, corresponding to 195.5 seconds after the third epoch, in which protons outnumbered neutrons three to one. The contribution to further reduction of the neutrons coming from their decay can be calculated using the above formula:

$$N(t) = \frac{N_0}{2^{195.5/614}} = 0.80 N_0$$

This shows that 20% of the neutrons present at the end of the third epoch decay by the beginning of the fifth epoch. If this were the only process converting neutrons to protons, we would have 20% less neutrons than at the end of the third epoch, i.e., 20% of the 25% decay, leaving only $0.25 \times 0.80 = 0.2$ or 20% neutrons, and the remaining 80% protons. Other processes contribute as well, and reduce the 20% further down to 13% by the beginning of the fifth epoch.

Change in H-He abundance in absence of radiation
(page 590 of the book)

Example D.38.6. If deuterons were made too early, say when the number of protons and neutrons were almost equal, the universe would consist only of He. That's far too soon. So the question is: In the absence of radiation, what is the least temperature at which deuterons can be formed without being disintegrated by the impact of the particle content of the universe? With 2.224 MeV as the binding energy of the deuteron, the question turns into: Which constituent of the universe has an average KE of 2.224 MeV at the least temperature? Electrons that have this much KE are relativistic and their KE is given by $2.7k_BT$, just like a photon, so that for electrons

$$\underbrace{(2.224 \times 10^6)(1.6 \times 10^{-19})}_{\text{KE in Joules}} = 2.7(1.38 \times 10^{-23})T \quad \text{or} \quad T = 9.55 \times 10^9 \text{ K}$$

For nucleons, 2.224 MeV is much less than their mass (times c^2), and therefore, they can be treated as nonrelativistic particles, whose KE is $\frac{3}{2}k_BT$. So

$$(2.224 \times 10^6)(1.6 \times 10^{-19}) = (\tfrac{3}{2})(1.38 \times 10^{-23})T \quad \text{or} \quad T = 4.3 \times 10^9 \text{ K}$$

which is less than the corresponding T for electrons.

We therefore conclude that, in the absence of radiation, for deuteron to be formed, the temperature must fall just below 4.3 billion K, corresponding to the middle of the fourth epoch. At this temperature, it turns out, there is approximately 1 neutron for every 4 protons. If deuteron (and therefore helium) were to be formed at this temperature, the H-He abundance would be 60%-40%, in complete violation of observation! Radiation is necessary to prevent deuteron formation at such an early time.

Estimating quark confinement temperature **(page 587 of the book)**

Example D.38.7. A typical hadron is a proton which has 3 quarks inside it. The radius of the proton is approximately 1 fm (1 femptometer or 1 fermi, equal to 10^{-15} m). This gives a volume of

$$\tfrac{4}{3}\pi r^3 = \tfrac{4}{3}\pi(10^{-15})^3 = 4.2 \times 10^{-45} \text{ m}^3$$

for the proton. Therefore, the quark number density n_q inside a proton is

$$n_q = \frac{3}{4.2 \times 10^{-45}} = 7.2 \times 10^{44} \text{ quarks/m}^3$$

As long as the quarks are separated by distances smaller than a typical hadron radius, they are free. In other words, as long as the number density of quarks is larger than the density calculated above, they are free. However, as soon as their separation equals this radius or larger, i.e., as soon as their number density falls below the above density, they become bound. Now the question is "At what temperature does the quark number density become the above n_q?" Since quarks are moving close to light speed, they act like photons, except for some factors similar to the those associated with the density of various particles discussed in Section 38.1. One of these factors is 3/4 (analogous to 7/8 in the calculation of densities), which arises from replacing the minus sign in the integral of Equation (E.145) with a plus sign due to the fermionic nature of the quarks. This replacement changes 2.404 to 1.8031, introducing a factor of 1.8031/2.404 which is about 3/4. A second factor is 2, because quarks have antiparticles; and the last factor is 2, the number of quarks (up and down) present near the end of the second epoch. Thus, the quark number density is the photon number density times $\frac{3}{4} \times 2 \times 2 = 3$, i.e., $n_q = 3n_\gamma$. Using Equation (E.146) and the n_q calculated above, we obtain

$$7.2 \times 10^{44} = 3 \times (2 \times 10^7 T^3) \quad \text{or} \quad T^3 = \frac{7.2 \times 10^{44}}{6 \times 10^7} = 1.2 \times 10^{37}$$

and $T = \sqrt[3]{1.2 \times 10^{37}} = 2.29 \times 10^{12}$ K. Thus, when the temperature of the universe gets down to about a trillion Kelvin, the quarks form hadronic traps in which they live for the rest of their existence.

D.39 Numerical Examples for Chapter 39

Blob formation and pressure **(page 596 of the book)**

Example D.39.1. Just before the decoupling of radiation from matter, when the universe was about 3000 K hot, the pressure in the universe was dominated by radiation and neutrino pressures. To see this, note that radiation pressure P_γ, is one third the radiation energy density [see Equation (E.31)], which is the equivalent mass density ρ_γ times c^2. Therefore,

$$P_\gamma = \tfrac{1}{3}\rho_\gamma c^2 = \tfrac{1}{3}(8.36 \times 10^{-33})(3 \times 10^8)^2 = 0.02 \text{ Pa}$$

where we used the fourth row of Table E.3 for ρ_γ. The neutrino pressure P_ν is 0.681 times P_γ (see Example D.38.4). Thus, the total pressure due to radiation and neutrinos is $0.02 \times 1.681 = 0.0336$ Pa.

The matter pressure P_m, is given by the ideal gas law, which can be written as $P = nk_BT$, with $n = N/V$ being the number density of matter particles. Assuming that matter particles are just nucleons, and that there are 0.25 nucleons per cubic meter now, we can calculate the nucleon number density at 3000 K. In fact, since nR^3 is a constant, and since R and T are inversely proportional, we get

$$\frac{n}{T^3} = \frac{n_0}{T_0^3} \quad \text{or} \quad n = n_0 \left(\frac{T}{T_0}\right)^3 = 0.25 \left(\frac{3000}{2.725}\right)^3 = 3.3 \times 10^8$$

and

$$P_m = nk_BT = 3.3 \times 10^8 (1.38 \times 10^{-23})(3000) = 1.38 \times 10^{-11} \text{ Pa}$$

which is about a billionth P_γ. Even if we include the dark matter, it only increases P_m by a factor of ten.

What was M_J before decoupling? We found above that the pressure is 0.0336 Pa. The density is given by Equation (38.1) with $\alpha = 1.681$:

$$\rho = 1.681(8.36 \times 10^{-33} T^4) = 1.405 \times 10^{-33}(3000)^4 = 1.14 \times 10^{-18}$$

Putting everything together gives

$$M_J = \frac{(0.0336/6.67 \times 10^{-11})^{3/2}}{(1.14 \times 10^{-18})^2} = 8.7 \times 10^{48} \text{ kg}$$

This is 4.35×10^{18} solar mass, or over 10 million times the mass of the Milky Way! Such a huge blobs were hard to find, and the clumping was next to impossible.

After the decoupling, the pressure dropped to a billionth its value before (only matter pressure was present after decoupling), so M_J dropped by a factor of $(10^9)^{3/2} = 3.16 \times 10^{13}$ giving $M_J = 2.75 \times 10^{35}$ kg or about 138,000 solar mass, which is the mass of a large globular cluster, making gravitational clumping feasible.

Example D.39.2. At the start of the dominance of matter, the universe was approximately 14800 K hot and 25,000 years old (see page 578 of the book). The horizon radius at that point was 50,000 light years (for the first 25,000 years, the universe was radiation-dominated, in which case $r_h = 2ct$). Thus, two points that were farther than 100,000 light years (at the two ends of a diameter of the horizon sphere) apart, could not communicate with one another. These two points have flown apart due to the expansion of the universe by the same factor as the scale $R(t)$ has. This factor is just the ratio of the temperature then to the present temperature: $14800/2.725 = 5430$. Thus, the two points are now

Causal disconnection **(page 599 of the book)**

$$100,000 \times 5430 = 5.43 \times 10^8 \text{ light years}$$

apart. The light from these two points has been traveling for almost the age of the universe. So the distance from these points to us is about 13.7 billion light years. The angular separation in radian is therefore

$$\frac{5.43 \times 10^8}{13.7 \times 10^9} = 0.04 \text{ radian} \quad \text{or} \quad 0.04 \times 57.3 = 2.27^\circ$$

A very small angle indeed!

Taking the decoupling event as the starting point, when the universe was approximately 3000 K hot and 375,000 years old,the horizon radius would have been 1,125,000 light years (the universe was clearly matter-dominated at the decoupling, therefore, $r_h = 3ct$). Thus, two points that were farther than 1,125,000 light years apart, could not communicate with one another. The present distance between these two points is 1,125,000 times the increase in the scale factor. This factor is just the ratio of the temperatures: $3000/2.725 = 1100$. Thus, the two points are now

$$1,225,000 \times 1100 = 1.24 \times 10^9 \text{ light years}$$

apart, giving an angular separation of

$$\frac{1.24 \times 10^9}{13.7 \times 10^9} = 0.09 \text{ radian} \quad \text{or} \quad 0.09 \times 57.3 = 5^\circ$$

Estimating how long Ω_{tot} remains 1 **(page 601 of the book)**

Example D.39.3. Write Equation (E.162) as

$$\frac{\Omega_{\text{tot}}(t_2) - 1}{\Omega_{\text{tot}}(t_1) - 1} = e^{-2H(t_2 - t_1)} \tag{D.26}$$

where t_1 and t_2 are two instants, between which inflation was operative. A typical value for the start of the inflation is $t_1 = 10^{-34}$ s. Since H is (to within a numerical factor) $1/t$, we take it to be 10^{34} s^{-1} between t_1 and t_2. Let t_2 be 10^{-32} s. Then Equation (D.26) gives

$$\frac{\Omega_{\text{tot}}(t_2) - 1}{\Omega_{\text{tot}}(t_1) - 1} = e^{-2\times 10^{34}(10^{-32} - 10^{-34})} = e^{-198} = 1.02 \times 10^{-86} \tag{D.27}$$

Now suppose that the universe starts substantially nonflat with $\Omega_{\text{tot}}(t_1) = 1.5$. Then a little later, at t_2, Ω_{tot} is given by

$$\frac{\Omega_{\text{tot}}(t_2) - 1}{1.5 - 1} = 1.02 \times 10^{-86} \quad \text{or} \quad \Omega_{\text{tot}}(t_2) - 1 = 5.1 \times 10^{-87}$$

which is unimaginably close to flat. How close? Suppose that after the end of inflation at t_2, the universe becomes radiation-dominated.[18] How long do we have to wait before the universe has an Ω_{tot} that is ever so slightly different from 1, say 1.0001? Denote by t_3 the time at which this happens. Then Equation (E.156) gives

$$\frac{\Omega_{\text{tot}}(t_3) - 1}{\Omega_{\text{tot}}(t_2) - 1} = \frac{t_3}{t_2} \quad \text{or} \quad \frac{0.0001}{5.1 \times 10^{-87}} = \frac{t_3}{10^{-32}}$$

yielding $t_3 = 1.96 \times 10^{50}$ s or 4.5×10^{32} times the present age of the universe!

Horizon expansion in inflationary universe **(page 601 of the book)**

Example D.39.4. Use the result of Math Note E.39.2 to calculate the expansion of the horizon radius during inflation. Assume, as in Example D.39.3, that inflation begins at $t_1 = 10^{-34}$ s and ends at $t_2 = 10^{-32}$ s. Assume also—as explained in the same example—that $H = 10^{34}$ s^{-1}. Then the horizon radius at the start of inflation is

$$r_h(t_1) = \frac{c}{H}(e^{Ht} - 1) = \frac{3 \times 10^8}{10^{34}}(e^{(10^{34})(10^{-34})} - 1) = 5.15 \times 10^{-26} \text{ m}$$

By the end of inflation this radius increases to

$$r_h(t_2) = \frac{3 \times 10^8}{10^{34}}(e^{(10^{34})(10^{-32})} - 1) = 3 \times 10^{-26}(e^{100} - 1) = 8 \times 10^{17} \text{ m}$$

which is over 85 light years!

[18] Matter-dominance yields essentially the same result.

Appendix E

Mathematical Notes

E.1 Math Notes for Chapter 1

Finding the Earth-Sun distance
(page 9 of the book)

Math Note E.1.1. Let us denote the period of the Moon by T, the time it take the Moon to go from M_1 to M_3 (see Figure 1.2) by T_{13}, and that from M_3 to M_1 by T_{31}. Assuming that the Moon is moving uniformly—and counterclockwise—around the Earth, we immediately conclude that the ratio of $T_{13} - T_{31}$ to T must be the same as the ratio of the difference between the two arc lengths M_1M_3 and M_3M_1 to the circumference of the Moon orbit. The last ratio is the same as the ratio of the difference between the two angles subtended by M_1M_3 and M_3M_1 to the total angle of a circle (360° or 2π radians). Figure 1.2 shows that this angular difference is 4α. So, we have

$$\frac{T_{13} - T_{31}}{T} = \frac{4\alpha}{2\pi} = \frac{2\alpha}{\pi} \quad \Rightarrow \quad \alpha = \frac{\pi}{2}\left(\frac{T_{13} - T_{31}}{T}\right)$$

Aristarchus believed (erroneously) that he had measured $T_{13} - T_{31}$ to be one day, and based on that belief, and the fact that the period of revolution of the Moon is approximately 30 days, he calculated α to be

$$\alpha = \frac{\pi}{2}\left(\frac{1}{30}\right) = \frac{\pi}{60} \text{ rad}$$

Since (small) angle in radian is "size" over distance, we have

$$\frac{\pi}{60} = \frac{\overline{ME}}{\overline{SE}} \quad \Rightarrow \quad \overline{SE} = \frac{60}{\pi}\overline{ME} = 19.1\overline{ME}$$

Details of the epicyclic motion
(page 13 of the book)

Math Note E.1.2. Figure E.1 shows snapshots of a planet M moving around the Earth E. For the sake of the argument, let's greatly simplify the actual motion of the planet and assume that it moves on the epicycle four times while the center of the epicycle moves on the deferent once. The figure shows only 12 snapshots of a complete revolution of M around E. Each revolution of M on its epicycle corresponds to a quarter of a revolution of the epicycle's center on the deferent. Therefore, M completes its epicyclic revolution in 3 snapshots, or a third of the revolution in one snapshot.

The planet starts at 1 (the 6 o'clock position on the epicycle); moves a third of the circumference of the epicycle to 2, while the center of the epicycle moves from a to b, one twelfth of the deferent. By the time M has made a complete revolution of the epicycle to 4 (back to the 6 o'clock position on the epicycle), the center of the epicycle has moved to d, a quarter of the deferent. Continuing thusly, you can locate the planet on its epicycle in the

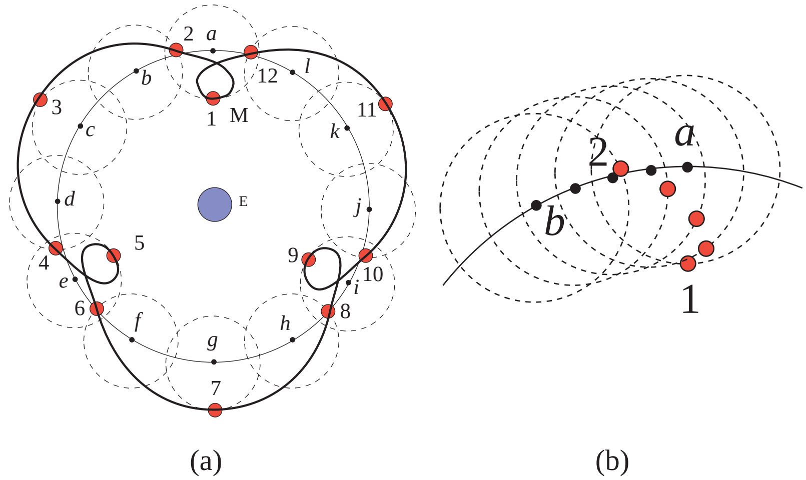


Figure E.1: (a) The combination of the two motions of the epicycle (smaller circle) on the deferent (larger circle) and the planet on the epicycle results in the path of M around E. (b) The detail of the motion between 1 and 2. The planet advances $\frac{1}{12}$ of the epicycle from one snapshot to the next.

second, third, and fourth quarter of the deferent as shown in Figure E.1(a). Connecting all the planet's positions with a smooth curve, you obtain its path around the Earth.

Now from the 12 positions in Figure E.1(a) it is not by any means obvious that the path of M around E is the curve drawn in heavy line. This is because—to avoid the cluttering of the figure—we have shown only three "snapshots" of the epicycle for each revolution of M. If you subdivide the motion into smaller intervals as done in Figure E.1(b), where the movement from 1 to 2 is subdivided into four parts each showing the slight advance ($\frac{1}{12}$ of the epicycle) of M toward 2, you will see that indeed the heavy curve is the correct path.

The actual paths of real planets are, of course, much more complicated than that depicted in Figure E.1. For example, no known planet goes around its epicycle an integer number of times when the center of its epicycle completes a single revolution around the Earth. Nevertheless, the simplified motion outlined above illustrates all the essential features of the planet's path, including the retrograde motion and the change in the brightness during this motion.

E.3 Math Notes for Chapter 3

The heliocentric theory applied to Earth and another planet (**page 38 of the book**)

Math Note E.3.1. The actual motion of planets as viewed from Earth is very complicated—that is why the geocentric model had to incorporate so many assumptions to reconcile the theory with observation. However, to illustrate the general feature of the observed motion of a planet from the Earth point of view, I make some simplifying assumptions. To be specific, I assume that a planet, say Mars, moves around the Sun once as the Earth moves around four times. This is an enormous simplification, as the actual periods of revolution of Mars and Earth do not have such a simple relation. However, the simplification does not alter the essential features of the heliocentric theory.

To plot the path of Mars relative to Earth, I need to find positions of Mars relative to Earth for different times. Since both planets are assumed to be moving uniformly, I can locate them at each instant starting with their initial location which I have designated as point 1 in Figure E.2. Now imagine taking snapshots of the location of the Earth periodically as it moves on its orbit. Figure E.2 shows six equally spaced snapshots, but one can take as many as one wishes, the larger the number of the snapshots, the more accurate the plot

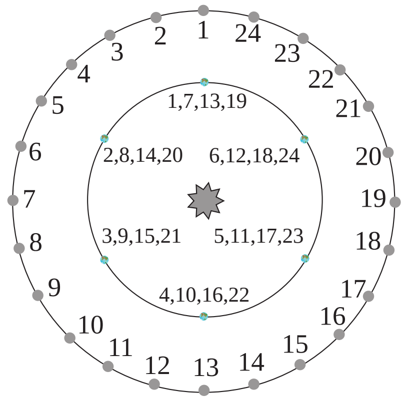


Figure E.2: The Earth orbit is divided into six equal parts by points numbered 1 through 6. The corresponding locations of Mars are also numbered 1 through 6. The rest of the numbers on Mars's orbit correspond to the second, third, and fourth revolutions of the Earth. Each location of Earth has four numbers corresponding to the four revolutions of the Earth.

of the path.[1] I have numbered these locations 1 through 6 on the Earth orbit. As Earth completes one revolution, Mars covers only a quarter of its orbit. So, I divide the first quarter of the orbit of Mars into six equal parts, again numbered 1 through 6. Thus, when Earth is, for instance, at location 4 on its orbit, Mars will be at location 4 on *its orbit*. When Earth returns to its original position, Mars will be at location 7, and when Earth goes through its second revolution—with snapshots taken at the previous locations—Mars will go through points 7 to 13. I write these numbers by the Earth locations as well to indicated their relation with those of Mars. I continue the process for the third and fourth revolution of the Earth, and obtain the numbering of Figure E.2. Each snapshot location of the Earth now has four numbers next to it, because Earth passes four time through each point in its four revolutions.

Next I connect each location of the Earth to the corresponding location of Mars with an arrow. This designates the location of Mars *relative to Earth*, or the "line of sight" of the planet. In other words, if I were to look at Mars from Earth, I would have to look in that direction. These arrows are drawn in Figure E.3(a), each carrying a number corresponding to the number of the snapshot location.

Now, as we look at Mars from Earth, the Earth does not appear moving. Although both Mars and Earth move from 1 to 2, the Earth inhabitants do not feel its motion. What they see is that Mars is "directly above" initially, and some time later—when both move to their corresponding locations numbered 2—Mars is farther and a little "to the right of directly above." This situation is depicted in Figure E.3(b), and from the Earth point of view, Mars has moved from the tip of arrow 1 to the tip of arrow 2. Therefore, to find the complete path of Mars, I draw all arrows from the same point and connect the tip of the consecutive ones. This will lead to the curve shown in Figure E.3(c).

Note that this curve is exactly the same as that in Figure E.1. Thus, the heliocentric model of Copernicus gives the same result as the geocentric model of Ptolemy, but the heliocentric model is much simpler, and has much fewer assumptions. It should therefore come as no surprise that Kepler, seeking a theoretical explanation for the recent observations of his time, chose the heliocentric model as his starting point.

[1]From a practical point of view, one does not want too many snapshots, because, as will be seen shortly, the diagram will be too cluttered.

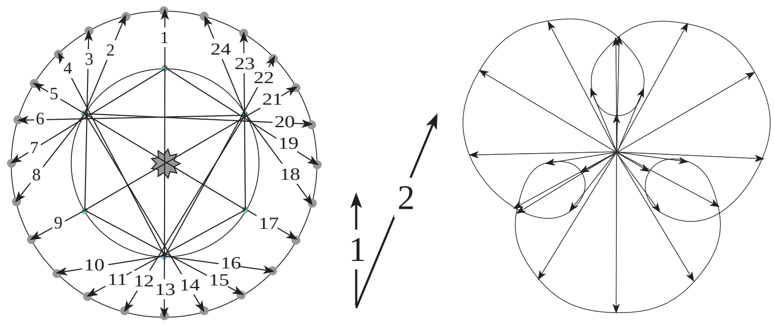


Figure E.3: (a) The positions of Mars relative to Earth are shown by arrows. (b) The first two positions as seen from the Earth. (c) The 24 positions of Mars as seen from the Earth, and Mars's path relative to Earth.

E.4 Math Notes for Chapter 4

Finding the distance formula for a uniformly accelerated motion **(page 58 of the book)**

Math Note E.4.1. As discussed in the text, the distance traveled in time t is the area enclosed by the curve, the t-axis, the v-axis, and the vertical line at t. This area is shown in Figure E.4(b) for the UAM, and consists of a rectangle and a triangle, the areas of both of which are easy to calculate once we know the dimensions of the figures.

The rectangle is easier; it has a width of v_0 and a length of t; so its area is $v_0 t$. The area of the triangle is simply $\frac{1}{2}(\overline{AB})(\overline{CB})$. But $\overline{CB}$ is just t. What about $\overline{AB}$? Since the slope of the line is a, we must have

$$a = \frac{\overline{AB}}{\overline{CB}} = \frac{\overline{AB}}{t} \quad \text{or} \quad \overline{AB} = at.$$

So, the area of the triangle is

$$\tfrac{1}{2}(at)(t) = \tfrac{1}{2}at^2$$

and the entire area, which is the sum of the area of the rectangle and that of the triangle, is as given in Equation (4.4).

Deriving the formula relating speed and distance **(page 59 of the book)**

Math Note E.4.2. Instead of using numbers as in Example 4.3.7, we use symbols and algebra to find the general formula connecting distance and speed directly. In fact, we can be even more general and leave v as nonzero. So, let us solve Equation (4.3) for t:

$$v = v_0 + at \quad \Rightarrow \quad at = v - v_0 \quad \Rightarrow \quad t = (v - v_0)/a.$$

Now substitute this in Equation (4.4) to find x:

$$\begin{aligned} x &= v_0\left(\frac{v - v_0}{a}\right) + \tfrac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 = \frac{v_0 v - v_0^2}{a} + \tfrac{1}{2}a\left(\frac{v^2 + v_0^2 - 2vv_0}{a^2}\right) \\ &= \frac{2v_0 v - 2v_0^2 + v^2 + v_0^2 - 2vv_0}{2a} = \frac{v^2 - v_0^2}{2a} \end{aligned}$$

This yields the useful relation

$$v^2 - v_0^2 = 2ax, \tag{E.1}$$

which holds for all values of v, v_0, a, and x. As a check, we recalculate the result of Example 4.3.7:

$$0 - (44.72)^2 = 2(-9.8)x \quad \Rightarrow \quad -1999.88 = -19.6x \quad \Rightarrow \quad x = \frac{-1999.88}{-19.6} = 102 \text{ m}.$$

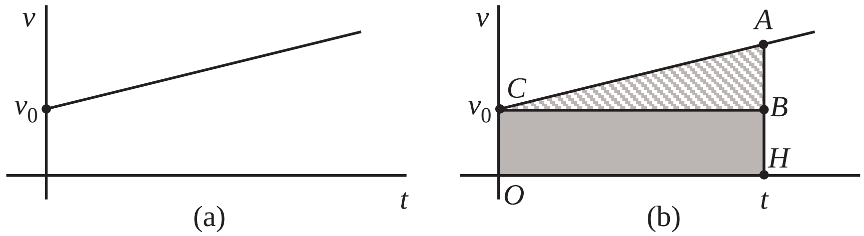


Figure E.4: (a) The graph of v versus t for a uniformly accelerated motion. (b) The area under the graph is the distance traveled in time t.

Equation (E.1) connects the distance to speed directly without involving time: For any value of x, (E.1) determines the speed there.

E.6 Math Notes for Chapter 6

Derivation of the formula for centripetal acceleration
(page 82 of the book)

Math Note E.6.1. Suppose that an object A moves on a circle with constant *speed* (not constant velocity!) as shown in Figure E.5(a). Let us find the acceleration at A_1 where the velocity is $\mathbf{v}_1$. We allow the object to move to a nearby point A_2 at which the velocity becomes $\mathbf{v}_2$. We need $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ to calculate the acceleration $\mathbf{a}$. First let us concentrate on the direction of $\mathbf{a}$, which is the same as the direction of $\Delta\mathbf{v}$ as long as A_2 is very close to A_1, a condition we have attempted to satisfy in the figure. To find $\Delta\mathbf{v}$ we draw $\mathbf{v}_1$ and $\mathbf{v}_2$ from a common point and connect the tip of $\mathbf{v}_1$ to the tip of $\mathbf{v}_2$. This is done in Figure E.5(b). We have also parallel-transported $\Delta\mathbf{v}$ to A_1 where $\mathbf{a}$ is to be calculated. It is clear that $\Delta\mathbf{v}$ points (almost) toward the center of the circle. The small discrepancy is due to the "large" size of the distance between A_1 and A_2. The smaller we make this distance, the more $\Delta\mathbf{v}$ will point toward the center. When this distance becomes infinitesimally small, $\Delta\mathbf{v}$ will point *exactly* toward the center.

What about the magnitude of the acceleration? The two isosceles triangles CA_1A_2 (with long sides r)[2] and BDF (with long sides v) are similar because they have the same angles. Therefore, we can write

$$\frac{\overline{A_1A_2}}{r} = \frac{\Delta v}{v}$$

But $\overline{A_1A_2}$ is the distance obtained when A moves with constant *speed* for a time Δt. So, $\overline{A_1A_2} = v\Delta t$, and the equation above becomes

$$\frac{v\Delta t}{r} = \frac{\Delta v}{v} \quad \text{or} \quad v^2\Delta t = r\Delta v \quad \text{or} \quad \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

which is identical to Equation (6.1), because $\Delta v/\Delta t = a$.

E.7 Math Notes for Chapter 7

Calculation of the critical speed for roller coaster looping
(**page 23** of *Appendix.pdf*)

Math Note E.7.1. Example D.7.7 used numbers; let us use symbols now. Denote the weight of the car plus the passenger by mg, the force of the track on the car by N (for

[2] CA_1A_2 is not really a triangle, because one of its sides is an arc of a circle. However, when A_2 is infinitesimally close to A_1, this arc is indistinguishable from a straight line.

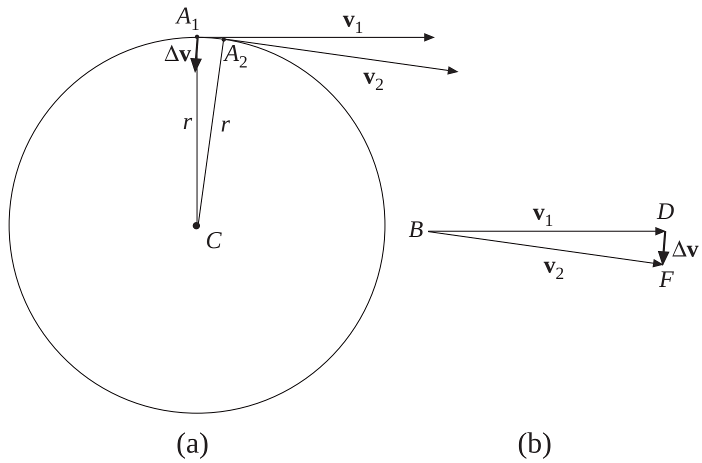


Figure E.5: (a) The object A moves from A_1 to the (very) nearby A_2. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same length, because the speed of the moving object is constant. (b) The change $\Delta\mathbf{v}$ in velocity is obtained as usual. It is also parallel-transported to A_1.

normal), the speed by v, and the radius by r. If the car is to remain on the track, N must be bigger than zero (it could be as small as possible). The net force on the car is

$$F_{\text{net}} = mg + N$$

The second law gives

$$mg + N = ma \quad \Rightarrow \quad mg + N = m\frac{v^2}{r} \quad \text{or} \quad v^2 = \left(\frac{mg+N}{m}\right) r$$

This gives

$$v = \sqrt{\left(\frac{mg+N}{m}\right) r}$$

If we insist that N be large, we need a large speed. However, theoretically N can be as small as we please. In particular, when $N = 0$, we get the critical speed. Thus,

$$v_{\text{crit}} = \sqrt{\left(\frac{mg}{m}\right) r} = \sqrt{gr}$$

which is independent of the mass. Once the speed is larger than this critical speed, *nobody* will fall!

E.8 Math Notes for Chapter 8

Finding the relation between work and KE **(page 108 of the book)**

Math Note E.8.1. We start with the second law of motion and assume that there is only one force acting on the object in question. Multiplying both sides of the second law by the displacement d, we obtain

$$Fd = mad \quad \text{or} \quad W = mad = m\frac{\Delta v}{\Delta t}d = m\frac{v_f - v_i}{\Delta t}d = m(v_f - v_i)\frac{d}{\Delta t}$$

because $\Delta v = v_f - v_i$, and the denominator can be moved under any one of the factors of a product. Now $d/\Delta t$ is the *average* velocity during the time Δt. Since, in that time interval, the initial velocity is v_i and the final velocity v_f, their average is their sum divided by two. Thus,

$$W = m(v_f - v_i)\frac{v_f + v_i}{2} = \tfrac{1}{2}m(v_f - v_i)(v_f + v_i) = \tfrac{1}{2}m(v_f^2 - v_i^2) = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

The right-hand side will be the change in KE if we define KE to be $\frac{1}{2}mv^2$.

Math Note E.8.2. Let us look at the car-passenger system from a general viewpoint, i.e., use symbols rather than numerical values. So, let m stand for the total mass. Then,

$$ME = KE_A + PE_A = 0 + mgh_0 \quad \text{or} \quad ME = mgh_0$$

This ME is the same at all other points of the track, such as the general point P where the height is assumed to be h and the speed v. Thus,

Showing that speed is the same for all cars of a roller coaster **(page 111 of the book)**

$$ME = KE_P + PE_P = \tfrac{1}{2}mv^2 + mgh$$

Combining the last two equations, we get

$$mgh_0 = \tfrac{1}{2}mv^2 + mgh \quad \text{or} \quad gh_0 = \tfrac{1}{2}v^2 + gh$$

where we divided both sides of the equation by m. It follows that

$$gh_0 - gh = \tfrac{1}{2}v^2 \quad \Rightarrow \quad v^2 = 2gh_0 - 2gh = 2g(h_0 - h) \quad \text{and} \quad v = \sqrt{2g(h_0 - h)}$$

This speed is *independent of the mass of the system.*

A relevant question in the design of a roller coaster like that in Figure D.9 is the height h_2 of the circular loop. This height cannot be too large, because the car may slow down too much at D and fall. In order for it not to fall, the speed at D must be larger than v_{crit} of Math Note E.7.1. This condition gives

$$v > v_{\text{crit}} \quad \Rightarrow \quad \sqrt{2g(h_0 - h_2)} > \sqrt{gr} \quad \text{or} \quad 2g(h_0 - h_2) > gr \quad \Rightarrow \quad 2(h_0 - h_2) > r$$

canceling g on both sides. But r, the radius of the circle, is just $\frac{1}{2}h_2$. This yields

$$2(h_0 - h_2) > \tfrac{1}{2}h_2 \quad \text{or} \quad 4(h_0 - h_2) > h_2 \quad \Rightarrow \quad 4h_0 - 4h_2 > h_2 \quad \Rightarrow \quad 4h_0 > 5h_2$$

or $h_2 < \frac{4}{5}h_0 = 0.8h_0$. So, circular loops of roller coasters cannot be taller than 80% of the height at the starting point.

Math Note E.8.3. We want to find the time that Santa has to spend in a typical chimney for his energy consumption to be minimum. There are two kinetic energies involved: One for chimney plunging and climbing, the other for hopping from one chimney to the next. For his trip down the chimney, he travels 4 m carrying a mass of 112 kg (himself plus toys). His speed is therefore $4/t$, where t is the time we are after. It follows that the KE of descent is

Finding condition for Santa's energy to be minimum **(page 113 of the book)**

$$KE_{\text{down}} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(112)(4/t)^2 = \frac{896}{t^2}.$$

When Santa climbs up the chimney, he is a little lighter (12 kg lighter for leaving the toys behind). Therefore, his KE of ascent is

$$KE_{\text{up}} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(100)(4/t)^2 = \frac{800}{t^2}.$$

and the total KE for his "chimney travel" is

$$KE_{\text{chim}} = KE_{\text{down}} + KE_{\text{up}} = \frac{896}{t^2} + \frac{800}{t^2} = \frac{1696}{t^2}.$$

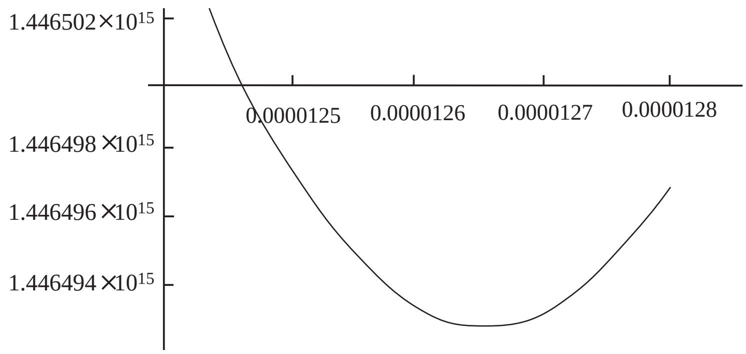


Figure E.6: The total KE that Santa consumes per house as a function of time.

To find the second KE, we note that the entire time available to Santa for going from one house to the next is the number of seconds in one day divided by the number of houses. This gives $86400/2.5\times10^7 = 3.456\times10^{-3}$ second. Out of this, $2t$ seconds is spent plunging down and climbing up the chimney. So, $3.456\times10^{-3}-2t$ is left to cover the distance of 15 m between two adjacent chimneys. Therefore, the second KE—the hopping kinetic energy—is

$$\begin{aligned} KE_{\text{hop}} &= \tfrac{1}{2}mv^2 = \tfrac{1}{2}(1.5\times10^8)\left(\frac{15}{3.456\times10^{-3}-2t}\right)^2 \\ &= (7.5\times10^7)\left[\frac{225}{(3.456\times10^{-3}-2t)^2}\right] = \frac{1.69\times10^{10}}{(3.456\times10^{-3}-2t)^2} \end{aligned}$$

and the total energy *as a function of* t is

$$KE(t) = KE_{\text{chim}} + KE_{\text{hop}} = \frac{1696}{t^2} + \frac{1.69\times10^{10}}{(3.456\times10^{-3}-2t)^2}.$$

What value of t minimizes this? Calculus answers this question through the process of differentiation. But we can manage without the help of calculus by simply plotting $KE(t)$ as a function of t and seeing where the minimum occurs. Figure E.6 shows $KE(t)$ as a function of t for points close to the minimum. It is clear that this minimum occurs at 0.00001265 or about 0.000013 s, or 13 microseconds, which is the same as the result one would obtain if one used calculus. The graph also shows the minimum KE, which appears to be about 1.4465×10^{15} Joules, close to 1.43×10^{15} Joules used in Section 8.1.3.

Derivation of $P = P_0 + \rho gh$ **(page 120 of the book)**

Math Note E.8.4. In Figure 8.7(a), let the area of the column of the liquid be A. The weight of the column exerts a force of mg on the base of the column. Furthermore, the weight of the column of air on top of the column of the liquid exerts another force which is simply P_0A. The pressure at depth h, therefore is

$$P = \frac{F}{A} = \frac{mg+P_0A}{A} = \frac{\rho Vg+P_0A}{A} = \frac{\rho Ahg+P_0A}{A} = P_0+\rho gh$$

because the volume of the column is the area of its base times its height. This equation clearly shows that pressure is independent of the area of the base, and depends only on density, gravitational acceleration, and height (depth). It also shows that when $h=0$, the pressure reduces to P_0, and that when P_0 changes, the pressure at *any* depth h changes by exactly the same amount.

Derivation of Bernoulli principle **(page 123 of the book)**

Math Note E.8.5. At the left end of the pipe in Figure E.7, the pressure is P_1, cross sectional area is A_1, and the speed of the liquid is v_1. The liquid between the two solid dark cross sections in the figure—which we call Λ—is displaced to a new position, and is now contained in the volume between the two light dashed cross sections. Suppose Λ is displaced by d_1 at the left end. Then the work done by the force of pressure on the left is simply $F_1 d_1 = P_1 A_1 d_1 = P_1 V_1$ where V_1 is the volume of the liquid displaced on the left. At the other end the pressure is acting against the flow,[3] so that the corresponding work is negative and equal to $-P_2 A_2 d_2$ or $-P_2 V_2$. In addition, because of the assumed incompressibility, the amount of liquid displaced at one end, must equal that at the other. This means that $V_1 = V_2$, and we name both volumes simply V. Therefore, the work done on Λ due to the pressure difference is

$$W_{\text{pressure}} = P_1 V_1 - P_2 V_2 = P_1 V - P_2 V = (P_1 - P_2)V$$

There is also the work due to the gravitational force mg. As the fluid moves, its particles are pulled down due to this force with the net effect that the mass on the left is moved from the height h_1 to height h_2 on the right. Thus, the work due to gravity is

$$W_{\text{gravity}} = mg(h_1 - h_2) = \rho V g(h_1 - h_2) = \rho V g h_1 - \rho V g h_2$$

Section 8.1 tells us that the *total* work must equal the change in KE of Λ. What is this change? If the flow is steady—as we always assume it is—the speed of the fluid at a given point will not change. This means that all fluid particles will have the same speed when they reach a given point. It follows that the KE of the portion of Λ between the dashed light cross section on the left and the solid dark cross section on the right will not change as Λ moves from its initial position to the final position. Therefore, the change in KE is the difference between the KE of the left displaced volume V_1 and the right volume V_2 (which are of course equal). Since the volume—and therefore the mass—is the same at both points, we get

$$W_{tot} = \Delta(KE) \quad \Rightarrow \quad (P_1 - P_2)V + \rho V g h_1 - \rho V g h_2 = \tfrac{1}{2} m v_2^2 - \tfrac{1}{2} m v_2^2 = \tfrac{1}{2} m \left(v_2^2 - v_1^2\right)$$

Dividing by V and noting that density ρ is m/V, we obtain

$$P_1 - P_2 + \rho g h_1 - \rho g h_2 = \tfrac{1}{2}\frac{m}{V}\left(v_2^2 - v_1^2\right) = \tfrac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

which can be rewritten as

Bernoulli's equation

$$P_1 + \rho g h_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \tfrac{1}{2}\rho v_2^2 \quad \text{or} \quad P + \rho g h + \tfrac{1}{2}\rho v^2 = \text{constant} \tag{E.2}$$

The second equation follows from the first, which states that the quantity $P + \rho g h + \frac{1}{2}\rho v^2$ at point 1 is equal to the same quantity at point 2. But these two points are completely arbitrary; so the quantity must remain constant throughout the motion of the fluid. In most cases of interest, the fluid moves horizontally, so that $h_1 = h_2$. In that case, we get

$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2 \quad \text{or} \quad P + \tfrac{1}{2}\rho v^2 = \text{constant} \tag{E.3}$$

It is interesting to note that when the fluid is not moving, we obtain $P_1 + \rho g h_1 = P_2 + \rho g h_2$; and if we take point 1 to be at the surface of the fluid (where pressure is now denoted as P_0), we regain the result of Math Note E.8.4, where $h = h_1 - h_2$, and P_2 is simply P.

Equation (E.2) is called **Bernoulli's equation** or **Bernoulli's principle** after the Swiss mathematical physicist who discovered it first in 1738. It is a very useful relation with many applications as illustrated in the text. In words, it says that the sum of pressure, $\rho g h$, and $\frac{1}{2}\rho v^2$ is the same for all points of a moving fluid.

[3]This can be seen by noting that the indicated portion of the fluid pushes the rest of the fluid to the right. By the third law of motion, the reaction must be to the left.

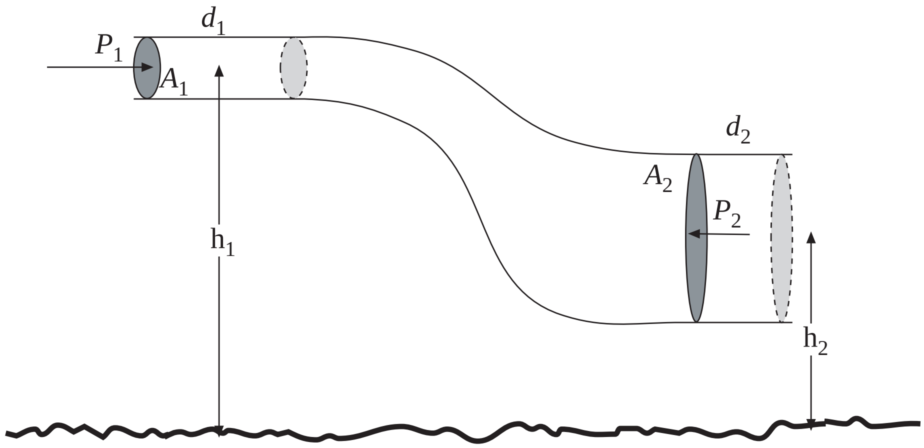


Figure E.7: Analysis of the flow of a fluid and Bernoulli's equation.

E.9 Math Notes for Chapter 9

Deriving Kepler's 3rd Law
(page 136 of the book)

Math Note E.9.1. We want to find a relation between the period of a circulating object and its distance from the gravitating center. We note that there is a relation between speed and period:

$$v = \frac{\text{circumference}}{\text{period}} = \frac{2\pi r}{T}$$

We now substitute this in Equation (9.3) and obtain

$$v = \sqrt{\frac{GM}{r}} \quad \Rightarrow \quad \frac{2\pi r}{T} = \sqrt{\frac{GM}{r}} \quad \Rightarrow \quad \frac{GM}{r} = \left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 r^2}{T^2}$$

which can be rewritten as

$$T^2 = \frac{4\pi^2}{GM} r^3 \tag{E.4}$$

In the case of the solar system, this is the third of the famous laws discovered by Kepler prior to Newton's discovery of the law of gravitation (see Box 3.3.1).

Showing absence of gravity in a spherical hole
(page 138 of the book)

Math Note E.9.2. Consider a *very thin* spherical shell within which is a mass m at some *arbitrary point* P [Figure E.8(a)]. Divide the surface of the shell into "squares"[4] of infinitesimally small sides, the cross section of one of which, labeled a, is shown in the figure. This square attracts m with a force Gmm_a/r_a^2, where m_a and r_a are, respectively, the mass of a and its distance from P. Extend the lines joining the vertices of the square and P until they meet the spherical shell at four new points forming another (larger) square labeled b in the figure. This square attracts m with a force Gmm_b/r_b^2, with m_b and r_b being the mass of b and its distance from P, respectively. We assume that the mass of the thin shell is distributed uniformly, so that m_a and m_b are proportional to the areas of a and b.

Now note that the sides of the square (the base of the pyramid whose vertex is P) changes proportionately to the distance of the square from P. For example, if the distance from b to P is three times the distance from a to P, then the sides of square b are three times longer than those of a. Therefore, the area of b is nine times the area of a; and m_b is also nine times m_a. In other words, the mass of the base of the pyramid whose vertex is P increases as the *square* of the distance from P to the base. This implies that $m_a/r_a^2 = m_b/r_b^2$; and

[4]The edges of these squares, being drawn on a sphere, are not perfectly straight lines; but if we make sure that squares are infinitesimally small, the sides can be made as straight at one pleases.

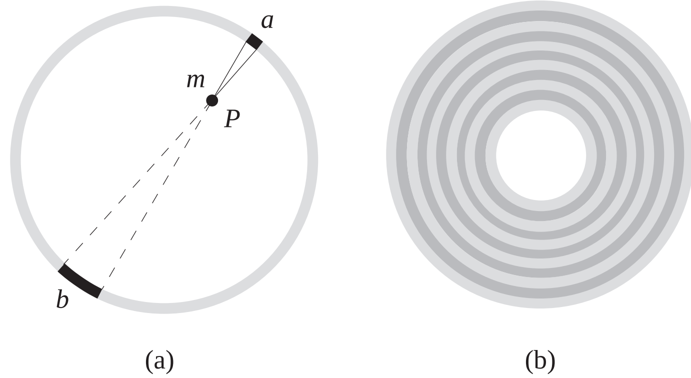


Figure E.8: (a) A thin spherical shell and a mass m located inside it at point P. (b) A thick shell is a collection of many thin shells.

since the force of gravity on m is Gm times this ratio, the force exerted by b on m is equal to the force exerted by a on m, and the two forces cancel each other. For each square (such as a) on one side of P, there is a square (such as b) on the other side canceling the force of the first square. Furthermore, the force of the shell on m is just the sum of the forces of all squares. Therefore, the total force of the shell on m is zero.

What about a *thick* shell with a spherical hole in the middle? Any thick shell can be thought of as a collection of many thin shells as illustrated in Figure E.8(b). The force of gravity on a mass m inside the cavity is the sum total of the forces due to all the thin shells. But the argument above showed that the forces coming from each thin shell is zero. We conclude that *the gravitational force inside a spherical cavity dug within a spherical distribution of mass is zero.*

Deriving formula for gravitational potential **(page 142 of the book)**

Math Note E.9.3. The gravitational potential *difference* for two points P_1 and P_2 is defined to be the difference between the gravitational potential energies of an object at those two points divided by the mass of the object. If the distance Δh between P_1 and P_2 is much smaller than the size of the gravitating body (the planet, the star, etc.), then the difference in the potential energy of the object is $mg\Delta h$, with g the gravitational acceleration (field) of the gravitating body at P_1 (or P_2, since the field is almost the same at the two points).[5] We then have

$$\Phi_2 - \Phi_1 = \frac{mg\Delta h}{m} = g\Delta h = gh_2 - gh_1$$

This equation tells us that $\Phi_2 = gh_2$ and $\Phi_1 = gh_1$, or in general, $\Phi = gh$. So, Φ is the PE (equal to mgh) divided by mass.

For a point that is far away at a distance r from the center of the celestial body, $PE = -GMm/r$, and dividing by the mass, $\Phi = -GM/r$. We summarize the discussion as follows:

$$\Phi = gh \qquad \text{if the point is close to celestial body and at height } h$$
$$\Phi = -\frac{GM}{r} \qquad \text{if the point is far and at distance } r \text{ from center}$$

It is clear that larger h or r corresponds to larger potential. Furthermore, because the direction of the gravitational field is toward the center [Figure 9.5(a)], or perpendicular to

[5] For instance, if P_1 (and, therefore, P_2) is close to the surface of the Earth, then g is simply the gravitational acceleration of the Earth, 9.8 m/s^2.

the surface [Figure 9.5(b)], an object falls naturally from a higher potential region to a lower one. Moreover, from the definition of the gravitational potential, $m(\Phi_2 - \Phi_1)$ is the energy *stored* in an object as it is displaced from P_1 to P_2. It is this stored energy that transforms into kinetic energy as m falls in a gravitational field.

Equivalence of orbiting and free fall **(page 145 of the book)**

Math Note E.9.4. In Figure 9.6, take the distance from satellite to the center of the Earth to be r. If the satellite were moving on a straight line, t seconds later it would be at a distance $r+h$ from the center of the Earth. Then, applying the Pythagorean theorem to the triangle of Figure 9.6, we obtain

$$(r+h)^2 = (vt)^2 + r^2 \quad \text{or} \quad r^2 + 2rh + h^2 = v^2t^2 + r^2 \quad \text{or} \quad 2rh + h^2 = \frac{GM}{r}t^2$$

where in the first step, we expanded the parentheses, and in the second step, we eliminated the r^2 term on both sides of the equation, and used Equation (9.3). Now on the left-hand side, $2rh$ is much larger than h^2, because r is much larger than h (r is about 6.4 million meters while h turns out to be only a few meters). So, we can ignore h^2 on the left-hand side and write

$$2rh = \frac{GM}{r}t^2 \quad \text{or} \quad 2h = \frac{GM}{r^2}t^2 \quad \text{or} \quad h = \tfrac{1}{2}\frac{GM}{r^2}t^2 \quad \text{or} \quad h = \tfrac{1}{2}gt^2$$

where g is the acceleration of the satellite [see Equation (9.2)]. In fact, if $r = R_\oplus$, then $g = 9.8$ m/s^2. In particular, after one second, $h = 4.9$ meters.

E.10 Math Notes for Chapter 10

Showing the determinism of Newtonian physics **(page 154 of the book)**

Math Note E.10.1. Assume that at time t_1, a particle is moving at a point Q_1 with velocity $\mathbf{v}_1$. We can determine its location and its velocity a short time later by applying Newton's second law assuming that the force $\mathbf{F}$ is known. Let us call this later time t_2, the position Q_2 and the velocity $\mathbf{v}_2$. Then from the second law in the form $\mathbf{a} = \mathbf{F}/m$,

$$\mathbf{F}(t_1) = m\mathbf{a}(t_1) = m\frac{\Delta\mathbf{v}}{\Delta t} = m\frac{\mathbf{v}_2 - \mathbf{v}_1}{\Delta t}, \quad \text{with } \Delta t = t_2 - t_1 \text{ very small,}$$

we obtain

$$\mathbf{v}_2 - \mathbf{v}_1 \approx \frac{\mathbf{F}(t_1)}{m}\Delta t \quad \Rightarrow \quad \mathbf{v}(t_2) \approx \mathbf{v}(t_1) + \mathbf{a}(t_1)\Delta t \tag{E.5}$$

In Equation (E.5), $\mathbf{F}(t_1)$ and $\mathbf{a}(t_1)$ give the values of the force and acceleration at time t_1, and by definition, $\mathbf{v}_1 = \mathbf{v}(t_1)$ and $\mathbf{v}_2 = \mathbf{v}(t_2)$. Similarly, from the definition of velocity, we obtain

$$\mathbf{v}(t_1) = \frac{\mathbf{r}_2 - \mathbf{r}_1}{\Delta t}, \quad \text{with } \Delta t \text{ very small.}$$

and

$$\mathbf{r}(t_2) \approx \mathbf{r}(t_1) + \mathbf{v}(t_1)\Delta t \tag{E.6}$$

The approximations (E.5) and (E.6) assume that $\mathbf{a}$ and $\mathbf{v}$ do not change between t_1 and t_2; i.e., that they are constant in that time interval. This is fine as long as t_2 is only infinitesimally larger than t_1, and $\mathbf{a}$ and $\mathbf{v}$ do not change abruptly. This last property is what we call the *continuity* of $\mathbf{a}$ and $\mathbf{v}$.

The objection to the assumption that $\mathbf{a}$ and $\mathbf{v}$ are constant during $t_2 - t_1$ becomes a practical problem. *Conceptually*, we can let t_2 to be as close to t_1 as we please. In higher mathematics, this is done regularly and a multitude of enriching conclusions ensue as a result. A good example is the approximation of the area of a circle by that of an inscribed regular polygon. As the number of sides of the polygon increases, its area approximates the circular area better and better. This was actually known to Greeks long before the

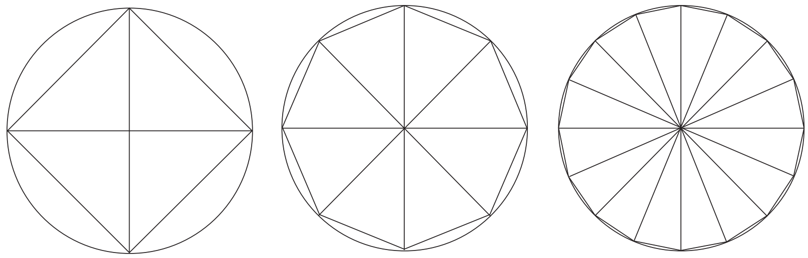


Figure E.9: The area of a circle is the limiting area of the sum of all the triangles as their number increases (and their area decreases, of course) indefinitely.

invention of calculus in the 17th century. By increasing the number of sides of the polygon beyond limits, they came up with the precise formula, area $= \pi r^2$, for a circle of radius r (see Figure E.9). In such a process the Greeks simultaneously dealt with infinitely large and infinitely small: The number of triangles into which the polygon is divided grows infinitely large while the area of each triangle diminishes to an infinitely small size. In some sense, the two infinities compensate each other to give a finite result, the area of the circle.

The same principle can be applied to the motion of a particle under the influence of a given force. Let us assume that we want to predict the position and the velocity of a particle an hour from now. The position and the velocity of the particle must be known *initially*, of course. This knowledge of the initial properties of the particle is called the **initial conditions**. We divide the hour into say 3600 seconds. We can predict the velocity a second later by using Equation (E.5) with $\Delta t = 1$ second. We can also predict the position a second later from Equation (E.6). Now that we have $\mathbf{r}(t_2)$ and $\mathbf{v}(t_2)$, we can predict $\mathbf{r}(t_3)$ and $\mathbf{v}(t_3)$, the position and velocity a second later than t_2 by the same procedure. From $\mathbf{r}(t_3)$ and $\mathbf{v}(t_3)$ we can calculate $\mathbf{r}(t_4)$ and $\mathbf{v}(t_4)$, and so forth, until we reach the end of one hour. Of course this would be only an approximation to the actual velocity and position at the end of the hour, just as the area of an inscribed polygon with 3600 sides is only an approximation to the actual area of the circle. However, just as the genius of the Greek solved the problem of infinity for polygons more than 2000 years ago, the genius of the 17th and 18th century mathematicians solved the problem of large scale motion as the limit of infinitely many infinitesimal motions. The result was the birth of a whole new branch of mathematics called **differential equations**.

Math Note E.10.2. To grasp the gist of the preceding Math Note, let us apply its procedure to the motion of a planet around the Sun. The force exerted on the planet by the Sun is $F = GMm/r^2$ where M is the mass of the Sun, m that of the planet, and r their separation. This force is always directed toward the Sun. The acceleration of the planet is therefore $a = F/m$ or $a = GM/r^2$, also directed toward the Sun.

How to find orbit of Earth around Sun **(page 154 of the book)**

The line that connects the *initial* location of the planet to the Sun and the line along which the planet moves *initially* form a plane. Call it the xy-plane, and draw the x and the y axes in such a way that the Sun coincides with the origin, and the line joining the Sun and the planet forms the x-axis. Since the planet has no initial speed perpendicular to this plane by construction, and the Sun exerts no force perpendicular to the plane, the planet will be confined to this plane for all time.

Figure E.10 shows a typical location of the planet on its path. It is convenient to resolve the acceleration vector into its components along the two axes, and consider the motion along these axes separately as in Equation (D.2). This means that the x coordinate, the velocity, and acceleration in the x direction are to be considered separately from the corresponding y quantities. In other words, Equations E.5 and E.6 are to be separated into

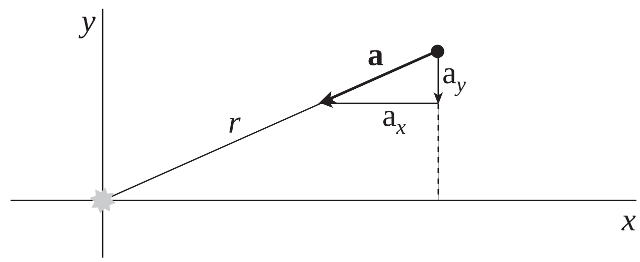


Figure E.10: A typical location of the planet on its path. Note that the acceleration is always pointing towards the Sun.

the following four equations:

$$\begin{aligned} v_x(t_2) &\approx v_x(t_1) + a_x(t_1)\Delta t, \quad x(t_2) \approx x(t_1) + v_x(t_1)\Delta t \\ v_y(t_2) &\approx v_y(t_1) + a_y(t_1)\Delta t, \quad y(t_2) \approx y(t_1) + v_y(t_1)\Delta t \end{aligned} \tag{E.7}$$

We now have to find $a_x(t)$ and $a_y(t)$.

From the similarity of the two right triangles in Figure E.10, we have

$$\frac{a_x(t)}{a(t)} = -\frac{x(t)}{r(t)} \quad \text{or} \quad a_x(t) = -\frac{x(t)}{r(t)}a(t) = -\frac{x(t)}{r(t)}\frac{GM}{[r(t)]^2} = -\frac{GMx(t)}{[r(t)]^3}$$

where we used $a = GM/r^2$, and the negative sign is introduced because a_x points in the negative x direction. The expression for $a_y(t)$ is obtained similarly:

$$a_y(t) = -\frac{GMy(t)}{[r(t)]^3}$$

If we substitute $GM = (6.67 \times 10^{-11}) \times (2 \times 10^{30}) = 1.33 \times 10^{20}$, the above expressions for the components of the acceleration, and $r(t) = \sqrt{[x(t)]^2 + [y(t)]^2}$ in Equation (E.7), we obtain

$$\begin{aligned} v_x(t_2) &\approx v_x(t_1) - \frac{1.33 \times 10^{20} x(t_1)}{\{[x(t_1)]^2 + [y(t_1)]^2\}^{3/2}}\Delta t, \quad x(t_2) \approx x(t_1) + v_x(t_1)\Delta t \\ v_y(t_2) &\approx v_y(t_1) - \frac{1.33 \times 10^{20} y(t_1)}{\{[x(t_1)]^2 + [y(t_1)]^2\}^{3/2}}\Delta t, \quad y(t_2) \approx y(t_1) + v_y(t_1)\Delta t \end{aligned} \tag{E.8}$$

Now apply the equations above to the motion of Earth. The initial location of Earth is described by $x(0) = 1.5 \times 10^{11}$ m, $y(0) = 0$, i.e., we place the Earth on the x-axis initially (or at $t = 0$). We also assume that Earth moves with a speed of 30,000 m/s *perpendicular* to the x-axis initially. Therefore, $v_x(0) = 0$, and $v_y(0) = 30,000$ m/s. In order to obtain accurate results, we have to make Δt as small as possible. For the motion of Earth, one minute (60 seconds) is a good value for Δt. Thus, Equation (E.8) becomes

$$\begin{aligned} v_x(t_2) &\approx v_x(t_1) - \frac{8 \times 10^{21} x(t_1)}{\{[x(t_1)]^2 + [y(t_1)]^2\}^{3/2}}, \quad x(t_2) \approx x(t_1) + 60 v_x(t_1) \\ v_y(t_2) &\approx v_y(t_1) - \frac{8 \times 10^{21} y(t_1)}{\{[x(t_1)]^2 + [y(t_1)]^2\}^{3/2}}, \quad y(t_2) \approx y(t_1) + 60 v_y(t_1) \end{aligned} \tag{E.9}$$

With Equation (E.9) as our master equation, we can now calculate the location of Earth at different times. We set $t_1 = 0$, $t_2 = 60$, $x(0) = 1.5 \times 10^{11}$ m, $y(0) = 0$, $v_x(0) = 0$, and

$v_y(0) = 30,000$ m/s in the master equation and obtain

$$\begin{aligned}
v_x(60) &\approx v_x(0) - \frac{8 \times 10^{21} \times 1.5 \times 10^{11}}{\{[1.5 \times 10^{11}]^2 + [0]^2\}^{3/2}} = 0 - 0.356 = -0.356 \text{ m/s}, \\
x(60) &\approx x(0) + 60v_x(0) = 1.5 \times 10^{11} + 0 = 1.5 \times 10^{11} \text{ m} \\
v_y(60) &\approx v_y(0) - \frac{8 \times 10^{21} \times 0}{\{[1.5 \times 10^{11}]^2 + [0]^2\}^{3/2}} = 30,000 \text{ m/s}, \\
y(60) &\approx y(0) + 60v_y(0) = 0 + 60 \times 30000 = 1800000 = 1.8 \times 10^6 \text{ m}
\end{aligned}$$

We have thus determined the location and the velocity of Earth one minute after the start of motion. From our new knowledge, we can locate Earth *two* minutes after the start of motion. We use Equation (E.9) with $t_1 = 60$, $t_2 = 120$, $x(60) = 1.5 \times 10^{11}$ m, $y(60) = 1.8 \times 10^6$ m, $v_x(60) = -0.356$, and $v_y(60) = 30,000$ m/s to find

$$\begin{aligned}
v_x(120) &\approx v_x(60) - \frac{8 \times 10^{21} \times 1.5 \times 10^{11}}{\{[1.5 \times 10^{11}]^2 + [1.8 \times 10^6]^2\}^{3/2}} \\
&= -0.356 - 0.356 = -0.711 \text{ m/s}, \\
x(120) &\approx x(60) + 60v_x(60) = 1.5 \times 10^{11} + 60 \times (-0.356) \\
&= 1.5 \times 10^{11} - 21.36 = 1.5 \times 10^{11} \text{ m} \\
v_y(120) &\approx v_y(0) - \frac{8 \times 10^{21} \times (1.8 \times 10^6)}{\{[1.5 \times 10^{11}]^2 + [1.8 \times 10^6]^2\}^{3/2}} \\
&= 30,000 - 4.3 \times 10^{-6} = 30,000 \text{ m/s}, \\
y(120) &\approx y(60) + 60v_y(60) = 1.8 \times 10^6 + 60 \times 30000 = 3.6 \times 10^6 \text{ m}
\end{aligned}$$

This process can obviously be continued: from the information we have obtained for $t = 2$ min, we determine the same information for $t = 3$ min, and from the information we obtain for $t = 3$ min, we determine the corresponding information for $t = 4$ min, and so on all the way to the end of Earth's journey after a year. We don't want to do this by hand, of course. It would take us weeks to calculate all the numbers! A calculator can help, but it is still too slow. However, in only a few minutes on a computer, some simple programming can calculate the location of Earth minute by minute for a whole year. Table E.1 shows the result of such a calculation, but instead of showing all the data (and making a long table with over 500,000 rows), we have shown the data in intervals of 5 days. In that table the x and y coordinates are measured in fractions of 1.5×10^{11} m.

It is instructive to analyze this table in some detail. First, using $r = \sqrt{x^2 + y^2}$, one can easily note that the Earth's distance from the Sun is not constant, an indication of the Keplerian fact that the path of the Earth is an ellipse. Second, note that sometime between the last two entries, where $x = 1$, the y-coordinate changes sign. This means that the Earth crosses the x-axis at the initial location of Earth. It follows that the "year" for such an Earth is between 373 and 375 days. The real Earth, of course, has a year that is approximately 365.25 days long. Why the disagreement? Because, the initial speed of 30,000 m/s is too large. The real Earth has a somewhat smaller speed. But how can a smaller initial speed, the reader may ask, lead to a *shorter* year? Should the Earth not cover its orbit in a *longer* time when it goes slower? It turns out that the slower the initial speed, the shorter the orbit; so much shorter that, in fact, it will take Earth less time to cover it. As an example, let us keep everything the same as before, but initially push the Earth with a speed of 20,000 m/s instead of 30,000 m/s. Then, following exactly the same procedure as before, we obtain Table E.2. By using $r = \sqrt{x^2 + y^2}$, one can easily note that the Earth's distance from the Sun varies considerably, indicating that the path of this Earth is definitely an ellipse. Figure E.11 shows the orbits of the two Earths discussed above.

t	x	y	t	x	y	t	x	y
0	1	0	130	−0.606	0.826	260	−0.380	−0.947
5	0.996	0.086	135	−0.672	0.774	265	−0.3	−0.975
10	0.985	0.172	140	−0.734	0.717	270	−0.217	−0.995
15	0.967	0.256	145	−0.791	0.655	275	−0.134	−1.008
20	0.941	0.339	150	−0.843	0.588	280	−0.049	−1.015
25	0.909	0.419	155	−0.889	0.518	285	0.036	−1.014
30	0.871	0.496	160	−0.928	0.444	290	0.121	−1.006
35	0.825	0.569	165	−0.962	0.367	295	0.205	−0.991
40	0.774	0.638	170	−0.989	0.288	300	0.287	−0.969
45	0.717	0.703	175	−1.009	0.206	305	0.368	−0.940
50	0.655	0.762	180	−1.023	0.124	310	0.446	−0.905
55	0.588	0.816	185	−1.03	0.04	315	0.520	−0.863
60	0.517	0.864	190	−1.03	0.044	320	0.591	−0.814
65	0.442	0.906	195	−1.023	−0.127	325	0.658	−0.760
70	0.364	0.942	200	−1.009	−0.210	330	0.72	−0.700
75	0.284	0.970	205	−0.988	−0.291	335	0.777	−0.636
80	0.201	0.992	210	−0.961	−0.371	340	0.828	−0.566
85	0.117	1.007	215	−0.927	−0.447	345	0.873	−0.493
90	0.032	1.014	220	−0.887	−0.521	350	0.911	−0.416
95	−0.053	1.014	225	−0.840	−0.591	355	0.943	−0.336
100	−0.138	1.008	230	−0.789	−0.658	360	0.968	−0.253
105	−0.221	0.994	235	−0.732	−0.719	365	0.986	−0.168
110	−0.304	0.973	240	−0.670	−0.776	370	0.997	−0.083
115	−0.384	0.946	245	−0.603	−0.828	371	0.998	−0.066
120	−0.461	0.912	250	−0.532	−0.874	373	1.00	−0.014
125	−0.535	0.872	255	−0.458	−0.914	375	1.00	0.004

Table E.1: The x and y coordinates—measured in fractions of 1.5×10^{11} m—of Earth around the Sun every five days (except for the last four entries). The year for the Earth of this table is a little longer, because its initial speed is a little larger than the real Earth.

t	x	y	t	x	y	t	x	y
0	1	0	70	0.222	0.528	140	0.616	−0.494
5	0.996	0.058	75	0.100	0.496	145	0.692	−0.461
10	0.985	0.114	80	−0.026	0.436	150	0.758	−0.422
15	0.967	0.171	85	−0.150	0.337	155	0.816	−0.378
20	0.941	0.226	90	−0.250	0.190	160	0.865	−0.331
25	0.907	0.279	95	−0.291	0.001	165	0.907	−0.280
30	0.866	0.330	100	−0.251	−0.188	170	0.941	−0.227
35	0.816	0.377	105	−1.151	−0.336	175	0.967	−0.173
40	0.759	0.421	110	−0.280	−0.435	180	0.986	−0.116
45	0.692	0.460	115	0.099	−0.496	184	0.995	−0.071
50	0.617	0.493	120	0.221	−0.528	186	0.998	−0.048
55	0.533	0.519	125	0.334	−0.540	188	1.000	−0.025
60	0.439	0.535	130	0.438	−0.536	190	1.001	−0.002
65	0.335	0.539	135	0.532	−0.519	191	1.001	0.010

Table E.2: The x and y coordinates—measured in fractions of 1.5×10^{11} m—of an Earth that is given an initial speed of 20,000 m/s. The year for such an Earth is about 190 days.

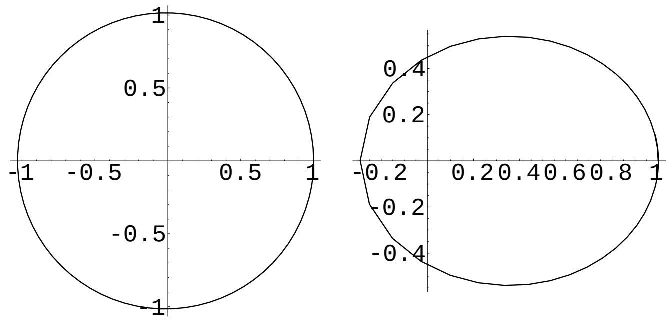


Figure E.11: The graph on the left shows Earth's orbit when its initial speed is 30,000 m/s. The graph on the right is Earth's orbit if its initial speed were changed to 20,000 m/s.

Earth is only one of the planets circling the Sun. Thus, the analysis above shows that the Newtonian law of gravity implies Kepler's first law of planetary motion, i.e., the fact that planetary orbits are elliptical. It turns out that the same gravitational law also implies the other two Keplerian laws.

E.11 Math Notes for Chapter 11

Analysis of interference **(page 166 of the book)**

Math Note E.11.1. To further understand interference patterns, consider two coherent sources S_1 and S_2 in Figure E.12 producing waves that are *in phase*, meaning that the crests or troughs of the waves occur at exactly the same time at the two sources. For concreteness, let us assume that S_1 and S_2 are light sources, although the discussion and its conclusions apply to all waves. These waves fall on a white screen, on which we can observe the interference pattern. Point C_0, which lies on the perpendicular bisector of $\overline{S_1S_2}$, is equidistant from the two sources. The crests (or the troughs) produced at the two sources *always* reach C_0 at exactly the same time and are added there. Therefore, C_0 is a point, at which the wave *always* oscillates with twice the amplitude of either waves. It follows that C_0 is a location of a constructive interference; we see a bright spot there.

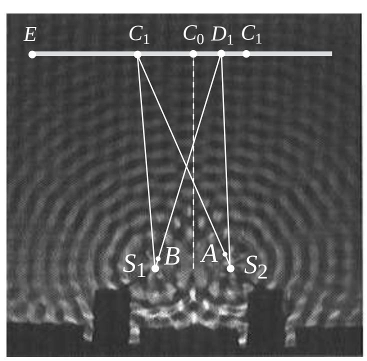


Figure E.12: Two coherent sources of wave produce interference, a pattern of high and low intensity that remain unchanged.

Now move away from C_0 on the screen until you reach the point C_1 where the difference between $\overline{S_1C_1}$ and $\overline{S_2C_1}$ is one wavelength. What do you get at C_1? Suppose that the two sources produce crests at a particular time. The crest from S_2 reaches A after one period

($\overline{C_1A}$ is drawn equal to $\overline{S_1C_1}$, so that $\overline{S_2A}$ is equal to one wavelength). At this time the sources produce the next crest. This new crest from S_1 and the previous crest from S_2 (which is now at A) travel the same distance and reach C_1 at the same time and get added there. Similarly, the trough of S_2 arrives at C_1 at exactly the same time that the *next* trough of S_1 arrives there. Therefore, C_1 is also a point, at which the wave *always* oscillates with twice the amplitude of either waves. It is a location of a constructive interference; we see a bright spot there also.

You can convince yourself that the location of the next bright spot on the screen is a point C_2, where the difference between $\overline{S_1C_2}$ and $\overline{S_2C_2}$ is *two* wavelengths. In general, the bright spots on the screen occur at points P where the difference between $\overline{S_1P}$ and $\overline{S_2P}$ is an *integer multiple* of the wavelength.

Somewhere between C_0 and C_1 on the screen there is a dark spot, because as you move away from C_0 you reach a point (call it D_1) for which the difference between $\overline{S_1D_1}$ and $\overline{S_2D_1}$ is *half* a wavelength. Once again suppose that the two sources produce crests at a particular time. The crest from S_1 reaches B after *half* a period ($\overline{D_1B}$ is drawn equal to $\overline{S_2D_1}$, so that $\overline{S_1B}$ is equal to half a wavelength). At this time the sources produce a trough. The trough from S_2 and the previous crest from S_1 (which is now at B) travel the same distance and reach D_1 at the same time and get *algebraically* added there, i.e., they cancel each other. Similarly, the crest of S_1 arrives at D_1 at exactly the same time that the *next* trough of S_2 arrives there. Therefore, D_1 is a point, at which the wave *never* oscillates. It is a location of a destructive interference, a dark spot.

The location of the next dark spot on the screen is a point D_2, where the difference between $\overline{S_1D_2}$ and $\overline{S_2D_2}$ is *three halves* wavelengths. In general, the dark spots on the screen occur at points Q where the difference between $\overline{S_1Q}$ and $\overline{S_2Q}$ is an *odd multiple* of half a wavelength.

I have talked about bright and dark "spots," while in reality I should talk about lines (if the two sources are long and narrow) or circles (if the two sources are round), because the screen is not a line as shown in Figure E.12, but a plane, of which the line in the figure is a cross section. Similarly, the "points" on the screen are cross sections of lines or circles. The succession of bright and dark lines or circles is called **interference fringes**. One characteristic of the fringes that is frequently used is how many of them one can find on a screen of a given length. **Example D.11.1** on **page 32** of *Appendix.pdf* shows how to find this number.

Derivation of the Doppler-shift when source moves **(page 169 of the book)**

Math Note E.11.2. Denote the speed of the source as v and that of the wave as c. Suppose that at a certain time the source sends its first pulse which spreads out into a sphere. The second pulse is sent T seconds later, where T is the period of oscillation of the wave. But during this time, the source has traveled a distance of vT. Thus, the second pulse will have partially caught up with the first, reducing its distance from the first in the forward direction. Similarly, the third pulse will reduce its distance from the second by exactly the same amount because of the motion of the source. The spherical pulses are squeezed together in the forward direction as shown in Figure 11.10(b).

The amount by which the spheres are squeezed is precisely the distance the source travels in T. Thus the *detected* wavelength, denoted by λ_{det}, is $\lambda_{\text{det}} = \lambda - vT$, or $\lambda_{\text{det}} = \lambda - v(\lambda/c)$, because $\lambda = cT$. This could also be written as $\lambda_{\text{det}} = \lambda - \lambda(v/c)$, or $\lambda_{\text{det}} = \lambda(1 - v/c)$. The *shift* $\Delta\lambda$ in the wavelength, defined as $\lambda_{\text{det}} - \lambda$ is simply $-vT$ (from the first equation in this paragraph). It is customary to define the **Doppler shift** as the *fractional change* in the wavelength: $\Delta\lambda/\lambda$, which is equal to $-vT/\lambda$, or $-v/c$ (because $\lambda/T = c$). If the source is receding from the detector, then $\lambda_{\text{det}} = \lambda + vT$, and the formula above will have a plus sign instead of minus: $\Delta\lambda/\lambda = +v/c$. Instead of carrying the $+$ and the $-$ in the formula, we agree to call v negative if the distance between the source and the detector *decreases*, i.e., if the source is approaching the detector. Similarly, $v > 0$ if the distance between the source and the detector *increases*, i.e., if the source is receding from the detector. We therefore have the simple but important Doppler shift formula

$$\lambda_{\text{det}} = \lambda\left(1 + \frac{v}{c}\right), \quad \frac{\Delta\lambda}{\lambda} = \frac{v}{c}, \qquad v > 0 \text{ for receding, } v < 0 \text{ for approaching} \tag{E.10}$$

If we are interested in the detected *frequency*, we use $c = \lambda f$, with λ being the *detected* wavelength.

Derivation of the Doppler-shift when detector moves **(page 169 of the book)**

Math Note E.11.3. Assume that the detector is moving with speed v relative to the medium while the source is stationary. It is easiest to calculate the detected *period* of the wave. First assume that the detector is approaching the source. Since the wave is moving with speed c relative to the medium, and the detector is moving *towards* the source, the wave fronts are approaching the detector with a speed of $c + v$, and the distance λ between two fronts is covered in

$$T_{\text{det}} = \frac{\lambda}{c+v} \quad \text{or} \quad cT_{\text{det}} = \frac{c\lambda}{c+v} = \frac{c\lambda}{c(1+v/c)} = \frac{\lambda}{1+v/c}$$

where in the first step we multiplied both sides of the equation by c, in the second step we factored out a c in the denominator, and in the last step we canceled the common c's in the numerator and denominator. If the detector is moving away from the source, the wave fronts will be approaching it with a speed of $c - v$.[6] Thus, a minus sign will appear in the above formulas. Now we note that cT_{det} is nothing but λ_{det}. As in Math Note E.11.2, we let v carry the algebraic sign and obtain the relation

$$\lambda_{\text{det}} = \frac{\lambda}{1 - v/c}, \qquad v > 0 \text{ for receding, } v < 0 \text{ for approaching} \tag{E.11}$$

If v is much much smaller than c, then to a very good approximation

$$\frac{1}{1+v/c} = 1 - \frac{v}{c} \quad \text{and} \quad \frac{1}{1-v/c} = 1 + \frac{v}{c},$$

i.e., the fraction is equal to just the denominator *with the opposite sign.* For example, if $v/c = 0.01$, then

$$\frac{1}{1-0.01} = 1.0101 \approx 1 + 0.01, \quad \text{and} \quad \frac{1}{1+0.01} = 0.99001 \approx 1 - 0.01$$

For $v/c = 0.001$, we get a better approximation; for $v/c = 0.0001$, the approximation gets even better, and so on. For such speeds, therefore, Equation (E.11) becomes identical to Equation (E.10), and it does not matter whether the source is moving or the detector.

Derivation of the Doppler-shift when both source and detector move **(page 170 of the book)**

Math Note E.11.4. If both the source S *and* the detector D are moving relative to the medium (Figure E.13), then the results of Math Notes E.11.2 and E.11.3 have to be combined. Suppose S is moving with a speed v_s and D with v_d. The detected wavelength is shortened by v_sT, and that shortened distance is covered with a speed $c - v_d$—because the detector is moving away from the wave, reducing the (relative) speed of the wave and the detector. Therefore, the detected period will be

$$T_{\text{det}} = \frac{\lambda - v_sT}{c - v_d} \quad \text{or} \quad cT_{\text{det}} = \frac{c\lambda - v_scT}{c - v_d} = \frac{c\lambda - v_s\lambda}{c - v_d}$$

where we multiplied both sides by c. The left-hand side of the last equation is λ_{det}, and if we divide the numerator and denominator of the right-hand side by c, we obtain

$$\lambda_{\text{det}} = \lambda\left(\frac{1 - v_s/c}{1 - v_d/c}\right)$$

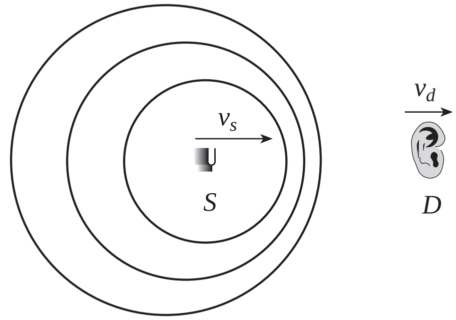


Figure E.13: Both the source and the detector are moving.

The important case when v_s and v_d are both much much smaller than c leads to

$$\lambda_{\text{det}} = \lambda\left(1 - \frac{v_s}{c}\right)\left(1 + \frac{v_d}{c}\right) = \lambda\left(1 - \frac{v_s}{c} + \frac{v_s}{c} - \frac{v_s v_d}{c^2}\right) = \lambda\left(1 + \frac{v_d - v_s}{c}\right)$$

where we ignored the last term because it was much smaller than the other terms. [For example if v_s/c and v_d/c are each 0.001, then $v_s v_d/c^2$—which is the same as $(v_s/c)(v_d/c)$—is 0.001^2 or 0.000001.] Now noting that $v_d - v_s$ is the speed of the source *relative to the detector*, we can write

$$\lambda_{\text{det}} = \lambda\left(1 + \frac{v_{\text{rel}}}{c}\right) \quad \text{or} \quad \frac{\Delta\lambda}{\lambda} = \frac{v_{\text{rel}}}{c} \tag{E.12}$$

where v_{rel} is positive if $v_d > v_s$, i.e., the detector runs away from the source, or *recedes* from it, and it is negative if $v_d < v_s$, i.e., if the source catches up with the detector, or *approaches* it.

In Equation (E.12) any reference to the medium has been eliminated! In relativity no medium is allowed for the propagation of electromagnetic waves. Therefore, only *relative* velocities should enter the formulas. It turns out that a full relativistic analysis of the Doppler effect will yield a result, whose limit—when the velocities of source and detector are much much smaller than the speed of light—is identical to Equation (E.12).

In some situations D is at S and it detects a wave *reflected* from a moving object A. It is desirable to find the wavelength of this reflected wave as measured by the detector at S. The wave reflected from A has the same wavelength as the incident wave as measured by A, namely that given by (E.12). When this wavelength—which we call λ_{ref}—is received at S it is shifted by the same factor. Therefore,

$$\lambda_{\text{ref}} = \lambda_{\text{det}}\left(1 + \frac{v_{\text{rel}}}{c}\right) = \lambda\left(1 + \frac{v_{\text{rel}}}{c}\right)^2 \approx \lambda\left(1 + 2\frac{v_{\text{rel}}}{c}\right)$$

assuming, as usual, that v_{rel}/c is very small compared to 1. This equation leads to the following fractional change in the wavelength:

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda_{\text{ref}} - \lambda}{\lambda} = 2\frac{v_{\text{rel}}}{c} \tag{E.13}$$

where, v_{rel} is negative if the source and the detector are approaching and positive if they are moving apart.

[6] To see this, simply note that if detector is moving with the same speed c as the waves, the wave will not catch up with it, because the wave fronts will be approaching the detector with a speed of $c - c = 0$.

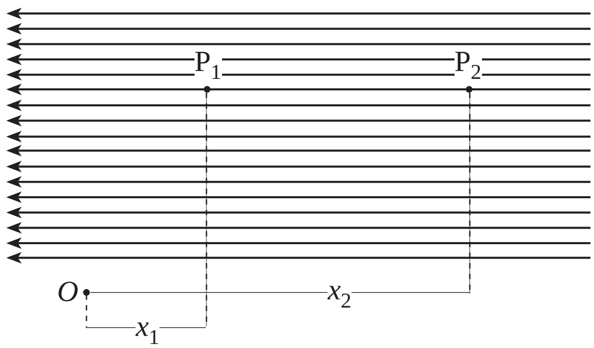


Figure E.14: Electric potential difference between two points is related to the work done by the electric field.

E.12 Math Notes for Chapter 12

Math Note E.12.1. To simplify the analysis suppose that the electric field is uniform (constant magnitude and direction) as shown in Figure E.14. Release a positive charge at P_2, and note that it will move towards P_1. The work done by the electric field in pushing the positive charge from P_2 to P_1 is

Connection between electric field and electric potential
(page 181 of the book)

$$W_e = F_e(x_2 - x_1) = qE(x_2 - x_1)$$

where x_1 and x_2 are the distances from an otherwise arbitrary origin O. This work is positive because the displacement and the force are in the same direction. But this work is precisely the difference in the EPE of the two points P_1 and P_2. So, we can write

$$EPE_2 - EPE_1 = qE(x_2 - x_1) \quad \text{or} \quad q(V_2 - V_1) = qE(x_2 - x_1)$$

using Equation (12.3). It now follows that

$$V_2 - V_1 = E(x_2 - x_1) = Ed \quad \text{where} \quad d = x_2 - x_1. \tag{E.14}$$

This is also a positive quantity, meaning that V_2 is larger than V_1, i.e., that P_2 is at a higher electric potential than P_1. We conclude that the electric field points from a higher potential to a lower potential. It is instructive to compare Equation (E.14) with the analogous expression for gravity: $\Phi_2 - \Phi_1 = g(h_2 - h_1)$ discussed in Math Note E.9.3.

Math Note E.12.2. If you divide the energy $q\mathcal{E}$ by time t, you obtain a useful expression for power:

Power delivered by a battery
(page 187 of the book)

$$P = \frac{q\mathcal{E}}{t} = \frac{q}{t}\mathcal{E} = i\mathcal{E} \tag{E.15}$$

Relation between power, current, and emf

Thus, the power supplied by a battery (its wattage) is the product of the current in the circuit and the voltage of the battery.

As a charge moves in a wire from a higher potential to a lower potential, it loses energy according to Equation (12.3). If you divide both sides of that equation by time t, the left-hand side becomes power loss P, while the q on the right-hand side gives the current as follows:

$$\underbrace{\frac{EPE_2 - EPE_1}{t}}_{=P} = \frac{q(V_2 - V_1)}{t} = \underbrace{\frac{q}{t}}_{=i}\underbrace{(V_2 - V_1)}_{=V} \quad \Rightarrow \quad P = iV \tag{E.16}$$

where V is defined to be the potential drop between the two points.

The power loss in a resistance can be written in terms of resistance and the current it carries. This is easily accomplished by combining Equations (E.16) and (12.5):

$$P = iV = i(Ri) = Ri^2 \tag{E.17}$$

E.13 Math Notes for Chapter 13

Deriving the ideal transformer formula (page 204 of the book)

Math Note E.13.1. Let us start with a transformer whose primary and secondary coils have only one turn each. Assuming that the transformer is **ideal**, i.e., no magnetic flux "escapes" the iron core, the emf in the two coils must be the same, because the two fluxes (as well as their rate of change) are identical.

Now let's add a third identical coil on the iron core. For exactly the same reason, we conclude that all three coils must have the same emf. In fact, however many coils we have on the iron core, the result is the same: *All identical single-turn coils have the same emf*, regardless of their location on the iron core. So, if we have 12 coils, and we move 7 of then to the left and 5 of them to the right, the conclusion is not altered.

Suppose there are N_1 coils on the left and N_2 coils on the right of the iron core, each coil having an emf of $\mathcal{E}$. Let us connect the left bunch together and do the same with the right bunch. Then we have a coil with N_1 turns and a total emf of $N_1\mathcal{E}$ on the left. Similarly, if we connect all the coils on the right, we have a coil with N_2 turns and a total emf of $N_2\mathcal{E}$. Then, we can write

$$\frac{\text{total emf on left}}{N_1} = \frac{N_1\mathcal{E}}{N_1} = \mathcal{E} = \frac{N_2\mathcal{E}}{N_2} = \frac{\text{total emf on right}}{N_2},$$

i.e., *emf per turn is equal on both sides.*

If you *start* with a coil of N_1 turns on the left having an emf $\mathcal{E}_1$, and a coil of N_2 turns on the right having an emf $\mathcal{E}_2$, then you can reason backwards: The coil on the left is equivalent to N_1 separate coils each having an emf of $\mathcal{E}_1/N_1$, and the one on the right is equivalent to N_2 separate coils each having an emf of $\mathcal{E}_2/N_2$; and these two quantities have to be equal:

$$\frac{\mathcal{E}_1}{N_1} = \frac{\mathcal{E}_2}{N_2} \quad \text{and} \quad i_1N_1 = i_2N_2 \tag{E.18}$$

What happens to the current in the coils? The answer lies in the conservation of energy.

Imagine bringing two single-turn coils together. Each coil carries a current—in general different from the other coil. These currents are in the same direction, and therefore exert an *attractive* force on one another [see Equation (D.11) and the comments after it]. This attraction lowers the energy of the system once the coils settle down in their final positions.[7] The lowering of energy translates into the reduction of currents. In fact, since the product of the current and the emf is the power, assuming conservation of energy, i.e., no heat loss (again an ideal transformer), we conclude that the power input of the primary should equal the power output of the secondary, or $i_1\mathcal{E}_1 = i_2\mathcal{E}_2$. This and the first equation in (E.18) lead to the equation for currents.

E.14 Math Notes for Chapter 14

Explaining Maxwell's first equation (page 212 of the book)

Math Note E.14.1. The left-hand side (LHS) of the equation in Figure 14.1(a) is called the *divergence* of the electric field. The symbol $\nabla \cdot \mathbf{E}$ has the following operational meaning. Pick a point in space; imagine a very small cube enclosing that point; calculate the total

[7] This lowering of energy is similar to the lowering of the energy of a ball—attracted to the Earth—when it hits the ground after falling from a height. A more detailed analysis (beyond the scope of this book) shows that, as a result of the motion toward one another, a magnetic force is exerted on the charges in each coil causing them to slow down.

outward electric flux through the six sides of the cube:[8] If the electric field lines point out of the cube for a side, give a positive sign to the flux through that side, and if they point inward, give it a negative sign. Add all the six fluxes to get the LHS.[9]

Suppose you did all of the above and you got a positive number. That means that the right-hand side (RHS) is also positive. But since $4\pi k_e$ is positive, this means that ρ, *which happens to be the electric charge (density) inside the cube*, is positive. Our discussion of electric fields and their connection with the signs of the charges (see Section 12.2) told us that a positive charge has *outgoing* electric field lines. This fits perfectly with the present discussion: If you have positive charges in the cube, their field lines *diverge outward* (thus the name divergence) and the outward flux is positive.

It is obvious that if the total flux is negative, the field lines must be *converging inward*, and the charge inside must be negative based on our discussion of the electric fields of negative charges.

If the LHS of the Maxwell's first equation turns out to be zero, then either you are in a region where there is no electric field ($\mathbf{E} = 0$), or there are as many field lines entering the volume as leaving it, in which case either the charges are somewhere else, or there are as many positive charges inside as there are negative. In all cases, the net charge inside will be zero.

So, the first Maxwell's equation connects the outward flux of the electric field through a volume to the electric charge enclosed in that volume.

Math Note E.14.2. The LHS of the equation in Figure 14.1(c) is called the *curl or circulation* of the electric field. How do you find the curl of a field? Take a point P in space and draw a little square around it with P at the center. Choose a direction to go around the square, say counterclockwise. In the middle of each side, measure the component of the field along that side. This component can be positive (if the angle between the field and the *directed* side is acute), negative (if the angle between the field and the *directed* side is obtuse), or zero (if the field is perpendicular to the side). If you add the components along the four sides, you get the curl (or circulation) of the field at P.[10]

Showing connection between Maxwell's third equation and Faraday's law
(page 212 of the book)

A good way to visualize the circulation is to imagine a whirlpool in which the water moves around a loop. The velocity field of the water in such a situation has a nonzero circulation. Figure E.15(a) shows a square in a whirlpool. It is clear that the components of the field along all sides are positive, so that the total circulation is positive.[11] On the other hand, the water flowing smoothly in a river has no circulation, because for any square traversed in some direction (say counterclockwise), and for any side of that square that gives a positive contribution to the circulation, there is another side which gives a negative contribution of equal magnitude. In Figure E.15(b) only two sides have nonzero contributions to the circulation; the other two give zero contribution because the field is perpendicular to both.

The RHS of Maxwell's third equation, symbolizes the rate at which the magnetic field at P changes with time. But this change in magnetic field accompanies a change in the magnetic flux through the little square at P. Thus, Maxwell's third equation says that the circulation of the electric field around a small (square) loop is the same as (the negative of) the rate of change of the magnetic flux through that loop.

Now consider a big loop (of arbitrary shape) L with magnetic field lines piercing the area formed by it. Divide the area into a large number of very small squares and apply the third equation to all these squares. Choose to go around the small squares counterclockwise.

[8] Recall from Subsection 13.4.1 that flux is the "number" of field lines piercing through an area.

[9] To be superprecise, you will have to divide the total flux by the *volume* of the cube to get the LHS, but that is not essential for our argument here.

[10] Actually you have to multiply each component by the length of the corresponding side and divide the sum by the area of the square to get the circulation, but these are minor details that do not affect the outcome of our discussion here.

[11] Had you chosen a clockwise direction for the square, the circulation would have been negative.

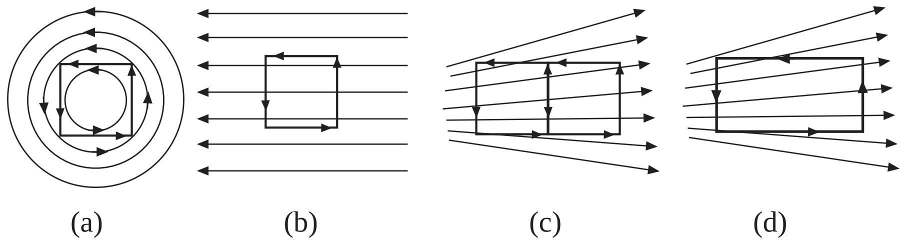


Figure E.15: (a) In a whirlpool the circulation around a square is nonzero. (b) In a smoothly flowing stream, the circulation is zero. (c) The common sides of two adjacent squares have contributions to the circulation that are opposite in sign, so that the total circulation is that of the rectangle in (d).

Then the component of the field along the side belonging to one square has the opposite sign to that along the same side considered as part of the adjacent square. Figure E.15(c) shows two such squares. The right side of the left square coincides with the left side of the right square. Since both squares are traversed in the counterclockwise direction, the common side's direction is up for the left square, and down for the right square. The component of the field will be positive for one and negative for the other. So, if you add the circulations of these two squares, the components along the common side will cancel each other, and the two squares can be replaced by the rectangle of Figure E.15(d).

Continuing pair by pair, sum up the circulations of all the little squares. The contribution from all the squares *inside* L will add up to zero due to this cancellation. Only the squares at the boundary, i.e., those one of whose sides is on L, will give a nonzero contribution. The result is *the circulation of the electric field on* L. On the other hand, adding all the fluxes of the little squares gives the total flux through the big loop L.

So far, we have obtained the result that if the magnetic flux through an imaginary loop changes with time, there will be a net electric circulation—i.e., some net component of the electric field—along that loop. Now replace the imaginary loop with a conducting wire of the same shape. Then the changing flux will create a net electric field in the wire which will cause the charges to move and create an electric current. We have regained Faraday's law as stated in Subsection 13.4.1!

Calculating speed of EM waves
(page 214 of the book)

Math Note E.14.3. Comparison of Figure 14.3 with Figure 11.1 shows that

$$\frac{1}{v^2} = \frac{k_m}{k_e} \quad \text{or} \quad v^2 = \frac{k_e}{k_m} \quad \text{and} \quad v = \sqrt{\frac{k_e}{k_m}}$$

Substituting the values $k_e = 8.988 \times 10^9$ and $k_m = 10^{-7}$, we obtain

$$v = \sqrt{\frac{8.988 \times 10^9}{10^{-7}}} = \sqrt{8.988 \times 10^{16}} = 2.998 \times 10^8 \text{ m/s}$$

which is precisely the speed of light!

I used the current values for k_e and k_m, and obtained the current—and accurate—value of the speed of light. The values Maxwell used were different, but accurate enough to give a speed for the waves very close to the speed of light as measured at that time.

E.16 Math Notes for Chapter 16

Probability of coin tosses
(page 230 of the book)

Math Note E.16.1. We consider the general case of n coins. The total number of outcomes is $2 \times 2 \times 2 \times ... \times 2(n\,\text{factors}) = 2^n$. This is because each coin has 2 possible outcomes and there are n coins.

The frequency f for zero H is clearly 1, and for one H is n (H can be coin #1, or #2, or #3, etc.). For two H's, an argument similar to that of the preceding example shows that $f(2) = n(n-1)/2$. For the frequency of 3 H's we note that there are n choices for the first H, $n-1$ choices for the second H, and $n-2$ choices for the third one. So, there are $n(n-1)(n-2)$ choices. However, we must not distinguish among the three coins, i.e., our result must not depend on which coin we called #1, which #2, and which #3. Another way of expressing this is to say that a permutation of the three heads must give the same result. Since there are $3 \times 2 \times 1 = 3!$ (read "three factorial") ways of rearranging the three heads, we get $f(3) = n(n-1)(n-2)/3!$. If we multiply the numerator and the denominator of this fraction by $(n-3)! = 1 \times 2 \times 3 \times ... \times (n-3)$, we obtain

$$f(3) = \frac{n(n-1)(n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1}{3!(n-3)!} \quad \text{or} \quad f(3) = \frac{n!}{3!(n-3)!}$$

It should now be clear that the frequency for four heads is $f(4) = n(n-1)(n-2)(n-3)/4!$, or multiplying the numerator and the denominator by $(n-4)!$,

$$f(4) = \frac{n!}{4!(n-4)!},$$

and in general, the frequency of m heads in a throw of n coins is denoted by $f_n(m)$ and is given by

$$f_n(m) = \frac{n!}{m!(n-m)!} \tag{E.19}$$

Probability is simply the ratio of the frequency to the number of outcomes, 2^n. So, writing $P_n(m)$ for this probability, we have

$$P_n(m) = \frac{f_n(m)}{2^n} = \frac{n!}{m!(n-m)!2^n} \tag{E.20}$$

This probability function is symmetric about $n/2$; i.e., it can be shown that $P_n(\frac{n}{2} - x) = P_n(\frac{n}{2} + x)$. In fact, if we let $m = \frac{n}{2} + x$, then $n - m = n - (\frac{n}{2} + x) = \frac{n}{2} - x$ and

$$P_n(m) = \frac{n!}{\left(\frac{n}{2} + x\right)! \left(\frac{n}{2} - x\right)! \, 2^n}$$

which is manifestly symmetric about $n/2$.

A convenient approximation to this equation is obtained when n is very large and m is very close to $n/2$, the average number of H's. Symbolically this means $n >> 1$, and $|n/2 - m| << n/2$. For such a situation, we have

$$P_n(m) = \sqrt{\frac{2}{n\pi}} e^{-\frac{(n-2m)^2}{2n}} \tag{E.21}$$

where $e = 2.7182818\ldots$ is the base of natural logarithm, and $\pi = 3.14159\ldots$. The larger the numbers m and n, the better the above approximation. For example, $P_{20}(9) = 0.160$ using the exact formula and $P_{20}(9) = 0.161$ using the approximate formula. These two results are already remarkably close although 20 and 9 are not large numbers. For 50 coins we get $P_{50}(24) = 0.107957$ from the exact formula, and $P_{50}(24) = 0.10841$ using the approximate formula. We see that for 50 coins the results agree much better. For $n = 1{,}000{,}000$ and $m = 498000$ the two results are essentially identical. For larger m and n, the results agree even better.

Math Note E.16.2. Recall that when both m and n are large, we can approximate the

Determination of m_- and m_+ **(page 232 of the book)**

probability $P_n(m)$ of getting m heads in a throw of n coins with Equation (E.21). For $m = n/2$, this gives the simple result

$$P_n\left(\frac{n}{2}\right) = \sqrt{\frac{2}{n\pi}} \tag{E.22}$$

As we mentioned earlier, this is the most probable outcome in the throw of n coins, yet for larger and larger n it becomes smaller and smaller due to the appearance of n in the denominator! This is surprising, because on the one hand we showed that for large n, the probability of getting anything much different from $n/2$ was negligible. Now, we say that the probability of getting $n/2$ is also negligible! So, what *do* we get when we throw a large number of coins? Nothing? Everything? We shall come back to this question later.

Let us compare (E.21) and (E.22) by constructing their ratio:

$$r = \frac{P_n(m)}{P_n(n/2)} = e^{-\frac{(n-2m)^2}{2n}}$$

This shows that r becomes smaller and smaller as the quantity $(n - 2m)^2/2n$ gets larger and larger. In other words, the ratio of probabilities decreases as the difference $n - 2m$ which is the same as $2(n/2 - m)$ increases, i.e., as m moves farther and farther away from $n/2$. This is the result we obtained qualitatively for a million coins.

How far do we have to move away from $n/2$ for the probability $P_n(m)$ to be negligible? To answer this question, we must first define the word "negligible." For us, negligible means small compared to the most probable. The latter is the the only thing we can compare our probability with.[12] Let us agree that if r is less than one millionth, 10^{-6}, we call the corresponding probability negligible. In words, we agree—in this book—to call the probability of m heads in a throw of n coins negligible if $P_n(m)$ is a million times smaller than $P_n(n/2)$. Notice that we are not talking about absolute probability, because, as mentioned above, even the most probable outcome would have a negligible probability if n is large enough. Note also that the number 10^{-6} is completely arbitrary. Any small enough number is equally plausible. It turns out, however, that the argument we are about to present will not change in quality, although quantitatively we may get slightly different results.

As before, let us denote by m_+ (respectively m_-) the number of heads above (respectively below) which r is less than 10^{-6}. We can then write

$$e^{-\frac{(n-2m_\pm)^2}{2n}} = 10^{-6} \quad \Rightarrow \quad -\frac{(n-2m_\pm)^2}{2n} = \ln(10^{-6}) = -13.82$$

or

$$(n-2m_\pm)^2 = 27.63n \quad \Rightarrow \quad n - 2m_\pm = \pm\sqrt{27.63n} = \pm 5.257\sqrt{n}$$

or

$$2m_\pm = n \pm 5.257\sqrt{n} \quad \Rightarrow \quad m_+ = \frac{n}{2} + 2.63\sqrt{n},\ m_- = \frac{n}{2} - 2.63\sqrt{n} \tag{E.23}$$

For example, for $n = 1{,}000{,}000$, we have

$$m_+ = 500{,}000 + 2.63\sqrt{1{,}000{,}000} = 500{,}000 + 2{,}630 = 502{,}630$$

and

$$m_- = 500{,}000 - 2.63\sqrt{1{,}000{,}000} = 500{,}000 - 2{,}630 = 497{,}370$$

which are very close to the numbers we obtained from the plot of the distribution of a million coins.

[12] In physics, words such as "negligible", "small", "fast", "large", etc. are relative words. Something that is small relative to the size of the Earth (Mount Everest) may seem huge to us, and something that is tiny relative to us (a small grain of sand) may be huge relative to a molecule.

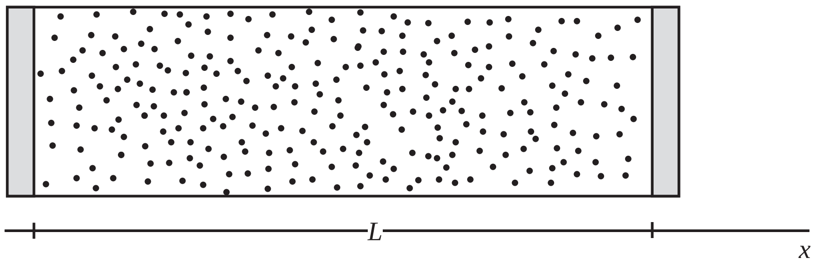


Figure E.16: The particles of the gas move randomly in all directions, but their motion along x-direction contributes to the pressure of the two end areas.

Math Note E.16.3. We want to estimate the time it takes the working population of the U.S.—around 100 million—to count the number of heads showing up in 10,000 throws of a trillion (10^{12}) coins, assuming that each adult can count 3 coins per second.

Estimating coin counting time (**page 234 of the book**)

The entire counters count 300 million coins per second. So, for each trial, we need

$$\frac{10^{12}}{300,000,000} = 3,333 \text{ seconds},$$

and for 10,000 trials, we need 3.333×10^7 seconds. Each working day consists of

$$8 \times 3600 = 28,800 \text{ seconds}.$$

Thus, the time needed is

$$\frac{3.333 \times 10^7}{28,800} = 1157 \text{ days}$$

or over three years!

E.17 Math Notes for Chapter 17

Deriving ideal gas law (**page 243 of the book**)

Math Note E.17.1. Consider a simple cylindrical container full of a gas as shown in Figure E.16. The particles of the gas move randomly in all directions, hitting the walls of the cylinder and imparting pressure on them. Let us concentrate on one of the bases of the cylinder, the one on the right. The force perpendicular to the base results from the change in the momentum of the particles *in the x-direction.* Therefore, we need to consider the motion in the x-direction only. Particle 1 has an x-momentum p_{1x}. It moves toward the right base, and after impact, bounces off and moves in the opposite direction with the same momentum (we are assuming an *elastic collision*, in which the magnitude of the momentum does not change). The change in the momentum of particle 1 is

$$\Delta p_1 = p_{1f} - p_{1i} = -p_{1x} - p_{1x} = -2p_{1x}$$

The momentum imparted to the cylinder is the *negative* of this momentum, because particle 1 and the cylinder constitute an isolated system for which the total momentum does not change. So, for the change in the momentum of the cylinder due to the impact of particle 1, we can write $\Delta p_{\text{cyl},1} = 2p_{1x}$.

How often does particle 1 impart this momentum to the cylinder? Right after each impact, particle 1 moves to the left base, bounces back, and moves to the right base again. So the time interval, Δt_1, it takes particle 1 to come back to the right cylinder after each impact is the time it takes the particle to travel *twice* the length of the cylinder. Therefore,

$\Delta t_1 = 2L/v_{1x}$; and the perpendicular force particle 1 exerts on the right base is (by the second law of motion as stated in Box 7.2.1)

$$F_{1x} = \frac{\Delta p_{\mathrm{cyl},1}}{\Delta t_1} = \frac{2p_{1x}}{(2L/v_{1x})} = \frac{p_{1x}v_{1x}}{L}$$

An identical result can be obtained for particles 2, 3, ... The total force on the right base is the sum of the forces exerted by all the particles in the gas. Assuming that there are N particles, we obtain

$$\begin{aligned} F_x &= F_{1x} + F_{2x} + F_{3x} + \ldots + F_{Nx} \\ &= \frac{p_{1x}v_{1x}}{L} + \frac{p_{2x}v_{2x}}{L} + \cdots + \frac{p_{Nx}v_{Nx}}{L} \\ &= \frac{1}{L}(p_{1x}v_{1x} + p_{2x}v_{2x} + \ldots + p_{Nx}v_{Nx}) \end{aligned} \tag{E.24}$$

We have so far used only the "mechanics" part of the "statistical mechanics." And if we were to apply solely the laws of mechanics, we would have to add the products of the x-components of the velocities and momenta of all (trillion trillion) particles in the gas! Fortunately, the "statistical" part comes to our rescue. We simply note that the *average* of a quantity for a sample is the sum of that quantity divided by the number of the constituents in the sample. Denoting the average by angle brackets, we can write

$$\langle p_x v_x \rangle = \frac{p_{1x}v_{1x} + p_{2x}v_{2x} + \ldots + p_{Nx}v_{Nx}}{N}$$

which can also be written as

$$p_{1x}v_{1x} + p_{2x}v_{2x} + \ldots + p_{Nx}v_{Nx} = N\langle p_x v_x \rangle$$

It follows from Equation (E.24) that

$$F_x = \frac{1}{L}N\langle p_x v_x \rangle = \frac{N}{L}\langle p_x v_x \rangle \tag{E.25}$$

Although we concentrated on the x-direction, there is really nothing special about it. In particular, the average of the product of components of momentum and velocity of all particles should not be biased about any directions. Thus, the product of y- and the z-components of momentum and velocity should give identical results as the x-component: $\langle p_y v_y \rangle = \langle p_z v_z \rangle = \langle p_x v_x \rangle$. Now, the product of the total momentum and total speed is the sum of the products of their components. Therefore,[13]

$$\langle pv \rangle = \langle p_x v_x + p_y v_y + p_z v_z \rangle = \langle p_x v_x \rangle + \langle p_y v_y \rangle + \langle p_z v_z \rangle \tag{E.26}$$

and since all these averages are equal, we have $\langle pv \rangle = 3\langle p_x v_x \rangle$ or $\langle p_x v_x \rangle = \frac{1}{3}\langle pv \rangle$, and Equation (E.25) becomes

$$F_x = \frac{N}{L}\left(\tfrac{1}{3}\langle pv \rangle\right) = \frac{N}{3L}\langle pv \rangle$$

To find the pressure, we have to divide this force by the area of the base. This area combined with the length L gives the volume V of the cylinder in the denominator:

$$P = \frac{F_x}{A} = \frac{N}{3LA}\langle pv \rangle = \frac{N}{3V}\langle pv \rangle \tag{E.27}$$

For ordinary (nonrelativistic) particles, $p = mv$, and (E.27) yields

$$P = \frac{N}{3V}\langle mv^2 \rangle = \frac{N}{3V}\langle 2KE \rangle = \frac{2N}{3V}\langle KE \rangle \tag{E.28}$$

[13] Readers familiar with the *dot product* of vectors recognize the right-hand side of the first equality of Equation (E.26) as $\mathbf{p} \cdot \mathbf{v}$, but since $\mathbf{p}$ and $\mathbf{v}$ are in the same direction, $\mathbf{p} \cdot \mathbf{v} = pv$, thus the left-hand side.

where in the last two steps we used the definition of the KE. Later, we shall see that the average energy (in this case, the *kinetic* energy) of a particle of the gas is proportional to temperature. An exact analysis of the statistical mechanics of the gas shows that

$$\langle KE\rangle = \tfrac{3}{2}k_BT \tag{E.29}$$

where T is the temperature in K (for Kelvin, the scientific unit of temperature) and k_B is the **Boltzmann constant**, whose value is 1.38×10^{-23} J/K. Combining Equations (E.28) and (E.29), we obtain

$$P = \frac{2N}{3V}\tfrac{3}{2}k_BT = \frac{N}{V}k_BT \quad \text{or} \quad PV = Nk_BT \tag{E.30}$$

This is called the **ideal gas law**.

For later reference, we also derive the EM **radiation pressure**. As we shall see in Section 20.3, EM waves consist of particles called *photons*, which (obviously) move at the speed of light; so $v = c$. Furthermore, relativity tells us that the energy E of a photon is related to its momentum via $E = pc$ (see Section 28.3.3). Combining these results with Equation (E.27) yields

$$P = \frac{N}{3V}\langle pc\rangle = \frac{N}{3V}\langle E\rangle$$

The total energy of the "photon gas" is the number of photons times the average energy of each photon. Thus, $N\langle E\rangle$ is the total energy. Dividing the total energy by the volume gives the **energy density** of the radiation, denoted by u. Therefore, we obtain the following succinct result from the last equation:

$$P = \tfrac{1}{3}u \tag{E.31}$$

Most probable state of a system in contact with another system **(page 247 of the book)**

Math Note E.17.2. Assume that system A has n coins, system B has N coins, and the number of positive coins in A and B are m and M, respectively. We then have

$$F(m) = \left\{\frac{n!}{m!(n-m)!}\right\}\left\{\frac{N!}{M!(N-M)!}\right\} \tag{E.32}$$

We have written the argument of the function on the LHS of Equation (E.32) as the single variable m because M can be calculated in terms of m. Here is how: The energy of the combined system, E, is fixed.[14] This means that the number of positive coins, $M+m$, minus the number of negative coins, $N+n-(M+m)$ must equal E. Thus,

$$E = M + m - \{N + n - (M+m)\} = 2(M+m) - N - n$$

and

$$M = \tfrac{1}{2}(E+N+n) - m \tag{E.33}$$

From this equation and (E.32), we can calculate $F(m)$ for given values of n, N, and E and plot the result as a function of m. Since we are interested in large numbers, we can approximate each factor of $F(m)$ by an exponential function as in Equations (E.19) and (E.21). The result is

$$F(m) \approx 2^{n+N}\sqrt{\frac{2}{n\pi}}\sqrt{\frac{2}{N\pi}}\,e^{-\left[\frac{(n-2m)^2}{2n}+\frac{(N-2M)^2}{2N}\right]} \tag{E.34}$$

[14] It would be more appropriate to introduce an elemental energy and write E as some *integer* times the elemental energy. However, this will unnecessarily complicates the formulas. Think of E as the net *number* of the elemental energies.

which after using (E.33) becomes

$$F(m) \approx 2^{n+N}\sqrt{\frac{2}{n\pi}}\sqrt{\frac{2}{N\pi}}e^{-\left[\frac{(n-2m)^2}{2n}+\frac{(E+n-2m)^2}{2N}\right]} \tag{E.35}$$

Such a function has a maximum at a certain value of m, denoted by $m_{\max}$, which is the value that yields the most probable configuration. Example 17.2.3 treats the case in which $n = 5000$, $N = 6000$, and $E = 1000$, and from the figure in that example you can approximate the peak to be at $m_{\max} = 2727$. This is the most probable value of m, respecting the constraint of $E = 1000$. Equation (E.33), then yields

$$M_{\max} = \tfrac{1}{2}(1000 + 6,000 + 5,000) - 2727 = 3273$$

As indicated in Example 17.2.3, the ratios $m_{\max}/n$ and $M_{\max}/N$ are very nearly equal:

$$\frac{m_{\max}}{n} = \frac{2727}{5,000} = 0.5454, \qquad \frac{M_{\max}}{N} = \frac{3273}{6,000} = 0.5455$$

As the numbers n, N, and E grow larger and larger, these ratios become more and more equal. In the extremely large ("almost" infinite) values, we get the equalities[15]

$$\frac{m_{\max}}{n} = \frac{M_{\max}}{N} = \frac{M_{\max}+m_{\max}}{N+n} \tag{E.36}$$

Equation (E.36) has a significant interpretation. To appreciate this interpretation, let us denote the energy of the system A by E_A and that of B by E_B. Then it is clear that

$$E_A = m - (n-m) = 2m - n \quad \Rightarrow \quad m = \tfrac{1}{2}(E_A + n)$$

and

$$E_B = M - (N-M) = 2M - N \quad \Rightarrow \quad M = \tfrac{1}{2}(E_B + N)$$

The most probable energy for the system A would be obtained when its number of positive coins is $m_{\max}$ (and for B, $M_{\max}$). Thus, writing E_{Amax} and E_{Bmax} for the most probable energies, we get

$$m_{\max} = \tfrac{1}{2}(E_{Amax} + n), \qquad M_{\max} = \tfrac{1}{2}(E_{Bmax} + N)$$

Substituting these and (E.33) in (E.36), we get

$$\frac{\frac{1}{2}(E_{Amax}+n)}{n} = \frac{\frac{1}{2}(E_{Bmax}+N)}{N} = \frac{\frac{1}{2}(E+N+n)}{N+n}$$

where E is the total energy of the combined system. This double equality can also be written as

$$\frac{E_{Amax}}{n} = \frac{E_{Bmax}}{N} = \frac{E}{N+n} \tag{E.37}$$

Noting that E_{Amax}/n is the average energy for system A (with a corresponding interpretation for the other ratios), Equation (E.37) says that the most probable configuration of two systems in contact is that for which the **average energy** per coin of the two systems are equal and both have the common value of the average energy of the combined system.

[15] If you are familiar with calculus, you may be interested to know that these equalities are obtained by differentiating Equation (E.35) and setting the derivative equal to zero.

Math Note E.17.3. Let systems A and B have initial temperatures T_A and T_B, respectively. What is the final equilibrium temperature after the two systems have been in contact for a long enough time? The answer is in the last equation of (E.37). To use that equation, we need the total energy, which is the sum of the two energies. Since this energy does not change, we can obtain it by adding the *initial* energies of the two systems. Box 17.2.5 tells us that

$$E_{Amax}/n = kT_A \quad \text{or} \quad E_{Amax} = nkT_A$$

where k is the proportionality factor. With a similar expression for system B, we obtain

$$E = E_{Amax} + E_{Bmax} = nkT_A + NkT_B \quad \text{and} \quad kT_f = \frac{E}{N+n} = \frac{nkT_A + NkT_B}{N+n}$$

where T_f is the final equilibrium temperature. Therefore,

Final temperature when two systems are brought together
(page 248 of the book)

$$T_f = \frac{nT_A + NT_B}{N+n} \tag{E.38}$$

Regardless of the size of the two systems, the final temperature lies between the two initial temperatures. To see this, let $t = T_A - T_B$ be the difference between the two initial temperatures. Then, $T_A = T_B + t$, and Equation (E.38) yields

$$T_f = \frac{n(T_B + t) + NT_B}{N+n} = \frac{(n+N)T_B + nt}{N+n} = T_B + \frac{n}{N+n}t \tag{E.39}$$

If $T_A > T_B$, then t is positive and $T_f > T_B$; but $T_f < T_A$, because $\frac{n}{N+n}t < t$. Therefore, $T_A > T_f > T_B$. If $T_A < T_B$, then t is negative and $T_f < T_B$; but $T_f > T_A$, because $\frac{n}{N+n}t > t$. Therefore, $T_B > T_f > T_A$. In either case, T_f is between T_A and T_B.

Math Note E.17.4. We want to compare the number of accessible states when the two systems of Math Note E.17.2 are in equilibrium with the number of accessible states when they depart from the equilibrium. To be specific, we want the ratio $F(m)/F(m_{\max})$—which is also the ratio of the probabilities—when m is different from $m_{\max}$. We shall depart slightly from Math Note E.17.2 in that we introduce an elemental energy ϵ. Then

Irreversibility of processes
(page 252 of the book)

$$E_A = (2m - n)\epsilon \quad \text{and} \quad E_B = (2M - N)\epsilon \tag{E.40}$$

and

$$M = \tfrac{1}{2}[(E/\epsilon) + N + n] - m \tag{E.41}$$

An interesting consequence of this equation is that $M + m$ is fixed, because E, N, and n are all fixed numbers. In particular, one can replace $M_{\max} + m_{\max}$ with $M + m$ in (E.36) and obtain

$$m_{\max} = \frac{n}{N+n}(M+m) \tag{E.42}$$

We shall use this result shortly.

To save writing, let $\alpha(m)$ stand for the (negative of the) exponent of (E.34):

$$\begin{aligned}\alpha(m) &\equiv \frac{(n-2m)^2}{2n} + \frac{(N-2M)^2}{2N} \\ &= \frac{N+n}{2} - 2(M+m) + 2\left(\frac{m^2}{n} + \frac{M^2}{N}\right)\end{aligned} \tag{E.43}$$

Then

$$\frac{F(m)}{F(m_{\max})} = e^{\alpha(m_{\max}) - \alpha(m)} \tag{E.44}$$

Since $N+n$ and $M+m$ are fixed, evaluating (E.43) at $m_{\max}$ and subtracting yields

$$\alpha(m_{\max}) - \alpha(m) = 2\frac{m_{\max}^2 - m^2}{n} + 2\frac{M_{\max}^2 - M^2}{N} \tag{E.45}$$

From (E.36), $M_{\max} = \frac{N}{n} m_{\max}$, and from (E.42)

$$M = \frac{N+n}{n} m_{\max} - m = \frac{N}{n} m_{\max} + m_{\max} - m$$

Substituting these in (E.45) yields for $\alpha(m_{\max}) - \alpha(m)$

$$\begin{aligned}
&\frac{2}{n}(m_{\max}^2 - m^2) + \frac{2}{N}\left[\frac{N^2}{n^2} m_{\max}^2 - \left(\frac{N}{n} m_{\max} + m_{\max} - m\right)^2\right] \\
&= \frac{2}{n}(m_{\max}^2 - m^2) - \frac{2}{N}\left[(m_{\max} - m)^2 + 2\frac{N}{n}(m_{\max} - m)\right] \\
&= \frac{2}{n}(m_{\max} - m)(m_{\max} + m) - \frac{2}{N}(m_{\max} - m)^2 - \frac{4}{n} m_{\max}(m_{\max} - m) \\
&= \frac{2}{n}(m_{\max} - m)(m_{\max} + m - 2m) - \frac{2}{N}(m_{\max} - m)^2 \\
&= -\left(\frac{2}{n} + \frac{2}{N}\right)(m_{\max} - m)^2
\end{aligned}$$

From Equation (E.42), we get

$$m_{\max} - m = \frac{n}{N+n}(M+m) - m = \frac{nM + nm - m(N+n)}{N+n} = \frac{nM - Nm}{N+n}$$

Substituting all these results in (E.45) and then in the exponent of (E.44), we obtain

$$\frac{F(m)}{F(m_{\max})} = \exp\left\{-\left(\frac{2}{n} + \frac{2}{N}\right)\left(\frac{Mn - mN}{n+N}\right)^2\right\} \tag{E.46}$$

We now want to measure the probability of the two systems attaining two different temperatures (while still in contact). To do this, we express the exponent of (E.46) in terms of temperatures. The specification of the temperature means that m is the most probable value of system A corresponding to a temperature T_A and M is the most probable value of system B corresponding to a temperature T_B. Then, using k as the proportionality constant relating temperature and energy, (E.40) yields

$$m = \frac{1}{2}\left(\frac{E_A}{\epsilon} + n\right) = \frac{n}{2}\left(\frac{kT_A}{\epsilon} + 1\right) \quad \text{and} \quad M = \frac{N}{2}\left(\frac{kT_B}{\epsilon} + 1\right)$$

so that

$$\frac{Mn - mN}{n+N} = \frac{nNk(T_B - T_A)}{2\epsilon(N+n)} = \frac{\frac{k}{\epsilon}(T_B - T_A)}{2(\frac{1}{N} + \frac{1}{n})}$$

Thus, the ratio of the probability of finding the system away from its equilibrium state [with temperature T_f given by (E.38)] to the probability of finding it at its final equilibrium state is

$$\frac{P(T_A)}{P(T_f)} = \frac{F(m)}{F(m_{\max})} = \exp\left[-\frac{k^2(T_B - T_A)^2/\epsilon^2}{2(1/N + 1/n)}\right] \equiv e^{-(\Delta T/\tau)^2} \tag{E.47}$$

where

$$\Delta T = T_B - T_A \quad \text{and} \quad \tau = \frac{\epsilon}{k}\sqrt{2\left(\frac{1}{N} + \frac{1}{n}\right)}$$

For typical values of $\epsilon = 10^{-19}$, $k = 10^{-23}$, and $n \approx N \approx 10^{24}$, we get $\tau \approx 10^{-8}$. Thus, for any reasonable finite value of ΔT, the ratio is completely negligible. For example, for as small a temperature difference as a millionth of a degree

$$e^{-(\Delta T/\tau)^2} = e^{-(10^{-6}/10^{-8})^2} = e^{-10000} \approx 1.14 \times 10^{-4343}$$

meaning that the odds of the occurrence of even such a minute temperature difference is about one in 10^{4343}!

E.18 Math Notes for Chapter 18

Specific heat of an ideal gas
(page 262 of the book)

Math Note E.18.1. As a guiding example, consider the simplest kind of ideal gas (called *monatomic* because its particles consist of a *single* atom), whose internal energy is only in the form of the kinetic energies of its particles. If there are N particles in this ideal gas, its internal energy U is $\frac{3}{2}Nk_BT$, because, by Equation (17.1), each particle has an average KE of $\frac{3}{2}k_BT$. It follows that any change in the internal energy is expressed as a change in the temperature of the ideal gas: $\Delta U = \frac{3}{2}Nk_B\Delta T$. Usually, the effects of work and heat on the internal energy are considered separately. When there is only a heat exchange (i.e., when $W = 0$), then the change in the internal energy is just the heat transferred: $Q = \frac{3}{2}Nk_B\Delta T$.

Now suppose that each particle of the gas has mass μ and the total mass of the gas is m, then $N = m/\mu$ and

$$Q = \frac{m}{\mu}\frac{3k_B}{2}\Delta T = m\left(\frac{3k_B}{2\mu}\right)(T_f - T_i) \equiv mc(T_f - T_i) \tag{E.48}$$

Specific heat of an ideal gas derived

where $T_f - T_i = \Delta T$, and $c = \frac{3k_B}{2\mu}$ is its **specific heat**. Although Equation (E.48) was derived for a specific ideal gas, it is a general formula that applies to many substances, gas, liquid, or solid.

Depending on whether the final temperature is greater or less than the initial temperature, ΔT in Equation (E.48) can be positive or negative, making Q positive (when heat is added to the system) or negative (when heat is extracted from the system).

Derivation of the efficiency of the Carnot engine
(page 269 of the book)

Math Note E.18.2. The theoretical limit imposed on the efficiency of an engine comes from the law of increase of entropy (Section 17.3.1). The engine and the environment in which it operates constitute a *closed* system. When the engine goes through one of its cycles, the total entropy of the engine plus the environment must not decrease. The change in the entropy of the engine is zero because the engine comes back to its original state after a full cycle.[16] Therefore, the total change in the entropy is that occurring in the environment.

It can be shown that the change in the entropy of any thermodynamic system at temperature T is

$$\Delta S = \frac{Q}{T} \tag{E.49}$$

where Q is the heat transferred to the system. Depending on the sign of Q, ΔS could be positive or negative. A negative ΔS does not violate the law of entropy increase, because the system may not be closed (it may be in contact with another system whose entropy change is more positive than the negative change of the original system).

What is the change in the entropy of the environment of an engine? The environment consists of just two reservoirs. The change in the entropy of the cold reservoir is Q_c/T_c and it is positive (because heat is dumped into the cold reservoir); and the change in the entropy of the hot reservoir is $-Q_h/T_h$. It follows that the change in the entropy of the

[16] That is how a cycle is *defined*.

environment is $Q_c/T_c - Q_h/T_h$. Setting this greater than or equal to zero (law of entropy increase) gives

$$\Delta S_{\text{total}} = \frac{Q_c}{T_c} - \frac{Q_h}{T_h} \geq 0 \quad \Rightarrow \quad \frac{Q_c}{T_c} \geq \frac{Q_h}{T_h} \quad \text{or} \quad \frac{Q_c}{Q_h} \geq \frac{T_c}{T_h}$$

Substituting this in Equation (18.2), we obtain

$$\epsilon = 1 - \frac{Q_c}{Q_h} \leq 1 - \frac{T_c}{T_h} \tag{E.50}$$

The right-hand side is a theoretical upper limit for the efficiency of *any* engine. Therefore, the most efficient (theoretical, ideal) engine, called the **Carnot engine** must have the efficiency *equal* to the right-hand side. Tracing back the derivation of (E.50), we note that this equality corresponds to $\Delta S_{\text{total}} = 0$.

E.20 Math Notes for Chapter 20

Derivations of Wien and Stefan-Boltzmann laws (**page 291 of the book**)

Math Note E.20.1. This Math Note is devoted to the derivation of Wien's displacement law and the Stefan-Boltzmann law, and requires calculus which is beyond the intended ability of most readers. However, the discovery of the connection between BBR formula and the other two laws is so satisfying that it is a loss for those with calculus background not to see it. The portions of this and the math notes that require higher mathematics are set in a different font style.

First, we derive the Wien's displacement law. If we substitute the numerical values for h, c, and k_B in Equation (20.5), we obtain

$$\Phi(\lambda, T) = \frac{3.747 \times 10^{-16}}{\lambda^5} \frac{1}{e^{0.0144/\lambda T} - 1} \tag{E.51}$$

Since, λ and T appear together, we define $x = \lambda T$, and write the above equation as

$$\Phi(\lambda, T) = \frac{3.747 \times 10^{-16} T^5}{x^5} \frac{1}{e^{0.0144/x} - 1} \equiv 3.747 \times 10^{-16} T^5 f(x)$$

where $f(x)$ is the abbreviation for $1/[x^5(e^{0.0144/x} - 1)]$. Since we are interested in the variation of the curve as a function of λ, we keep T constant. It follows, that the maximum of $\Phi(\lambda, T)$ can be obtained by finding the maximum of $f(x)$. Writing $f(x)$ as $x^{-5}(e^{0.0144/x} - 1)^{-1}$, it is straightforward to show that the derivative of $f(x)$ is

$$f'(x) = \frac{0.0144 e^{0.0144/x}}{x^7(e^{0.0144/x} - 1)^2} - \frac{5}{x^6(e^{0.0144/x} - 1)}$$

Setting this equal to zero and simplifying, gives

$$\frac{0.0144 e^{0.0144/x}}{x(e^{0.0144/x} - 1)} - 5 = 0 \quad \text{or} \quad e^{0.0144/x} = 347.2x(e^{0.0144/x} - 1)$$

A numerical solution—obtained by using a graphing calculator, for instance—of this last equation yields $x_{max} = 0.00290043$, which, recalling that $x = \lambda T$, is identical to the Wien's displacement law.

For the Stefan-Boltzmann law, we need to integrate (E.51) over all wavelengths. This gives

$$J_e = 3.747 \times 10^{-16} \int_0^\infty \frac{d\lambda}{\lambda^5(e^{0.0144/\lambda T} - 1)}$$

Once again, we use $x = \lambda T$, and therefore, $dx = T d\lambda$ to obtain

$$\begin{aligned} J_e &= 3.747 \times 10^{-16} \int_0^\infty \frac{dx/T}{(x^5/T^5)(e^{0.0144/x} - 1)} \\ &= 3.747 \times 10^{-16} T^4 \int_0^\infty \frac{dx}{x^5(e^{0.0144/x} - 1)} \end{aligned}$$

The last integral can be evaluated numerically to yield 1.51×10^8. Then we get

$$J_e = (3.747 \times 10^{-16}) T^4 (1.51 \times 10^8) = 5.66 \times 10^{-8} T^4$$

which is the Stefan-Boltzmann law. The slight difference between this equation and (20.1) is due to the fact that we did not keep sufficient significant figures in the values of h, c, and k_B.

If instead of (E.51), we use (20.5), and define $y = (k_B T/hc)\lambda$, then the integral over λ could be changed to an integral over y as follows

$$\begin{aligned} J_e &= 2\pi c^2 h \int_0^\infty \frac{d\lambda}{\lambda^5(e^{hc/k_B\lambda T} - 1)} = 2\pi c^2 h \int_0^\infty \frac{\frac{hc}{k_B T} dy}{(\frac{hc}{k_B T})^5 y^5 (e^{1/y} - 1)} \\ &= \frac{2\pi k_B^4}{h^3 c^2} T^4 \underbrace{\int_0^\infty \frac{dy}{y^5(e^{1/y} - 1)}}_{=\pi^4/15} = \frac{2\pi^5 k_B^4}{15 h^3 c^2} T^4 \end{aligned} \tag{E.52}$$

If we substitute the numerical values of the constants in the coefficient of T^4, we obtain 5.67×10^{-8} as in Equation (20.1). In fact, Planck used the last equation above and the then known *Stefan-Boltzmann constant to estimate h. He obtained $h = 6.56 \times 10^{-34}$, which is remarkably close to the current accepted value.*

Derivations of Planck relation **(page 290 of the book)**

Math Note E.20.2. The probability of finding a particle with energy E is proportional to the Boltzmann factor introduced in Section 17.2.2, which can also be written as $1/e^{\frac{E}{k_B T}}$. Comparing this with the denominator of Equation (20.3) gives a clue that EM radiation should probably be considered as "particles," and that a/λ should be identified with E/k_B:

$$\frac{a}{\lambda} = \frac{E}{k_B} \quad \text{or} \quad E = \frac{ak_b}{\lambda}$$

On the other hand, the wavelength of an EM wave is given by $\lambda = c/f$ in terms of frequency. Therefore, the last equation can be expressed as

$$E = \frac{ak_b f}{c} = \left(\frac{ak_b}{c}\right) f$$

The term in the parentheses is constant, implying that the energy of an EM "particle" is proportional to its frequency.

E.21 Math Notes for Chapter 21

Finding the speed and total energy of electron in H-atom **(page 304 of the book)**

Math Note E.21.1. We assume that the electron moves on a circle of radius r around the nucleus. Then Newton's second law of motion gives

$$F = ma_{\text{cent}} \quad \text{or} \quad \frac{k_e e^2}{r^2} = m\frac{v^2}{r} \quad \Rightarrow \quad mv^2 = \frac{k_e e^2}{r} \tag{E.53}$$

where e is the charge of the electron (and the nucleus of the H-atom) and m is the electronic mass. We can use the last equation in (E.53) to find two important quantities: speed and KE. The speed is

$$v^2 = \frac{k_e e^2}{mr} \quad \text{or} \quad v = \sqrt{\frac{k_e e^2}{mr}}, \tag{E.54}$$

and the KE is

$$KE = \tfrac{1}{2}mv^2 = \frac{k_e e^2}{2r} \tag{E.55}$$

From the speed, we can find the frequency of rotation of the electron around the nucleus

$$v = \frac{\text{distance}}{\text{time}} = \frac{2\pi r}{T} = 2\pi r f \quad \Rightarrow \quad f = \frac{v}{2\pi r} \tag{E.56}$$

where T is the period of revolution and f the corresponding frequency. The total energy, which is the sum of the KE and electric potential energy, can be obtained from Equations (E.55) and (D.10). The electric potential energy is negative because the electric force is attractive. Therefore,

$$E = KE + PE = \frac{k_e e^2}{2r} - \frac{k_e e^2}{r} = -\frac{k_e e^2}{2r} \tag{E.57}$$

Energy of bound systems

Note that the total energy is negative. This is a characteristic of a bound system (see also Example 9.2.5), and the energy (without the sign) is called the **binding energy**.

Let's put in some numbers. Suppose that the electron is moving on a circle of radius 10^{-10} m, a typical atomic size. Then substituting the numerical values of the charge and mass of the electron in Equation (E.54), we get

$$v = \sqrt{\frac{(9 \times 10^9)(1.6 \times 10^{-19})^2}{(9.1 \times 10^{-31})(10^{-10})}} = 1.59 \times 10^6 \text{ m/s}$$

Equation (E.56) then gives the frequency of the motion of the electron:

$$f = \frac{v}{2\pi r} = \frac{1.59 \times 10^6}{6.28 \times 10^{-10}} = 2.53 \times 10^{15} \text{ Hz}$$

The total energy of the electron at the given distance is

$$E = -\frac{k_e e^2}{2r} = -\frac{(9 \times 10^9)(1.6 \times 10^{-19})^2}{2(10^{-10})} = -1.15 \times 10^{-18} \text{ J}$$

Solution of Bohr model of the H-atom **(page 305 of the book)**

Math Note E.21.2. Bohr's basic assumption, for which he had absolutely no reason or motivation, was that the angular momentum of the electron is an integer multiple of the Planck constant (divided by 2π). Combining this assumption with Equation (8.5), we obtain

$$rmv = nh/2\pi \quad \Rightarrow \quad r^2m^2v^2 = n^2h^2/4\pi^2, \quad n = 1, 2, 3, \ldots$$

If we now substitute for v^2 from Equation (E.54) of Math Note E.21.1, we obtain

$$r^2m^2\left(\frac{k_e e^2}{mr}\right) = n^2h^2/4\pi^2 \quad \Rightarrow \quad r = \frac{h^2}{4\pi^2 k_e m e^2}n^2,$$

Quantization of electron orbits and Bohr radius

which shows that the electron has multiple orbits, and that these orbits are quantized. The radius of the smallest orbit is denoted by a_0, and is called the **Bohr radius**. Its value is

$$a_0 = \frac{h^2}{4\pi^2 k_e m e^2} = \frac{(6.626 \times 10^{-34})^2}{4(3.1416)^2(9 \times 10^9)(9.1 \times 10^{-31})(1.6 \times 10^{-19})^2}$$

equal to 5.3×10^{-11} m. If we denote the radius of the nth orbit by r_n, then

$$r_n = n^2 a_0, \quad n = 1, 2, 3, \ldots \tag{E.58}$$

Now that we have the radii of the orbits, we can calculate the important quantity, energy. The formula for energy is given by Equation (E.57) of Math Note E.21.1. Thus the energy of the nth orbit is

$$E_n = -\frac{k_e e^2}{2n^2 a_0} = -\frac{2.17 \times 10^{-18}}{n^2} \text{ J} \quad \text{or} \quad E_n = -\frac{13.6}{n^2} \text{ eV} \tag{E.59}$$

where we inserted the numerical values of the constants to get the energy first in Joules, and then (using the conversion factor) in eV.

E.22 Math Notes for Chapter 22

De Broglie relation implies Bohr assumption (**page 313 of the book**)

Math Note E.22.1. De Broglie pointed out in 1924 that his assumption could explain the mysterious starting point of the Bohr theory. He argued that if electron is a wave and it happens to be in one of the Bohr orbits, then the circumference of that orbit must be an integer multiple of the electron's wavelength. Figure E.17 shows why. Consider point P on a Bohr orbit of a hydrogen atom. Imagine taking a mental snapshot of the electron at a particular time. If the circumference of the orbit is not an integer multiple of the wavelength as in Figure E.17(a), then the electronic wave will have two different values at P as in Figure E.17(b). This is logically impossible. This condition is written as $2\pi r = n\lambda$. Using Equation (22.1), we get

$$2\pi r = n\frac{h}{p} \quad \text{or} \quad 2\pi r p = nh \quad \Rightarrow \quad rmv = n\frac{h}{2\pi}$$

This is precisely the starting point of Math Note E.21.2.

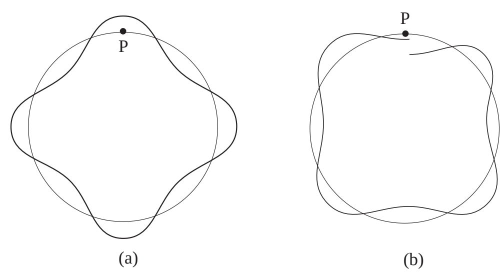


Figure E.17: (a) Electron has a unique existence at P. (b) Electron has a "double personality" at P.

E.26 Math Notes for Chapter 26

Calculation of the tick of a moving MM clock (**page 381 of the book**)

Math Note E.26.1. An MM clock is placed on the train and observed by our two observers O (on the ground) and O' (on the train). Since there are two observers, it is convenient to specify events and see how these observers perceive the events. In our case, we have three events: The emission of a light beam at S, its reflection at M, and its reception at S. These three events constitute one tick. Let us denote them by E_1, E_2, and E_3, respectively. How does O' see the ticking of the clock? The clock is sitting right beside her, and she observes the whole process of ticking as the light going straight up and coming straight down. She concludes that the time interval, denoted by $\Delta\tau$, is simply twice the time it takes light to travel the distance $\overline{SM} = L$:

$$\Delta\tau = 2\frac{L}{c} \tag{E.60}$$

Now, let us see how O perceives the succession of these three events. Since the clock is moving to the right, the light signal that leaves S will reach M only after M has moved to the right. Thus, to O, the events E_1 and E_2 are separated not only by a vertical distance, but also by a horizontal distance [see Figure E.18(a)]. To further clarify this, suppose that the light signal sent by S and reflected by M is represented by a black dot. Figure E.18(a) shows five snapshots of the clock. In the first snapshot the dot is produced at E_1. A little later (therefore a little to the right) the ball is at the middle of the clock tube. Still a little later (and a little further to the right) the ball reaches the mirror at the top.

We argued in Section 25.4.2 that the vertical distance is unaffected by motion. So, the length of the MM clock is still L, and because of the addition of the horizontal distance—the

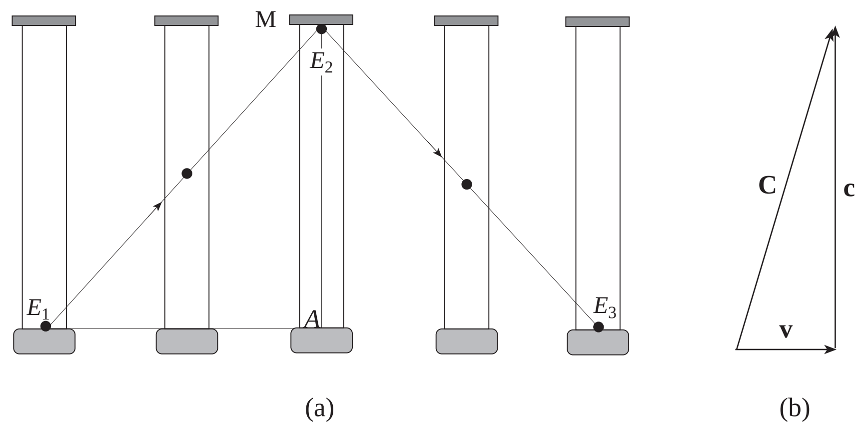


Figure E.18: (a) A moving Michelson–Morley clock. The path of light (represented by a black dot) is not a vertical line but a slanted one due to the motion of M. (b) Law of addition of velocities applied to the light signal in the MM clock.

line segment $\overline{E_1A}$ in Figure E.18(a)—O decides that $\overline{E_1E_2} > \overline{AM} = L$. The *speed of light being the same for all observers*, he concludes that it takes light more than L/c to travel $\overline{E_1E_2}$. The other leg, $\overline{E_2E_3}$, has exactly the same length as $\overline{E_1E_2}$. Thus, to O, the total travel time from S to M and back takes *longer* than $2L/c$. Therefore, he concludes that the *clock on the train must tick slower!*

Moving clocks slow down.

We can quantify the above statement by referring to the triangle E_1AE_2 of Figure E.18. Pythagorean theorem implies

$$\left(\overline{E_1E_2}\right)^2 = \left(\overline{E_1A}\right)^2 + \left(\overline{AE_2}\right)^2$$

Let the speed of the train be v and the light beam's travel time from S to M be t according to O. Then $\overline{E_1A} = vt$ and $\overline{E_1E_2} = ct$ with c the (universal) speed of light. Putting all of this in the above equation gives

$$(ct)^2 = (vt)^2 + L^2 \quad \Rightarrow \quad c^2t^2 = v^2t^2 + L^2 \tag{E.61}$$

or

$$t^2 = \frac{v^2}{c^2}t^2 + \frac{L^2}{c^2} \quad \Rightarrow \quad t^2 - \frac{v^2}{c^2}t^2 = \frac{L^2}{c^2} \quad \Rightarrow \quad t^2\left(1 - \frac{v^2}{c^2}\right) = \frac{L^2}{c^2}$$

This yields

$$t^2 = \frac{L^2/c^2}{1 - v^2/c^2} \quad \Rightarrow \quad t = \frac{L/c}{\sqrt{1 - v^2/c^2}}$$

Let us denote by Δt the duration of the light's round trip as seen by O. Then

$$\Delta t = 2t = \frac{2L/c}{\sqrt{1 - v^2/c^2}} = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} = \frac{\Delta\tau}{\sqrt{1 - (v/c)^2}} \tag{E.62}$$

where we have used Equation (E.60).

It is instructive to investigate the consequence of abandoning the second postulate of relativity and restoring the law of addition of velocity. Figure E.18(b) shows the "new" speed of light C as seen by O. If this law were true, then O would measure the speed of light—as it goes from E_1 to E_2—to be C. Then Equation (E.61) would become

$$(Ct)^2 = (vt)^2 + L^2 \quad \Rightarrow \quad C^2t^2 = v^2t^2 + L^2$$

But Figure E.18(b) and the Pythagorean theorem tell us that $C^2 = v^2 + c^2$. Therefore,

$$C^2t^2 = v^2t^2 + L^2 \quad \Rightarrow \quad (v^2 + c^2)t^2 = v^2t^2 + L^2 \quad \text{or} \quad c^2t^2 = L^2$$

or $ct = L$ or $t = L/c$, leading to

$$\Delta t = 2t = 2L/c = \Delta\tau,$$

i.e., the two observers would measure the same time interval: time would *not* be relative but universal! It should now be clear that the second postulate of relativity is at the heart of the relativity of time.

Although Equation (E.62) is derived for a single tick, it really applies to all time intervals because any such interval is a multiple of a single tick. For instance a second is simply 10^6 ticks, an hour is 3.6×10^9 ticks, a week is 6×10^{11} ticks and a year is 3.15×10^{13} ticks. If each tick is altered by a factor of $\sqrt{1 - v^2/c^2}$, then a second, an hour, or a year is also altered by the same factor. What is the difference between Δt and $\Delta\tau$? Both measure the time interval between two events, E_1 and E_3, but $\Delta\tau$ measures the time interval in a RF in which E_1 and E_3 occur at the *same spatial point*: The emission and reception of light occur at the same point S. For this reason $\Delta\tau$ is called **proper time**. We now rewrite Equation (E.62), realizing that $\Delta\tau$ is the proper time between *any two events*, while Δt is the time measured by a clock relative to which the two events occur at two different spatial points

proper time defined

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} \tag{E.63}$$

Math Note E.26.2. Consider two points P_1 and P_2 both at rest relative to Karl. These two points can be locations of two stars, locations of two cities, or merely the two ends of a meter stick. Karl measures the distance between the two points and calls it L_0; thus, $L_0 = \overline{P_1P_2}$.

Deriving length contraction formula **(page 384 of the book)**

Now consider Emmy moving relative to Karl with speed v (see Figure E.19). Since the length L_0 is moving along her direction of motion, the distance between P_1 and P_2 will appear smaller to Emmy. To find how much smaller, we resort to time dilation. Consider the time interval it takes Emmy to go from P_1 to P_2. This time interval is a proper time interval, because her clock is present at the two events "passing by P_1" and "passing by P_2." She thus concludes that the distance between P_1 and P_2 is $L = v\Delta\tau$. On the other hand, Karl measures Emmy's time of flight from P_1 to P_2 to be Δt and concludes that $L_0 = v\Delta t$. Equation (26.1) now gives $\Delta\tau = \Delta t\sqrt{1 - (v/c)^2}$, and therefore, $L = v[\Delta t\sqrt{1 - (v/c)^2}]$ or $L = (v\Delta t)\sqrt{1 - (v/c)^2}$, which yields

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} \tag{E.64}$$

L_0 is the **rest length**, and L the **moving length**.

Math Note E.26.3. The spaceship clock measures the proper time for the following two events: Lift-off, when Karl is zero year old, and departure of signal, when Karl is one year old. Not aware of relativistic effects, the crew members assume that Emmy is also one; so, they send the message "Happy first, Emmy!" on Karl's first birthday. Let us denote the time passed according to the Earth due to the time dilation by Δt_1. Then

Connecting traveling twin's birthday signals to ground twin's birthday **(page 385 of the book)**

$$\Delta t_1 = \gamma\Delta\tau = \frac{\Delta\tau}{\sqrt{1 - (v/c)^2}}$$

Thus, Δt_1 is when—according to the Earth—the crew in Marinarus sends the birthday message. However, this message is sent at a very far distance. In fact, the spaceship is

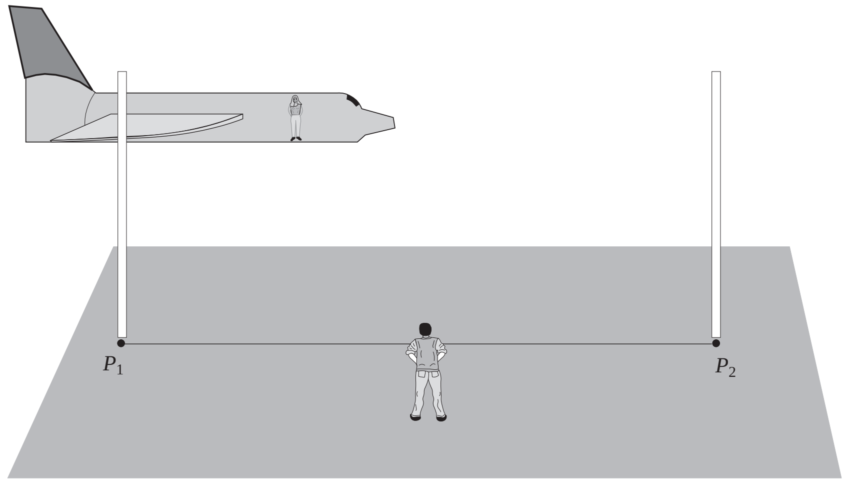


Figure E.19: The distance between P_1 and P_2 is stationary relative to Karl, but in motion relative to Emmy.

precisely at a distance of $v\Delta t_1$, when the signal is issued. This signal, being an EM wave, travels this distance with speed c. So, it takes the signal

$$\Delta t_2 = \frac{v\Delta t_1}{c} = \frac{v}{c}\Delta t_1 = \frac{v}{c}\frac{\Delta\tau}{\sqrt{1-(v/c)^2}}$$

to reach the Earth *after* it is issued. The Earth observers will, therefore, receive the signal after the time interval

$$\begin{aligned}\Delta t = \Delta t_1 + \Delta t_2 &= \frac{\Delta\tau}{\sqrt{1-(v/c)^2}} + \frac{v}{c}\frac{\Delta\tau}{\sqrt{1-(v/c)^2}} \\ &= \left(1+\frac{v}{c}\right)\frac{\Delta\tau}{\sqrt{1-(v/c)^2}} = \frac{1+v/c}{\sqrt{1-(v/c)^2}}\Delta\tau\end{aligned}$$

which can be simplified as

$$\Delta t = \frac{\sqrt{1+v/c}\sqrt{1+v/c}}{\sqrt{(1-v/c)(1+v/c)}}\Delta\tau = \sqrt{\frac{1+v/c}{1-v/c}}\Delta\tau \tag{E.65}$$

If we substitute $v/c = 0.9$ and $\Delta\tau = 1$ year, we get

$$\Delta t = \sqrt{\frac{1+0.9}{1-0.9}} \times 1 \text{ year} = 4.3589 \text{ years} = 4 \text{ years and } 131 \text{ days}$$

which is the time interval mentioned in the text.

Time calculations
(page 385 of the book)

Math Note E.26.4. In the language of the previous Math Note, we want to calculate Δt_1 in terms of Δt. But $\Delta t_1 = \gamma\Delta\tau$ and Equation (E.65) can be solved for $\Delta\tau$ in terms of Δt: $\Delta\tau = \sqrt{(1-v/c)/(1+v/c)}\,\Delta t$. It follows that

$$\Delta t_1 = \frac{1}{\sqrt{1-(v/c)^2}}\sqrt{\frac{1-v/c}{1+v/c}}\Delta t = \frac{1}{\sqrt{(1-v/c)(1+v/c)}}\sqrt{\frac{1-v/c}{1+v/c}}\Delta t$$

or, finally

$$\Delta t_1 = \frac{\Delta t}{1+v/c}$$

In the case of the example of the twins, this yields

$$\Delta t_1 = \frac{4.3589 \text{ years}}{1 + 0.9} = 2.2942 \text{ years}$$

As a check, note that

$$\Delta t_1 = \frac{\Delta\tau}{\sqrt{1-(v/c)^2}} = \frac{1 \text{ year}}{\sqrt{1-(0.9)^2}} = \frac{1 \text{ year}}{\sqrt{0.19}} = 2.2942 \text{ years}$$

One year of the spaceship is dilated to 2.2942 years according to the Earth clock. The remaining time, $4.3589 - 2.2942 = 2.0647$ years, is the time it takes light to reach Earth once released from the spaceship.

Approximation formulas for gamma factor **(page 386 of the book)**

Math Note E.26.5. Using a calculator, the reader may verify that $\sqrt{1-x}$ can be approximated very well by $1 - \frac{1}{2}x$ when x is very small. Similarly, $1/\sqrt{1-x}$ and $\sqrt{1+x}$ can be approximated by $1 + \frac{1}{2}x$ and $1/\sqrt{1+x}$ by $1 - \frac{1}{2}x$. We write

$$\begin{aligned} &\sqrt{1-x} \approx 1 - \tfrac{1}{2}x, \quad \sqrt{1+x} \approx 1 + \tfrac{1}{2}x \\ &\frac{1}{\sqrt{1-x}} \approx 1 + \tfrac{1}{2}x, \quad \frac{1}{\sqrt{1+x}} \approx 1 - \tfrac{1}{2}x \end{aligned} \tag{E.66}$$

For example, when $x = 0.1$, we have (using a calculator)

$$\sqrt{0.9} = \sqrt{1-0.1} = 0.948683298, \quad \sqrt{1.1} = \sqrt{1+0.1} = 1.048808848$$

for the first pair, and

$$\frac{1}{\sqrt{0.9}} = \frac{1}{\sqrt{1-0.1}} = 1.054092553, \quad \frac{1}{\sqrt{1.1}} = \frac{1}{\sqrt{1+0.1}} = 0.953462589$$

for the second pair of Equation (E.66). These values are very nearly equal to $1 - \frac{0.1}{2} = 0.95$ and $1 + \frac{0.1}{2} = 1.05$. If we decrease x further to 0.01, we get

$$\sqrt{0.99} = \sqrt{1-0.01} = 0.994987437, \quad \sqrt{1.01} = \sqrt{1+0.01} = 1.004987562$$

for the first pair, and

$$\frac{1}{\sqrt{0.99}} = \frac{1}{\sqrt{1-0.01}} = 1.005037815, \quad \frac{1}{\sqrt{1.01}} = \frac{1}{\sqrt{1+0.01}} = 0.99503719$$

for the second pair of Equation (E.66). These values are even closer to $1 - \frac{0.01}{2} = 0.995$ and $1 + \frac{0.01}{2} = 1.005$.

Now let us obtain a formula for how much an object shrinks when moving with speed v. We are seeking the difference between L_0 and L. This is given by

$$L_0 - L = L_0 - L_0\sqrt{1-(v/c)^2} = L_0[1 - \sqrt{1-(v/c)^2}] \tag{E.67}$$

When v is very small compared to light speed c, we can simplify the expression above. In fact, applying the first equation in (E.66) to Equation (E.67), we obtain

$$L_0 - L = L_0[1 - \sqrt{1-(v/c)^2}] \approx L_0\left[1 - \left(1 - \tfrac{1}{2}(v/c)^2\right)\right] = \tfrac{1}{2}(v/c)^2 L_0 \tag{E.68}$$

This equation expresses the length shrinkage in terms of L_0. We can write the same shrinkage in terms of L by using the third equation in (E.66). Here is how:

$$L_0 - L = \frac{L}{\sqrt{1-(v/c)^2}} - L \approx L\left(1 + \tfrac{1}{2}(v/c)^2\right) - L = \tfrac{1}{2}(v/c)^2 L \tag{E.69}$$

Therefore, it does not matter which length we use on the right hand side. For all man-made vehicles, the approximation in (E.68) or (E.69) works extremely well.

The same procedure can be used to find the difference between the proper and improper time intervals. Thus,

$$\begin{aligned}\Delta t - \Delta\tau &= \Delta t - \Delta t\sqrt{1-(v/c)^2} = \Delta t[1-\sqrt{1-(v/c)^2}] \\ &\approx \Delta t\left[1-\left(1-\tfrac{1}{2}(v/c)^2\right)\right] = \tfrac{1}{2}(v/c)^2\Delta t \end{aligned} \tag{E.70}$$

A similar procedure as the one used in deriving Equation (E.69) shows that Δt on the right-hand side can be replaced by $\Delta\tau$.

Finally, applying the second of the approximations to $\gamma - 1$ (what we have called *relativisticity*), we obtain

$$\gamma - 1 = \frac{1}{\sqrt{1-(v/c)^2}} - 1 \approx 1 + \tfrac{1}{2}(v/c)^2 - 1 = \tfrac{1}{2}(v/c)^2, \tag{E.71}$$

a useful result when we want to calculate the relativisticity of the motion of ordinary objects. Let's put all these formulas together:

$$L_0 - L = \tfrac{1}{2}(v/c)^2 L_0, \quad \Delta t - \Delta\tau = \tfrac{1}{2}(v/c)^2\Delta t, \quad \gamma - 1 = \tfrac{1}{2}(v/c)^2 \tag{E.72}$$

where L_0 on the right hand side of the first equation and Δt on the right hand side of the second equation could be replaced by L and $\Delta\tau$, respectively.

E.27 Math Notes for Chapter 27

Transformation rule connecting a pair of coordinates **(page 397 of the book)**

Math Note E.27.1. Figure E.20 shows a point P with coordinates (x, y) in the system O and (x', y') in O'. We want to express (x', y') in terms of (x, y). The origin of O has coordinates (a, b) in O', and its x-axis makes an angle α with x'-axis corresponding to a slope m. All the angles marked off can be shown to be equal to α using simple high school geometry.

Let us find x' first. From the figure it is clear that

$$x' = a + \overline{AA'} = a + \overline{OC} = a + \overline{OD} - \overline{CD} = a + \overline{OD} - \overline{QR} \tag{E.73}$$

We want to calculate $\overline{OD}$ and $\overline{QR}$ in terms of the coordinates in O as well as the quantity that distinguishes O from O', namely the slope m of the x-axis relative to x'-axis. The right triangle ODR and the definition of the slope give

$$(\overline{OR})^2 = (\overline{OD})^2 + (\overline{DR})^2 \quad \text{and} \quad m = \frac{\overline{DR}}{\overline{OD}} \quad \text{or} \quad \overline{DR} = m(\overline{OD}) \tag{E.74}$$

Thus, substituting the last equation in the first, we get

$$(\overline{OR})^2 = (\overline{OD})^2 + [m(\overline{OD})]^2 = (\overline{OD})^2(1+m^2) \quad \text{or} \quad \overline{OR} = \overline{OD}\sqrt{1+m^2}$$

But $\overline{OR} = x$; so we get

$$\overline{OD} = \frac{x}{\sqrt{1+m^2}} \tag{E.75}$$

The similarity of the triangles QRP and DRO implies similar relations between its corresponding sides:

$$\overline{PQ} = \frac{\overline{PR}}{\sqrt{1+m^2}} \quad \text{and} \quad \overline{QR} = m(\overline{PQ}) \quad \text{or} \quad \overline{PQ} = \frac{\overline{QR}}{m} \tag{E.76}$$

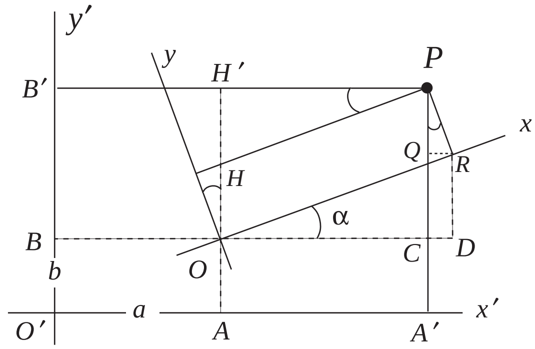


Figure E.20: The same point P has different pairs of coordinates in different coordinate systems.

It follows that

$$\frac{\overline{QR}}{m} = \frac{\overline{PR}}{\sqrt{1+m^2}} \quad \text{or} \quad \overline{QR} = \frac{m\,\overline{PR}}{\sqrt{1+m^2}} = \frac{my}{\sqrt{1+m^2}} \tag{E.77}$$

because $\overline{PR} = y$. Substituting all these in Equation (E.73), we obtain

$$x' = a + \frac{x}{\sqrt{1+m^2}} - \frac{my}{\sqrt{1+m^2}} = a + \frac{1}{\sqrt{1+m^2}}(x - my)$$

As for y', we have a similar result:

$$y' = b + \overline{BB'} = b + \overline{CP} = b + \overline{CQ} + \overline{QP} = b + \overline{DR} + \overline{PQ} \tag{E.78}$$

But $\overline{DR} = m(\overline{OD})$ by Equation (E.74) and $\overline{QP} = \overline{QR}/m$ by Equation (E.76). Substituting in (E.78) for $\overline{OD}$ and $\overline{QR}$ in terms of coordinates as found in (E.75) and (E.77), respectively, we get

$$y' = b + \frac{mx}{\sqrt{1+m^2}} + \frac{my/\sqrt{1+m^2}}{m} = b + \frac{1}{\sqrt{1+m^2}}(mx + y)$$

Let us put these two transformation rules together:

$$\begin{aligned} x' &= a + \frac{1}{\sqrt{1+m^2}}(x - my) \\ y' &= b + \frac{1}{\sqrt{1+m^2}}(mx + y) \end{aligned} \tag{E.79}$$

As a check, we note that if $a = 0$, $b = 0$ (so that the two origins coincide), and $m = 0$ (so that $1/\sqrt{1+m^2} = 1$), then

$$x' = 0 + 1(x - 0) = x \quad \text{and} \quad y' = 0 + 1(0 + y) = y$$

as we should, because the two coordinate systems are really the same!

In most applications we assume that the two origins coincide. This corresponds to a rotation of the axes without displacing the origin. When we are dealing with the distance between two points—a quantity that is at the root of all coordinated geometries—the displacement of the origin, expressed by a and b above, does not enter in the formulas (see Math Note E.27.3). Therefore, most of the times we write the transformation rules of

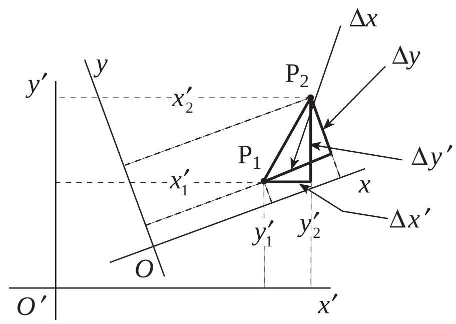


Figure E.21: The distance $\overline{P_1 P_2}$ is the same for both O and O'.

Equation (E.79) as

$$
\begin{aligned}
x' &= \frac{1}{\sqrt{1+m^2}}(x - my) \\
y' &= \frac{1}{\sqrt{1+m^2}}(mx + y)
\end{aligned}
\tag{E.80}
$$

Readers familiar with trigonometry may identify m with $\tan\alpha$. Then

$$
\frac{1}{\sqrt{1+m^2}} = \cos\alpha \quad \text{and} \quad \frac{m}{\sqrt{1+m^2}} = \sin\alpha,
$$

A familiarity with trigonometry is needed for the rest of this Math Note.

and Equation (E.80) can be written as

$$
\begin{aligned}
x' &= x\cos\alpha - y\sin\alpha \\
y' &= x\sin\alpha + y\cos\alpha
\end{aligned}
\tag{E.81}
$$

To check the correctness of this formula let $\alpha = 90°$. Then we *know* that x turns into y' and y into *negative* x'. Does Equation (E.81) agree with this conclusion? To see this, substitute $90°$ for α:

$$
\begin{aligned}
x' &= x\cos(90°) - y\sin(90°) = -y \\
y' &= x\sin(90°) + y\cos(90°) = x
\end{aligned}
$$

because $\cos(90°) = 0$ and $\sin(90°) = 1$.

Finding the Euclidean distance **(page 397 of the book)**

Math Note E.27.2. In this example, we want to calculate the distance between two points in terms of the coordinates of the points. Figure E.21 shows two points P_1 and P_2; P_1 has coordinates (x_1, y_1) in O and (x'_1, y'_1) in O' (only the coordinates in O' are labeled to avoid excessive cluttering of the figure); P_2 has coordinates (x_2, y_2) in O and (x'_2, y'_2) in O'. P_1 and P_2 form the line segment $\overline{P_1 P_2}$, whose length we denote by Δr, which—using Pythagoras' theorem—can be expressed in terms of $\Delta x \equiv x_2 - x_1$ and $\Delta y \equiv y_2 - y_1$ or in terms of $\Delta x' \equiv x'_2 - x'_1$ and $\Delta y' \equiv y'_2 - y'_1$:

Euclidean distance

$$
\Delta r \equiv \overline{P_1 P_2} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x')^2 + (\Delta y')^2}
\tag{E.82}
$$

This is a very important relation. It states the fact that the (straight) distance between any two points is independent of the coordinate system we choose.

Math Note E.27.3. Let P_1 and P_2 be any two points. Suppose that P_1 has coordinates (x_1, y_1) in O and (x'_1, y'_1) in O' and P_2 has coordinates (x_2, y_2) in O and (x'_2, y'_2) in O'. We use the coordinate transformation rules (E.79) of Math Note E.27.1 to write all the primed coordinates in terms of the unprimed coordinates. To save writing, denote $1/\sqrt{1+m^2}$ by C and $m/\sqrt{1+m^2}$ by S (note that $S^2 + C^2 = 1$). Then we have

$$\begin{aligned} x'_1 &= a + Cx_1 - Sy_1, \quad x'_2 = a + Cx_2 - Sy_2 \\ y'_1 &= b + Sx_1 + Cy_1, \quad y'_2 = b + Sx_2 + Cy_2 \end{aligned} \tag{E.83}$$

Space transformation does not affect distance (**page 397 of the book**)

The next step is to find $\Delta x' \equiv x'_2 - x'_1$ and $\Delta y' \equiv y'_2 - y'_1$ in terms of $\Delta x \equiv x_2 - x_1$ and $\Delta y \equiv y_2 - y_1$. But these are easy to do. For example,

$$\begin{aligned} \Delta x' \equiv x'_2 - x'_1 &= a + Cx_2 - Sy_2 - (a + Cx_1 - Sy_1) \\ &= C(x_2 - x_1) - S(y_2 - y_1) = C(\Delta x) - S(\Delta y) \end{aligned}$$

Similarly, $\Delta y' = S(\Delta x) + C(\Delta y)$. Note that the constants a and b have disappeared. That is why we ignored them in Equation (E.80) of Math Note E.27.1.

To find $\overline{P_1P_2}$ in O' (which we denote by $\Delta r'$, because we "don't know" yet that it is equal to Δr, the length of $\overline{P_1P_2}$ in O), we square $\Delta x'$, add it to the square of $\Delta y'$, and take the square root. To avoid introducing long square roots, let us calculate $(\Delta r')^2$ instead:

$$\begin{aligned} (\Delta r')^2 &= (\Delta x')^2 + (\Delta y')^2 = [C(\Delta x) - S(\Delta y)]^2 + [S(\Delta x) + C(\Delta y)]^2 \\ &= C^2(\Delta x)^2 + S^2(\Delta y)^2 - 2SC(\Delta x)(\Delta y) \\ &\quad + S^2(\Delta x)^2 + C^2(\Delta y)^2 + 2CS(\Delta x)(\Delta y) \\ &= (\Delta x)^2 \underbrace{(C^2 + S^2)}_{=1} + (\Delta y)^2 \underbrace{(S^2 + C^2)}_{=1} = (\Delta x)^2 + (\Delta y)^2 = (\Delta r)^2 \end{aligned}$$

Thus, whether we coordinatize the points in O or O', the distance $\overline{P_1P_2}$ comes out the same.

Math Note E.27.4. Let us start with

Derivation of Equation (E.79) using only algebra (**page 397 of the book**)

$$x' = a + cx + dy, \quad y' = b + ex + fy \tag{E.84}$$

where a, c, etc. are unknowns to be determined. We have assumed a linear relation (no x^2, y^3, or any other functions), because we want straight lines in O to remain straight lines in O'. For example, if $x' = 2x + y$ and $y' = 18x^2 + 3y$, then the straight line $y = x$ in O would transform into a parabola as the following equation shows:

$$\begin{aligned} x' &= 2x + y = 2x + x = 3x \quad \Rightarrow \quad x = \frac{x'}{3} \\ y' &= 18x^2 + 3y = 18x^2 + 3x = 18\left(\frac{x'}{3}\right)^2 + 3\left(\frac{x'}{3}\right) = 2x'^2 + x' \end{aligned}$$

So, the equation $y = x$ in O turns into the equation $y' = 2x'^2 + x'$ in O', which is a parabola. We don't want this to happen.

Now take any two points P_1 and P_2, where P_1 has coordinates (x_1, y_1) in O and (x'_1, y'_1) in O' and P_2 has coordinates (x_2, y_2) in O and (x'_2, y'_2) in O'. By assumption these coordinates are related via Equation (E.84) as follows:

$$\begin{aligned} x'_1 &= a + cx_1 + dy_1, \quad x'_2 = a + cx_2 + dy_2 \\ y'_1 &= b + ex_1 + fy_1, \quad y'_2 = b + ex_2 + fy_2 \end{aligned} \tag{E.85}$$

Next, find $\Delta x' \equiv x'_2 - x'_1$ and $\Delta y' \equiv y'_2 - y'_1$ in terms of $\Delta x \equiv x_2 - x_1$ and $\Delta y \equiv y_2 - y_1$. But these are easy to do. For example,

$$\Delta x' \equiv x'_2 - x'_1 = a + cx_2 + dy_2 - (a + cx_1 + dy_1) = c(\Delta x) + d(\Delta y)$$

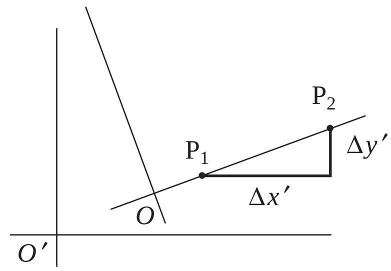


Figure E.22: Two points on the x-axis and their coordinate differences in O'.

Similarly, $\Delta y' = e(\Delta x) + f(\Delta y)$. Put these together for future reference:

$$\begin{aligned} \Delta x' &= c(\Delta x) + d(\Delta y) \\ \Delta y' &= e(\Delta x) + f(\Delta y) \end{aligned} \tag{E.86}$$

Note again that the constants a and b have disappeared.

The next step is to demand that the two distances calculated in O and O' be equal; i.e., that $(\Delta x')^2 + (\Delta y')^2 = (\Delta x)^2 + (\Delta y)^2$. Substituting for $\Delta x'$ and $\Delta y'$ in terms of Δx and Δy, we get

$$[c(\Delta x) + d(\Delta y)]^2 + [e(\Delta x) + f(\Delta y)]^2 = (\Delta x)^2 + (\Delta y)^2$$

or

$$\begin{aligned} &c^2(\Delta x)^2 + d^2(\Delta y)^2 + 2cd(\Delta x)(\Delta y) + e^2(\Delta x)^2 + f^2(\Delta y)^2 + 2ef(\Delta x)(\Delta y) \\ &= (\Delta x)^2 + (\Delta y)^2 \end{aligned}$$

or

$$(c^2 + e^2)(\Delta x)^2 + (d^2 + f^2)(\Delta y)^2 + 2(cd + ef)(\Delta x)(\Delta y) = (\Delta x)^2 + (\Delta y)^2$$

If the two sides of this equation are to be equal for *any* values of Δx and Δy (corresponding to *any* two points in the plane), we must have

$$c^2 + e^2 = 1, \quad d^2 + f^2 = 1, \quad cd + ef = 0 \tag{E.87}$$

When $\Delta y = 0$, i.e., when P_1 and P_2 lie along the x-axis, we must get $\Delta y'/\Delta x' = m$ as Figure E.22 demonstrates. On the other hand, Equation (E.86) gives

$$\Delta x' = c(\Delta x), \quad \Delta y' = e(\Delta x) \quad \Rightarrow \quad \frac{\Delta y'}{\Delta x'} = \frac{e(\Delta x)}{c(\Delta x)} = \frac{e}{c}$$

Thus, $e/c = m$ or $e = mc$. The first equation in (E.87) now gives

$$c^2 + (mc)^2 = 1, \quad \text{or} \quad c^2(1 + m^2) = 1 \quad \Rightarrow \quad c = \pm\frac{1}{\sqrt{1 + m^2}}$$

With c so determined, we can determine e: $e = \pm m/\sqrt{1 + m^2}$. Using the values of c and e (either plus or minus) in the last equation of (E.87), we get $d = -mf$. Now we can decide which sign to choose. If we choose the negative sign, then Equation (E.84) would yield $x' = -x$ for $m = 0$, $a = 0$, and $b = 0$ ($d = -mf$ is also zero). But this is impossible, because for these parameters, the two coordinates should coincide. Putting $d = -mf$ in the middle equation, we obtain

$$m^2 f^2 + f^2 = 1 \quad \text{or} \quad f^2 = \frac{1}{1 + m^2} \quad \Rightarrow \quad f = \pm\frac{1}{\sqrt{1 + m^2}}$$

Figure E.23: Three Euclidean coordinate systems O, O', and O''. The thickness of the axes emphasize the difference between the systems.

Again, we have to choose the plus sign, because otherwise Equation (E.84) would yield $y' = -y$ for $m = 0$, $a = 0$, and $b = 0$. Substituting all the unknowns in Equation (E.84) yields Equation (E.79).

Math Note E.27.5. To find the invariant distance of the spacetime geometry, we must bring the Euclidean geometry closer to the spacetime geometry. Since time is a "hidden" dimension, we "hide" one of the dimensions in the familiar Euclidean 2D space. Let us represent this hidden, but otherwise measurable coordinate by the y-axis. We can draw this axis in a plane (just as we can draw a time axis in a spacetime plane); we can also draw points in the plane and assign coordinates to them, but, for instance, we don't know what the length of the line segment connecting two points is and how it is written in terms of their coordinates, because we don't have a measuring device that can determine this distance. However, if the two points happen to lie on either of the two axes, we can tell how far apart they are, i.e., what their distance is.

Deriving spacetime interval using a Euclidean analogy **(page 398 of the book)**

How can you measure a quantity that is "hidden?" Suppose that y represents height. Remember, we can't see the height of an object, but we can have an "*acrometer*" that measures the height of a point. This acrometer could be built on the principle that the strength of the impact of a falling object on a stationary object is proportional to the height from which the first object is dropped. To measure the height of a point, simply drop a standard object (say a mass of 1 kg) from that point on the acrometer. This enables us to measure the height of a point without having to be aware of "height" as a dimension. This is exactly what happens with time. We can measure time with a clock without being aware of it as an extra dimension.

With the second dimension absent, any preconceived geometric notion that requires two dimensions is out of our reach. For example, the notion of an angle does not exist because it requires two crossing lines. Of course, there are many different coordinate systems that measure different "heights" for the same point, but we are not allowed to draw these systems as we did in Figure 27.5. The best we can do is draw *one axis* per "observer" and label them differently. We can emphasize this difference by drawing the axes differently (say with different thicknesses) as shown in Figure E.23, but in most cases we simply label them with different letters.

How do we know *physically* that O' is different from O if they are both a single horizontal line? The property that distinguishes among different CSs is their *relative slopes.* How can we determine the slope without seeing the angle that the two axes make with one another? Remember, we don't need y to determine a height; we have acrometers to do that. So, observer O' can pick two points, P_1 and P_2, on the (only real) axis of O, determine their coordinates x'_1 and x'_2, measure their heights y'_1 and y'_2 using his own acrometer, and take the ratio $(x'_2 - x'_1)/(y'_2 - y'_1) = \Delta x'/\Delta y'$.[17]

Now we are ready to "derive" an expression for the distance in a Euclidean plane. First we note that

[17] The slope is normally defined as $\Delta y'/\Delta x'$. However, in preparation for the derivation of the spacetime "distance" (our main interest here), we have switched the numerator and the denominator.

Box E.27.6. *If two points have the same x-coordinates (so that $\Delta x = 0$), the distance is the difference in their heights as measured by an acrometer.*

We need one more piece of information before deriving the distance formula. Suppose that the "one-dimensional" creatures living in such Euclidean spaces have discovered the following fact:

> Take points P_1 and P_2, which *happen to lie on the y-axis of O*, who measures Δy to be the y-difference between these two points. Another observer O' measures the y-difference between the same two points to be $\Delta y'$. If the slope of these observers is m, then the relation between Δy and $\Delta y'$ is given by

$$\Delta y' = \frac{\Delta y}{\sqrt{1+m^2}} \quad \text{or} \quad \Delta y' \sqrt{1+m^2} = \Delta y \tag{E.88}$$

With this extra information we can finally derive the distance equation. By the previous paragraph, $m = \Delta x'/\Delta y'$. Substituting this in Equation (E.88) gives

$$\Delta y' \sqrt{1+m^2} = \Delta y \quad \Rightarrow \quad \Delta y = \Delta y' \sqrt{1 + \left(\frac{\Delta x'}{\Delta y'}\right)^2} = \sqrt{(\Delta y')^2 + (\Delta x')^2}$$

Recalling that Δy *is* the distance between P_1 and P_2,[18] we have

$$\text{Euclidean distance } \overline{P_1P_2} = \sqrt{(\Delta x')^2 + (\Delta y')^2} \tag{E.89}$$

Since O' could be *any* observer, this relation holds true for all observers (they measure different $\Delta x'$ and $\Delta y'$, but when they square them and add them, the result will be the same).

spacetime interval

Having gone through the steps leading to Equation (E.89), it is easy to find the distance in a spacetime plane. We call it the **spacetime interval** and denote it by Δs. Two RFs are distinguished by their relative speed v (this replaces the relative slope of the Euclidean plane). Let E_1 and E_2 be two events in our spacetime plane. We need an observer O for whom the two events occur either at the same location or at the same time. The first choice is impossible—and this is the crucial difference between the spacetime plane and the Euclidean plane—because O has to be present at two different places at the same time! Thus, we have to be content with the second choice: we look for an observer O for whom these events occur at the same location; i.e., an observer that *measures the proper time* $\Delta\tau$ (see Section 26.1) between the two events.

Our Euclidean analogy stated in Box E.27.6 tells us that $\Delta s = c\Delta\tau$ (we introduce c to make Δs a distance). Now take any other observer O' relative to whom O moves with speed v. Equation (26.1) then yields

$$\Delta s = c\Delta\tau = c\Delta t' \sqrt{1 - v^2/c^2} = \sqrt{c^2(\Delta t')^2(1 - v^2/c^2)} = \sqrt{c^2(\Delta t')^2 - v^2(\Delta t')^2)}$$

But since O is present at both events, v (being the speed of O relative to O') is precisely the distance $\Delta x'$ covered by O in O' divided by the time $\Delta t'$. Therefore,

$$v^2(\Delta t')^2 = (v\Delta t')^2 = (\Delta x')^2$$

and we obtain one of the most radical equations in physics (and mathematics):

$$\Delta s = c\Delta\tau = \sqrt{c^2(\Delta t')^2 - (\Delta x')^2} \tag{E.90}$$

We can ignore the primes, as the formula holds for *all* RFs.

[18] Because they both lie on the y-axis of O.

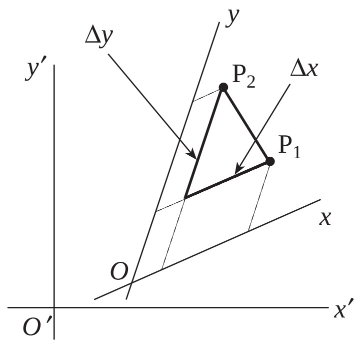


Figure E.24: If the Euclidean distance formula is to hold, i.e., if $\overline{P_1P_2}^2$ is to equal $(\Delta x)^2 + (\Delta y)^2$, then the x-axis must be perpendicular to the y-axis.

Math Note E.27.7. Two points P_1 and P_2 are located in the plane of the paper (see Figure E.24). We can always take one coordinate system (CS) to have perpendicular axes. The question is whether the other CS's are *necessarily* perpendicular. Take x'- and y'-axes to be perpendicular to each other. Suppose we draw the y-axis. What is the relation of the x-axis to this y-axis? Draw an arbitrary line intersecting the y-axis and call it the x-axis. In this (slanted) x-y coordinate system, Δx and Δy can be drawn by projecting P_1 and P_2 on the axes as shown.

Now by assumption, $\overline{P_1P_2}^2$ must equal $(\Delta x)^2 + (\Delta y)^2$. This can happen only if $\overline{P_1P_2}$ is the hypotenuse of a right triangle with perpendicular sides Δx and Δy; i.e., if the x-axis is perpendicular to the y-axis. Although this conclusion may seem trivial in ordinary plane geometry, it becomes highly nontrivial when applied to the spacetime geometry.

Math Note E.27.8. We note that $\overline{CD} = \overline{BD} + \overline{CB} = \overline{BD} + \overline{AB} - \overline{AC}$. But, because light travels on a worldline that makes and angle of 45° with the axes, $\overline{BD} = \overline{BE_2}$ and $\overline{AC} = \overline{AE_1}$. So $\overline{CD} = \overline{BE_2} + \overline{AB} - \overline{AE_1} = (\overline{BE_2} - \overline{AE_1}) + \overline{AB}$, or $\overline{CD} = (\overline{BE_2} - \overline{BF}) + \overline{AB}$. Thus, $\overline{CD} = \overline{FE_2} + \overline{AB}$. Furthermore, $\overline{FE_2}/\overline{FE_1} = \beta$ by rule 2 of Box F.0.2, and $\overline{FE_1} = \overline{AB}$, yielding $\overline{FE_2} = \beta\,\overline{AB}$. We, therefore, have $\overline{CD} = \beta\,\overline{AB} + \overline{AB}$. Invoking rule 4 of Box F.0.3 we get $\overline{AB} = \gamma\overline{E_1E_2}$ and

Finding $\overline{CD}$ of Figure E.25(a) in terms of v and T **(page 401 of the book)**

$$\overline{CD} = \beta\,\overline{AB} + \overline{AB} = (1+\beta)\overline{AB} = (1+\beta)\gamma\overline{E_1E_2}$$

But $\overline{CD} = cT' = \lambda'$ and $\overline{E_1E_2} = cT = \lambda$; therefore

$$\lambda' = (1+\beta)\gamma\lambda = (1+\beta)\frac{1}{\sqrt{1-\beta^2}}\lambda = \frac{1+\beta}{\sqrt{(1-\beta)(1+\beta)}}\lambda$$

or

$$\lambda' = \sqrt{\frac{1-\beta}{1+\beta}}\,\lambda \tag{E.91}$$

Math Note E.27.9. From C in Figure E.25(b) draw an upward ray parallel to ct'-axis. This ray intersects $\overline{E_2D}$ at F and $\overline{DB}$ at H. Also drop the perpendicular $\overline{DG}$ onto $\overline{CH}$. The four heavily marked angles are 45° and the other two marked angles have a slope β. It is quite obvious that $\overline{CF} = \overline{E_1E_2}$ and $\overline{CH} = \overline{AB}$. Therefore, we can write

Finding $\overline{AB}$ in terms of $\overline{E_1E_2}$ in Figure E.25(b) **(page 401 of the book)**

$$\overline{AB} = \overline{CH} = \overline{CF} + \overline{FH} = \overline{CF} + 2(\overline{FG}) = \overline{E_1E_2} + 2(\overline{FG}) \tag{E.92}$$

All that is left to do is to write $\overline{FG}$ in terms of $\overline{E_1E_2}$. But $\overline{FG} = \overline{GD}$, because the triangle DFG is isosceles. Furthermore, $\overline{GD}/\overline{CG} = \beta$. Thus,

$$\overline{FG} = \beta\overline{CG} = \beta(\overline{CF} + \overline{FG}) = \beta\overline{E_1E_2} + \beta\overline{FG}$$

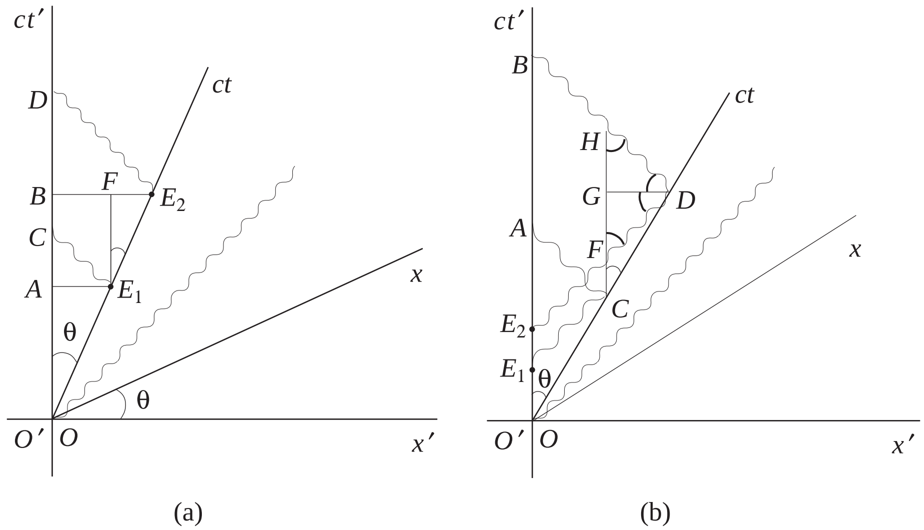


Figure E.25: (a) The spacetime diagram of the Doppler effect. (b) The spacetime diagram of the Doppler effect including reflection from a moving object. In both diagrams, the dashed lines represent light signals.

This can be rewritten as

$$\overline{FG} - \beta\overline{FG} = \beta\overline{E_1E_2} \quad \text{or} \quad (1-\beta)\overline{FG} = \beta\overline{E_1E_2} \quad \Rightarrow \quad \overline{FG} = \frac{\beta}{1-\beta}\overline{E_1E_2}$$

Substituting this in (E.92) we get our desired result:

$$\overline{AB} = \overline{E_1E_2} + 2(\overline{FG}) = \overline{E_1E_2} + \frac{2\beta}{1-\beta}\overline{E_1E_2} = \frac{1+\beta}{1-\beta}\overline{E_1E_2}$$

Since $\overline{AB} = cT_{\text{ref}} = \lambda_{\text{ref}}$ and $\overline{E_1E_2} = cT = \lambda$, we have

$$\lambda_{\text{ref}} = \frac{1+\beta}{1-\beta}\,\lambda \tag{E.93}$$

When Emmy's speed becomes much smaller than light speed, β is very small and we can make the following approximations:

$$\frac{1}{1-\beta} \approx 1+\beta \quad \text{and} \quad (1+\beta)^2 \approx 1+2\beta$$

Then Equation (E.93) reduces to

$$\lambda_{\text{ref}} = \frac{1+\beta}{1-\beta}\,\lambda = (1+\beta)\frac{1}{1-\beta}\lambda \approx (1+\beta)(1+\beta)\lambda = (1+\beta)^2\lambda = (1+2\beta)\lambda$$

This is precisely Equation (E.13) when we note that $\beta = v/c$ and v is the *relative* velocity of the two observers.

Deriving Lorentz transformations from spacetime diagrams **(page 163 of the book)**

Math Note E.27.10. Figure E.26 shows an event E with coordinates (x, ct) in O. We want to calculate its coordinates (x', ct') in O' in terms of x and ct. It is clear that $x' = \overline{O'A} + \overline{AB}$. But $\overline{O'A}$ is the projection of $\overline{OD}$—lying on the x-axis of O—onto the x'-axis. By rule 4 of Box F.0.3, $\overline{O'A} = \gamma\,\overline{OD} = \gamma x$. As for $\overline{AB}$, the figure shows that $\overline{AB} = \overline{CD}$. But rule 2 of Box F.0.2 gives $\beta = \overline{CD}/\overline{CE}$, because the marked angles are all equal. Now $\overline{CE} = \overline{O'M}$

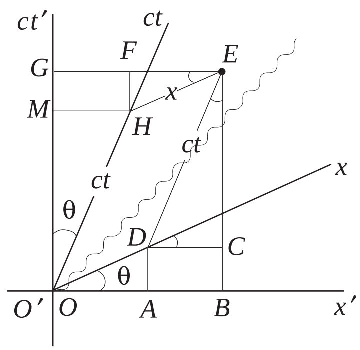


Figure E.26: Deriving the Lorentz transformations from the diagram.

and $\overline{O'M}$ is the projection of $\overline{OH}$ onto the time axis of O'. Again by rule 2 of Box F.0.2, $\overline{O'M} = \gamma\,\overline{OH} = \gamma ct$; so

$$\overline{AB} = \overline{CD} = \beta\overline{CE} = \beta\overline{O'M} = \beta\gamma ct$$

Therefore, $x' = \overline{O'A} + \overline{AB} = \gamma x + \beta\gamma ct = \gamma(x + \beta ct)$.

How is ct' related to x and ct? We note that $ct' = \overline{O'M} + \overline{MG}$. We have already calculated $\overline{O'M}$: it is equal to γct. To find $\overline{MG}$, we note that $\overline{MG} = \overline{FH} = \overline{DA} = \beta\overline{O'A}$ (the last equality follows from rule 2 again). But we have calculated $\overline{O'A}$ above: it is equal to γx. Therefore, $\overline{MG} = \beta\gamma x$, and

$$ct' = \overline{O'M} + \overline{MG} = \gamma ct + \beta\gamma x = \gamma(\beta x + ct)$$

We have obtained the celebrated **Lorentz transformation**:

$$\begin{aligned} x' &= \gamma(x + \beta ct) \\ ct' &= \gamma(\beta x + ct) \end{aligned} \tag{E.94}$$

If we solve these two equations for x in terms of x', t, and t' and set them equal, we get an equation involving x', t, and t'. We can solve this equation for t in terms of x' and t'. The result is $ct = \gamma(-\beta x' + ct')$. We can do the same with x and get a similar result. We put these two equations together:

$$\begin{aligned} x &= \gamma(x' - \beta ct') \\ ct &= \gamma(-\beta x' + ct') \end{aligned} \tag{E.95}$$

This is called the Lorentz transformation **inverse** to Equation (E.94).

inverse Lorentz transformation

We could have guessed (E.95) if we had noted that the only difference between O and O' is that O moves in the *positive* direction of O', while O' moves in the *negative* direction of O. This means that the only difference in the Lorentz transformation relating one RF to the other should be in the sign of speed v, and consequently of β. We therefore make the following convention: In Equation (E.94), β is positive if O moves along the positive direction of O'; it is negative if O moves along the negative direction of O'.

Most often we are interested in space and time *intervals* between two events. If events E_1 and E_2 have respective coordinates (x_1, ct_1) and (x_2, ct_2) relative to O and (x'_1, ct'_1) and (x'_2, ct'_2) relative to O', then

$$x'_1 = \gamma(x_1 + \beta ct_1), \quad \text{and} \quad x'_2 = \gamma(x_2 + \beta ct_2)$$

and

$$\begin{aligned}\Delta x' &= x_2' - x_1' = \gamma(x_2 + \beta c t_2) - \gamma(x_1 + \beta c t_1)\\ &= \gamma[(x_2 - x_1) + \beta c(t_2 - t_1)] = \gamma(\Delta x + \beta c \Delta t)\end{aligned}$$

with a corresponding equation for time. We write both below:

$$\begin{aligned}\Delta x' &= \gamma(\Delta x + \beta c \Delta t)\\ c\Delta t' &= \gamma(\beta \Delta x + c\Delta t)\end{aligned} \tag{E.96}$$

To appreciate the ease and power of the geometric and diagrammatic reasoning, consult Math Note E.27.11 for an algebraic derivation of the Lorentz transformations!

Math Note E.27.11. As in the case of Math Note E.27.4, we start with[19]

$$x' = a + dx + et, \quad t' = b + fx + gt \tag{E.97}$$

where a, d, etc. are unknowns to be determined. For the same reason as in Math Note E.27.4 we have assumed a linear relation.

Now take any two events E_1 and E_2, where E_1 has coordinates (x_1, ct_1) in O and (x_1', ct_1') in O' and E_2 has coordinates (x_2, ct_2) in O and (x_2', ct_2') in O'. By assumption these coordinates are related via Equation (E.97) as follows:

$$\begin{aligned}x_1' &= a + dx_1 + et_1, \quad x_2' = a + dx_2 + et_2\\ t_1' &= b + fx_1 + gt_1, \quad t_2' = b + fx_2 + gt_2\end{aligned} \tag{E.98}$$

As in Math Note E.27.4, we next find $\Delta x'$ and $\Delta t'$ in terms of Δx and Δt:

$$\begin{aligned}\Delta x' &= d(\Delta x) + e(\Delta t)\\ \Delta t' &= f(\Delta x) + g(\Delta t)\end{aligned} \tag{E.99}$$

Note again that the constants a and b have disappeared.

The next step is to demand that the two spacetime intervals calculated in O and O' be equal; i.e., that $(c\Delta t')^2 - (\Delta x')^2 = (c\Delta t)^2 - (\Delta x)^2$. Substituting for $\Delta x'$ and $\Delta t'$ in terms of Δx and Δt, we get

$$[cf(\Delta x) + cg(\Delta t)]^2 - [d(\Delta x) + e(\Delta t)]^2 = (c\Delta t)^2 - (\Delta x)^2$$

or

$$\begin{aligned}c^2f^2(\Delta x)^2 + c^2g^2(\Delta t)^2 + 2c^2fg(\Delta x)(\Delta t) - d^2(\Delta x)^2 - e^2(\Delta t)^2 - 2de(\Delta x)(\Delta t)\\ = c^2(\Delta t)^2 - (\Delta x)^2\end{aligned}$$

or

$$(c^2f^2 - d^2)(\Delta x)^2 + (c^2g^2 - e^2)(\Delta t)^2 + 2(c^2fg - de)(\Delta x)(\Delta t) = c^2(\Delta t)^2 - (\Delta x)^2$$

If the two sides of this equation are to be equal for *any* values of Δx and Δt (corresponding to *any* two events in the spacetime plane), we must have

$$c^2f^2 - d^2 = -1, \quad c^2g^2 - e^2 = c^2, \quad c^2fg - de = 0 \tag{E.100}$$

[19] We have been using ct as the second coordinate. However, to keep the algebra simple, we prefer to leave c out for the time being. We will reintroduce c at the end.

When $\Delta x = 0$, i.e., when E_1 and E_2 occur at the same point along x-axis, we must get $\Delta x'/\Delta t' = v$ the speed of that point on the x-axis relative to O'. On the other hand, Equation (E.99) gives

$$\Delta x' = e(\Delta t), \quad \Delta t' = g(\Delta t) \quad \Rightarrow \quad \frac{\Delta x'}{\Delta t'} = \frac{e(\Delta t)}{g(\Delta t)} = \frac{e}{g} \tag{E.101}$$

Thus, $e/g = v$ or $e = gv$. Furthermore, since E_1 and E_2 occur at the same point, Δt is the proper time. It follows from Equation (26.1) and the second equation in (E.101) that $g = 1/\sqrt{1-(v/c)^2}$. With e and g so determined, the last equation in (E.100) yields

$$c^2 fg - dgv = 0, \quad \text{or} \quad c^2 f = dv \quad \text{or} \quad f = \frac{dv}{c^2}$$

Substituting this in the first equation of (E.100), we get

$$c^2\left(\frac{dv}{c^2}\right)^2 - d^2 = -1 \quad \Rightarrow \quad d^2 - \frac{d^2v^2}{c^2} = 1 \quad \text{or} \quad d = \pm\frac{1}{\sqrt{1-(v/c)^2}} = \pm\gamma$$

Only the positive sign is acceptable, because when $a = 0$, $b = 0$, and $v = 0$, x' must equal x (the two RFs are not moving relative to one another). But, choosing the negative sign, d becomes -1 at $v = 0$, and Equation (E.97) (along with the fact that $e = gv = 0$) leads to $x' = -x$, which is not acceptable.

We can now write Equation (E.97) as

$$x' = a + \gamma x + \gamma vt, \quad t' = b + \gamma(v/c^2)x + \gamma t$$

Specializing to $a = 0$ and $b = 0$ and using $\beta = v/c$ (or $v = \beta c$), we can write the first equation as

$$x' = \gamma x + \gamma\beta ct = \gamma(x + \beta ct)$$

Multiply both sides of the second equation by c to get

$$ct' = c\gamma(v/c^2)x + c\gamma t = \gamma(v/c)x + \gamma ct = \gamma(\beta x + ct)$$

Impossibility of traveling back in time

Math Note E.27.12. Let E_1 be now on Earth, E_2 be the event we want to get to, and E_3 the event in the past of E_2 to which we first go to wait for E_2. We can think of O—the origin of the RF (call it Diracus) we are seeking—as an event as well [(see Figure E.27(a)]. In the following, we use prime for the coordinates of events in O' and label the *intervals* by the events they connect. For example, $\Delta x'_{10}$ is the space interval between E_1 and O as measured by O' (Earth); similarly, Δt_{23} is the time interval between E_2 and E_3 as measured by O (Diracus).

First we find the speed β of O relative to O' using the fact that E_3 is taking place NOW in O (so that $\Delta t_{03} = 0$). By dividing both sides of the second of the two equations

$$\begin{aligned} \Delta x'_{03} &= \gamma(\Delta x_{03} + \beta c\Delta t_{03}) = \gamma(\Delta x_{03} + 0) = \gamma\Delta x_{03} \\ c\Delta t'_{03} &= \gamma(\beta\Delta x_{03} + c\Delta t_{03}) = \gamma(\beta\Delta x_{03} + 0) = \gamma\beta\Delta x_{03} \end{aligned} \tag{E.102}$$

by the first, we find β:

$$\frac{c\Delta t'_{03}}{\Delta x'_{03}} = \frac{\gamma\beta\Delta x_{03}}{\gamma\Delta x_{03}} \quad \text{or} \quad \beta = \frac{c\Delta t'_{03}}{\Delta x'_{03}}$$

For convenience, use X for $\Delta x'_{03}$, T for $\Delta t'_{03}$ (which is also the time interval between E_1 and E_3, the amount of time we want to go back in the past), T_1 for $\Delta t'_{12}$, and T_2 for $\Delta t'_{23}$. Then β can be written as

$$\beta = \frac{cT}{X} \quad \text{from which we get} \quad X = \frac{cT}{\beta} \tag{E.103}$$

Now let us figure out how O sees the events of interest. Figure E.27(b) shows the situation for O. Event E_3 is on Diracus's x-axis. Its distance can be calculated from the first equation of (E.102):

$$\Delta x_{03} = -\frac{\Delta x'_{03}}{\gamma} = -\frac{X}{\gamma}$$

The negative sign indicates that E_3 is occurring left of Diracus's origin. The intervals between E_2 and E_3 *as measured by Diracus* are given by the inverse Lorentz transformations obtained from Lorentz transformations by changing β to $-\beta$:

$$\begin{aligned} \Delta x_{23} &= \gamma(\Delta x'_{23} - \beta c\Delta t'_{23}) = \gamma(0 - \beta c\Delta t'_{23}) = -\gamma\beta cT_2 \\ c\Delta t_{23} &= \gamma(-\beta\Delta x'_{23} + c\Delta t'_{23}) = \gamma(0 + c\Delta t'_{23}) = \gamma cT_2 \end{aligned} \tag{E.104}$$

Therefore, the space interval Δx_{20} between E_2 and Diracus (which is the distance between E_3 and Diracus plus the space interval between E_2 and E_3) is

$$\Delta x_{20} = \Delta x_{30} + \Delta x_{23} = -\frac{X}{\gamma} - \gamma\beta cT_2$$

and the time interval Δt_{20} between E_2 and Diracus's NOW, i.e., the time that E_2 will take place in Diracus's future is just γcT_2, because Δt_{20} is the same as Δt_{23}.

Summarizing, Diracus wants to send a probe to Earth so that it will reach Earth at the exact time that E_2 is happening. This probe has to travel a distance of $-\frac{X}{\gamma} - \gamma\beta cT_2$ in a time interval of γcT_2. Therefore, its speed should be (we ignore the negative sign of the speed, because we are just interested in seeing how fast the probe should be moving once its direction is determined)

$$\beta_{\text{probe}} = \frac{X/\gamma + \gamma\beta cT_2}{\gamma cT_2} = \beta + \frac{X}{\gamma^2 cT_2}$$

Substituting X from Equation (E.103) in this equation and using $T = T_1 + T_2$ and $1/\gamma^2 = 1 - \beta^2$ yields

$$\begin{aligned} \beta_{\text{probe}} &= \beta + \frac{cT/\beta}{\gamma^2 cT_2} = \beta + \frac{T_1 + T_2}{\beta\gamma^2 T_2} = \beta + \frac{1}{\beta}(1-\beta^2)\left(1 + \frac{T_1}{T_2}\right) \\ &= \beta + \left(\frac{1}{\beta} - \beta\right)\left(1 + \frac{T_1}{T_2}\right) = \beta + \frac{1}{\beta} + \frac{1}{\beta}\frac{T_1}{T_2} - \beta - \beta\frac{T_1}{T_2} \\ &= \frac{1}{\beta} + \frac{T_1}{T_2}\left(\frac{1}{\beta} - \beta\right) \end{aligned}$$

The last line shows that $\beta_{\text{probe}} > 1$ because the first term is greater than 1 and the second term is positive.

We have shown very generally that it is impossible to be present at an event that occurred in the past without violating relativity. We had to undergo a lot of algebraic torture in this Math Note to prove this. On the other hand, the proof given on page 402 uses hardly any equations. This goes to show how much geometric and diagrammatic approach facilitates relativistic discussions.

Relativistic law of addition of velocities **(page 405 of the book)**

Math Note E.27.13. Suppose that the speed of an object (a bullet) relative to observer O is v_b. How does she measure this speed? She *locates* the bullet at two different *times*, measures the distance between those locations, and divides by the time interval. In other words, she picks two events $E_1 = (x_1, ct_1)$ and $E_2 = (x_2, ct_2)$ and takes the ratio of $\Delta x = x_2 - x_1$ to $\Delta t = t_2 - t_1$.

Observer O' looks at O as she makes her speed measurement. He sees the two events as (x'_1, ct'_1) and (x'_2, ct'_2) and concludes that the speed of the bullet v'_b is the ratio of $\Delta x'$ to

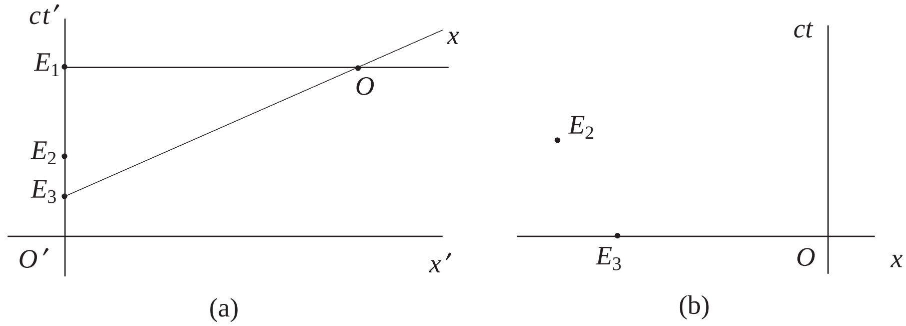


Figure E.27: (a) The events as they appear to O'. (b) The relevant events as they appear to O.

$\Delta t'$, where $\Delta x'$ and $\Delta t'$ are related to Δx and Δt via the Lorentz transformations. Thus, by Equation (E.96) we can write

$$\begin{aligned}\frac{v_b'}{c} &= \frac{\Delta x'}{c\Delta t'} = \frac{\gamma(\Delta x + \beta c\Delta t)}{\gamma(\beta\Delta x + c\Delta t)} = \frac{\Delta x + \beta c\Delta t}{\beta\Delta x + c\Delta t}\\ &= \frac{\Delta x/(c\Delta t) + \beta c\Delta t/(c\Delta t)}{\beta\Delta x/(c\Delta t) + c\Delta t/(c\Delta t)} = \frac{v_b/c + \beta}{\beta(v_b/c) + 1}\end{aligned}$$

where we divided the last equation of the first line by $c\Delta t$ to get to the first equation of the second line. Let us introduce two new but obvious notations: $\beta_b' \equiv v_b'/c$ and $\beta_b \equiv v_b/c$. Then the relation above can be written as

$$\beta_b' = \frac{\beta_b + \beta}{1 + \beta\beta_b} \tag{E.105}$$

If we multiply both sides by c and substitute for β and β_b in terms of v and v_b, Equation (E.105) becomes

$$v_b' = \frac{v_b + v}{1 + vv_b/c^2} \tag{E.106}$$

Now we can show quite generally that it is impossible to surpass the speed of light by "adding" two smaller speeds, regardless of how close they are to light speed. Consider the inequality $\beta_b < 1$. If you multiply both sides of this inequality by the positive quantity $1 - \beta$, the inequality still holds. So,

$$\beta_b(1-\beta) < 1 - \beta \quad \text{or} \quad \beta_b - \beta_b\beta < 1 - \beta$$

Now add $\beta + \beta\beta_b$ to both sides to get $\beta + \beta_b < 1 + \beta\beta_b$.

It follows that the numerator of Equation (E.105) is *always* smaller that the denominator; therefore, $\beta_b' < 1$. Although the bullet is moving close to the speed of light in the train and the train is also moving close to light speed, the observer on the platform will measure a combined speed for the bullet that is smaller than light speed.

Proof of spacetime triangle inequality **(page 406 of the book)**

Math Note E.27.14. We want to prove that the sum of two sides of a spacetime triangle is *less than* the third side. Figure E.28 shows a spacetime triangle whose vertices are the three events E_1, E_2, and E_3. The space and time intervals between any two events are shown on the corresponding axes.[20] For example, the space interval between E_1 and E_2 is x_{12} and the time interval between E_2 and E_3 is t_{23}. It should be clear that $x_{13} = x_{12} + x_{23}$ and $t_{13} = t_{12} + t_{23}$. Denote the spacetime interval between any two events similarly. Our

[20]To avoid the cluttering of symbols, we are not using the Δ-notation here.

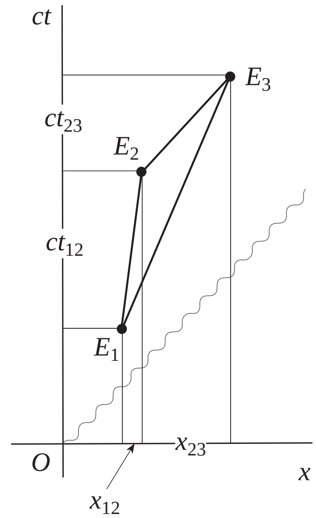


Figure E.28: The three events E_1, E_2, and E_3 form a spacetime triangle.

goal is to show that $s_{13} > s_{12} + s_{23}$. We first obtain the following relation among the three spacetime intervals:

$$\begin{aligned}
s_{13}^2 &= (ct_{13})^2 - x_{13}^2 = (ct_{12} + ct_{23})^2 - (x_{12} + x_{23})^2 \\
&= (ct_{12})^2 + (ct_{23})^2 + 2(ct_{12})(ct_{23}) - x_{12}^2 - x_{23}^2 - 2x_{12}x_{23} \\
&= \underbrace{(ct_{12})^2 - x_{12}^2}_{=s_{12}^2} + \underbrace{(ct_{23})^2 - x_{23}^2}_{=s_{23}^2} + 2[(ct_{12})(ct_{23}) - x_{12}x_{23}] \\
&= s_{12}^2 + s_{23}^2 + 2[(ct_{12})(ct_{23}) - x_{12}x_{23}]
\end{aligned}$$

Since $s_{12}^2 + s_{23}^2 = (s_{12} + s_{23})^2 - 2s_{12}s_{23}$, the last equation can be written as

$$s_{13}^2 = (s_{12} + s_{23})^2 + 2[(ct_{12})(ct_{23}) - x_{12}x_{23} - s_{12}s_{23}] \tag{E.107}$$

Our task has now been reduced to showing that the expression in the square brackets is positive. For this we manipulate the (square of the) last term in the square brackets:

$$\begin{aligned}
(s_{12}s_{23})^2 &= s_{12}^2 s_{23}^2 = [(ct_{12})^2 - x_{12}^2][(ct_{23})^2 - x_{23}^2] \\
&= \underbrace{(ct_{12})^2(ct_{23})^2 + x_{12}^2 x_{23}^2}_{\text{Call this term1}} - [(ct_{12})^2 x_{23}^2 + (ct_{23})^2 x_{12}^2] \\
&= \underbrace{[(ct_{12})(ct_{23}) - x_{12}x_{23}]^2 + 2(ct_{12})(ct_{23})x_{12}x_{23}}_{\text{This is also term1}} - [(ct_{12})^2 x_{23}^2 + (ct_{23})^2 x_{12}^2] \\
&= [(ct_{12})(ct_{23}) - x_{12}x_{23}]^2 - [(ct_{12})x_{23} - (ct_{23})x_{12}]^2
\end{aligned}$$

The last line tells us that

$$[(ct_{12})(ct_{23}) - x_{12}x_{23}]^2 = (s_{12}s_{23})^2 + [(ct_{12})x_{23} - (ct_{23})x_{12}]^2$$

or, since the second term on the right-hand side is positive,

$$[(ct_{12})(ct_{23}) - x_{12}x_{23}]^2 > (s_{12}s_{23})^2$$

This shows that

$$(ct_{12})(ct_{23}) - x_{12}x_{23} > s_{12}s_{23}$$

proving that the expression in the square brackets of Equation (E.107) is positive.

E.28 Math Notes for Chapter 28

Findingspacetime velocity in terms of ordinary velocity (**page 415 of the book**)

Math Note E.28.1. By the definition of the spacetime velocity, we have

$$u_{bx} = \frac{\Delta x_b}{\Delta \tau_b} = \frac{\Delta x_b/\Delta t_b}{\Delta \tau_b/\Delta t_b} = \frac{v_b}{\sqrt{1-(v_b/c)^2}} = \gamma_b v_b$$
$$u_{bt} = \frac{c\Delta t_b}{\Delta \tau_b} = \frac{c}{\sqrt{1-(v_b/c)^2}} = \gamma_b c \tag{E.108}$$

where we used Equation (26.1) noting that the speed v_b is that of the bullet as measured by Emmy. It should be clear that "bullet" represents *any* moving object of interest, including a light signal, for example. An important property of Equation (E.108) is

$$u_{bt}^2 - u_{bx}^2 = c^2 \tag{E.109}$$

which follows from the definition of the two components:

$$u_{bt}^2 - u_{bx}^2 = \left(\frac{c\Delta t_b}{\Delta \tau_b}\right)^2 - \left(\frac{\Delta x_b}{\Delta \tau_b}\right)^2 = \frac{(c\Delta t_b)^2 - (\Delta x_b)^2}{(\Delta \tau_b)^2} = \frac{(\Delta s_b)^2}{(\Delta \tau_b)^2} = c^2$$

The components of spacetime velocity satisfy Equation (E.109) in all RFs. For instance in the O' frame, we get

$$u_{bt}'^2 - u_{bx}'^2 = \left(\frac{c\Delta t_b'}{\Delta \tau_b}\right)^2 - \left(\frac{\Delta x_b'}{\Delta \tau_b}\right)^2 = \frac{(c\Delta t_b')^2 - (\Delta x_b')^2}{(\Delta \tau_b)^2} = \frac{(\Delta s_b)^2}{(\Delta \tau_b)^2} = c^2$$

One can think of $u_{bt}^2 - u_{bx}^2$ as the **invariant length** of the spacetime velocity in the 2D spacetime geometry.

Lorentz transformations of spacetime velocity leads to the relativistic law of addition of velocities (**page 416 of the book**)

Math Note E.28.2. All we need to do is to write the first equation in (28.2) in terms of v_b' and v_b. It is actually more convenient to write both sides in terms of the corresponding β's. Then the first equation in (28.2) yields

$$\gamma_b' v_b' = \gamma(\gamma_b v_b + \beta\gamma_b c) = \gamma\gamma_b(v_b + \beta c)$$

or

$$\frac{\beta_b' c}{\sqrt{1-\beta_b'^2}} = \frac{1}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{1-\beta_b^2}}(\beta_b c + \beta c)$$

dividing both sides by c and squaring both sides yields

$$\frac{\beta_b'^2}{1-\beta_b'^2} = \frac{(\beta_b+\beta)^2}{(1-\beta^2)(1-\beta_b^2)}$$

We now solve for $\beta_b'^2$ by first cross multiplying:

$$\beta_b'^2(1-\beta^2)(1-\beta_b^2) = (1-\beta_b'^2)(\beta_b+\beta)^2$$

or

$$\beta_b'^2(1-\beta^2)(1-\beta_b^2) = (\beta_b+\beta)^2 - \beta_b'^2(\beta_b+\beta)^2$$

Next we add $\beta_b'^2(\beta_b+\beta)^2$ to both sides:

$$\beta_b'^2 \underbrace{[(1-\beta^2)(1-\beta_b^2) + (\beta_b+\beta)^2]}_{=[1+\beta^2\beta_b^2+2\beta_b\beta]=(1+\beta\beta_b)^2} = (\beta_b+\beta)^2$$

Finally, we solve for $\beta_b'^2$ and take its square root:

$$\beta_b'^2(1+\beta\beta_b)^2 = (\beta_b+\beta)^2 \quad \Rightarrow \quad \beta_b'^2 = \frac{(\beta_b+\beta)^2}{(1+\beta\beta_b)^2} \quad \text{or} \quad \beta_b' = \frac{\beta_b+\beta}{1+\beta\beta_b}$$

This is identical to Equation (E.105), giving the relativistic law of addition of velocities in terms of β's.

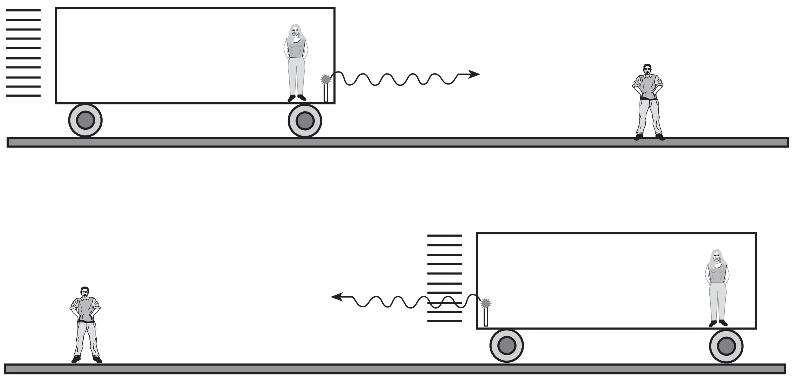


Figure E.29: The light emitted by Emmy has a different energy and momentum than that received by Karl.

Identification of p_{bt} with a known classical quantity
(**page 416 of the book**)

Math Note E.28.3. If we completely neglect v/c and set γ_b equal to 1, we obtain $p_{bt} = mc$, which we cannot identify with any Newtonian quantity.[21] So, we do the next best thing: we approximate $1/\sqrt{1-\beta_b^2}$ using Math Note E.26.5. Then, we obtain

$$p_{bt} = \frac{mc}{\sqrt{1-\beta_b^2}} \approx mc\left(1+\tfrac{1}{2}\beta_b^2\right) = mc + \tfrac{1}{2}mc\left(v_b^2/c^2\right) = mc + \tfrac{1}{2}m\frac{v_b^2}{c}$$

Multiplying both sides by c, we get

$$p_{bt}c \approx mc^2 + \tfrac{1}{2}mv_b^2$$

This tells us that in the Newtonian limit, $p_{bt}c - mc^2$ is the kinetic energy. We extrapolate to relativity and call $p_{bt}c - mc^2$ the *relativistic kinetic energy*, and write

$$p_{bt}c = mc^2 + \text{relativistic KE} \quad \text{or} \quad p_{bt}c = E_b$$

where E_b is called simply the "energy" of the bullet. It consists of a kinetic energy part and mc^2, the "rest energy" part.

Derivation of relativistic Doppler formula
(**page 419 of the book**)

Math Note E.28.4. Emmy (observer O) is moving with speed v toward Karl (observer O') in his positive direction as shown in Figure E.29(a). She emits a beam of light, whose energy and momentum she measures to be E and p with $E = pc$. Karl receives this signal and measures its energy and momentum to be E' and p', related to E and p via Lorentz transformations of Equation (28.5):

$$\begin{aligned} p' &= \gamma(p+\beta E/c) = \gamma(p+\beta p) = \frac{p+\beta p}{\sqrt{1-\beta^2}} = \sqrt{\frac{1+\beta}{1-\beta}}\,p \\ E' &= \gamma(\beta pc + E) = \gamma(\beta E + E) = \sqrt{\frac{1+\beta}{1-\beta}}\,E \end{aligned} \tag{E.110}$$

Some interesting results come out of this equation. First we note that, since $E = pc$, $E' = p'c$ as well. This is what we expect, as the energy of a photon is its momentum times its speed in all RFs. The second result is that $p' \neq p$. Thus, although a photon moves with the same speed in all RFs, its momentum is different. The third (and the most important)

[21] We can interpret the result as the momentum of the bullet when it moves with light speed, but that is prohibited by relativity.

result is obtained by using the Planck-Einstein relation $E = hc/\lambda$ on both sides of the second equation. The result is

$$\frac{hc}{\lambda'} = \sqrt{\frac{1+\beta}{1-\beta}}\frac{hc}{\lambda} \quad \text{or} \quad \lambda' = \sqrt{\frac{1-\beta}{1+\beta}}\,\lambda \tag{E.111}$$

The wavelength that Karl receives is smaller than that emitted by Emmy. This is the **relativistic Doppler formula**, which was obtained in Math Note E.27.8 using a completely different approach. Thus we can now refute the original suspicion: because of the wave property of light, a photon has different wavelengths for different observers due to the Doppler effect. Since the wavelength is related to the energy and momentum of the photon, we should expect these quantities to be different for different observers.

relativistic Doppler formula

As Emmy passes Karl and moves away from him, we expect the wavelength of the photon detected by Karl to increase. Let us see if Equation (E.110) gives us the correct answer. The photons that Karl receives are moving in the *negative* direction [see Figure E.29(b)]. Therefore, we need to introduce a negative sign for p, and write (E.110) as

$$p' = \gamma(-p + \beta E/c) = \gamma(-p + \beta p) = -\frac{p - \beta p}{\sqrt{1-\beta^2}} = -\sqrt{\frac{1-\beta}{1+\beta}}\,p$$

$$E' = \gamma(-\beta pc + E) = \gamma(-\beta E + E) = \frac{E(1-\beta)}{\sqrt{1-\beta^2}} = \sqrt{\frac{1-\beta}{1+\beta}}\,E$$

The second equation now gives us

$$\frac{hc}{\lambda'} = \sqrt{\frac{1-\beta}{1+\beta}}\frac{hc}{\lambda} \quad \text{or} \quad \lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda \tag{E.112}$$

showing an *increase* in the wavelength. You can use this formula for *both* approach and recession by assigning a positive value to β when the source and the detector are receding from each other, and a negative value when they are approaching one another.

Whenever we obtain a formula in relativity, it is instructive to see if it reduces to a familiar formula in classical physics in the limit of small velocities. Math Note E.28.5 shows that Equations (E.111) and (E.112) indeed reduce to Equation (E.10) of Chapter 11.

Math Note E.28.5. When β is small, we can use the results of Math Note E.26.5, namely

Reduction of relativistic Doppler formula to classical formula **(page 61 of the book)**

$$\begin{aligned} &\sqrt{1-\beta} \approx 1 - \tfrac{1}{2}\beta, \quad \sqrt{1+\beta} \approx 1 + \tfrac{1}{2}\beta \\ &\frac{1}{\sqrt{1-\beta}} \approx 1 + \tfrac{1}{2}\beta, \quad \frac{1}{\sqrt{1+\beta}} \approx 1 - \tfrac{1}{2}\beta \end{aligned} \tag{E.113}$$

Then

$$\sqrt{\frac{1-\beta}{1+\beta}} = \frac{1}{\sqrt{1+\beta}}\sqrt{1-\beta} \approx (1 - \tfrac{1}{2}\beta)(1 - \tfrac{1}{2}\beta) \approx 1 - \beta$$

where in the last step we neglected $\frac{1}{4}\beta^2$, because it is so much smaller than β.[22] With this approximation, Equation (E.111) becomes

$$\lambda' = \sqrt{\frac{1-\beta}{1+\beta}}\,\lambda \approx (1-\beta)\lambda = \left(1 - \frac{v}{c}\right)\lambda$$

[22]If β is 0.001, then β^2 is 0.000001.

which is Equation (E.10) with a minus sign (for approach). Similarly, if we use the approximation

$$\sqrt{\frac{1+\beta}{1-\beta}} = \frac{1}{\sqrt{1-\beta}}\sqrt{1+\beta} \approx (1+\tfrac{1}{2}\beta)(1+\tfrac{1}{2}\beta) \approx 1+\beta$$

and use it in Equation (E.112), we get

$$\lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda \approx (1+\beta)\lambda = \left(1+\frac{v}{c}\right)\lambda$$

which is Equation (E.10) with a plus sign (for recession).

Conservation of relativistic momentum and energy **(page 421 of the book)**

Math Note E.28.6. Referring to Figure 28.3, let us call the momenta before collision p_1 and p_2, the energies before collision E_1 and E_2, the momenta after collision P_1 and P_2 and the *relativistic* energies $\mathcal{E}_1$, $\mathcal{E}_2$. Then Emmy's conservation of momentum becomes $p_1 + p_2 = P_1 + P_2$. For Karl, the conservation of momentum yields $p_1' + p_2' = P_1' + P_2'$. The primed quantities are related to the unprimed quantities via Equation (28.5) (with b replaced by 1 or 2):

$$\begin{aligned} p_1' &= \gamma(p_1 + \beta E_1/c), \quad p_2' = \gamma(p_2 + \beta E_2/c),\\ P_1' &= \gamma(P_1 + \beta \mathcal{E}_1/c), \quad P_2' = \gamma(P_2 + \beta \mathcal{E}_2/c) \end{aligned}$$

Substituting these in Karl's conservation law yields

$$\gamma(p_1 + \beta E_1/c) + \gamma(p_2 + \beta E_2/c) = \gamma(P_1 + \beta\mathcal{E}_1/c) + \gamma(P_2 + \beta\mathcal{E}_2/c)$$

dividing all terms by γ gives

$$p_1 + p_2 + \beta(E_1/c + E_2/c) = P_1 + P_2 + \beta(\mathcal{E}_1/c) + \mathcal{E}_2/c)$$

Emmy's conservation law equates the first two terms on the left-hand side to the first two terms on the right-hand side. It follows that

$$\beta(E_1/c + E_2/c) = \beta(\mathcal{E}_1/c) + \mathcal{E}_2/c) \quad \text{or} \quad E_1 + E_2 = \mathcal{E}_1 + \mathcal{E}_2$$

i.e., the total *relativistic energy* does not change in a collision.

E.29 Math Notes for Chapter 29

Deflection angle of light in a uniform field **(page 431 of the book)**

Math Note E.29.1. The direction of motion is specified by the velocity vector, and as Figure E.30(b) shows, the angle *in radian* is given by the velocity in the y-direction divided by the initial velocity, which is c [see Equation (B.1) and Figure B.2]. But the motion in the y-direction is a uniformly accelerated motion, whose velocity is given by gt [see Equation (4.3)]. Therefore, $\varphi = gt/c$; and if we denote the length traveled in the x-direction by w (for "width"), then $t = w/c$ and

$$\varphi = \frac{gw}{c^2}\ \text{radian} \quad \text{or} \quad \varphi = 57.3\frac{gw}{c^2}\ \text{degree} \quad \text{or} \quad \varphi = 206280\frac{gw}{c^2}\ \text{arcsecond} \tag{E.114}$$

where an arcsecond is 1/3600 degree. For $g = 9.8\ \text{m/s}^2$ and $w = 10$ m,

$$\varphi = 206280\frac{9.8\times 10}{(3\times 10^8)^2} = 2.25\times 10^{-10}\ \text{arcsecond}$$

which is immeasurably small.

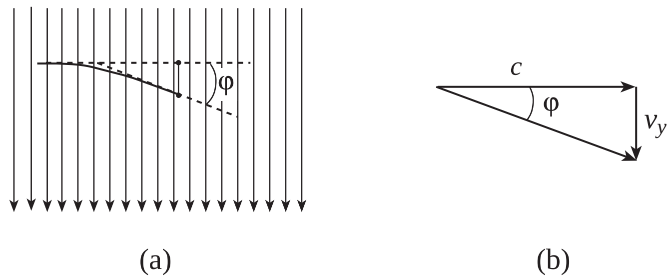


Figure E.30: (a) A light beam bends in a gravitational field. (b) The deflection angle is the angle between initial and final velocity vectors. The angle is exaggerated a great deal; it is very small for ordinary gravitational fields.

On the other hand, a white dwarf is as massive as Sun, while its radius is comparable to the Earth's radius. The gravitational acceleration at the surface of such an object is of the order of 10^6 m/s^2. Suppose we send a beam of light to a point 1000 km away. Then, with $w = 10^6$ m, Equation (E.114) gives a deflection angle of

$$\varphi = 206280\frac{gw}{c^2} = 206280\frac{10^6 \times 10^6}{(3 \times 10^8)^2} = 2.3 \text{ arcseconds}$$

which, although still very small, is much larger than the value obtained above.

Math Note E.29.2. A light beam coming from far away goes through regions of varying gravitational field of a spherical body as shown in Figure E.31(a). At large distances, the field is very weak; as the light beam approaches the body, the field becomes stronger and stronger; when it grazes the surface, the field is the strongest; and as the beam recedes from the body, the field gets weaker and weaker again.

Derivation of deflection angle of a light beam **(page 431 of the book)**

The problem of finding the deflection angle in the field of Figure E.31(a), although complicated, can be solved using techniques of differential equation. I am not going to solve the problem exactly. Instead, let me make a wild approximation: I replace the inhomogeneous field of the spherical body with a homogeneous field which is zero everywhere except for a region equal to the diameter of the body [see Figure E.31(b)]. The strength of this field is the same as that at the point of closest approach. By letting the homogeneous field have the largest value, I am compensating for the fact that outside the region of width $2R$, I have set the strength of the gravity equal to zero, while in reality it is not. Now recall from Equation (9.2) that for a celestial body of radius R and mass M, the gravitational acceleration at the surface is $g = GM/R^2$. This, plus the fact that $w = 2R$, turns Equation (E.114) into

$$\varphi = \frac{(GM/R^2)2R}{c^2} = \frac{2GM}{Rc^2} \tag{E.115}$$

where φ is measured in radian. It is a remarkable coincidence that the exact calculation gives an identical result!

Math Note E.29.3. We know from our discussion of the Doppler effect in Section 11.4 that $\Delta\lambda/\lambda = v_{\text{rel}}/c$ [see Equation (E.12) in Math Note E.11.4], where v_{rel} is the speed of the detector relative to the source; i.e., the amount by which the detector changes its speed in the time it takes light to reach it after it leaves the flashlight. But this is precisely Δv, the change in the rocket speed during the light's flight.

Doppler shift in an accelerating rocket **(page 433 of the book)**

Let us call Δh (the change in height) the distance that light travels, or the distance between the flashlight and the detector; and call Δt the time it takes light to travel this

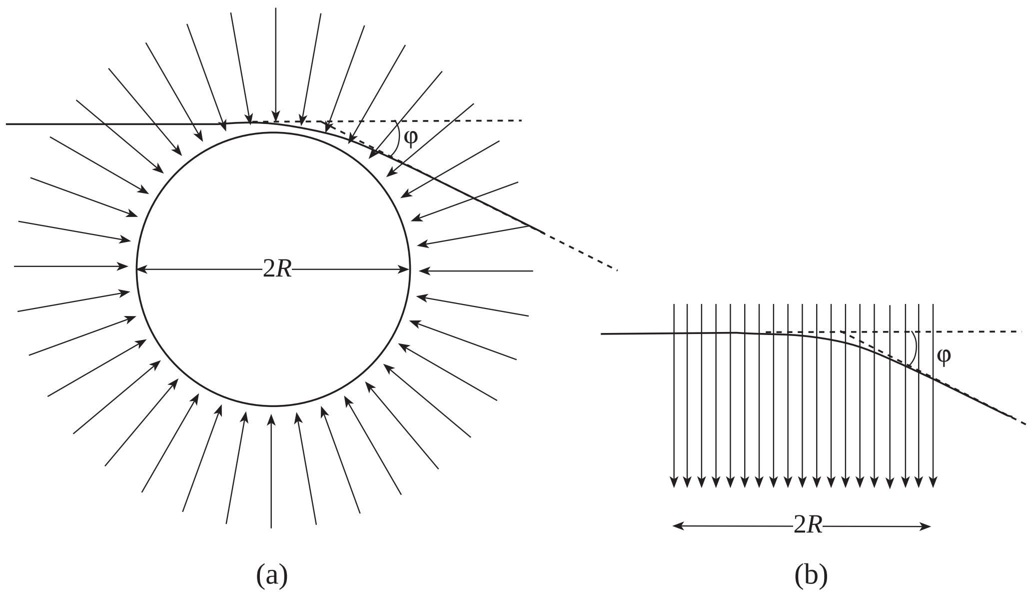


Figure E.31: (a) A light beam bending in the weak gravitational field of a spherical body. (b) The equivalent homogeneous gravitational field.

distance. Then it is clear that $\Delta t = \Delta h/c$. During this same time the speed of the rocket has changed by $\Delta v = a\Delta t = a\Delta h/c$. Putting all this together, we get

$$\frac{\Delta\lambda}{\lambda} = \frac{v_{\text{rel}}}{c} = \frac{\Delta v}{c} = \frac{a\Delta h/c}{c} = \frac{a\Delta h}{c^2}$$

The equivalence principle allows us to change the acceleration to the corresponding gravitational field g:

$$\frac{\Delta\lambda}{\lambda} = \frac{g\Delta h}{c^2} \quad \text{or} \quad \frac{\lambda_{\text{det}} - \lambda_{\text{src}}}{\lambda} = \frac{g(h_{\text{det}} - h_{\text{src}})}{c^2} \tag{E.116}$$

where λ_{det} and h_{det} are the wavelength and height at the detector, and λ_{src} and h_{src} those at the source. By definition, gravitational field lines are directed toward lower heights.

Derivation of gravitational time dilation **(page 433 of the book)**

Math Note E.29.4. To see how much gravity affects time, divide the numerator and denominator of the left-hand side of Equation (E.116) by c and note that $c = \lambda/T$ or $T = \lambda/c$. Then, denoting by $\Delta\lambda$ the difference in wavelength on the left, and by Δh, the difference in height on the right, we obtain

$$\frac{\Delta\lambda/c}{\lambda/c} = \frac{g\Delta h}{c^2} \quad \text{or} \quad \frac{\Delta T}{T} = \frac{g\Delta h}{c^2}$$

where T is the period of the wave at the source and ΔT is the change in the period when it reaches the detector. The right-hand side of the last equation can be rewritten. Recall that $mg\Delta h$ is the gravitational *potential energy* difference between two points separated by a height Δh. Therefore, $g\Delta h$ is this difference divided by the mass m. We denote the ratio of the gravitational potential energy to mass by Φ and call it the **gravitational potential** (see Math Note E.9.3 for further detail). Then, the second part of the above equation can be written as

$$\frac{\Delta T}{T} = \frac{g\Delta h}{c^2} = \frac{\Delta\Phi}{c^2}$$

This equation can be expressed more suggestively as follows. Since ΔT is the difference in the period of the wave at two different points, we write it as $\Delta T = T_2 - T_1$ where the

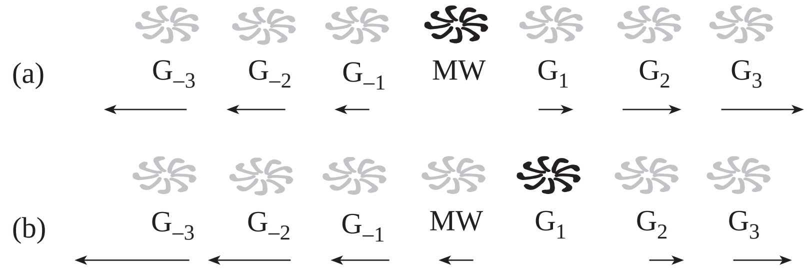


Figure E.32: (a) A Milky Way (MW) observer sees galaxies G_{-3} through G_3 move away. (b) Because of homogeneity and isotropy, an observer on G_1 must see an identical situation as MW.

subscripts 1 and 2 refer to the detector and the source, but they are really any two points in space.[23] Similarly, $\Delta h = h_2 - h_1$, where h_1 and h_2 are, respectively, the heights at which the periods are T_1 and T_2. Finally, $\Delta\Phi = \Phi_2 - \Phi_1$, with Φ_1 and Φ_2 relating to T_1 and T_2. The last equation then becomes

$$\frac{T_2 - T_1}{T} = \frac{g(h_2 - h_1)}{c^2} = \frac{\Phi_2 - \Phi_1}{c^2}$$

where T is either T_1 or T_2 (the difference is so small that it does not matter which one you use in the denominator).

Now, period is a measure of time. For some radio waves the period is 10^{-8} s. So a second is 100 million periods, a minute 6 billion, an hour 360 billion and a year 3.15×10^{15} periods. And if period is affected by gravity, time itself is as well. We rewrite the equation above as

$$\frac{t_2 - t_1}{t} = \frac{g(h_2 - h_1)}{c^2}, \quad \text{or} \quad \frac{t_2 - t_1}{t} = \frac{\Phi_2 - \Phi_1}{c^2}, \qquad \Phi \equiv -\frac{GM}{r} \tag{E.117}$$

where again t is either t_1 or t_2. For points near the Earth's (or any other gravitating body's) surface,[24] there is no difference between the two ways of calculating $t_2 - t_1$, but the first formula is easier to use. If the height difference between the two points is comparable with the radius of the gravitating body, we have to use the second formula.

Math Note E.29.5. We want to show that in an expanding homogeneous and isotropic universe, the speed of a distant galaxy relative to another is proportional to their separation. To simplify the argument, consider the motion of galaxies along a line. Figure E.32(a) shows only six out of billions of galaxies moving away from the Milky Way (MW). Assume that G_1 is at a distance d to the right of MW, and the distances of all the other galaxies from MW are multiples of d: G_2 is $2d$ to the right, G_3 is $3d$ to the right, and all the negatively indexed galaxies have corresponding distances to the left. Let the speed of G_1 be v. Then isotropy implies that the speed of G_{-1} is $-v$, i.e., same as G_1, but in the opposite direction. [If an observer rotates 180 degrees from G_1, she sees G_{-1}.]

Derivation of proportionality of galactic speed and distance
(page 445 of the book)

How do the galaxies move as seen from another galaxy, say G_1? Figure E.32(b) shows the recession of galaxies as seen by G_1. Since G_1 moves away to the right as seen by MW observers, MW at a distance d, must be moving to the left as seen by G_1 observers. Isotropy now implies that G_2, which is at the same distance from G_1 as is MW, must be moving away from G_1 with speed v. For this to happen, G_2 must be moving at speed $2v$ away from MW. Isotropy implies that G_{-2} must also be moving away from MW with a speed of $2v$ in

[23] You can move your source and detector to any point you desire.

[24] That is, points whose heights above the surface are much smaller than the radius of the gravitating body.

the negative direction. By jumping on G_2 and observing the runaway galaxies, an observer must measure the speed of G_3 to be v relative to G_2, because of isotropy and the fact that G_1 on the left is moving away with that speed. This puts the speed of G_3—and by isotropy, G_{-3}—at $3v$ relative to MW. It should now be clear that galactic speeds relative to G_1 are proportional to their separation from G_1.

E.31 Math Notes for Chapter 31

Rigorous calculation of mass defect
(**page 464 of the book**)

Math Note E.31.1. Apply the conservation of energy to a very general fission process:

$$ {}^{1}_{0}\mathrm{n} + \mathrm{A} \;\rightarrow\; \mathrm{X} + \mathrm{Y} + \text{neutrons}. $$

With obvious notation write

$$ E_n + E_A = E_X + E_Y + E_{ns} $$

where the subscript ns refers to the total number of neutrons. Each energy consists of the KE and the rest energy: $E = KE + mc^2$. So, rewrite the equation above as

$$ KE_n + m_n c^2 + KE_A + M_A c^2 = KE_X + M_X c^2 + KE_Y + M_Y c^2 + KE_{ns} + rM_n c^2 $$

where r is the number of neutrons produced. Now use Equation (31.1) for each nucleus,

$$ \begin{aligned} KE_n + m_n c^2 + KE_A + Z_A m_p c^2 + N_A m_n c^2 - BE_A &= KE_X + Z_X m_p c^2 + N_X m_n c^2 - BE_X \\ &\quad + KE_Y + Z_Y m_p c^2 + N_Y m_n c^2 - BE_Y + KE_{ns} + rM_n c^2 \end{aligned} $$

The number of protons on the left should equal the umber of protons on the right; similarly for neutrons. Therefore,

$$ KE_n + KE_A - BE_A = KE_X - BE_X + KE_Y - BE_Y + KE_{ns} $$

which can be rewritten as

$$ BE_X + BE_Y - BE_A = KE_X + KE_Y + KE_{ns} - KE_n - KE_A $$

The KE of the initial neutron and parent nucleus is usually very small compared to other energies. So, we can write the last equation as

$$ BE_X + BE_Y - BE_A = KE_X + KE_Y + KE_{ns} $$

Calculating energy of electron in a neutron decay
(**page 469 of the book**)

Math Note E.31.2. The initial energy of the decay of a neutron is the rest energy of the neutron, $m_n c^2$. The final energy is the energy of the proton E_p plus the energy of the electron E_e. But these energies are related to the corresponding (equal) momenta via the relativistic energy-momentum formula (28.7) which can be written as $E^2 = p^2c^2 + m^2c^4$. Conservation of energy gives

$$ m_n c^2 = E_p + E_e \quad \text{or} \quad E_p = m_n c^2 - E_e $$

Squaring both sides yields

$$ E_p^2 = (m_n c^2 - E_e)^2 = m_n^2 c^4 + E_e^2 - 2m_n c^2 E_e $$

Substituting $E^2 = p^2c^2 + m^2c^4$ for E_p^2 on the left and E_e^2 on the right gives

$$ p^2c^2 + m_p^2 c^4 = m_n^2 c^4 + p^2 c^2 + m_e^2 c^4 - 2m_n c^2 E_e \quad \text{or} \quad 2m_n E_e = m_n^2 c^2 - m_p^2 c^2 + m_e^2 c^2 $$

It follows that the energy of the electron is

$$E_e = \frac{m_n^2 - m_p^2 + m_e^2}{2m_n} c^2,$$

and gives the unique value of 1.289 MeV once all the mass-values and the speed of light are substituted in the right-hand side.

If instead of the neutron beta decay $\mathrm{n} \to \mathrm{p}^+ + \mathrm{e}^-$ we consider a general nuclear beta decay $\mathrm{X} \to \mathrm{Y}^+ + \mathrm{e}^-$, where X and Y are nuclei, then the electron energy will be

$$E_e = \frac{m_X^2 - m_Y^2 + m_e^2}{2m_X} c^2,$$

which is completely determined by the masses of the nuclei.

E.32 Math Notes for Chapter 32

Energetics of an antiproton production **(page 481 of the book)**

Math Note E.32.1. We want to calculate the minimum energy required of an impinging proton to produce an antiproton upon impact with a stationary proton. One cannot convert energy simply into an antiproton; creation of matter always accompanies creation of similar antimatter. Thus, the antiproton *must* accompany a proton. Furthermore, protons cannot turn into other particles; so the initial net number of protons must equal the final net number. Since we start with two protons, we must end up with two *net* protons. The proton and the antiproton "cancel" each other. Therefore, the final products must include two extra protons. The reaction corresponding to the *minimum* energy is thus

$$p + p \to p + p + p + \overline{p}$$

where a bar over the symbol of a particle denotes its antiparticle.

To calculate the actual minimum energy, we observe the reaction in the center of mass (CM). What does this mean? Suppose we move in the same direction as the initial moving proton with half its speed. Then, the moving proton will appear to move at half its speed in the same direction, while the stationery proton will appear to move at half its speed *in the opposite direction*: in the CM reference frame, the total momentum is zero. Since momentum does not change in a collision, the momenta of the four end particles must add up to zero. The minimum energy corresponds to the case where all four particles remain stationary (in the CM reference frame). In the original (laboratory) RF, they, of course, move with the same speed—that of the center of mass RF. Since they all have the same mass (antiparticles have identical mass to their corresponding particles), they must have equal momenta in the lab RF. Thus, the minimum energy corresponds to the case when the initial momentum of the moving proton is divided equally among the four final particles.

Now we are ready to calculate the minimum energy. First we note that the energy-momentum relation, $E^2 = P^2c^2 + m^2c^4$, implies that the final particles have equal energy, because they have equal masses and momenta. Label this energy e_f and the corresponding momentum p_f. Label the energy and momentum of the initial moving proton E and P, respectively. Equating the initial total energy (including the rest energy of the stationary proton) and momentum to the final total energy and momentum gives

$$E + mc^2 = 4e_f, \qquad P = 4p_f \tag{E.118}$$

where m is the mass of the proton (or antiproton). Squaring both sides of the second equation yields $P^2 = 16p_f^2$, or $P^2c^2 = 16p_f^2c^2$. If we replace the momentum on each side with its corresponding energy using the energy-momentum relation, we obtain

$$E^2 - m^2c^4 = 16(e_f^2 - m^2c^4) \quad \text{or} \quad E^2 = 16e_f^2 - 15m^2c^4 \tag{E.119}$$

The first equation in (E.118) gives $(E+mc^2)^2 = 16e_f^2$. Substituting this in the last equation of (E.119) yields $E^2 = (E + mc^2)^2 - 15m^2c^4$ or

$$E^2 = E^2 + 2mc^2E + m^2c^4 - 15m^2c^4 \quad \Rightarrow \quad 0 = 2mc^2E - 14m^2c^4 \tag{E.120}$$

whose solution is $E = 7mc^2$. This is the total energy of the impinging proton. Its kinetic energy—its total minus its rest energy—is $KE = 6mc^2$. For a proton, $mc^2 = 938.28$ MeV; therefore, the minimum KE required for the production of an antiproton is 6 times this or 5630 MeV, which is close to 6 billion eV.

Relation between angular momentum and magnetic moment **(page 488 of the book)**

Math Note E.32.2. When a particle of mass m and charge q moves with speed v on a circle of radius r, its angular momentum L is rmv. On the other hand, the magnetic moment μ, defined as *the product of the electric current and the area of the circle*, is $i\pi r^2$. Now we note that $v = 2\pi r/T$, where T is the period of the revolution of the charge. Furthermore, since q moves around the circle in T, the current is q/T. Therefore,

$$\begin{aligned} L &= mrv = mr\frac{2\pi r}{T} \qquad \text{or} \qquad LT = 2m\pi r^2 \\ \mu &= i\pi r^2 = \frac{q}{T}\pi r^2 \qquad \text{or} \qquad \mu T = q\pi r^2 \end{aligned}$$

and dividing the second equation by the first yields $\mu/L = q/(2m)$ or $\mu = (q/2m)L$.

E.37 Math Notes for Chapter 37

The Math Notes for this chapter are a little different from the other ones in that I have been a little more liberal in using slightly more sophisticated mathematics, mostly calculus. As I was writing these notes, the intimidation of the difficulty of the math faded next to the sublimity of the subject matter. I thought to myself "If I have to use some simple higher mathematics to derive an important relation, I'll do it." After all, we are talking about the universe itself. And those readers who are familiar with calculus will see the power of mathematics as seen nowhere else. It is for these readers that I have included some integrals and differential equations. The other readers can simply skip the higher math and move on to the rest of the math note. For your convenience, I have set the mathematical discussions in a different font style.

Derivation of the Friedmann equation **(page 570 of the book)**

Math Note E.37.1. The total energy E of m moving away with speed v at a distance R from the center of sphere in Figure E.33 is

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{R}$$

where M is the mass of the sphere. This mass can be written in terms of the *uniform* density ρ of the material filling the sphere.[25] The density ρ includes both matter and radiation, because both contribute to the gravitational force; matter due to its mass, and radiation, due to its energy, which by $E = mc^2$, has an equivalent mass.

Now recall (see Section 8.3.1) that density is defined as mass divided by volume, and therefore, mass is density times volume. For a sphere of radius R, the volume is $\frac{4}{3}\pi R^3$. Thus, $M = \frac{4}{3}\pi R^3\rho$. Putting this in the equation above yields

$$E = \tfrac{1}{2}mv^2 - \tfrac{4}{3}Gm\pi R^2\rho$$

Multiply both sides of this equation by 2 and divide by mR^2 and denote E/m by e to get

$$\frac{2e}{R^2} = \left(\frac{v}{R}\right)^2 - \frac{8\pi G}{3}\rho$$

[25] Since the universe is assumed homogeneous and isotropic, its density cannot change from point to point, or from one angle to another.

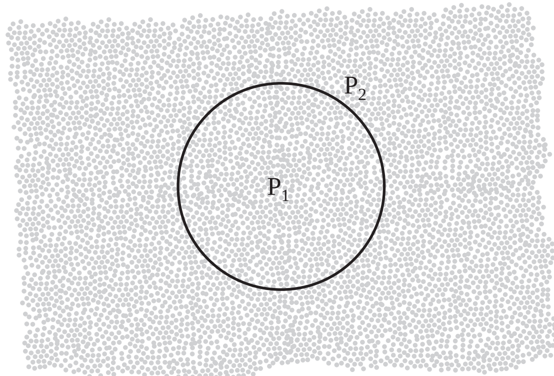


Figure E.33: Two points P_1 and P_2 and the sphere filled with matter and radiation between them.

When the general theory of relativity is applied to the universe, the same equation is obtained except that the numerator of the left-hand side is replaced by $-kc^2$, where k, like e, is a constant, which is related to the geometry of the universe, and is called the **curvature**. Borrowing this piece of information from GTR and rearranging the last equation slightly, we get the **Friedmann equation**:

$$\left(\frac{v}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} \tag{E.121}$$

Depending on the sign of k, we have three kinds of universes. If k is negative, then e and E are both positive, and Box 9.2.3 tells us that the two points P_1 and P_2 of Figure E.33 will move away from each other forever. Since P_1 and P_2 are two typical (far) points of the universe, the universe will expand forever. This universe is called **open** or **hyperbolic**.

If k is positive, then e and E are both negative, and once again Box 9.2.3 tells us that P_2 will move away from P_1 for a while, but eventually it will stop and start coming back towards P_1. Such a universe will eventually stop expanding and a contracting era will start sometime in the future. This universe is called **closed** or **spherical**.

Finally, if $k = 0$, then e and E are both zero, and although P_2 will move away from P_1 forever, it will constantly slow down, and only at infinity will it come to a complete stop. This universe is called **flat**.

Derivation of $\rho_m(t)$ and $R(t)$ for a matter-dominated universe **(page 572 of the book)**

Math Note E.37.2. Let M denote the mass inside the sphere of radius R_0. Then $M = \frac{4}{3}\pi R_0^3\rho_{m0}$. This mass remains the same as the size of the sphere changes. Thus, at a later time, $M = \frac{4}{3}\pi[R(t)]^3\rho_m(t)$. Equating these two expressions for M and simplifying gives

$$[R(t)]^3\rho_m(t) = R_0^3\rho_{m0} \quad \text{or} \quad \rho_m(t) = \frac{R_0^3\rho_{m0}}{[R(t)]^3}, \tag{E.122}$$

which shows that the density of matter increases for a contracting universe as the inverse third power of the size of the universe.

Substituting this in the Friedmann equation for a flat universe, we get

$$\left[\frac{v(t)}{R(t)}\right]^2 = \frac{8\pi G}{3}\frac{R_0^3\rho_{m0}}{[R(t)]^3} \quad \text{or} \quad [v(t)]^2 = \frac{8\pi G R_0^3\rho_{m0}}{3}\frac{1}{R(t)}$$

and

$$v(t) = \sqrt{\frac{8\pi G R_0^3\rho_{m0}}{3}}\,\frac{1}{\sqrt{R(t)}} \tag{E.123}$$

Equation (E.123) is a differential equation, whose solution can be obtained very easily. Ordinarily, I would just quote the solution, but since we are talking about the fate of the universe itself, I feel obligated to share with you the simple method that *finds* the solution to this "colossal" result. If you are not familiar with calculus, skip to the next paragraph. But if you know some calculus, follow the steps and *feel* the universal power of math and physics!

Anyway, let's go back to Equation (E.123) and note that $v(t) = dR/dt$. *Lumping all the constants into one and calling it* A, *Equation (E.123) becomes*

$$\frac{dR}{dt} = \frac{A}{\sqrt{R}}, \quad \text{or} \quad R^{1/2} dR = A\, dt$$

Integrating both sides and introducing a constant of integration gives

$$\tfrac{2}{3} R^{3/2} = At + C \tag{E.124}$$

Since at the moment of the big bang (i.e., $t = 0$*) the universe had zero size,* C *must be zero. Therefore,*

$$R(t) = \left(\tfrac{3}{2} At\right)^{2/3} = A^{2/3} \left(\tfrac{3}{2} t\right)^{2/3} = \left(\sqrt{\frac{8\pi G R_0^3 \rho_{m0}}{3}}\right)^{2/3} \left(\tfrac{3}{2} t\right)^{2/3} = \left(\frac{8\pi G R_0^3 \rho_{m0}}{3}\right)^{1/3} \left(\tfrac{3}{2} t\right)^{2/3}$$

which can be simplified to

$$\frac{R(t)}{R_0} = \left(6\pi G \rho_{m0}\right)^{1/3} t^{2/3} \tag{E.125}$$

Now substitute t_0, the age of the universe, for t and note that $R(t_0) = R_0$. Then

$$\frac{R_0}{R_0} = \left(6\pi G \rho_{m0}\right)^{1/3} t_0^{2/3} \quad \text{or} \quad \left(6\pi G \rho_{m0}\right)^{1/3} = \frac{1}{t_0^{2/3}}$$

Substitute this in Equation (E.125) to get a useful formula:

$$\frac{R(t)}{R_0} = \frac{1}{t_0^{2/3}} t^{2/3} = \left(\frac{t}{t_0}\right)^{2/3} \tag{E.126}$$

where $t_0 = 13.7$ billion years, the age of the universe.

Cubing both sides of Equation (E.125) and using (E.122), we obtain

$$\frac{[R(t)]^3}{R_0^3} = 6\pi G \rho_{m0} t^2 \quad \text{or} \quad \frac{[R(t)]^3}{R_0^3 \rho_{m0}} = 6\pi G t^2$$

The left-hand side of the second equation is the inverse of matter density. Therefore,

$$\rho_m(t) = \frac{1}{6\pi G t^2} \tag{E.127}$$

This is usually written with t given in terms of ρ_m:

$$t = \frac{1}{\sqrt{6\pi G \rho_m}} \tag{E.128}$$

Finally, we can find a simple equation that gives the variation of H with time. Equation (E.123) in terms of the constant A introduced above is $v = A/\sqrt{R}$. Hence, $H = A/(R\sqrt{R}) = A/R^{3/2}$. But Equation (E.124) with $C = 0$ gives us the denominator. Thus,

$$H = \frac{A}{R^{3/2}} = \frac{A}{\tfrac{3}{2} At} = \frac{2}{3t} \tag{E.129}$$

Note that H decreases with time, *approaching* zero for very large t, but never becoming zero. The first row of Table E.3 summarizes the results of this Math Note.

Matter dominated	$\frac{R(t)}{R_0} = (6\pi G\rho_{m0})^{1/3}\, t^{2/3} = \left(\frac{t}{t_0}\right)^{2/3},\ t = \frac{1}{\sqrt{6\pi G\rho_m}} = \frac{126863}{\sqrt{\rho_{m0}}\ T^{3/2}},\quad H = \frac{2}{3t}$
Radiation dominated	$\frac{R(t)}{R_0} = \left(\frac{32\pi G\rho_{\gamma 0}}{3}\right)^{1/4} t^{1/2},\quad t = \sqrt{\frac{3}{32\pi G\rho_\gamma}} = \frac{2.3\times 10^{20}}{T^2},\quad H = \frac{1}{2t}$
Matter or radiation dominated	$\frac{\lambda(t)}{\lambda_0} = \frac{R(t)}{R_0},\qquad \frac{T(t)}{T_0} = \frac{R_0}{R(t)}$ $\rho_\gamma(T) = 8.36\times 10^{-33} T^4\ \text{kg/m}^3,\quad \rho = \alpha\rho_\gamma,\quad t = \frac{2.3\times 10^{20}}{\sqrt{\alpha}\ T^2}$ $\rho_\gamma(t) = \rho_{\gamma 0}\left[\frac{R_0}{R(t)}\right]^4,\quad n_\gamma(T) = 2\times 10^7 T^3\ \text{photons/m}^3,\quad \langle E_\gamma\rangle = 2.7 k_B T$

Table E.3: The collection of formulas used frequently in cosmology. Note how time is written in terms of densities instead of densities as a function of time. The second and third equations of the fourth row apply to the very early universe when all particles were relativistic.

Derivation of relation $\lambda(t)$ and $T(t)$ **(page 573 of the book)**

Math Note E.37.3. Consider two nearby points P_1 and P_2 in a perfectly homogeneous and isotropic universe separated by a distance r. By a perfectly homogeneous and isotropic universe I mean one which is so even at small distances. Since I am interested in the change in the wavelength *due to expansion*, I do not want to worry about inhomogeneity caused by the presence of matter. Let $\lambda(t)$ denote the wavelength of the EM wave as it passes by P_1 at time t after the big bang. This same wave reaches P_2 at time $t + \Delta t$ after the big bang. This means that the wave travels the distance r in Δt seconds, so that $r = c\Delta t$. Let v be the relative speed of the two points due to expansion, i.e., $v = Hr$.

Now recall from our discussion of Doppler effect that the fractional change in the wavelength of an EM wave is simply v/c [see the second equation in (E.10)]. Thus, we can write

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{Hr}{c} = H\frac{r}{c} = H\Delta t$$

The Hubble parameter is related to the *scale* of the universe. It is the ratio of the rate of change of the scale divided by the scale. Denoting by V the speed of increase of the scale R, we have $H = V/R$ and $H\Delta t = (V\Delta t)/R$. But $V\Delta t$ is simply how much R increases in Δt. Call this increase ΔR, and write the last equation as

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta R}{R} \tag{E.130}$$

Let's denote the quantities at the moment that the EM wave is at P_1 by a subscript 1, and similarly for P_2. Then Equation (E.131) becomes

$$\frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{R_2 - R_1}{R_1} \quad \text{or} \quad \frac{\lambda_2}{\lambda_1} - 1 = \frac{R_2}{R_1} - 1 \quad \text{or} \quad \frac{\lambda_2}{\lambda_1} = \frac{R_2}{R_1} \tag{E.131}$$

Therefore, the wavelength of an EM wave increases in proportion to the scale of the universe. Although we have derived Equation (E.131) for two nearby points, it holds true for any two points. The following derivation <u>*using calculus*</u> shows this.

Write Equation (E.130) in differential form: $d\lambda/\lambda = dR/R$. *Now integrate both sides to get* $\ln\lambda = \ln R + \ln C$, *where the constant of integration has been written as the natural log of some number. Using the properties of the logarithms, we have* $\ln\lambda = \ln(CR)$, *or* $\lambda = CR$, *which shows that* λ *is proportional to* R.

To see how the temperature of the once black-body radiation changes after decoupling, I first modify Equation (20.5). As it stands, (20.5) gives the spectral energy *flux*, i.e., when multiplied by $\Delta\lambda$, it gives the amount of energy crossing a unit area per unit time. Flux is related to density: if I multiply (20.5) by 4 and divide it by c, I get the spectral energy *density*. Denote by $\Delta u(\lambda, T)$ the spectral energy density i.e., the amount of energy emitted by the black body in a cubic meter per unit time. Then

$$\Delta u(\lambda, T) = \frac{4}{c}\Phi(\lambda, T)\Delta\lambda = \frac{8\pi hc}{\lambda^5}\frac{\Delta\lambda}{e^{hc/\lambda k_B T} - 1} \tag{E.132}$$

Note that I have explicitly indicated the dependence of Δu on λ and T for later use.

Now assume that λ' is a multiple of λ, i.e., $\lambda' = a\lambda$, so that $\lambda = \lambda'/a$ with a some constant. Substituting λ'/a for λ on the right, we get

$$\begin{aligned}\Delta u(\lambda, T) &= \frac{8\pi hc}{(\lambda'/a)^5}\frac{\Delta\lambda'/a}{e^{hc/(\lambda'/a)k_B T} - 1} = a^4\frac{8\pi hc}{\lambda'^5}\frac{\Delta\lambda'}{e^{hc/(\lambda'/a)k_B T} - 1}\\ &= a^4\frac{8\pi hc}{\lambda'^5}\frac{\Delta\lambda'}{e^{hc/\lambda' k_B (T/a)} - 1} = a^4\Delta u(\lambda', T/a) = a^4\Delta u(a\lambda, T/a)\end{aligned}$$

which can be summarized as

$$\Delta u(a\lambda, T/a) = \frac{1}{a^4}\Delta u(\lambda, T) \tag{E.133}$$

Equation (E.133) carries a very significant information. Remember that $\Delta u(\lambda, T)$ describes the spectral energy density of a *black-body* radiator. So the left-hand side of Equation (E.133) is the spectral energy density of a black-body radiator with wavelength $a\lambda$ and temperature T/a, and the message of Equation (E.133) is: a black-body radiator whose wavelength is increased by a factor *a remains a black-body radiator*, but its temperature *decreases* by a factor a, and its spectral energy density *decreases* by a factor a^4. As the universe expands, both its scale and the wavelength of the EM radiation increase by the same factor. Denoting by t the time after the big bang and by t_0 the present time, we can express the wavelength and temperature of radiation as

$$\frac{\lambda(t)}{\lambda(t_0)} = \frac{R(t)}{R(t_0)}, \qquad \frac{T(t)}{T(t_0)} = \frac{R(t_0)}{R(t)} \tag{E.134}$$

Derivation of radiation density
(page 574 of the book)

Math Note E.37.4. In Math Note E.20.1 we calculated the total energy *flux* of a black-body radiator [see Equation (E.52)]. It turns out that energy density can be obtained from energy flux by multiplying the latter by 4 and dividing it by c. Thus, to find the total energy density, all we need to do is multiply the result of Math Note E.20.1 by $4/c$:

$$u = \frac{4}{c}J_e = \frac{4}{c}\frac{2\pi^5 k_B^4}{15h^3c^2}T^4 = \frac{8\pi^5 k_B^4}{15h^3c^3}T^4 \tag{E.135}$$

This is the (average) amount of *energy* that radiation carries in every cubic meter of the universe. We are interested in the equivalent *mass* density, which we denote by ρ_γ, because γ usually refers to EM radiation. Thus, we divide (E.135) by c^2 ($E = mc^2$ can also be written as $m = E/c^2$):

$$\rho_\gamma(T) = u/c^2 = \frac{8\pi^5 k_B^4}{15h^3c^5}T^4 \tag{E.136}$$

How does the radiation density change with the scale of the universe? The last equation in combination with Equation (E.134) gives the answer. First note that $\rho_\gamma(T)/\rho_\gamma(T_0) =$

$(T/T_0)^4$. But $(T/T_0)^4 = (R_0/R)^4$ by (E.134). Thus, $\rho_\gamma(T)/\rho_\gamma(T_0) = [R_0/R]^4$. Let's write this in terms of time:

$$\frac{\rho_\gamma(t)}{\rho_\gamma(t_0)} = \left[\frac{R(t_0)}{R(t)}\right]^4 \quad \text{or} \quad \rho_\gamma(t) = \rho_{\gamma 0}\left[\frac{R_0}{R(t)}\right]^4 \tag{E.137}$$

where $\rho_{\gamma 0}$ is the same as $\rho_\gamma(t_0)$, the present radiation density. We expect the density to be proportional to inverse R^3, as is indeed the case with matter density. However, while it is true that the *number* density in both cases varies in inverse proportion to R^3, the energy of a photon is hc/λ—and λ is proportional to R—giving an extra factor of R in the denominator for ρ_γ.

Having found $\rho_\gamma(t)$, we can now use the Friedmann equation to find the variation of the scale of the universe with time in a radiation-dominated universe. Substitute $\rho_\gamma(t)$ of (E.137) in Equation (37.1) with $k = 0$ to get

$$\left[\frac{v(t)}{R(t)}\right]^2 = \frac{8\pi G}{3}\frac{R_0^4\rho_{\gamma 0}}{[R(t)]^4} \quad \text{or} \quad [v(t)]^2 = \frac{8\pi G R_0^4\rho_{\gamma 0}}{3}\frac{1}{[R(t)]^2}$$

and

$$v(t) = \sqrt{\frac{8\pi G R_0^4\rho_{\gamma 0}}{3}}\,\frac{1}{R(t)} \tag{E.138}$$

The differential equation (E.138) can be solved as easily as was done for Equation (E.123). Call the constant on the right-hand side A, and write

$$\frac{dR}{dt} = \frac{A}{R}, \quad \text{or} \quad R\,dR = A\,dt$$

Integrate both sides, and note that the constant of integration is zero as before to get

$$\tfrac{1}{2}R^2 = At \quad \text{or} \quad R = \sqrt{2At} = \sqrt{2}\left(\frac{8\pi G R_0^4\rho_{\gamma 0}}{3}\right)^{1/4}\sqrt{t} \tag{E.139}$$

which can be written as

$$\frac{R(t)}{R_0} = \left(\frac{32\pi G\rho_{\gamma 0}}{3}\right)^{1/4} t^{1/2} \tag{E.140}$$

Furthermore, this, combined with Equation (E.137), gives the density of the universe as a function of time:

$$\rho_\gamma(t) = \rho_{\gamma 0}\frac{R_0^4}{[R(t)]^4} = \rho_{\gamma 0}\left[\left(\frac{3}{32\pi G\rho_{\gamma 0}}\right)^{1/4}\frac{1}{t^{1/2}}\right]^4, \quad \text{or} \quad \rho_\gamma(t) = \frac{3}{32\pi G t^2} \tag{E.141}$$

which is usually written with t given as a function of ρ_γ:

$$t = \sqrt{\frac{3}{32\pi G\rho_\gamma}} \tag{E.142}$$

The variation of H with time can also be found in a radiation-dominated universe. Equation (E.138) in terms of the constant A introduced above is $v = A/R$. Hence, $H = v/R = A/R^2$. But Equation (E.139) gives us the denominator as $2At$. Thus,

$$H = \frac{A}{R^2} = \frac{A}{2At} = \frac{1}{2t} \tag{E.143}$$

Note once again that H is inversely proportional to time as in the case of a matter-dominated universe, although with a different constant of proportionality.

Math Note E.37.5. We have already found the spectral energy density of a black-body radiator in Equation (E.132) of Math Note E.37.3. Since the energy of each photon is hc/λ, if we divide Equation (E.132) by hc/λ we should get the spectral number density Δn_γ of the black body:

$$\Delta n_\gamma(\lambda, T) = \frac{\Delta u(\lambda, T)}{hc/\lambda} = \frac{8\pi}{\lambda^4}\frac{\Delta\lambda}{e^{hc/\lambda k_B T} - 1} \tag{E.144}$$

Derivation of photon number density and average energy **(page 574 of the book)**

The total number density is given by adding all the Δn_γ's for all the wavelengths, i.e., integrating. The procedure is the same as that done in Math Note E.20.1 and yields

$$n_\gamma(T) = 8\pi\left(\frac{k_B T}{hc}\right)^3 \underbrace{\int_0^\infty \frac{dy}{y^4(e^{1/y} - 1)}}_{=2.404} \tag{E.145}$$

the numerical value of the integral is 2.404 (as indicated in the equation), so that

$$n_\gamma(T) = 8\pi\left(\frac{k_B T}{hc}\right)^3 (2.404) = 19.232\pi\left(\frac{k_B}{hc}\right)^3 T^3 = 2\times 10^7 T^3 \text{ photons/m}^3 \tag{E.146}$$

where in the last step, I substituted the values of all physical and numerical constants.

Now, if the energy density is given by Equation (E.135) and the number density by Equation (E.146), then the average energy of a photon must be the first divided by the second. After all, the energy density is nothing but the number of photons in a cubic meter times the average energy of each photon. Therefore,

$$\langle E_\gamma\rangle = \frac{\frac{8\pi^5 k_B^4}{15h^3c^3}T^4}{(2.404)8\pi\left(\frac{k_B}{hc}\right)^3 T^3} = \frac{k_B T\pi^4/15}{2.404} = 2.7k_B T \tag{E.147}$$

where as usual, angle brackets surrounding a quantity indicate the average of that quantity.

Sometimes we are interested in the number density of photons whose wavelength lies between two given values, say λ_1 and λ_2. This is obtained by integrating the spectral number density from λ_1 to λ_2. Denoting this number density by $n_\gamma(\lambda_1, \lambda_2, T)$, we get

$$n_\gamma(\lambda_1, \lambda_2, T) = 8\pi\int_{\lambda_1}^{\lambda_2}\frac{d\lambda}{\lambda^4\left(e^{hc/\lambda k_B T} - 1\right)} = 8\pi\left(\frac{k_B T}{hc}\right)^3\int_{y_1}^{y_2}\frac{dy}{y^4(e^{1/y} - 1)} \tag{E.148}$$

where $y_1 = k_B T\lambda_1/hc$ and $y_2 = k_B T\lambda_2/hc$. Equation (E.146) is a special case of Equation (E.148): $n_\gamma(T) = n_\gamma(0, \infty, T)$. As an example, we can ask "At a temperature of 1000 K, what fraction of photons are visible?" Since the visible range is 0.4 μm to 0.7 μm, the two wavelengths are $\lambda_1 = 0.4$ μm and $\lambda_2 = 0.7$ μm, and the question wants to know the ratio $n_\gamma(\lambda_1, \lambda_2, T)/n_\gamma(T)$, which is simply the ratio of the two integrals in y. The integral of the denominator was found to be 2.404 in Equation (E.145). For the numerator, we need

$$y_1 = \frac{k_B T\lambda_1}{hc} = \frac{(1.38\times 10^{-23})(1000)(0.4\times 10^{-6})}{(6.63\times 10^{-34})(3\times 10^8)} = 0.02775$$

and $y_2 = 0.048567$, which is found similarly to y_1. Then, using a graphing calculator, we find

$$\int_{0.02775}^{0.048567}\frac{dy}{y^4(e^{1/y} - 1)} = 5.34\times 10^{-7}$$

Thus, only about 0.534 millionth of the photons at 1000 K are visible.

Another question we can ask and answer is "At what temperature the number of photons having an energy of 13.6 eV or higher is 6×10^{-10} the total number?" This may seem a *random question, but it is relevant to the decoupling of radiation from matter. The wavelength corresponding to 13.6 eV is (using the Planck formula)*

$$\lambda = \frac{hc}{E} = \frac{(6.63 \times 10^{-34})(3 \times 10^{8})}{13.6(1.6 \times 10^{-19})} = 9.14 \times 10^{-8} \text{ m}$$

Any wavelength shorter *than this has an energy larger than 13.6 eV. Therefore,* $\lambda_1 = 0$ *and* $\lambda_2 = 9.14 \times 10^{-8}$ *m. The corresponding* y*'s are* $y_1 = 0$ *and*

$$y_2 = \frac{k_B T \lambda_2}{hc} = \frac{(1.38 \times 10^{-23})T(9.14 \times 10^{-8})}{(6.63 \times 10^{-34})(3 \times 10^{8})} = 0.0000063T \tag{E.149}$$

We are looking for a temperature at which $n_\gamma(0, \lambda_2, T)/n_\gamma(T)$ *is* 6×10^{-10}*. Since the constants multiplying the* y*-integrals are the same, and the integral of the denominator is 2.204, we want to solve the equation*

$$\frac{n_\gamma(0, \lambda_2, T)}{n_\gamma(T)} = 6 \times 10^{-10} \quad \text{or} \quad \int_0^{y_2} \frac{dy}{y^4(e^{1/y} - 1)} = 2.404(6 \times 10^{-10}) = 1.44 \times 10^{-9}$$

By substituting various (small) values for y_2 *and calculating the integral above numerically (on a graphing calculator, for example), you can get* $y_2 = 0.037$*. Equation (E.149) now gives* $T = 5873$ *K.*

Derivation of horizon radius
(page 574 of the book)

Math Note E.37.6. In Section 27.4, we saw that the spacetime interval Δs for light was zero. This led to $\Delta x = c\Delta t$, which simply indicated that light traveled with speed c. All this was in the absence of gravity and universal expansion. If you include expansion, Δx stretches by the scale factor. It is common to denote the ratio of the scale $R(t)$ at time t to the present scale R_0 by $a(t)$, so that $a(t) = R(t)/R_0$. Then the stretched Δx is $a(t)\Delta x$, and, in an expanding universe, time travels in such a way that $a(t)\Delta x = c\Delta t$. The rest of the discussion uses calculus to find the horizon radius.

The differential form of the last equation is $a(t)dx = cdt$ *or* $dx = cdt/a(t)$*, which can be integrated to some final time* t_f *to find the* x *at that time:*

$$x_f = \int_0^{t_f} \frac{cdt}{a(t)}$$

To find the actual radius at time t_f*, you must multiply* x_f *by* $a(t_f)$*. This yields*

$$r_h(t_f) = ca(t_f)\int_0^{t_f} \frac{dt}{a(t)} \quad \text{or} \quad r_h(t_f) = cR(t_f)\int_0^{t_f} \frac{dt}{R(t)} \tag{E.150}$$

where in the last step, I multiplied the numerator and the denominator by R_0*. For the special—but important—case in which* $a(t)$ *is proportional to* t^n *with* $n < 1$*, Equation (E.150) can be easily solved:*

$$r_h(t_f) = ct_f^n \int_0^{t_f} \frac{dt}{t^n} = ct_f^n \left.\frac{t^{1-n}}{1-n}\right|_0^{t_f} = \frac{ct_f}{1-n} \quad \text{or} \quad r_h(t) = \frac{ct}{1-n} \tag{E.151}$$

where in the last equation the subscript f *has been dropped for convenience.* For a matter-dominated universe $n = 2/3$ and $r_h(t) = 3ct$; for a radiation-dominated universe $n = 1/2$ and $r_h(t) = 2ct$.

Deriving a formula for $\rho_m(T)$
(page 578 of the book)

Math Note E.37.7. The constancy of the mass in a given (expanding) volume—expressed as $\rho_m(T)[R(T)]^3 = \rho_{m0}R_0^3$—plus the second equation in the third row of Table E.3, with $\rho_{m0} = 2.5 \times 10^{-27}$ and $t_0 = 13.7$ billion years (equal to 4.3×10^{17} seconds), give the desired function:

$$\rho_m(T) = \rho_{m0}\frac{R_0^3}{R^3} = \rho_{m0}\frac{T^3}{T_0^3} = 1.24 \times 10^{-28}T^3 \text{ kg/m}^3 \tag{E.152}$$

E.38 Math Notes for Chapter 38

Explaining the numerical factors
(page 584 of the book)

Math Note E.38.1. As mentioned in the last chapter, the energy density is the single most important quantity that determines the development of the universe. When the universe was very young, all particles contributing to its energy density were relativistic, namely, their kinetic energy was so much larger than their rest energy that one could assume that they were massless. Therefore, their contribution could be calculated using the penultimate massless example, photon.

The spectral energy density of photons is given in Equation (E.132) of Math Note E.37.3. All the relativistic particles will have almost the same function for their spectral energy density, except for the following two changes:

- The number 8 in (E.37.3) is replaced by 4 multiplied by the number of spin projections. This is because for photons that 8 includes a 2 for the spin (or polarization) of photons. Thus for a massive spin-s particle 8 changes to $4(2s+1)$, and for massless particles (except neutrinos) it does not change. For neutrinos, 8 changes to 4 because they have only one spin orientation (see Section 33.3.3).
- The expression in the denominator, $1/(e^{hc/\lambda k_B T}-1)$, is related to the average number of particles in a state with energy hc/λ, and is valid for bosons. The negative sign in the denominator allows the fraction to be infinite, which is fine as long as we are dealing with bosons. Pauli's exclusion principle does not permit more that one particle per state for fermions. Therefore, the average number of fermions cannot have a negative sign in the denominator. A detailed analysis leads to a surprisingly simple solution: change the minus to a plus. Thus for fermions, the expression above changes to $1/(e^{hc/\lambda k_B T}+1)$.

The total energy density of a relativistic fermion is obtained by integrating the spectral energy density over all wavelengths. This is identical to what was done in Equation (E.52) of Math Note E.20.1, except that the integral in y, whose value turned out to be $\pi^4/15$, is replaced by

$$\int_0^\infty \frac{dy}{y^5(e^{1/y}+1)} = \frac{7\pi^4}{120} = \frac{7}{8}\left(\frac{\pi^4}{15}\right)$$

and that is where the fermionic factor of 7/8 comes from.

The factor of 2 associated with the distinctness of the antiparticle simply includes the latter in the calculation. One could treat the antiparticle as a different particle and calculate its contribution separately; but since this contribution is identical to that of the particle, it is more convenient to talk of the particle "species," which includes both the particle and its antiparticle.

Let's summarize the discussion above as follows. A relativistic particle P has a density ρ_p, which is a multiple of the photon density: $\rho_p = \alpha_p\rho_\gamma$, where α_p is the product of three factors, $\alpha_p = a_{bf}a_s a_{\text{anti}}$, where a_{bf} is 1 if P is a boson and $\frac{7}{8}$ if it is a fermion; a_s is $\frac{1}{2}$ if P is a neutrino, 1 if P is a massless particle (but not a neutrino), and $\frac{2s+1}{2}$ if P is a massive spin-s particle; a_{anti} is 1 if P has no antiparticle and 2 if it does.

T_γ and T_ν after e^+e^- annihilation
(page 588 of the book)

Math Note E.38.2. The calculation of the photon and neutrino temperatures after the electrons and positrons annihilate each other requires the concept of entropy. The entropy *density* (entropy per unit volume) of a gas of relativistic particles with density ρ is $4\rho/3T$. Thus, the entropy in a cube of side R (scale of the universe) is $S=(4\rho/3T)R^3$, and this quantity does not change as the universe evolves. Since $\rho = \alpha\rho_\gamma$, we see that

$$S = \frac{4}{3}\frac{\rho}{T}R^3 = \frac{4}{3}R^3\frac{\alpha\rho_\gamma}{T} = \frac{4}{3}R^3\frac{\alpha(8.36\times 10^{-33}T^4)}{T} = \frac{4}{3}\alpha(8.36\times 10^{-33})(TR)^3$$

and the quantity $\alpha(TR)^3$ is constant:

$$\alpha(TR)^3 = \text{constant} \tag{E.153}$$

Just before the e^+e^- annihilation, the neutrinos decouple, and therefore only electrons, positrons, and photons are in thermal equilibrium.[26] The α for this combination is

$$\alpha = \tfrac{7}{8} \times 2 + 1 = \tfrac{11}{4}$$

After the annihilation, only photons remain, and Equation (E.153) yields

$$\frac{11}{4}(T_\gamma R)^3_{\text{bef}} = (T_\gamma R)^3_{\text{aft}} \quad \text{or} \quad \frac{(T_\gamma R)_{\text{aft}}}{(T_\gamma R)_{\text{bef}}} = \sqrt[3]{\tfrac{11}{4}} = 1.401$$

The decoupled neutrinos cool down as the universe expands so that the product $T_\nu R$ is constant (the temperature falls in inverse proportion to the scale of the universe). Thus, $(T_\nu R)_{\text{bef}} = (T_\nu R)_{\text{aft}}$. But before e^+e^- annihilation, $T_\nu = T_\gamma$. Therefore, $(T_\nu R)_{\text{aft}} = (T_\gamma R)_{\text{bef}}$. Now, we can write

$$(T_\gamma/T_\nu)_{\text{aft}} = \frac{(T_\gamma R)_{\text{aft}}}{(T_\nu R)_{\text{aft}}} = \frac{(T_\gamma R)_{\text{aft}}}{(T_\gamma R)_{\text{bef}}} = \sqrt[3]{\tfrac{11}{4}}$$

This ratio will be maintained throughout the history of the universe, because both photon and neutrino temperatures fall in exactly the same way.

Calculation of the threshold T for deuteron formation **(page 589 of the book)**

Math Note E.38.3. We want to find the temperature at which we have sufficient number of energetic photons to break up the deuterons that may be formed. "Energetic enough" means having an energy larger than the binding energy of the deuteron, which is 2.224 MeV. The photon-nucleon number ratio is 1.6 billion, and it does not change in the course of the evolution of the universe. So if just $1/(1.6 \times 10^9)$ or 6×10^{-10} of the population of photons has energies 2.224 MeV or higher, we have one energetic photon for every nucleon.

We refer to the end of Math Note E.37.5 where we calculated the temperature at which the ratio of the number of photons with wavelengths shorter than λ_2 to the total number of photons was 6×10^{-10}. Here λ_2 corresponds to an energy of 2.224 MeV or 2224000 eV. Thus,

$$\lambda_2 = \frac{hc}{E} = \frac{(6.63 \times 10^{-34})(3 \times 10^8)}{2224000(1.6 \times 10^{-19})} = 5.59 \times 10^{-13} \text{ m}$$

and

$$y_2 = \frac{k_B T \lambda_2}{hc} = \frac{(1.38 \times 10^{-23})T(5.59 \times 10^{-13})}{(6.63 \times 10^{-34})(3 \times 10^8)} = 3.88 \times 10^{-11} T$$

Math Note E.37.5 calculated y_2 to be 0.037. Hence,

$$0.037 = 3.88 \times 10^{-11} T \quad \text{or} \quad T = \frac{0.037}{3.88 \times 10^{-11}} = 9.5 \times 10^8$$

E.39 Math Notes for Chapter 39

Flatness problem **(page 597 of the book)**

Math Note E.39.1. The left-hand side of Equation (37.1) is H^2. Divide both sides of the equation by H^2 to obtain

$$1 = \underbrace{\frac{8\pi G}{3H^2}}_{=1/\rho_c} \rho - \frac{kc^2}{R^2 H^2} = \Omega_{\text{tot}}(t) - \frac{kc^2}{R^2 H^2}$$

[26]The protons and neutrons are too few, and thus negligible.

by the definition of ρ_c and the fact that by ρ is meant ρ_{tot}. A little rearrangement of terms now yields

$$\Omega_{\text{tot}}(t) - 1 = \frac{kc^2}{R^2 H^2} \tag{E.154}$$

If the universe is flat, the right-hand side is zero and $\Omega_{\text{tot}}(t) = 1$, i.e., $\Omega_{\text{tot}}(t)$ is always 1, and the universe has always been flat. If the universe is not flat, then the right-hand side is not zero, and it varies with time. The variation with time depends on which component of the universe is dominant. For a matter domination, the first row of Table E.3 yields

$$R^2 = R_0^2 \left(\frac{t}{t_0}\right)^{4/3}, \qquad H^2 = \frac{4}{9t^2}$$

which upon substitution in Equation (E.154) gives

$$\Omega_{\text{tot}}(t) - 1 = \frac{kc^2}{\left[R_0^2 (t/t_0)^{4/3}\right][4/9t^2]} = \frac{9kc^2 t_0^{4/3}}{4R_0^2} t^{2/3}$$

Suppose that we know Ω_{tot} at some time t', then

$$\Omega_{\text{tot}}(t') - 1 = \frac{9kc^2 t_0^{4/3}}{4R_0^2} t'^{2/3}$$

and dividing both sides of the last two equations, we obtain

$$\frac{\Omega_{\text{tot}}(t) - 1}{\Omega_{\text{tot}}(t') - 1} = \left(\frac{t}{t'}\right)^{2/3} \quad \text{or} \quad \Omega_{\text{tot}}(t) - 1 = [\Omega_{\text{tot}}(t') - 1]\left(\frac{t}{t'}\right)^{2/3} \tag{E.155}$$

For a radiation domination, Table E.3 yields

$$R^2 = R_0^2 \left(\frac{32\pi G \rho_{\gamma 0}}{3}\right)^{1/2} t, \qquad H^2 = \frac{1}{4t^2}$$

which upon substitution in Equation (E.154) gives

$$\Omega_{\text{tot}}(t) - 1 = \frac{kc^2}{\left[R_0^2 (32\pi G \rho_{\gamma 0}/3)^{1/2} t\right][1/4t^2]} = \frac{kc^2}{R_0^2 (2\pi G \rho_{\gamma 0}/3)^{1/2}} t$$

Once again a knowledge of Ω_{tot} at some time t' leads to

$$\Omega_{\text{tot}}(t') - 1 = \frac{kc^2}{R_0^2 (2\pi G \rho_{\gamma 0}/3)^{1/2}} t'$$

and dividing both sides of the last two equations, we obtain

$$\frac{\Omega_{\text{tot}}(t) - 1}{\Omega_{\text{tot}}(t') - 1} = \frac{t}{t'} \quad \text{or} \quad \Omega_{\text{tot}}(t) - 1 = [\Omega_{\text{tot}}(t') - 1]\frac{t}{t'} \tag{E.156}$$

Now suppose that the present universe is not flat. In fact, let Ω_{tot} be 1.5 now. Then, with $t' = t_0$ and $\Omega_{\text{tot}}(t') - 1 = 0.5$, Equation (E.155) yields[27]

$$\Omega_{\text{tot}}(t) - 1 = 0.5 \left(\frac{t}{t_0}\right)^{2/3} \tag{E.157}$$

[27] Equations (E.155) and (E.156) are valid only if the universe is flat. However, if the deviation from flatness is small, those equations are still good approximations to the exact equations that we have not derived.

for the matter-dominated history of the universe. At the decoupling time, $t = 375,000$ years, Equation (E.157) yields

$$\Omega_{\text{tot}}(t) - 1 = 0.5 \left(\frac{375000}{1.37 \times 10^{10}} \right)^{2/3} = 0.000908$$

or $\Omega_{\text{tot}} = 1.000908$. Further back, at the dawn of matter domination, 33000 years after the big bang,

$$\Omega_{\text{tot}}(t) - 1 = 0.5 \left(\frac{33000}{1.37 \times 10^{10}} \right)^{2/3} = 0.00018$$

or $\Omega_{\text{tot}} = 1.00018$.

Earlier than 33,000 years after the big bang, the universe was radiation-dominated, and Equation (E.156) must be used. For t' take the onset of matter domination, so that $t' = 33,000$ years or approximately 10^{12} s, and $\Omega_{\text{tot}}(t') - 1 = 0.00018$. Then Equation (E.156) yields

$$\Omega_{\text{tot}}(t) - 1 = 0.00018 \left(\frac{t}{10^{12}} \right) \qquad \text{with } t \text{ in seconds} \tag{E.158}$$

At the time of helium formation, when the universe was 196 seconds old,

$$\Omega_{\text{tot}}(t) - 1 = 0.00018 \left(\frac{196}{10^{12}} \right) = 3.53 \times 10^{-14}$$

or $\Omega_{\text{tot}} = 1.0000000000000353$.

Math Note E.39.2. If $\Lambda/3$ is the only term on the right-hand side of Equation (39.2), then

Inflationary expansion **(page 600 of the book)**

$$\left(\frac{v}{R} \right)^2 = \frac{\Lambda}{3} \quad \text{or} \quad H^2 = \frac{\Lambda}{3} \quad \text{or} \quad H = \sqrt{\frac{\Lambda}{3}} \tag{E.159}$$

because $v/R = H$. The following derivation involving calculus shows how R depends on time.

Since $v = dR/dt$, the last equation in (E.159) becomes

$$\frac{dR/dt}{R} = \sqrt{\frac{\Lambda}{3}} \quad \text{or} \quad \frac{dR}{R} = \sqrt{\frac{\Lambda}{3}}\, dt = H dt$$

Integrating both sides gives

$$\ln(R) = Ht + \text{constant} = Ht + \ln(r)$$

where r is a constant, which could be interpreted as the scale of the universe just before inflation. The last equation can now be written as

$$\ln(R) - \ln(r) = Ht \quad \text{or} \quad \ln\left(\frac{R}{r}\right) = Ht \quad \text{or} \quad R = re^{Ht} \tag{E.160}$$

which shows that R grows exponentially, because H is a constant.

Now that $R(t)$ is known, Equation (E.150) can be used to find the horizon radius. Substitution of (E.160) in that equation yields

$$\begin{aligned} r_h(t_f) &= cre^{Ht_f} \int_0^{t_f} \frac{dt}{re^{Ht}} = ce^{Ht_f} \int_0^{t_f} e^{-Ht}\, dt \\ &= ce^{Ht_f} \left(- \frac{e^{-Ht}}{H} \Bigg|_0^{t_f} \right) = \frac{c}{H} e^{Ht_f}(-e^{-Ht_f} + 1) \\ &= \frac{c}{H}(e^{Ht_f} - 1) \end{aligned}$$

Neglecting the subscript for time, we get

$$r_h(t) = \frac{c}{H}(e^{Ht} - 1) \tag{E.161}$$

which gives the horizon radius as a function of time during inflation. Furthermore, with $H = \sqrt{\Lambda/3}$ and R given by Equation (E.160), Equation (E.154) becomes

$$\Omega_{\text{tot}}(t) - 1 = \frac{kc^2}{H^2 r^2 e^{2Ht}} = \frac{kc^2}{r^2 H^2} e^{-2Ht} \tag{E.162}$$

Mass needed for a star ignition
(page 602 of the book)

Math Note E.39.3. To ignite the hydrogen fusion, the gravitational force must be strong enough to push the protons sufficiently close so that they can penetrate the Coulomb repulsive barrier for the strong nuclear force to take over. But the gravitational force cannot be too strong (and the star too compact) because the electrons, which obey Pauli exclusion principle, cannot get too close to each other. The quantity that measures how close two electrons can be is their (average) wavelength λ_e. So, let's assume that a typical electron encloses itself in a cube of length λ_e, inside of which no other electron is allowed. Let's also assume that the electrons are spread uniformly throughout the star. Then the number of the electrons N times λ_e^3 should give the volume of the star (with radius R):

$$N\lambda_e^3 = \tfrac{4}{3}\pi R^3 \quad \text{or} \quad R = 0.62\lambda_e N^{1/3} \tag{E.163}$$

Given the average wavelength of the electrons, their average momentum is determined by the de Broglie relation: $p_e = h/\lambda_e$. This gives rise to a speed $v_e = p_e/m_e = h/m_e\lambda_e$ and an average kinetic energy per electron of

$$\langle KE_e \rangle = \tfrac{1}{2}m_e v_e^2 = \tfrac{1}{2}m_e \left(\frac{h}{m_e\lambda_e}\right)^2 = \frac{h^2}{2m_e\lambda_e^2}$$

The total KE is then

$$KE_{\text{tot}} = N\langle KE_e \rangle = \frac{Nh^2}{2m_e\lambda_e^2} \tag{E.164}$$

I have to emphasize that KE_{tot} comes directly from exclusion principle. In the absence of that principle, R and λ of Equation (E.163) would be zero, and there would not be any minimum size or mass for the star.

What gives rise to this KE? The gravitational potential energy PE of the star. Assuming that the total energy starts out as zero (and, therefore, remains zero by energy conservation) yields $KE_{\text{tot}} + PE = 0$. The PE of a star of mass M turns out to be $-\frac{3}{5}GM^2/R$. It follows that

$$KE_{\text{tot}} - \tfrac{3}{5}\frac{GM^2}{R} \quad \text{or} \quad \frac{Nh^2}{2m_e\lambda_e^2} = \tfrac{3}{5}\frac{G(Nm_n)^2}{0.62\lambda_e N^{1/3}} \tag{E.165}$$

where M is just the number of nucleons (assumed to be the same as N)[28] times the mass of each nucleon m_n. Equation (E.165) can be simplified to

$$\lambda_e N^{2/3} = \frac{h^2}{2Gm_n^2 m_e} \tag{E.166}$$

$\langle KE_e \rangle$ is related to the T by $\langle KE_e \rangle = \frac{3}{2}k_B T$ [see Equation (17.1)] or

$$\tfrac{3}{2}k_B T = \frac{h^2}{2m_e\lambda_e^2}$$

[28]The number of protons alone is equal to N; and that is only 87% of the nucleons. I am ignoring the neutrons because the extra 13% neutrons will only slightly affect the final outcome of our discussion.

Since $\frac{3}{2}k_BT$ is also the average KE of the protons, it has to be larger than the Coulomb potential barrier E_{bar} for fusion to occur. This gives

$$\frac{h^2}{2m_e\lambda_e^2} \geq E_{\text{bar}} \quad \text{or} \quad \lambda_e \leq \frac{h}{\sqrt{2m_eE_{\text{bar}}}} = 1.7 \times 10^{-11} \text{ m} \tag{E.167}$$

Substituting this in Equation (E.166) yields

$$N^{2/3} = \frac{h^2}{2Gm_n^2m_e\lambda_e} \geq \frac{h\sqrt{2m_eE_{\text{bar}}}}{2Gm_n^2m_e} = \frac{h\sqrt{2E_{\text{bar}}}}{2Gm_n^2\sqrt{m_e}} \tag{E.168}$$

Inserting the numerical values of all quantities in this equation, including $E_{\text{bar}} = 5$ keV (see Example 31.2.6), yields

$$N^{2/3} \geq \frac{h\sqrt{2E_{\text{bar}}}}{2Gm_n^2\sqrt{m_e}} = \frac{6.63 \times 10^{-34}\sqrt{2 \times 5000 \times (1.6 \times 10^{-19})}}{2(6.67 \times 10^{-11})(1.67 \times 10^{-27})^2\sqrt{9.1 \times 10^{-31}}} = 7.47 \times 10^{37}$$

or $N = (7.47 \times 10^{37})^{3/2} = 6.46 \times 10^{56}$. The minimum mass is therefore

$$M = Nm_n = (6.46 \times 10^{56})(1.67 \times 10^{-27}) = 1.08 \times 10^{30} \text{ kg} = 0.5M_\odot$$

The radius is obtained by combining the value for N, (E.163), and (E.167):

$$R = 0.62(1.7 \times 10^{-11})(6.46 \times 10^{56})^{1/3} = 9.1 \times 10^7 \text{ m} = 0.13R_\odot$$

Math Note E.39.4. Math Note E.39.3 calculated the *minimum* mass required for the hydrogen fusion to start. Once it starts, the fusion produces radiation at the core, which on its way out, tends to push layers of the star outward. If the star is too massive, it will overcome this pressure and collapse. The balance of the radiation and gravitational pressures keeps the star alive.

Maximum mass supported by radiation **(page 602 of the book)**

The radiation pressure is given by Equation (E.31), where u is the energy density:

$$P_\gamma = \tfrac{1}{3}u = \frac{8\pi^5k_B^4}{45h^3c^3}T^4 = 2.5 \times 10^{-16}T^4 \tag{E.169}$$

using Equation (E.135) and then substituting all the numerical constants. The gravitational pressure is given by the (inward) pressure of other particles:

$$P_G = \frac{Nk_BT}{V} = \frac{Nk_BT}{\frac{4}{3}\pi R^3} = 3.3 \times 10^{-24}\frac{NT}{R^3}$$

Equating the two pressures gives

$$2.5 \times 10^{-16}T^4 = 3.3 \times 10^{-24}\frac{NT}{R^3} \quad \text{or} \quad N = 7.6 \times 10^7(TR)^3 \tag{E.170}$$

Equality of the total KE (in terms of the temperature) and the gravitational PE (see Math Note E.39.3) yields

$$\tfrac{3}{2}Nk_BT = \tfrac{3}{5}\frac{GM^2}{R} = \tfrac{3}{5}\frac{G(Nm_n)^2}{R} \quad \text{or} \quad RT = \tfrac{2}{5}\frac{Gm_n^2N}{k_B} = 5.4 \times 10^{-42}N \tag{E.171}$$

Using Equations (E.170) and (E.171), we get

$$N = 7.6 \times 10^7(5.4 \times 10^{-42}N)^3 \quad \text{or} \quad 1 = 1.2 \times 10^{-116}N^2 \quad \text{or} \quad N = 9.1 \times 10^{57} \tag{E.172}$$

and a mass of

$$M = Nm_n = (9.1 \times 10^{57})(1.67 \times 10^{-27}) = 1.5 \times 10^{31} \text{ kg} = 7.5M_\odot \tag{E.173}$$

Math Note E.39.5. When gravity pushes on the star undergoing collapse, the electrons acquire enough energy to become relativistic. The energy of such an electron is of the order of its rest energy, m_ec^2. As in the case of a normal star, this energy results from the conversion of the potential energy of the collapsing star, as discussed in Math Note E.39.3. With N electrons in the star, the equivalent of Equation (E.165) is

$$Nm_ec^2 = \tfrac{3}{5}\frac{G(Nm_n)^2}{R} = \tfrac{3}{5}\frac{G(Nm_n)^2}{0.62\lambda_e N^{1/3}} = \tfrac{3}{5}\frac{Gm_n^2N^{5/3}}{0.62\lambda_e} \quad \text{or} \quad m_ec^2 = \tfrac{3}{5}\frac{Gm_n^2N^{2/3}}{0.62\lambda_e} \tag{E.174}$$

Physical properties of white dwarfs (**page 603 of the book**)

where we used (E.163) for the radius of the star.

A relativistic particle has a special wavelength, called the **Compton wavelength**. A heuristic argument is that v in the de Broglie relation $\lambda = h/mv$ is to be replaced by c, the special speed of relativity, in which case we obtain the Compton wavelength: $\lambda = h/mc$. Substituting Compton wavelength for an electron in Equation (E.174), we get

$$m_ec^2 = \tfrac{3}{5}\frac{Gm_n^2N^{2/3}}{0.62(h/m_ec)} \quad \text{or} \quad N = \left(\frac{0.62hc}{0.6Gm_n^2}\right)^{3/2} = 3.67\times 10^{58} \tag{E.175}$$

Substituting this and $\lambda_e = h/m_ec = 2.4\times 10^{-12}$ m in (E.163) yields

$$R = 0.62(2.4\times 10^{-12})(3.67\times 10^{58})^{1/3} = 5\times 10^7 \text{ m} \tag{E.176}$$

which is of the order of the radius of a planet.

The numbers obtained above are very rough estimates. A detailed and more precise analysis yields a maximum number $N = 1.68\times 10^{57}$ for a white dwarf corresponding to a maximum mass of $1.4M_\odot$, called the **Chandrasekhar mass**. Stars more massive than this will turn into neutron stars and black holes. The same precise analysis yields a radius that is more comparable to the Earth's.

The physics of a neutron star is very similar to that of the white dwarf. In fact, all the formulas above apply to the neutrons with suitable substitution. For example, Equation (E.175), being independent of the properties of the electron, directly gives the number of neutrons. in the star. For the radius, we use Equation (E.163), except that λ_e should be replaced by the corresponding neutron Compton wavelength:

$$\lambda_n = \frac{h}{m_nc} = \frac{6.63\times 10^{-34}}{(1.67\times 10^{-27})(3\times 10^8)} = 1.3\times 10^{-15} \text{ m}$$

This yields

$$R = 0.62(1.3\times 10^{-15})(3.67\times 10^{58})^{1/3} = 27266 \text{ m}$$

or about 17 miles, a very compact object indeed!

E.43 Math Notes for Chapter 43

Chances of monkeys typing a Hamlet phrase (**page 646 of the book**)

Math Note E.43.1. To make the problem of the monkey and the typewriter simple, assume that the typewriter has 26 letters and a space bar. The sentence has 28 characters (23 letters and 5 spaces). For each of the 28 characters, the monkey has to choose from 27 keys. By randomly striking the keys on this typewriter, in how many different ways can a monkey produce a 28-character sentence? There are 27 choices for the first letter. For each of these 27 choices there are 27 ways of choosing the second letter. So, if the sentence were two letters long, there would be 27×27 ways of choosing letters. For a three-letter sentence, the number of choices would be $27\times 27\times 27$. So, for a 28-letter long sentence there are

$$\underbrace{27\times 27\times 27\times\cdots\times 27}_{28 \text{ times}} = 1.197\times 10^{40}$$

choices. So, the odds for a monkey to type the given sentence is less than one in 10,000 trillion trillion trillion!

Now assume that each keystroke takes half a second. Then, on the average, the total amount of time elapsed before the monkey gets to the correct combination is about[29]

$$\tfrac{1}{2} \times 1.197 \times 10^{40} \times 0.5 \approx 3 \times 10^{39}$$

seconds or 9.5×10^{31} years. The age of the universe is about 1.4×10^{10} years! So, there is no hope! What if we relegate the task of the monkey to a computer. Assuming that the computer can perform each instruction in a billionth of a second, the total time required will be

$$\tfrac{1}{2} \times 1.197 \times 10^{40} \times 10^{-9} \approx 6 \times 10^{30}$$

seconds or 1.9×10^{23} years, or over 12 trillion times the age of the universe!

[29]The factor $\frac{1}{2}$ in front of the expression is due to the fact that the success may be on the first try (very unlikely!) or on the last one. We take the middle ground.

Appendix F

Spacetime Geometry

An event with coordinates (x, ct) [which are, by the way, determined by drawing *parallel lines* to the axes (see Appendix C)] relative to O has a different set of coordinates (x', ct') relative to a second observer O'. Is there a geometric (diagrammatic) way of relating these two pairs of coordinates?

Go back to the Euclidean case and concentrate on the orientation of the axes. Figure 27.5 shows two different Euclidean coordinate systems and how one system is oriented relative to the other. In particular, it is seen that whatever angle the x-axis of one system makes with the x-axis of the other, the same angle is made by the corresponding y-axes, *in such a way that the x- and y-axes of each system are perpendicular.* By picking two points and using the invariance of the Euclidean distance [Equation (E.82)], one can actually "prove" that the y- and x-axes are perpendicular. **Math Note E.27.7** on **page 123** of *Appendix.pdf* explains how to do this. Of course, we *assumed* this property to derive Equation (E.82). The point is that distance rule [Equation (E.82)] dictates the angle which the axes make with one another.

Relativity requires nonperpendicular set of axes!

How do the axes of a spacetime observer O appear in the spacetime plane of another observer O' (whose axes are assumed perpendicular)? Suppose that O moves in the positive direction of O'. Then the worldline of O, which is inside the light cone of O', is the ct-axis by Box 27.2.1. Figure F.1(a) shows this axis and notes that it makes an angle of θ with the ct'-axis. Now apply the cornerstone of relativity: that the speed of light—whose worldline is drawn as a wavy line—is the same for all observers. For O', this line makes a $45°$ angle with both axes. For any other observer, the light worldline must make equal angles with both axes.[1] Therefore, the x-axis must make an angle of θ with the x'-axis as shown in Figure F.1(a). We thus see that if the axes of an observer O' are drawn perpendicular, then the axes of another observer O moving relative to O' cannot be perpendicular.

The slope of the ct-axis *relative to the ct'-axis* (which by definition, is $\Delta x'/c\Delta t'$) is β, the (fractional) speed of O relative to O'. Similarly, the slope of the x-axis relative to the x'-axis is also β. Invoking the rules of finding coordinates in a nonperpendicular system of axes—as discussed in Section C.1—we find the first three rules of spacetime geometry:

[1]This is because $c\Delta t = \Delta x$ for any two events (points) that lie on the light worldline.

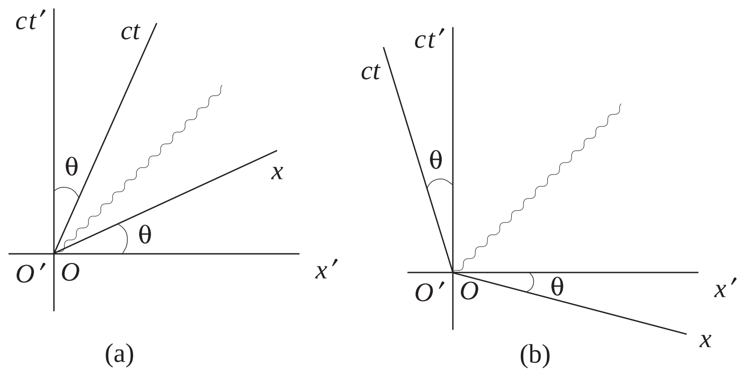


Figure F.1: (a) O is moving in the positive direction of O'. (b) O is moving in the negative direction of O'.

Box F.0.2. (**Rules of spacetime geometry**) *Suppose that O moves relative to O' and the axes of O' are drawn perpendicular.*

1. The axes of O form an acute angle if O moves in the positive direction, and an obtuse angle if O moves in the negative direction.

2. The slope between the axes of O and the corresponding axes of O' is $\beta = v/c$.

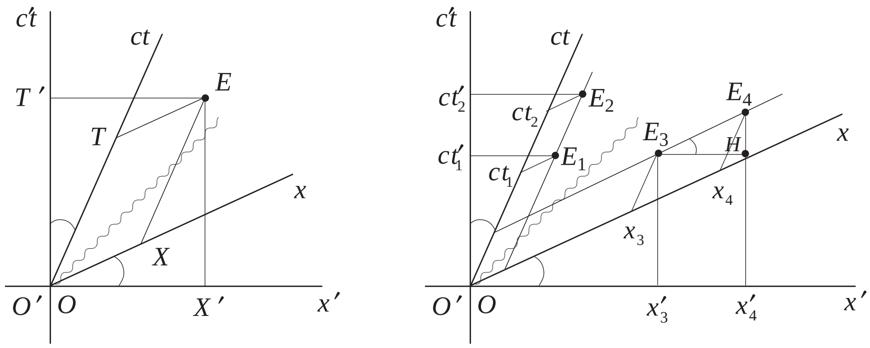
3. Suppose that the lines drawn parallel to the axes from an event E intersect the time axis at T and the space axis at X; then $\overline{ET}$ is the space coordinate and $\overline{EX}$ is the time coordinate of E [Figure F.2(a)].

Our ultimate goal is to be able to find the spacetime coordinates of an event in a frame O' if we know its coordinates in another frame O, and vice versa. For this, we need to relate the time intervals and lengths measured in the two coordinate systems. First consider events E_1 and E_2 in Figure F.2(b), which occur at the same point *relative to* Emmy (observer O), i.e., for which (one of) Emmy's clock(s) is present at both events. It follows that $t_2 - t_1$ is the *proper time*, and must be related to $t'_2 - t'_1$—the time interval between the same two events according to Karl (observer O')—via (26.1), our first relativistic equation.

Lengths are not what they appear to be!

Here we encounter the first strange phenomenon of spacetime geometry: Although $\overline{E_1E_2}$ *appears* longer to Karl than its projection, the projection is actually longer! In general, we can apply the rules of ordinary geometry only to lengths *measured by a single observer*. Lengths measured by two *different* observers *are not* related by the rules of ordinary geometry.

What about lengths along (or parallel to) the space axes? The events E_3 and E_4 in Figure F.2(b) occur simultaneously according to Emmy, i.e., $\overline{E_3E_4} = x_4 - x_3$ is parallel to Emmy's x-axis, and as such represents a length (say of her spaceship if E_3 and E_4 are explosions of two firecrackers at the ends of the spaceship). The same two events are separated by $x'_4 - x'_3$ according to Karl, and we might think that, since moving lengths shrink, $x'_4 - x'_3 = (x_4 - x_3)/\gamma$. But as Example F.0.4 shows, $x'_4 - x'_3 = \gamma(x_4 - x_3)$. The reason for the discrepancy is that $x'_4 - x'_3$ *does not represent the length* of the spaceship for O'. For two explosions to represent the length of the spaceship as measured by Karl, they must occur simultaneously *for Karl.* E_3 and E_4 are not simultaneous for Karl. Example F.0.4 also derives a rule that connects the length of the line segment $\overline{E_1E_2}$ (or $\overline{E_3E_4}$) as measured by Karl (using a ruler, for example) and that measured by Emmy. We are now ready to state the remaining rules of space time geometry.



(a) (b)

Figure F.2: (a) Coordinates (X, cT) and (X', cT') of E in O and O'. (b) Events E_1 and E_2 occur at the same location relative to O, which is moving in the positive direction of O'. Events E_3 and E_4 occur simultaneously relative to O.

Box F.0.3. (**Rules of spacetime geometry**) *Suppose that O moves relative to O' and the axes of O' are drawn perpendicular.*

4. If E_1 and E_2 occur on a worldline parallel to an axis of one observer—for whom $\overline{E_1E_2}$ is the interval on that axis—and A_1 and A_2 are their (parallel) projections onto the ***corresponding*** *axis of the other observer, then $\overline{A_1A_2} = \gamma\overline{E_1E_2}$.*

5. The (Euclidean) length $\overline{E_1E_2}$, as measured by O', is longer than the interval itself (as measured by O) by a factor of $\gamma\sqrt{1+\beta^2}$. This factor is called the ***stretch factor****.*

Example F.0.4. In Figure F.2(b), let $\Delta x = x_4 - x_3$, $\Delta x' = x'_4 - x'_3$, and $\Delta t' = t'_4 - t'_3$. Then O calculates $(\Delta s)^2$ and gets $(\Delta s)^2 = -(\Delta x)^2$, because $\Delta t = 0$, as the two events are simultaneous for O. On the other hand, O' calculates $(\Delta s)^2$ and gets

$$(\Delta s)^2 = (c\Delta t')^2 - (\Delta x')^2 = (\beta \Delta x')^2 - (\Delta x')^2 = (\beta^2 - 1)(\Delta x')^2$$

where we used the fact that $c\Delta t' = \overline{E_4H}$, $\beta = \overline{E_4H}/\overline{E_3H}$ (by rule 2 of Box F.0.2), and $\Delta x' = \overline{E_3H}$. Since $(\Delta s)^2$ is an invariant quantity, we must have

$$-(\Delta x)^2 = (\beta^2 - 1)(\Delta x')^2 \quad \text{or} \quad (\Delta x')^2 = \frac{(\Delta x)^2}{1-\beta^2} = \gamma^2 (\Delta x)^2 \quad \text{or} \quad \Delta x' = \gamma \Delta x$$

as we had for the *ct*-axis.

It is also interesting to calculate the relation between *Euclidean* lengths as measured by O' (using a ruler) and lengths measured by O. Take $\overline{E_3E_4}$ for example (the argument for $\overline{E_1E_2}$ is identical). O' sees this length as the hypotenuse of a right triangle and uses Pythagoras' theorem to find its length:

$$(\overline{E_3E_4})^2_{O'} = \overline{E_3H}^2 + \overline{HE_4}^2 = (\gamma \overline{E_3E_4})^2 + [\beta(\gamma \overline{E_3E_4})]^2 = (1+\beta^2)\gamma^2 \overline{E_3E_4}^2$$

where use was made of the facts discussed in the previous paragraph. Taking the square root of the equation above gives

$$(\overline{E_3E_4})_{O'} = \gamma\sqrt{1+\beta^2}\ \overline{E_3E_4}$$

It needs to be emphasized that $(\overline{E_3E_4})_{O'}$ is the *Euclidean* length measured by O'—by placing a ruler on the two *points* E_3 and E_4. On the other hand, $\overline{E_3E_4}$ is the difference between the x-coordinates of E_3 and E_4 as measured by O. ■

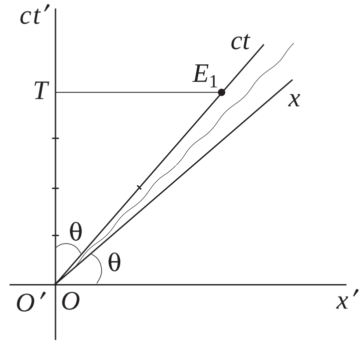


Figure F.3: The event E_1 occurs 2 hours after departure according to O, but 4 hours according to O'. The length of the line segment $\overline{O'E_1}$ is 5.29 light hours. (b) Deriving the Lorentz transformations from the diagram.

F.1 Examples of Spacetime Diagrams

The best way to understand the implications of Boxes F.0.2 and F.0.3 is to look at some examples. First let us elaborate on the last item in the second Box. Emmy (reference frame O) is moving at $0.866c$ in the positive direction of Karl (reference frame O'). At the very moment that Emmy passes Karl, they both start counting time; thus the origins of the two spacetime coordinate systems coincide (see Figure F.3). Event E_1 occurs two years later according to Emmy. So on her time axis, E_1 is two years away from the origin. This time interval is proper. The interval that Karl measures is given by

$$\Delta t' = \frac{\Delta t}{\sqrt{1-(v/c)^2}} = \frac{2}{\sqrt{1-(0.866)^2}} = 4 \text{ years}$$

This verifies the first part of rule 4 (which here translates to $\overline{O'T} = \gamma\overline{OE_1}$), because $\gamma = 1/\sqrt{1-(0.866)^2} = 2$ and $\overline{OE_1} = 2$ years.

The slope of the angle θ is $\beta = v/c = 0.866$. Therefore, the length of the line segment $\overline{TE_1}$ is 0.866 times the length of $\overline{O'T}$, which is $c\Delta t' = c \times 4$ years or 4 light years (each vertical tick represents one light year). Thus, $\overline{TE_1} = 4 \times 0.866 = 3.464$ light years. Now we can calculate the *Euclidean* length of $\overline{O'E_1}$ by Pythagoras' theorem:

$$\overline{O'E_1} = \sqrt{(\overline{TE_1})^2 + (\overline{O'T})^2} = \sqrt{(3.464)^2 + (4)^2} = 5.29$$

So the actual length of the line segment $\overline{O'E_1}$ that Karl measures (when he places a ruler on the page of the book) is 5.29 light years. The real length (as measured by Emmy) is 2 light years, of course. Therefore, the stretch factor is 5.29/2=2.645, which is identical to what Box F.0.3 says it should be: $\gamma\sqrt{1+\beta^2} = 2\sqrt{1+0.866^2} = 2.645$.

F.2 Simultaneity Revisited

The diagrammatic approach to relativity can elucidate some of the notions we discussed earlier. Take the relativity of simultaneity, which was one of the first topics we encountered. Chapter 25 showed a picture identical to Figure F.4(a), in which Karl (observer O') detects a simultaneous explosion of two firecrackers A and B. Emmy, on the other hand, describes the situation as B happening before A. Let's see if we can further unravel the succession of these events.

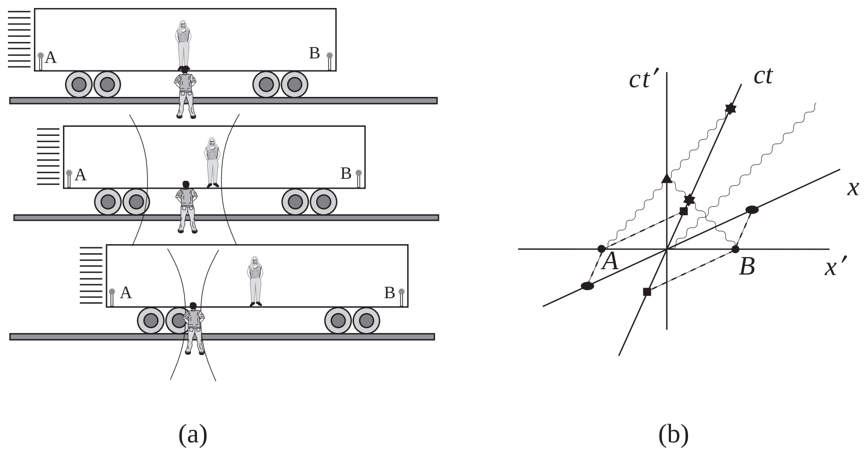


Figure F.4: (a) Karl sees the explosion of the two firecrackers at the same time that Emmy passes him by. (b) The spacetime geometry of the events as seen by Karl (O') and Emmy (O).

For simplicity, assume that the explosions occur—according to Karl—at exactly the same time that Emmy passes him, and all these happen at Karl's time zero. This zero of time is also Emmy's zero of time. (At the moment that they pass each other, Emmy and Karl start their stop watches). How do we describe these events in Karl's and Emmy's spacetime coordinate systems (CS)?

Start with Karl, whose CS is assumed perpendicular. Since A, B, and the passage of Emmy all occur at time $t' = 0$, they must all lie on the x'-axis (Emmy's worldline is the ct-axis, which crosses the x'-axis at the origin). The light signals (the wavy lines) from A and B reach Karl at a later time; this time is shown as a solid triangle on the ct'-axis. Note that Karl, being in the middle of the two events, receives the two signal at the same time, as he should.

How does Emmy perceive the occurrence of the events and the reception of their signals? Draw parallel lines to Emmy's axes from A and B to find the coordinates of the two events. The locations on her x-axis are designated as ovals, and she is in the middle of them as is clear in Figure F.4(a). The times are represented by squares on the ct-axis. Note that B occurs first, as indicated in the middle picture of Figure F.4(a). But Figure F.4(b) tells us that B occurred *before* Emmy reached Karl. The middle picture of Figure F.4(a) shows the *reception* of B's light signal, not its explosion. Emmy's reception of the light signals are denoted by stars on her ct-axis. We see that although B explodes *before* Emmy reaches Karl, its signal reaches Emmy *after* she passes Karl, consistent with the middle picture of Figure F.4(a). Finally, the signal from A reaches Emmy after the reception of the signal from B.

The discussion above, although qualitative, sheds some light on the notion of simultaneity as perceived by two different observers. If one knows Emmy's speed relative to Karl and Karl's measurement of the length $\overline{AB}$, one can use the figure to calculate the time difference between the explosions, the separation between the two firecrackers, and the time of the reception of the two signals all according to Emmy.

F.3 The Train and the Tunnel

In Figure F.5(a), Emmy's 756-meter-long train moves at $0.75c$ as it approaches a tunnel. Karl measures the contracted length of the train to be 500 m, and concludes that it should nicely fit the 500-meter tunnel he is standing by. Emmy, on the other hand, sees a contracted

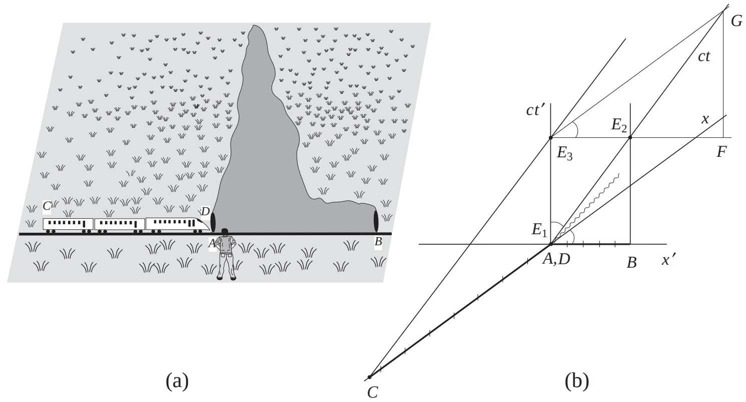


Figure F.5: (a) Emmy's train and the tunnel appear to have the same length according to Karl. (b) The spacetime diagram of the train-tunnel paradox.

tunnel[2] of only 331 m and concludes that there is no way she can fit a 750-meter long train in that tunnel. What is going on?

With $v = 0.75c$, $\gamma = 1/\sqrt{1-0.75^2} = 1.51$, and lengths shrink by this factor. To analyze the relative motion of the two RFs on a spacetime diagram, we label the ends of the tunnel and the train with A, B, C, and D, as shown in Figure F.5(a). There are three conspicuous events that we have to specify on the diagram: the coincidence of A and D corresponding to the front of the train entering the tunnel (call it E_1, assumed to be the origin of the two RFs), the coincidence of D and B corresponding to the front of the train exiting the tunnel (call it E_2), and the coincidence of A and C corresponding to the end of the train entering the tunnel (call it E_3). To construct these events on the diagram we proceed as follows.

Draw Karl's axes as two perpendicular lines on a sheet of paper. From Karl's origin draw a line making a slope of 0.75 with Karl's x'-axis. This is Emmy's x-axis. Emmy's ct-axis is the line passing through the origin and making the same slope with Karl's ct'-axis [see Figure F.5(b)]. Now draw the worldlines of A and B as two vertical lines, with A's worldline coinciding with the ct'-axis (we are assuming that Karl is standing at A). These worldlines are separated by 500 m, the length of the tunnel *according to Karl*. The worldlines of C and D are two parallel lines (because, being the two ends of the train, they move with the same velocity, i.e., same slope), with D's worldline coinciding with the ct-axis (we are assuming that Emmy is sitting in the front of the train at D). Here is how to draw C's worldline: From the intersection of the ct-axis and the B's worldline (event E_2) draw a line parallel to the x'-axis. This line meets the A's worldline (the ct'-axis) at E_3 (we know this because Karl sees E_2 and E_3 as simultaneous). We know that C's worldline must pass through E_3. We also know that C's worldline must be parallel to D's (the ct-axis). So we draw a line through E_3 parallel to the ct-axis. This is the C's worldline.

The C's worldline meets the x-axis at a point, which we naturally call C. The points C and D on the x-axis are separated by 750 m *as measured by Emmy*. The reason that the line segment $\overline{CD}$ appears much longer than 750 m (remember that $\overline{AB}$ is 500 m) is due to the stretch factor mentioned in rule 4 of Box F.0.3.

It is clear from Figure F.5(b) that E_3 has a time coordinate in Emmy's RF equal to $\overline{DG}$,[3] which is obviously larger than $\overline{DE_2}$, the time of the coincidence of B and D according to Emmy. Therefore, the coincidence of C and A occurs *after* the coincidence of D and B (event E_2). This means that, although Karl sees the train completely in the tunnel, with the end points of the two coinciding at the same time, Emmy notices that the back of the

[2] Because the tunnel is moving relative to Emmy; so, its length should shrink for her.

[3] Remember that to find the coordinates of an event, we draw lines from that event parallel to the axes.

train is outside the tunnel when the front of the train has reached the end of the tunnel. This is a diagrammatic representation of the relativity of simultaneity discussed in Chapter 25. The quantitative analysis of this discussion can be found in Example D.27.2.

F.4 Lorentz Transformations

It is worthwhile to introduce an *algebraic* recipe, which gives the coordinates of an event or the time interval and distance between two events as seen by one observer in terms of the same quantities of another observer. **Math Note E.27.10** on **page 124** of *Appendix.pdf* derives this recipe,[4] and obtains the following result:

Lorentz transformations

Box F.4.1. *If O moves with speed v relative to O' and the origin of the two RFs coincide at $t = 0 = t'$, then the coordinates of events in the two RFs are related by the following **Lorentz transformations**:*

$$\begin{aligned} x' &= \gamma(x + \beta ct) & \Delta x' &= \gamma(\Delta x + \beta c\Delta t) & \beta &\equiv v/c \\ ct' &= \gamma(\beta x + ct) & c\Delta t' &= \gamma(\beta\Delta x + c\Delta t) & \gamma &\equiv 1/\sqrt{1-\beta^2} \end{aligned}$$

$\beta > 0$ (< 0) if O moves along the positive (negative) direction of O'.

The first set of equations gives the relation between the coordinates of a single event; the second set gives the relation between the space and time intervals of two events. Δx and Δt could each be positive or negative. The Δ-equations can always be used whether the origins coincide at time zero or not; but the first set of equations does not apply when O and O' do not coincide. In such cases one has to use the Δ-equations for intervals and add appropriate intervals to get to the actual coordinate values.

Suppose E_1 and E_2 have respective coordinates (x_1, ct_1) and (x_2, ct_2) in O and (x'_1, ct'_1) and (x'_2, ct'_2) in O'. We can take Δx to be $x_2 - x_1$, in which case Δt is *necessarily* $t_2 - t_1$, $\Delta x'$ is *necessarily* $x'_2 - x'_1$, and $\Delta t'$ is *necessarily* $t'_2 - t'_1$. Or we can take Δx to be $x_1 - x_2$, in which case $\Delta t = t_1 - t_2$, $\Delta x' = x'_1 - x'_2$, and $\Delta t' = t'_1 - t'_2$. Sometimes to emphasize the order of subtraction, we use a double subscript for the Δ quantities. For example, $\Delta x_{21} = x_2 - x_1$ and $\Delta t_{12} = t_1 - t_2$. It should be clear that *if we use double subscripts, then all the Δ quantities in the Δ equations of Box F.4.1 ought to have exactly the same subscripts.*

[4] To appreciate the ease and power of geometric reasoning, I have also derived the Lorentz transformations in Math Note E.27.11 using *algebraic* reasoning.

Appendix G

Numerical Exercises

G.1 Numerical Exercises for Chapter 1

Exercise G.1.1. You are standing 500 meter away from a tall building. You measure the angle that your line of sight to the top make with the line of sight to the bottom. This angle turns out to be 15 degrees. What is the height of the building? Hint: Draw a 15° angle; label its vertex O; from a point P on one side of the angle, draw a perpendicular line to meet the other side at Q. The ratio of $\overline{PQ}$ to the building height is the same as the ratio of $\overline{OP}$ to 500 (see Figure G.1).

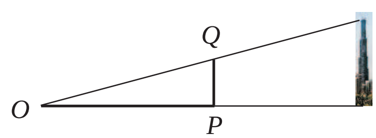


Figure G.1: The height of the building can be found from angles and distance from the building.

Exercise G.1.2. The north-south distance between Miami and New York is about 1200 miles. The circumference of the Earth is 25,000 miles.
(a) What is the difference between the angles that shadows make in Miami and in New York?
(b) Which angle is bigger?

Exercise G.1.3. On planet Neemaz, the city of Shaar is 300 miles directly north of the city of Havaz. The angle of the shadows in Shaar is 10 degrees larger than those in Havaz.
(a) What is the circumference of Neemaz?
(b) What is its radius?

Exercise G.1.4. Something like Figure G.2(a) was used by Aristarchus to find the Earth–Sun distance. Suppose that the period of counterclockwise revolution of the Moon around the Earth is 30 days, and that the time interval between the first and the third quarter Moon (arc AB) is 1.25 hours longer than the time interval between the third and the first quarter Moon (arc BA).
(a) How many *radians* do 30 days correspond to?
(b) From the figure determine how many α's 1.25 hours correspond to.
(c) Use proportions to find α in radians.

(d) From this value of α and the fact that the Earth–Moon distance is about 400,000 km, determine the Earth–Sun distance.
(e) Now use the fact that the angular size of the Sun [Figure G.2(b)] is 0.5 degree to find the diameter of the Sun.

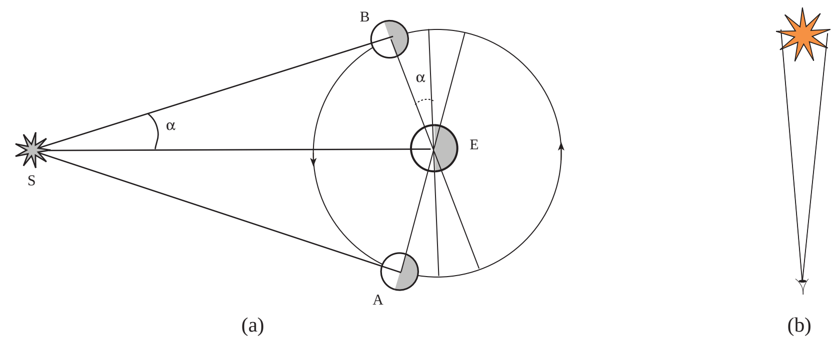


Figure G.2: (a) The difference between the arc length AB and the arc length BA is used to find the angle α. (b) The angular size of the Sun on Earth.

Exercise G.1.5. Figure 1.5 shows snapshots of the epicycle of planet M as it goes around the Earth E counterclockwise. Suppose that M moves around the small circle four times as the center of the small circle moves on the big circle once. M starts at the position shown in the figure; call it the first snapshot.
(a) On which small circle will you find M after its first revolution? Draw M on that circle.
(b) On which small circle will you find M after its second and third revolutions? Draw M on those circles.
(c) What fraction of its epicycle does M cover from one snapshot to the next?
(d) Starting with the second snapshot, draw the location of M on all the remaining small circles. Connect all locations of M *smoothly* to find its path around E.

Exercise G.1.6. A planet M moves 4 times on its epicycle while the center of the epicycle goes around Earth once on the deferent. M starts at the 6 o'clock position of its epicycle when the center of the epicycle is at 12 o'clock position of the deferent. All motions are counterclockwise.
(a) When the center of the epicycle reaches the 11 o'clock position of the deferent, at what position of its epicycle is M?
(b) When the center of the epicycle reaches the 7 o'clock position of the deferent, at what position of its epicycle is M?
(c) When the center of the epicycle reaches the 2 o'clock position of the deferent, at what position of its epicycle is M?
(d) When M is at 4 o'clock position on its epicycle during its first revolution, at what position on the deferent is the center of the epicycle?
(e) When M is at 4 o'clock position on its epicycle during its second revolution, at what position on the deferent is the center of the epicycle?
(f) When M is at 8 o'clock position on its epicycle during its third revolution, at what position on the deferent is the center of the epicycle?
(g) When M is at 4 o'clock position on its epicycle during its fourth revolution, at what position on the deferent is the center of the epicycle?

G.3 Numerical Exercises for Chapter 3

Copernicus Model

Exercise G.3.1. Figure G.3 shows the circular orbits of Mars (M) and Earth (E). Suppose that Earth's period is 1/3 that of Mars, and that they start in the location shown in the graph (label it 1).
(a) On M's orbit draw the locations of M when E completes its first, second, and third revolutions.
(b) Divide E's orbit into 6 equal parts and label the new locations 2 through 6.
(c) Locate the position of M on its orbit when E is in positions 2 through 6.
(d) Do the same for the second and third revolutions of E, labeling the positions of M and E on their orbits 7 through 18. E will have multiple labels for its locations.
(e) Now draw arrows from E to M on the graph.
(f) Redraw these arrows from a common center and connect the tips of the arrows to see how M's motion appears to E.

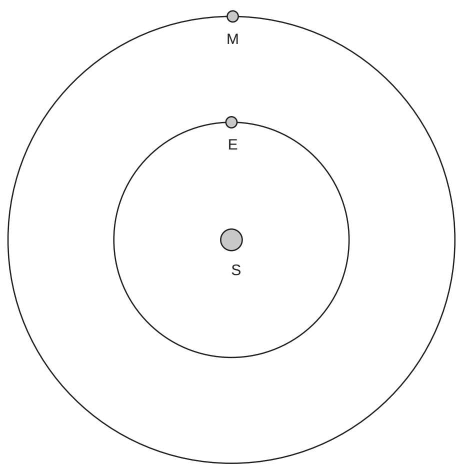


Figure G.3: The orbits of Mars (M) and Earth (E) around the Sun.

Exercise G.3.2. Figure G.3 shows the circular orbits of Mars (M) and Earth (E). Suppose that Earth's period is 1/4 that of Mars, and that they start in the location shown in the graph (label it 1).
(a) On M's orbit draw the locations of M when E completes its first, second, and third and fourth revolutions.
(b) Divide E's orbit into 6 equal parts and label the new locations 2 through 6.
(c) Locate the position of M on its orbit when E is in positions 2 through 6.
(d) Do the same for the second, third, and fourth revolutions of E, labeling the positions of M and E on their orbits 7 through 24. E will have multiple labels for its locations.
(e) Now draw arrows from E to M on the graph.
(f) Redraw these arrows from a common center and connect the tips of the arrows to see how M's motion appears to E.

Exercise G.3.3. Figure G.4 shows the circular orbits of Venus (V) and Earth (E). Suppose that Earth's period is 4 times that of Venus, and that they start in the location shown in the graph (label it 0).
(a) On E's orbit locate the positions of E when V completes its first, second, third, and

fourth revolutions.
(b) Divide V's orbit into 6 equal parts and label the new five locations 1, 2, 3, 4, and 5.
(c) Locate the position of E on its orbit when V is in positions 1 through 5.
(d) Do the same for the second through sixth revolutions of V, labeling the positions of V and E on their orbits 6 through 23. V will have multiple labels for its locations.
(e) Now draw arrows from E to V on the graph.
(f) Redraw these arrows from a common center and connect the tips of the arrows to see how V's motion appears to E.

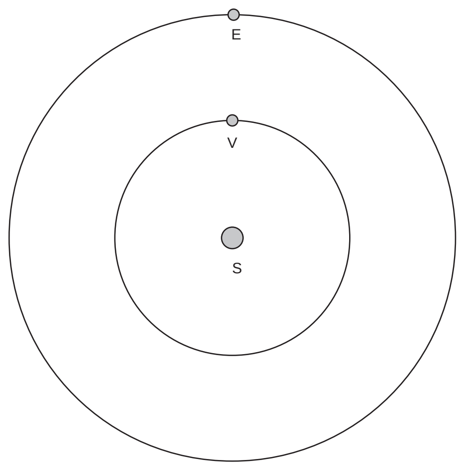


Figure G.4: The orbits of Venus (V) and Earth (E) around the Sun.

Kepler's Laws

Exercise G.3.4. We want to determine the constant of proportionality in Kepler's third law from the motion of the Earth whose orbit is almost circular and is 150 million km away from the Sun.
(a) What is T in seconds?
(b) What is a in meters?
(c) What is the constant of proportionality?

Exercise G.3.5. The semimajor axis of the planet Mars is 228 million km.
(a) Use the constant of the previous exercise to find the period of Mars in seconds?
(b) How many Earth days are there in a Martian year?
(c) Use ratios to find the answer in (b).

Exercise G.3.6. Mercury is seen to go around Sun in 87.8 days.
(a) What is the period of Mercury in seconds?
(b) How far is Mercury from the Sun (use the constant k for the solar system)?
(c) Use ratios to find the answer in (b).

Exercise G.3.7. Pluto is seen to go around Sun in 246.7 years.
(a) What is the period of Pluto in seconds?
(b) How far is Pluto from the Sun (use the constant k for the solar system)?
(c) Use ratios to find the answer in (b).
(d) The angular size of an object decreases in proportion to the distance. How much smaller is the angular size of the Sun as seen from Pluto than from Earth? Earth is 150 million km away from the Sun.

Exercise G.3.8. It takes Comet Hale-Bopp 2380 years to go around Sun. (a) What is the semi-major axis of the orbit of Comet Hale-Bopp?
(b) Suppose that it almost grazes Sun as it approaches it. What is the farthest it gets from the Sun?
(c) Does the comet ever go beyond Pluto's orbit?

Exercise G.3.9. It takes Comet Faye 7.5 years to go around Sun.
(a) What is the semi-major axis of the orbit of this comet?
(b) Suppose that its distance of closest approach to Sun is 2.5×10^{11} m. What is the farthest it gets from the Sun?

G.4 Numerical Exercises for Chapter 4

General Rectilinear Motion

Exercise G.4.1. Figure G.5 shows the plot of the velocity of an object versus time. The units on the time axis are seconds and those on the velocity axis are m/s.
(a) What is the initial velocity of the object?
(b) What is the velocity after 4 seconds?
(c) What is the velocity after 7, 12, 13, 19, and 20 seconds?
(d) What is the average acceleration during the time interval $t = 6$ s to $t = 8$ s (include sign)?
(e) What is the average acceleration during the time interval $t = 11$ s to $t = 14$ s (include sign)?
(e) What is the average acceleration during the time interval $t = 15$ s to $t = 18$ s (include sign)?
(f) What is the instantaneous acceleration at $t = 7$ s and $t = 16$ s (include sign)?
(g) What is the distance traveled in the first 4 seconds?
(h) What is the distance traveled in the first 8 seconds?
(i) What is the distance traveled between 5 and 12.5 seconds?
(j) What is the distance traveled between 12.5 and 20 seconds?
(k) What is the distance traveled between 18 seconds and the end of motion?
(l) What is the distance traveled during the entire motion?

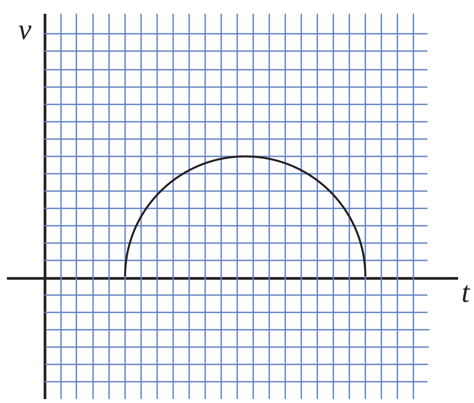


Figure G.5: The plot of velocity versus time. Each tick on the horizontal axis represents one second. Each tick on the vertical axis represents one m/s.

Exercise G.4.2. Figure G.6 shows the plot of the velocity of an object versus time. The units on the time axis are seconds and those on the velocity axis are m/s.

(a) What is the initial velocity of the object?
(b) What is the velocity after 4 seconds?
(c) What is the velocity after 7 seconds? After 8 seconds? After 14 seconds?
(d) What is the acceleration during the first 7 seconds (include sign)?
(e) What is the acceleration between 14 and 18 seconds (include sign)?
(f) What is the acceleration between 18 seconds and the end of motion (include sign)?
(g) What is the distance traveled in the first 4 seconds?
(h) What is the distance traveled in the first 7 seconds?
(i) What is the distance traveled between 7 and 14 seconds?
(j) What is the distance traveled between 14 and 19 seconds?
(k) What is the distance traveled between 7 seconds and the end of motion?
(l) What is the distance traveled during the entire motion?

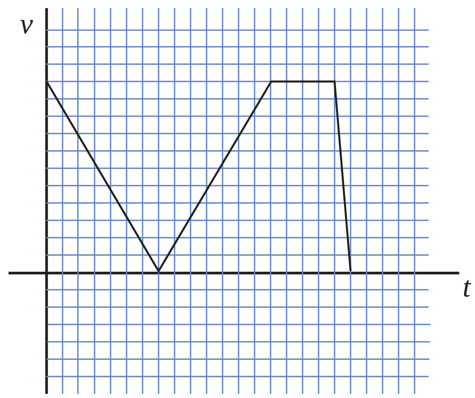


Figure G.6: The plot of velocity versus time. Each tick on the horizontal axis represents one second. Each tick on the vertical axis represents one m/s.

Exercise G.4.3. Figure G.7 shows the plot of the velocity of a car versus time. Suppose that motion takes place on a straight east–west highway, and that the car is moving eastward initially. The units on the time axis are seconds and those on the velocity axis are m/s.
(a) What is the initial velocity of the object?
(b) What is the velocity after 4 seconds? Which direction?
(c) What is the velocity after 9.5 seconds? After 12 seconds? After 19 seconds? Give directions for all velocities.
(d) What is the acceleration during the first 6 seconds (include sign)?
(e) What is the acceleration between 6 and 12 seconds (include sign)?
(f) What is the acceleration between 9.5 seconds and the end of motion (include sign)?
(g) What is the distance and the displacement traveled in the first 4 seconds? Give a direction for the diplacement.
(h) What is the distance and the displacement traveled in the first 9.5 seconds? Give a direction for the diplacement.
(i) What is the distance and the displacement traveled between 7 and 14 seconds? Give a direction for the diplacement.
(j) What is the distance and the displacement traveled between 14 and 19 seconds? Give a direction for the diplacement.
(k) What is the distance and the displacement traveled between 7 seconds and the end of motion? Give a direction for the diplacement.
(l) What is the distance and the displacement traveled during the entire motion? Give a direction for the diplacement.

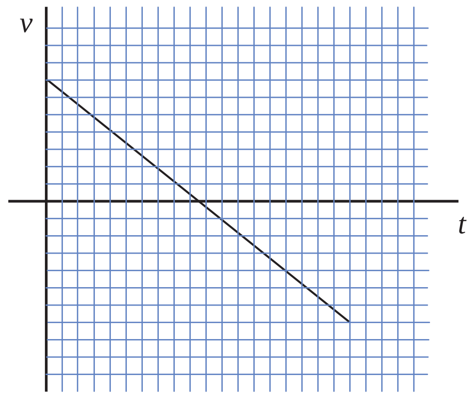


Figure G.7: The plot of velocity versus time. Each tick on the horizontal axis represents one second. Each tick on the vertical axis represents one m/s.

Uniformly Accelerated Motion

Exercise G.4.4. A car accelerates uniformly from rest to 65 mph in 5 seconds.
(a) What is the acceleration of the car in m/s^2?
(b) How far does the car travel in the process?

Exercise G.4.5. A ball is thrown vertically upward with a speed of 90 mph.
(a) How long does it take the ball to reach its maximum height?
(b) What is this maximum heigh?

Exercise G.4.6. A car is moving at 50 mph on a street. A cat jumps in front of it in the middle of the street 30 meters away. The driver, whose reflex time is 0.2 second, immediately slams on the brakes causing a deceleration of 8 m/s^2. Is the cat dead or alive? Hint: Follow the steps of Numerical Exercise 4.4 in the text.

Exercise G.4.7. A car is moving at 40 mph on a street. A squirrel jumps in front of it in the middle of the street 30 meters away. The driver, whose reflex time is 0.2 second, immediately slams on the brakes causing a deceleration of 6 m/s^2. Is the squirrel dead or alive? Hint: Follow the steps of Numerical Exercise 4.4 in the text.

Exercise G.4.8. A crazy driver is moving at 70 mph on a street. A rabbit jumps in front of the car in the middle of the street 50 meters away. The driver, whose reflex time is 0.1 second, immediately slams on the brakes causing a deceleration of 6 m/s^2. Is the rabbit dead or alive? Hint: Follow the steps of Numerical Exercise 4.4 in the text.

Exercise G.4.9. St. Speedsburgh is a little country in central Atlantis. There is no speed limit in St. Speedsburgh, but there is an "acceleration limit." Anyone whose magnitude of acceleration is more than 5 m/s^2 gets a ticket. Sharon is moving at 100 mph and speeds up to 150 mph in 5 seconds. Sherlock is moving at 60 mph when he spots a goose crossing the highway. He slams on the brakes and slows down to 20 mph in 3 seconds. Which car, if any, will be ticketed by a lurking police at the side of the highway?

G.6 Numerical Exercises for Chapter 6

Exercise G.6.1. Consider Figure G.8 in which a car moves on a curved path.
(a) Draw the position vectors of the driver relative to the observer O when the car is in the numbered locations.

(b) Draw the displacement vectors $\mathbf{r}_{13}$, $\mathbf{r}_{25}$, $\mathbf{r}_{37}$, $\mathbf{r}_{56}$, and $\mathbf{r}_{18}$.
(c) In a separate diagram, draw the position vectors **of the observer** O **relative to the driver** when the car is in the numbered locations.
(d) Sketch the path of O relative to the driver.

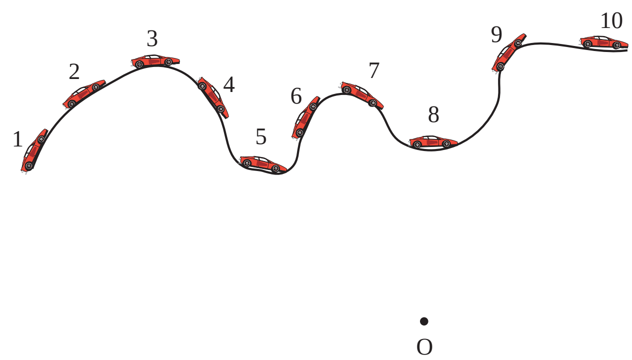


Figure G.8: The car moving on a path with certain locations specified.

Exercise G.6.2. Consider Figure G.9 in which observers A and B move relative to a third observer (not shown). Suppose that they both start at the beginning of their corresponding paths (which are equal in length) and reach the end 10 seconds later each moving at constant speed.
(a) Draw points on A's path showing his position 2, 4, 6, and 8 seconds after he starts his motion.
(b) Draw points on B's path showing her position 2, 4, 6, and 8 seconds after she starts her motion.
(c) Draw arrows from A to B at 0, 2, 4, 6, 8, and 10 second into the motion.
(d) With A as the observer, draw the arrows of the prvious part and determine how B moves realtive to A.
(e) Draw arrows from B to A at 0, 2, 4, 6, 8, and 10 second into the motion.
(f) With B as the observer, draw the arrows of the prvious part and determine how A moves realtive to B.

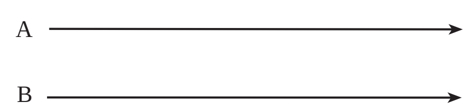


Figure G.9: Observers A and B move relative to a third observer (not shown).

Exercise G.6.3. Consider Figure G.10 in which observers A and B move relative to a third observer (not shown). Suppose that they both start at the beginning of their corresponding paths (which are equal in length) and reach the end 10 seconds later each moving at constant speed.
(a) Draw points on A's path showing his position 2, 4, 6, and 8 seconds after he starts his motion.
(b) Draw points on B's path showing her position 2, 4, 6, and 8 seconds after she starts her motion. (c) Draw arrows from A to B at 0, 2, 4, 6, 8, and 10 second into the motion.
(d) With A as the observer, draw the arrows of the prvious part and determine how B

moves realtive to A.
(e) Draw arrows from B to A at 0, 2, 4, 6, 8, and 10 second into the motion.
(f) With B as the observer, draw the arrows of the prvious part and determine how A moves realtive to B.

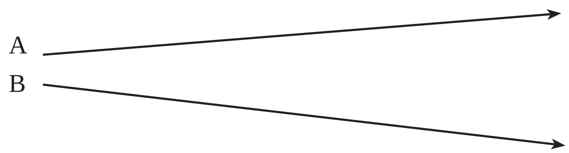


Figure G.10: Observers A and B move relative to a third observer (not shown).

Exercise G.6.4. Consider Figure G.11 showing the motion of Earth around the Sun. Assuming that the location of the Earth in the figure corresponds to January first,
(a) draw points on the Earth's orbit corresponding to the first of each of the remaining 11 months.
(b) Draw arrows from Earth to Sun at each location of the Earth.
(c) With Earth as the observer, draw the arrows of the prvious part and determine how Sun moves realtive to Earth.

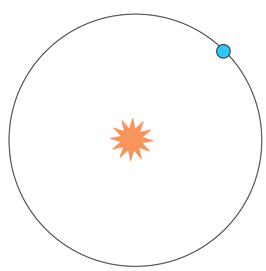


Figure G.11: Motion of Earth as seen from the Sun.

Exercise G.6.5. An ant is moving on a table top. It moves up for 4 seconds at the rate of 5 cm/s. It suddenly changes its direction and moves to the right 10 cm for 5 seconds. It again changes its direction and moves down 20 cm for 6 seconds.
(a) What is the total distance the ant covers?
(b) What is the displacement of the ant?
(c) What is its average speed in cm/s?
(d) What is its average velocity in cm/s?
(e) What is its average speed in cm/s as it is moving to the right?
(f) What is its average velocity in cm/s as it is moving to the right?
(g) What is its average speed in cm/s as it is moving down?
(h) What is its average velocity in cm/s as it is moving down?

G.7 Numerical Exercises for Chapter 7

Exercise G.7.1. A 100-gram bullet traveling with the speed of 100 m/s hits a 2-kg block of wood resting on a smooth floor and sticks to it (see Figure G.12).
(a) What is the appropriate system and what does it consist of?

(b) What is the initial momentum of the bullet? Draw an arrow!
(c) What is the initial momentum of the block?
(d) What is the momentum of the system before the bullet hits the wood? Draw an arrow!
(e) What is the momentum of the system after the bullet hits the wood? Draw an arrow!
(f) What is the speed of the system of block and the bullet?

Figure G.12: The bullet is about to hit the stationary block.

Exercise G.7.2. A 100-gram bullet traveling with the speed of 80 m/s hits a 2-kg block of wood which is already in motion in the opposite direction on a smooth floor (see Figure G.13) with a speed of 2 m/s. The bullet sticks to the block.
(a) What is the initial momentum of the bullet? Draw an arrow!
(b) What is the initial momentum of the block? Draw an arrow!
(c) What is the momentum of the system before the bullet hits the wood? Draw an arrow!
(d) What is the momentum of the system after the bullet hits the wood? Draw an arrow!
(e) What is the speed of the system of block and the bullet?

Figure G.13: The bullet is about to hit the block that is moving in the opposite direction.

Exercise G.7.3. A 200-gram bullet moving to the left hits a stationary 8-kg block of wood. The bullet sticks to the block and the two move with a speed of 2 m/s.
(a) What is the momentum of the system after collision? Draw an arrow!
(b) What is the momentum of the system before the bullet hits the wood?
(c) What was the initial momentum of the bullet? Draw an arrow!
(d) What was the speed of the bullet before it hit the block?

Exercise G.7.4. A 200-gram tennis ball moving with the unknown speed of v_{in} to the left, hits a 5-kg stationary bowling ball and bounces back from it with the same speed. After collision, the bowling ball is seen to move with a speed of 2m/s.
(a) What is the initial momentum of the tennis ball in terms of v_{in}? Draw an arrow!
(b) What is the initial momentum of the bowling ball?
(c) What is the momentum of the system before collision in terms of v_{in}?
(d) What is the momentum of the system after collision in terms of v_{in}?
(e) Write an equation involving v_{in} that describes the constancy of the total momentum before and after collision. Solve this equation to find the initial speed of the tennis ball.

Exercise G.7.5. A 300-gram tennis ball moving with the unknown speed of v_{in} to the left, hits a 6-kg bowling ball moving to the right with a speed of 1 m/s and bounces back from it with the same speed of v_{in}. After collision, the bowling ball is seen to move with a speed of 2 m/s to the left. Take left to be the positive direction.

(a) What is the initial momentum of the tennis ball in terms of v_{in}? Draw an arrow!
(b) What is the initial momentum of the bowling ball? Draw an arrow!
(c) What is the momentum of the system before collision in terms of v_{in}?
(d) What is the momentum of the system after collision in terms of v_{in}?
(e) Write an equation involving v_{in} that describes the constancy of the total momentum before and after collision. Solve this equation to find the initial speed of the tennis ball.

Exercise G.7.6. A 100-gram bullet traveling with the speed of 100 m/s hits a 10000-kg block resting on the frictionless floor of an indoor hockey stadium 50 m long at one end of the stadium. How long does it take the block to reach the other end after the bullet penetrates it?

Exercise G.7.7. An astronaut with a total mass (astronaut plus wrench) of 75 kg is detached from her spaceship and moves with a speed of 0.2 m/s away from it at a distance of 20 m. The commander tells the astronaut to throw the 0.5-kg wrench she is holding as hard as she can. The astronaut follows the order, throwing the wrench at a speed of 20 m/s. Drawing arrows for momenta will help!
(a) Which direction does the astronaut throw the wrench?
(b) What is the momentum of the system before she throws the wrench? Draw an arrow!
(c) What is the momentum of the system after she throws the wrench? Draw an arrow!
(d) What is the momentum of the wrench? Draw an arrow!
(e) What is the momentum of the astronaut? Draw an arrow!
(f) What is the speed of the astronaut?
(g) Will the astronaut get back to the spaceship? If so, how long does it take her to reach the spaceship?

Exercise G.7.8. An astronaut with a total mass (astronaut plus wrench) of 95 kg is detached from his spaceship and moves with a speed of 0.4 m/s *twoards* it at a distance of 30 m. He needs to get to the ship in less than a minute. The commander tells the astronaut to throw the 1-kg wrench he is holding as hard as she can. The astronaut follows the order, throwing the wrench at a speed of 10 m/s. Drawing arrows for momenta will help!
(a) Which direction does the astronaut throw the wrench?
(b) What is the momentum of the system before he throws the wrench?
(c) What is the momentum of the system after she throws the wrench?
(d) What is the momentum of the wrench? Draw an arrow!
(e) What is the momentum of the astronaut? Draw an arrow!
(f) What is the speed of the astronaut? Will the astronaut get back to the spaceship in time?

Exercise G.7.9. A 200-gram mass is tied to one end of a string 1.5 m long. Emmy holds the other end and whirls the string above her head at the rate of 2 revolutions per second. Assume that the string is almost horizontal.
(a) What is the speed of the mass?
(b) What is the acceleration of the mass?
(c) What is the force acting on the mass?
(d) What is the source of this force?

Exercise G.7.10. An 18-ton truck is moving around a curve at a speed of 60 mph in opposite direction to that of the car shown in Figure 7.7. The radius of the curve is 150 m.
(a) What is the acceleration of the truck in m/s^2?
(b) What is the net force acting on the truck? Draw an arrow on the truck indicating the direction of the net force.
(c) How is the direction of this arrow related to that of the car of Figure 7.7?
(d) What applies this force on the truck?

Exercise G.7.11. The ferris wheel of Figure G.14 has a radius of 10 m and makes a complete revolution every 6.5 seconds.
(a) What is the distance covered by a person on the wheel in one revolution?
(b) What is the speed of that person?
(c) Find the magnitude of the acceleration of people sitting in positions A, C, F, and G.
(d) A person has a mass of 70 kg. What is the net force acting on her when she is at C? At G? Draw an arrow on Figure G.14 (a) for each of these net forces.
(e) What is the weight of that person? Draw her weight at C and at G of Figure G.14 (a).
(f) Is there any other force acting on the person? If so, is it a contact or an action-at-a-distance force? Draw this force at C and at G of Figure G.14 (a), and compare it with the weight of the person.

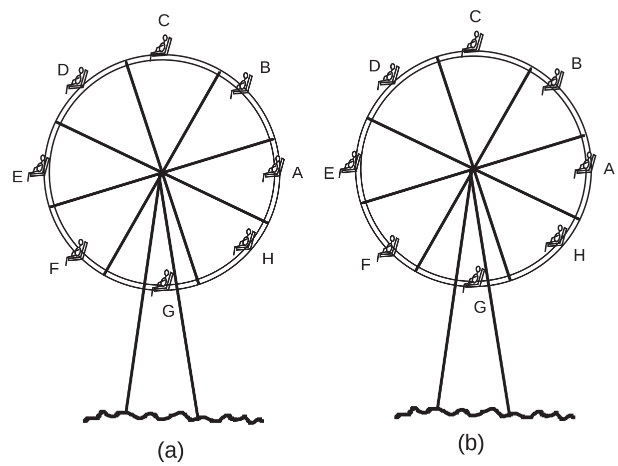


Figure G.14: The Ferris wheel and the people riding on it.

Exercise G.7.12. Suppose now that the ferris wheel slows down, making a complete revolution every 30 seconds.
(a) What is the distance covered by a person on the wheel in one revolution?
(b) What is the speed of that person?
(c) Find the magnitude of the acceleration of people sitting in positions A, C, F, and G.
(d) A person has a mass of 70 kg. What is the net force acting on him when he is at C? At G? Draw an arrow on Figure G.14 (b) for each of these net forces.
(e) What is the weight of that person? Draw his weight at C and at G of Figure G.14 (b). Draw the normal force at C and at G of Figure G.14 (b), and compare it with the weight of the person.

G.8 Numerical Exercises for Chapter 8

Exercise G.8.1. Calculate the kinetic energy (in Joules) of the following objects:
(a) A 15-ton truck moving with a speed of 70 mph.
(b) An 80-kg biker moving with a speed of 25 mph.
(c) A 0.1-kg bullet moving with a speed of 200 mph.
(d) A 0.5-kg hammer head moving with a speed of 60 mph.

Exercise G.8.2. A 30-kg block is given an initial speed of 15 m/s on a rough floor. The block stops after 20 meters.

(a) What is the initial kinetic energy of the block?
(b) What is the final kinetic energy of the block?
(c) What is the change in the kinetic energy of the block?
(d) How much work is done by the force of friction?
(e) How large is the force of friction?

Exercise G.8.3. A baseball pitcher throws a 0.2 kg ball at 100 mph straight up. We want to find how high the ball will rise.
(a) What is the speed in m/s?
(b) What is the KE at the beginning?
(c) What is the PE at the beginning? Consider the pitcher's hand level as the reference level.
(d) What is the total ME at the beginning?
(e) What is the ME, KE, and PE at the highest point?
(f) What is the maximum height?

Exercise G.8.4. A roller coaster starts at point B with a speed of 32 m/s and moves towards C. The mass of the car plus the passenger is 250 kg (see Figure 8.12). Assume that $h_1 = 70$ m, $h_2 = 50$ m, and $h_3 = 25$ m.
(a) What is the total mechanical energy at B?
(b) What is the total mechanical energy at C? What is PE at C? What is KE at C? Speed at C?
(c) What is the total mechanical energy at D? What is PE at D? What is KE at D? Speed at D?

Exercise G.8.5. A roller coaster starts at point C with a speed of 30 m/s and moves towards A. The mass of the car plus the passenger is 250 kg (see Figure 8.12). Assume that $h_1 = 60$ m, $h_2 = 40$ m, and $h_3 = 20$ m.
(a) What is the total mechanical energy at C?
(b) What is the total mechanical energy at B? What is KE at B? Speed at B?
(c) Will the roller coaster be able to make it to the top of the hill at A? If so, what is the total mechanical energy at A? What is KE at A? Speed at A?

Exercise G.8.6. A roller coaster starts at point D with a speed of 30 m/s and moves towards A. The mass of the car plus the passenger is 200 kg (see Figure 8.12). Assume that $h_1 = 80$ m, $h_2 = 60$ m, and $h_3 = 30$ m.
(a) What is the total mechanical energy at D?
(b) What is the total mechanical energy at C? What is KE at C? Speed at C?
(c) What is the total mechanical energy at B? What is KE at B? Speed at B?
(d) Will the roller coaster be able to make it to the top of the hill at A? If so, what is the total mechanical energy at A? What is KE at A? Speed at A?

Exercise G.8.7. Two trucks 70,000 lb (32 metric tons) each, moving at 70 mph collide head-on, with the wreckage being at rest after collision. Each kilogram of TNT has 4 million Joules of destructive energy.
(a) What is the KE of each truck?
(b) What is the total KE?
(c) How many kg of TNT is the destructive energy of collision?

Exercise G.8.8. Santa Claus has to visit 200 million children, and on the average, there are 8 children per home.
(a) How many chimneys does Santa have to climb down and up in 24 hours?
(b) Assume that Santa can climb down and up 20,000 chimneys every second. How long does Santa spend climbing down and up all the chimneys?
(c) How much time is left for Santa to hop from chimney to chimney?

(d) The houses Santa visits are 20 m apart. What is the total distance Santa has to cover?
(e) How fast should Santa travel to be able to deliver all the toys?
(f) Take the effective cargo mass for Santa's entire trip to be half the mass Santa starts with. If each toy is approximately 2 kg, what is the effective mass?
(g) How much energy does Santa spend to reach the hopping speed?
(h) This energy turns into heat in a fraction of a second when Santa lands at the next chimney. How many tons of TNT is the "bomb" associated with Santa's landing at each chimney?
(i) The "Little Boy," dropped on Hiroshima in 1945, was 15000 tons of TNT. How many "Little Boy"s is associated with Santa's landing at each chimney?
(j) What is the total energy Santa uses in 24 hours?
(k) The total yearly energy consumption of the world is approximately 4×10^{20} J. For how many years should the entire population of Earth stop using *any* form of energy before Santa can make his one-day trip?

Exercise G.8.9. Santa Claus has to visit 180 million children, and on the average, there are 6 children per home.
(a) How many houses does Santa have to visit?
(b) Assuming he has 30 hours to visit all the houses, how much time does he have for each house?
(c) Assume that Santa can climb down and up 30,000 chimneys every second. How long does Santa spend climbing down and up each chimney?
(d) How much time is left for Santa to spend between two chimneys?
(e) The houses Santa visits are 25 m apart. How fast should Santa travel to be able to deliver all the toys?
(f) Take the effective cargo mass for Santa's entire trip to be half the mass Santa starts with. If each toy is approximately 1.5 kg, what is the effective mass?
(g) How much energy does Santa spend to reach the hopping speed?
(h) This energy turns into heat in a fraction of a second when Santa lands at the next chimney. How many tons of TNT is the "bomb" associated with Santa's landing at each chimney?
(i) The "Fat Man," dropped on Nagasaki in 1945, was 22000 tons of TNT. How many "Fat Man"s is associated with Santa's landing at each chimney?
(j) What is the total energy Santa uses in 30 hours?
(k) The total yearly energy consumption of the world is approximately 4×10^{20} J. For how many years should the entire population of Earth stop using *any* form of energy before Santa can make his one-day trip?

G.9 Numerical Exercises for Chapter 9

Exercise G.9.1. Atlas, one of Saturn's moons, is seen to move around Saturn once every 14.5 hours (the period of the moon) on a circular orbit. The distance of Atlas from Saturn is 137,000 km.
(a) What is the distance covered by Atlas in one period?
(b) What is Atlas' period in seconds?
(c) What is the speed (in m/s) of Atlas as it moves around Saturn?
(d) What is Atlas' (centripetal) acceleration?
(e) From the knowledge of the centripetal acceleration, find Saturn's mass.
(f) Using Kepler's third law, find Saturn's mass.

Exercise G.9.2. Phobos, one of the two Mars' moons, is seen to move around Mars in 7.7 hours (the period of the moon) on a circular orbit. The distance of Phobos from Mars is 9,400 km.

(a) What is the distance covered by Phobos in one period?
(b) What is Phobos' period in seconds?
(c) What is the speed (in m/s) of Phobos as it moves around Mars?
(d) What is Phobos' (centripetal) acceleration?
(e) From the knowledge of the centripetal acceleration, find Mars' mass.
(f) Using Kepler's third law, find Mars' mass.
(g) Deimos, the other Moon of Mars, is seen to go around its mother planet every 30 hours. Use Kepler's third law to find Deimos' distancce from Mars.

Exercise G.9.3. The radius of the Earth $R_{\oplus}$ is 6,400 km. An apple is circling the Earth at an altitude of 20 km (see Figure G.15). (a) How far is the apple from the center of the Earth? (b) What is the gravitational acceleration of the apple assuming that the mass of the Earth is 6×10^{24} kg? How is this related to the centripetal acceleration of the apple? (c) Calculate the speed of the apple.

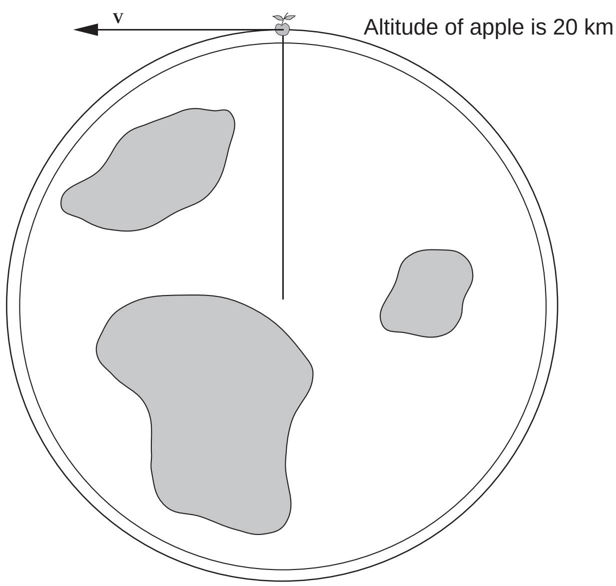


Figure G.15: The Earth with an apple circling it!

Exercise G.9.4. A 0.1 kg apple is launched with a speed of 200 m/s from the surface of a planet in some exotic star system. The planet has a mass of 5×10^{18} kg and a radius of 400 km.
(a) What is the initial KE of the apple? Its initial PE? Its total energy?
(b) What is the total energy, the PE, and the KE of the apple 200 km above the surface of the planet?
(c) What is the total energy, the PE, and the KE of the apple 400 km above the surface of the planet?
(d) Will the apple ever return?

Exercise G.9.5. A 0.1 kg apple is launched with a speed of 800 m/s from the surface of a planet in some exotic star system. The planet has a mass of 3×10^{20} kg and a radius of 40 km.
(a) What is the initial KE of the apple? Its PE? Its total energy?
(b) What is the total energy, the PE, and the KE of the apple 50 km above the surface of the planet?

(c) What is the total energy, the PE, and the KE of the apple 100 km above the surface of the planet? What is wrong?

Exercise G.9.6. Asteroid Vesta has a radius of 275 km and a mass of 2.4×10^{20} kg. What is Vesta's escape velocity.

Exercise G.9.7. Mercury has a radius of 2440 km and a mass of 3.3×10^{23} kg. What is Mercury's escape velocity.

Exercise G.9.8. Jupiter has a radius of 71800 km and a mass of 1.9×10^{27} kg. What is Jupiter's escape velocity.

Exercise G.9.9. Neptune has a radius of 24300 km and a mass of 1.03×10^{26} kg. What is Neptune's escape velocity.

Exercise G.9.10. A small planet in a remote star system has a radius of 75 km and a mass of 5×10^{17} kg. A baseball is thrown vertically upward on this planet with a speed of 75 mph.
(a) What is the escape velocity of the planet?
(b) Will the baseball come back? If so what is the maximum *height* reached by the baseball?

Exercise G.9.11. A small planet in a remote star system has a radius of 120 km and a mass of 8×10^{19} kg. A bullet is fired vertically upward on this planet with a speed of 500 mph.
(a) What is the escape velocity of the planet?
(b) Will the bullet come back? If so what is the maximum *height* reached by the bullet?

G.11 Numerical Exercises for Chapter 11

Exercise G.11.1. You see a lightning and 10 seconds later you hear its thunder. Assume that the light from lightning reaches you instantly.
(a) How long did it take the sound to reach you?
(b) How far away is the cloud?

Exercise G.11.2. The range of AM radio station frequencies is 530 to 1700 kHz. Radio waves travel at the speed of light $c = 300,000$ km/s. What is the range of AM wavelengths?

Exercise G.11.3. The range of FM radio station frequencies is 88 to 108 MHz. Radio waves travel at the speed of light $c = 300,000$ km/s. What is the range of FM wavelengths?

Exercise G.11.4. Figure G.16 shows two coherent sources producing waves in phase, i.e., when S_1 produces a crest or a trough, so does S_2. The distance between the sources, exaggerated for clarity, is comparable with the wavelength. Point A is equidistant from the two sources, $\overline{BS_1}$ is half a wavelength longer than $\overline{BS_2}$, and $\overline{CS_2}$ is one wavelength longer than $\overline{CS_1}$.
(a) Is there a constructive or destructive interference at A? Explain!
(b) Is there a constructive or destructive interference at B? Explain!
(c) Is there a constructive or destructive interference at C? Explain!

Exercise G.11.5. Figure G.16 shows two coherent sources producing waves in phase, i.e., when S_1 produces a crest or a trough, so does S_2. The distance between the sources, exaggerated for clarity, is comparable with the wavelength. Point A is equidistant from the two sources, $\overline{BS_1}$ is one wavelength longer than $\overline{BS_2}$, and $\overline{CS_2}$ is two wavelengths longer than $\overline{CS_1}$.
(a) Is there a constructive or destructive interference at A? Explain!
(b) Is there a constructive or destructive interference at B? Explain!
(c) Is there a constructive or destructive interference at C? Explain!

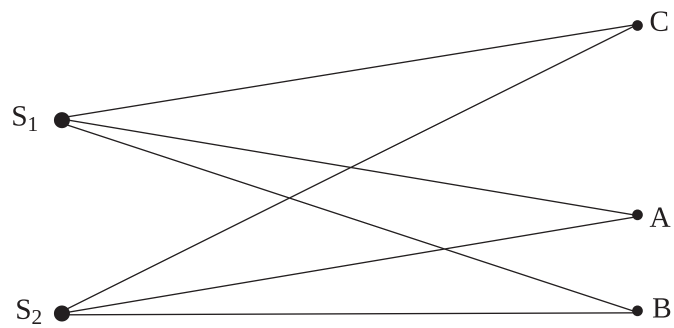


Figure G.16: The two coherent sources producing waves. The distance between the sources is comparable with the wavelength.

Exercise G.11.6. Figure G.16 shows two coherent sources producing waves in phase, i.e., when S_1 produces a crest or a trough, so does S_2. The distance between the sources, exaggerated for clarity, is comparable with the wavelength. Point A is equidistant from the two sources, $\overline{BS_1}$ is 1.5λ longer than $\overline{BS_2}$, and $\overline{CS_2}$ is 3λ longer than $\overline{CS_1}$.
(a) Is there a constructive or destructive interference at A? Explain!
(b) Is there a constructive or destructive interference at B? Explain!
(c) Is there a constructive or destructive interference at C? Explain!

Exercise G.11.7. Figure G.16 shows two coherent sources producing waves in phase, i.e., when S_1 produces a crest or a trough, so does S_2. The distance between the sources, exaggerated for clarity, is comparable with the wavelength. Point A is equidistant from the two sources, $\overline{BS_1}$ is 2λ longer than $\overline{BS_2}$, and $\overline{CS_2}$ is 4λ longer than $\overline{CS_1}$.
(a) Is there a constructive or destructive interference at A? Explain!
(b) Is there a constructive or destructive interference at B? Explain!
(c) Is there a constructive or destructive interference at C? Explain!

Exercise G.11.8. Figure G.16 shows two coherent sources producing waves in phase, i.e., when S_1 produces a crest or a trough, so does S_2. The distance between the sources, exaggerated for clarity, is comparable with the wavelength. Point A is equidistant from the two sources, $\overline{BS_1}$ is 2.5λ longer than $\overline{BS_2}$, and $\overline{CS_2}$ is 5λ longer than $\overline{CS_1}$.
(a) Is there a constructive or destructive interference at A? Explain!
(b) Is there a constructive or destructive interference at B? Explain!
(c) Is there a constructive or destructive interference at C? Explain!

Exercise G.11.9. Two coherent loud speakers producing a sound wave of frequency 9000 Hz are 5 m apart.
(a) Will these sources produce an inteference pattern?
(b) The distance is now reduced to 0.5 m. Is interference possible now?

Exercise G.11.10. Two coherent sources of electromagnetic waves producing waves of frequency 5×10^{10} Hz are 1 m apart. Electromagnetic waves travel at the speed of light $c = 3 \times 10^8$ m/s.
(a) Will these sources produce an inteference pattern?
(b) The distance is now reduced to 50 cm. Is interference possible now?
(c) The distance is further reduced to 1 cm. Is interference possible now?

Exercise G.11.11. A loud speaker produces a sound wave of frequency 10,000 Hz. This wave approaches a circular aperture one meter in diameter.
(a) Will the wave produce a diffraction pattern?

(b) The diameter is now reduced to 8 cm. Is diffraction possible now?
(c) The diameter is further reduced to 5 cm. Is diffraction possible now?

Exercise G.11.12. A source of electromagnetic waves produces microwaves of frequency 5×10^7 Hz. This wave approaches a circular aperture 10 cm in diameter. Electromagnetic waves travel at the speed of light $c = 3 \times 10^8$ m/s.
(a) Will the wave produce a diffraction pattern?
(b) The diameter is now reduced to 8 cm. Is diffraction possible now?

Exercise G.11.13. The yellow light with a wavelength of 6×10^{-7} m coming from a galaxy is seen to be shifted to 5.5×10^{-7} m.
(a) Is the galaxy approaching or receding from us?
(b) What is the fractional change in the wavelength $(\Delta\lambda/\lambda)$?
(c) What is the speed of the galaxy?

Exercise G.11.14. The speedometer of a police car shows a speed of 110 mph as the policeman chases a speeder. He sends a radar wave with a wavelength of 5 m to the speeder and receives a signal whose wavelength has decreased by 1.5×10^{-7} m.
(a) Is the police car approaching or receding from the speeder?
(b) What is the fractional change in the wavelength $(\Delta\lambda/\lambda)$?
(c) What is the speed of the police car *relative to the speeder*?
(d) How fast is the speeder going?

G.12 Numerical Exercises for Chapter 12

Exercise G.12.1. The mass of an electron is 9.1×10^{-31} kg and its charge is (negative) 1.6×10^{-19} Coulomb.
(a) Find the gravitational attraction (call it F_g) between two electrons separated by 1 m.
(b) Find the electrical repulsion (call it F_e) between two electrons separated by 1 m.
(c) What is the ratio $\dfrac{F_e}{F_g}$? How does this ratio change if the electrons above were separated by 2 m?

Exercise G.12.2. Two trucks each having a mass of 10 metric tons carry one millicoulomb of negative charge each. They are parked 3 m apart.
(a) What is the gravitational attraction between the two trucks?
(b) What is the electrical repulsion between the two trucks?
(c) What is the ratio of the two forces?
(d) What is the ratio of the two forces if the trucks are 30 m apart?

G.14 Numerical Exercises for Chapter 14

Exercise G.14.1. An electric charge is oscillating with a frequency of 5 Hz.
(a) What is the wavelength of the EM wave produced? How does it compare with the circumference of the Earth (40,000 km)? Does it make sense to call such an EM wave a "wave"?
(b) The frequency is now increased to 5 kHz. What is the wavelength now? Does it make sense to call such an EM wave a "wave"?
(c) The frequency is further increased to 5 MHz. What is the wavelength now? Does it make sense to call such an EM wave a "wave"?

Exercise G.14.2. One of the fastest mechanical rotators—with an electric charge on it—spins at a rate of 60,000 rpm (revolutions per minute).
(a) What is the frequency of the rotation?

(b) What is the wavelength of the EM wave so produced?
(c) Does it make sense to call such an EM wave a "wave"?

Exercise G.14.3. An electron (a negatively charged subatomic particle) moves around the nucleus of an atom on a circular orbit with a period of 10^{-15} second.
(a) What is the frequency of the electron orbital motion?
(b) What is the wavelength of the EM wave produced?
(c) Does it make sense to call such an EM wave a "wave"?

G.16 Numerical Exercises for Chapter 16

Exercise G.16.1. For the toss of 16 coins, use $P_n(m) = \dfrac{n!}{m!(n-m)!2^n}$ to
(a) find the frequency for 0 through 16 heads.
(b) Find the probability for 0 through 16 heads.
(c) How much is the probability of getting 8 heads bigger than the probability of getting 2 heads?

Exercise G.16.2. The figure on the next page contains eight plots of the probability of the occurrence of varying number of heads. Each plot corresponds to a certain total number of coins. Find the following information from each plot and write it on that plot.
(a) The total number of coins.
(b) The right zero m_+ and the left zero m_-.
(c) The width of the curve Δ_n.
(d) The relative width of the curve δ_n.
(e) Which curve is the narrowest? Which one the widest? (This question is not to test your visual ability! Think!)
(f) Can you see a trend from the last question?

G.17 Numerical Exercises for Chapter 17

Exercise G.17.1. A mixture of nitrogen and hydrogen molecules is held at a temperature of 50 °C. Nitrogen is 14 times heavier than hydrogen.
(a) How many times larger is the average KE of the hydrogen molecules than that of the nitrogen molecules?
(b) How many times larger is the average speed (root mean square of velocity) of the hydrogen molecules than that of the nitrogen molecules?
(c) The temperature is now doubled to 100 °C. How do the answers to (a) and (b) change?
(d) Hydrogen molecule has a mass of 3.32×10^{-27} kg. What is the average speed of the hydrogen molecules when the temperature is 50 °C? When it is 100 °C?
(e) What are the average speeds of the nitrogen molecules at these two temperatures?

Exercise G.17.2. Molecular nitrogen gas is in a vessel at a temperature of 150 °C. The gas is now heated so that its temperature is tripled to 450 °C.
(a) How many times larger is the average KE of the nitrogen molecules after heating than before heating?
(b) How many times larger is the average speed of the nitrogen molecules after heating than before heating?
(c) Nitrogen molecule has a mass of 4.65×10^{-26} kg. What is the average speed of the nitrogen molecules when the temperature is 150 °C?
(d) What is the average speed of the nitrogen molecules when it is 450 °C?

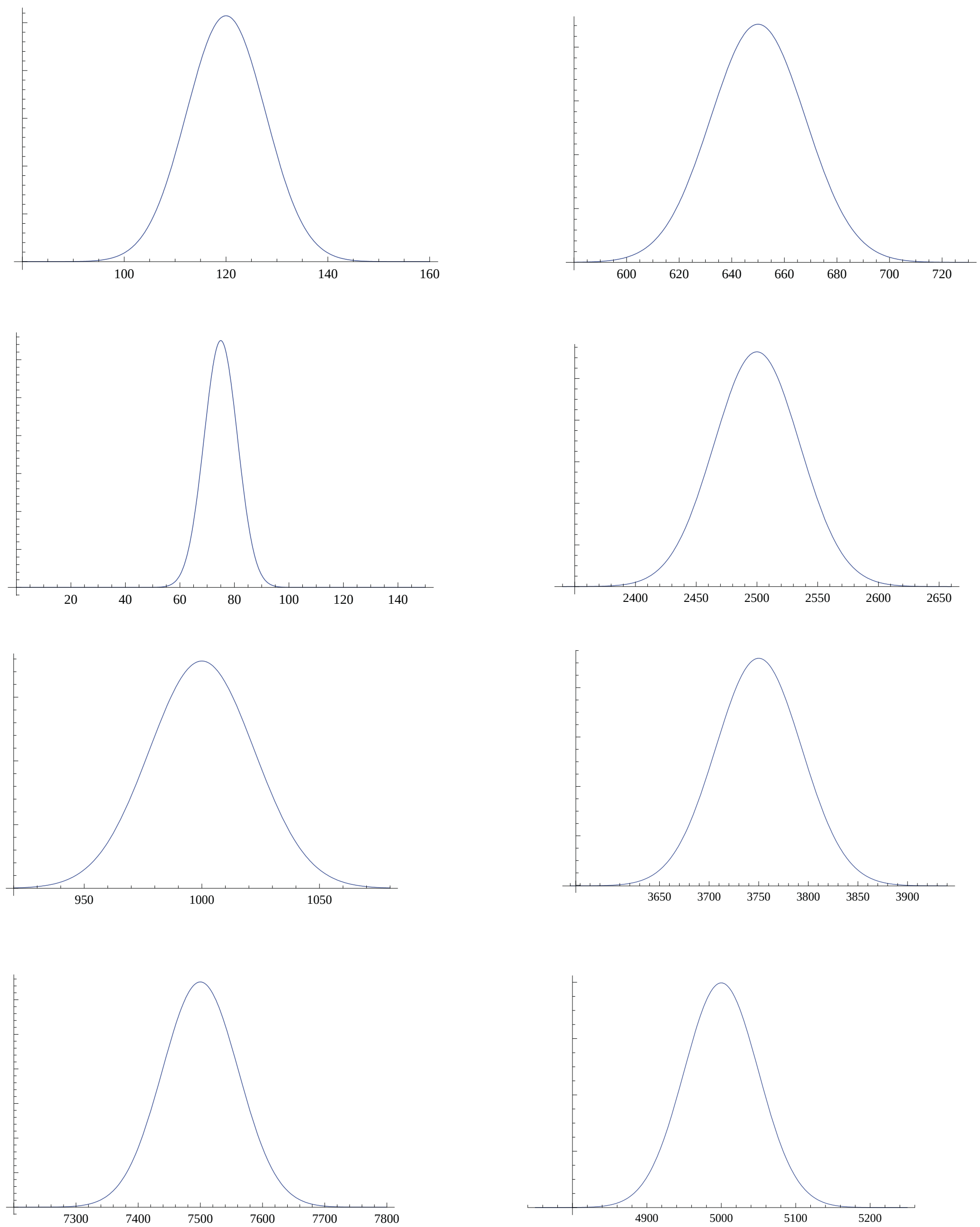


Figure G.17: Plots of number of heads versus probability.

Exercise G.17.3. Argon gas is in a vessel at a temperature of 200 °C. The gas is now heated so that its temperature is quadrupled to 800 °C.
(a) How many times larger is the average KE of the argon atoms after heating than before heating?
(b) How many times larger is the average speed of the argon atoms after heating than before heating?
(c) Nitrogen molecule has a mass of 4.65×10^{-26} kg. What is the average speed of the nitrogen molecules when the temperature is 150 °C? When it is 450 °C?

Exercise G.17.4. How many "molecules" are there in a liter of an ideal gas at room temperature (300 °K) and atmospheric pressure?

Exercise G.17.5. A car flat tire has a volume of 25 liters and a pressure of 1 atm (about 14 psi) when the temperature is 15 °C.
(a) How many "air molecules" are there in the tire?
(b) How many more molecules of air should we add to the tire to raise its pressure to 2.5 atm (about 36 psi) at the same temperature? Assume that the volume increases to 28 liters.

Exercise G.17.6. 18 grams (1 mole) of water is boiled completely into steam in a four-liter pressure cooker at a temperature of 400 °C.
(a) What is the pressure in the cooker?
(b) Suppose the temperature is doubled to 800 °C. What is the pressure now?

G.18 Numerical Exercises for Chapter 18

Exercise G.18.1. This exercise shows you the difference between heat and temperature. Take 100 grams of water (a cup) and add enough heat to raise its temperature from 20 $°C$ to the boiling point, 100 $°C$. The specific heat of water is 4186.
(a) How much heat is required for this process?
(b) Now take a bath tub full of water ($m = 200$ kg). How much heat is needed to raise its temperature from 20 $°C$ to 30 $°C$?

G.20 Numerical Exercises for Chapter 20

Exercise G.20.1. Figure G.18 shows four pairs of figures depicting the BBR curve as predicted by classical theory (left) and the actual observation (right). The horizontal axis is the wavelength in μm, and the vertical axis is intensity in some arbitrary units. How well does the classical prediction agree with observation in each of the following cases? Find the actual values from the corresponding graph and compare them!
(a) When the wavelength is 28 μm, 21 μm, 9 μm, 5.5 μm, 2.5 μm, and 0.17 μm.
(b) While the general shapes of the two curves in the first three pairs are the same, the last pair shows a drastic difference! What can you say about the behavior of intensity when the wavelength decreases from 0.3 μm to 0.1 μm according to classical prediction? According to observation?
(c) Based on the graphs on the left, what do you think the classical theory predicts about intensity when the wavelength is reduced further beyond 0.1 μm?
(d) Based on the graphs on the right, what do you think actually happens to the intensity when the wavelength is reduced further beyond 0.1 μm?
(e) Based on the graphs on the right, try to *sketch* a graph of intensity versus wavelength with a horizontal range starting at 0.1 μm and ending at 30 μm.

Exercise G.20.2. The surface temperature of human body is about 300 °K.
(a) What is the wavelength corresponding to the maximum intensity of EM waves we

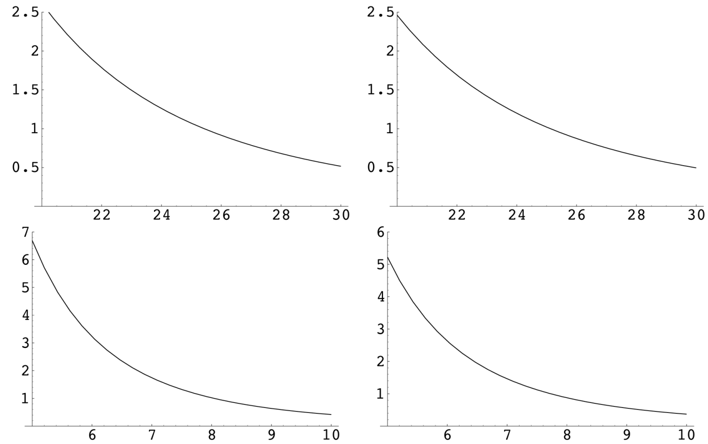


Figure G.18: Comparison of classical prediction (left) versus observation (right) of BBR curves.

produce as a BBR?
(b) What category of EM spectrum does this correspond to?

Exercise G.20.3. Figure G.19 shows a black body radiation curve of a star in which the wavelength is given in units of μm.
(a) From the graph, read off the (approximate) wavelength corresponding to the maximum intensity.
(b) What is the surface temperature of the star?

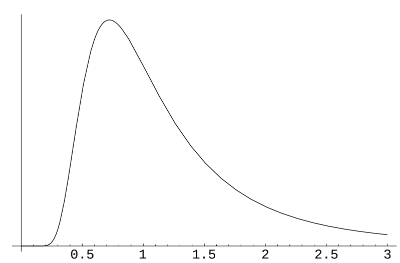


Figure G.19: The BBR curve of a star. The values of the wavelength on the horizontal axis are in μm.

Exercise G.20.4. The Sun has a surface temperature of 6000 °K and a radius of 700,000 km.
(a) Calculate the brightness of the Sun.
(b) What is the total power output of the Sun?
(c) How much energy is received by each m^2 of the surface of the Earth which is 150 million km away? Hint: The Earth is on a big sphere of radius 150 million km. This sphere receives the entire power output of the Sun.

Exercise G.20.5. A star 10 light years away has a surface temperature of 12000 °K and a radius of 1 million km.
(a) Calculate the brightness of the star.
(b) What is the total power output of the star?
(c) How much energy is received by each m^2 of the surface of the Earth? Hint: The Earth

is on a big sphere of radius 10 light years. This sphere receives the entire power output of the star.

Exercise G.20.6. Figure G.20 shows a black body radiation curve of a star in which the wavelength is given in units of μm.
(a) From the graph, read off the (approximate) wavelength corresponding to the maximum intensity.
(b) What is the surface temperature of the star?
(c) What is the brightness of the star?
(d) How much power (joules per second) is given off by the star if its radius is one million km?

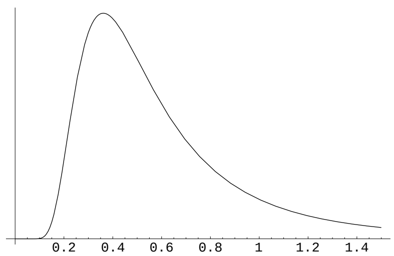


Figure G.20: The BBR curve of a star. The values of the wavelength on the horizontal axis are in μm.

Exercise G.20.7. Figure G.21 shows a black body radiation curve of a star in which the wavelength is given in units of μm.
(a) From the graph, read off the (approximate) wavelength corresponding to the maximum intensity.
(b) What is the surface temperature of the star?
(c) What is the brightness of the star?
(d) How much power (joules per second) is given off by the star if its radius is 150 million km?

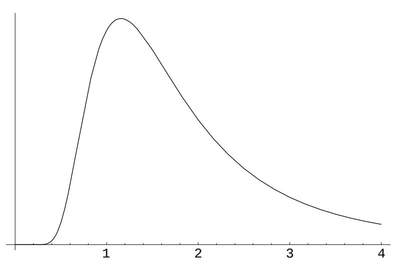


Figure G.21: The BBR curve of a star. The values of the wavelength on the horizontal axis are in μm.

Exercise G.20.8. Figure G.22 shows a black body radiation curve of a star in which the wavelength is given in units of μm.
(a) From the graph, read off the (approximate) wavelength corresponding to the maximum intensity.
(b) What is the surface temperature of the star?
(c) What is the brightness of the star?
(d) How much power (joules per second) is given off by the star if its radius is 3 million km?

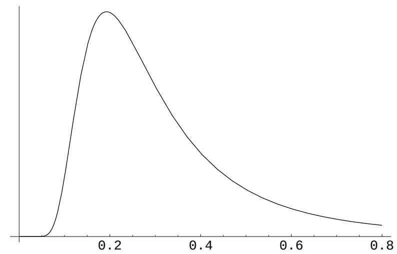


Figure G.22: The BBR curve of a star. The values of the wavelength on the horizontal axis are in μm.

Exercise G.20.9. Figure G.23 shows a black body radiation curve of a star in which the wavelength is given in units of μm.
(a) From the graph, read off the (approximate) wavelength corresponding to the maximum intensity.
(b) What is the surface temperature of the star?
(c) What is the brightness of the star?
(d) How much power (joules per second) is given off by the star if its radius is 10 million km?

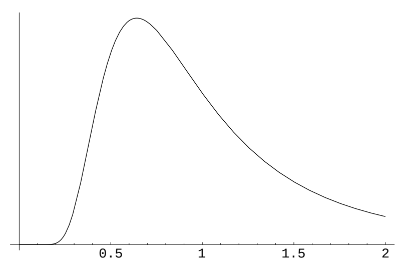


Figure G.23: The BBR curve of a star. The values of the wavelength on the horizontal axis are in μm.

Exercise G.20.10. Figure G.24 shows a black body radiation curve of a star in which the wavelength is given in units of μm.
(a) From the graph, read off the (approximate) wavelength corresponding to the maximum intensity.
(b) What is the surface temperature of the star?
(c) What is the brightness of the star?
(d) How much power (joules per second) is given off by the star if its radius is 15 million km?

Exercise G.20.11. A spherical 100-Watt light bulb has a radius of 2.5 cm.
(a) What is the area of the sphere in m^2?
(b) What is the brightness of the bulb?
(c) Use the Stefan–Boltzmann law to estimate the bulb's surface temperature.
(d) What is this temperature in degrees Celsius?

Exercise G.20.12. Suppose that the filament of a spherical 100-Watt light bulb can be approximated as a sphere of radius of 0.75 cm.
(a) What is the area of the sphere in m^2?
(b) What is the brightness of the bulb?

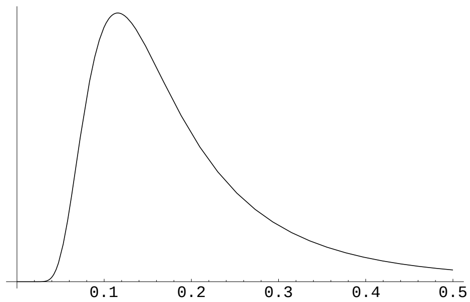


Figure G.24: The BBR curve of a star. The values of the wavelength on the horizontal axis are in μm.

(c) Use the Stefan–Boltzmann law to estimate the bulb's surface temperature.
(d) What is this temperature in degrees Celsius?

Exercise G.20.13. Recall that the wavelength range of visible light is 0.4 to 0.7 μm.
(a) Which color of light has the highest energy?
(b) What is the energy (in eV) of this color?
(c) Which color of light has the lowest energy?
(d) What is the energy (in eV) of this color?

Exercise G.20.14. A gamma ray has a wavelength of 10^{-15} m.
(a) What is the energy (in eV) of this gamma ray?
(b) What is its frequency?

Exercise G.20.15. A hypothetical metal has a work function of 1.5 eV.
(a) What is the longest-wavelength photon that can release the photoelectrons?
(b) Is it visible?

Exercise G.20.16. Blue light with wavelength of 0.45 μm is incident on a hypothetical metal containing electrons which are bound to the metal with an energy of 2 eV.
(a) What is the energy of the photon in eV?
(b) Will the photoelectrons be released?
(c) If so, what will their (maximum) kinetic energy be?

G.21 Numerical Exercises for Chapter 21

Exercise G.21.1. The electron of a hydrogen atom makes a transition from the tenth orbit to the third orbit.
(a) What is the energy of the photon released?
(b) What is the photon's frequency?
(c) What is the photon's wavelength?
(d) Is it visible? If so, what color does it have? If not, which category of the EM spectrum does it belong to?

Exercise G.21.2. The electron of a hydrogen atom makes a transition from the fourth orbit to the third orbit.
(a) What is the energy of the photon released?
(b) What is the photon's frequency?
(c) What is the photon's wavelength?
(d) Is it visible? If so, what color does it have? If not, which category of the EM spectrum does it belong to?

Exercise G.21.3. The electron of a hydrogen atom makes a transition from the tenth orbit to the first orbit.
(a) What is the energy of the photon released?
(b) What is the photon's frequency?
(c) What is the photon's wavelength?
(d) Is it visible? If so, what color does it have? If not, which category of the EM spectrum does it belong to?

Exercise G.21.4. A hydrogen atom makes a transition from the $n = 5$ to $n = 3$ state.
(a) What is the energy of the photon released?
(b) What is the photon's frequency?
(c) What is the photon's wavelength?
(d) Is it visible? If so, what color does it have? If not, which category of the EM spectrum does it belong to?

Exercise G.21.5. A hydrogen atom makes a transition from the n=2 to n=1 state.
(a) What is the energy of the photon released?
(b) What is the photon's frequency?
(c) What is the photon's wavelength?
(d) Is it visible? If so, what color does it have? If not, which category of the EM spectrum does it belong to?

Exercise G.21.6. A hydrogen atom makes a transition from the n=10 to n=5 state.
(a) What is the energy of the photon released?
(b) What is the photon's frequency?
(c) What is the photon's wavelength?
(d) Is it visible? If so, what color does it have? If not, which category of the EM spectrum does it belong to?

Exercise G.21.7. A hydrogen atom makes a transition from the n=4 to n=2 state.
(a) What is the energy of the photon released?
(b) What is the photon's frequency?
(c) What is the photon's wavelength?
(d) Is it visible? If so, what color does it have? If not, which category of the EM spectrum does it belong to?

G.22 Numerical Exercises for Chapter 22

Exercise G.22.1. An electron has a mass of 9.1×10^{-31} kg, a diameter which is smaller than 10^{-18} m and moves with a speed of 10^5 m/s.
(a) What is the wavelength of this electron?
(b) Is it possible to see the diffraction of such electrons?

Exercise G.22.2. Figure G.25 shows the probability and the wave function (probability amplitude) of a hydrogen atom when $n = 2$. The unit of length on the horizontal axis is a_0, the Bohr radius.
(a) Which plot is the probability and which one is the probability amplitude?
(b) At what distance(s) in units of a_0 is the electron least likely to be found?
(c) Is the atom stable? I.e., is there a chance that the electron collapses to the nucleus?
(d) At what distance(s) (in units of a_0) is the electron most likely to be found?
(e) What are the chances that the electron is found at a distance greater than $15a_0$?
(f) Compare the probability of finding the electron between the nucleus and a sphere of radius $2a_0$ with the probability of finding it between the sphere of radius $2a_0$ and a sphere of radius $15a_0$. Hint: Approximate the region with a familiar geometrical shape!

(g) Compare the probability of finding the electron between a sphere of radius $2a_0$ and a sphere of radius $10a_0$ with the probability of finding it between the sphere of radius $10a_0$ and a sphere of radius $100a_0$.

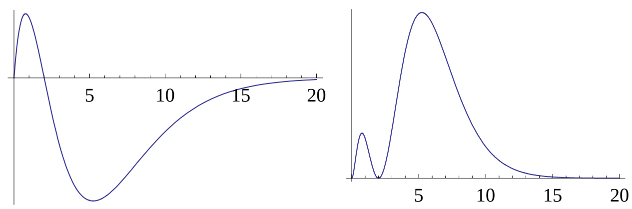


Figure G.25: The hydrogen wave function and its probability for $n = 2$. The horizontal axis gives the distance of the electron from the nucleus in units of a_0.

Exercise G.22.3. Figure G.26 shows the probability and the wave function (probability amplitude) of a hydrogen atom when $n = 3$. The unit of length on the horizontal axis is a_0, the Bohr radius.
(a) Which plot is the probability and which one is the probability amplitude?
(b) At what distance(s) in units of a_0 is the electron least likely to be found?
(c) Is the atom stable? I.e., is there a chance that the electron collapses to the nucleus?
(d) At what distance(s) (in units of a_0) is the electron most likely to be found?
(e) What are the chances that the electron is found at a distance grater than $30a_0$?
(f) Compare the probability of finding the electron between the nucleus and a sphere of radius $2a_0$ with the probability of finding it between the sphere of radius $2a_0$ and a sphere of radius $7a_0$. Hint: Approximate the region with a familiar geometrical shape!
(g) Compare the probability of finding the electron between the nucleus and a sphere of radius $7a_0$ with the probability of finding it between the sphere of radius $7a_0$ and a sphere of radius $30a_0$.

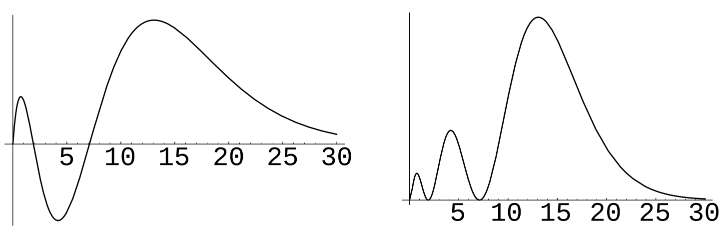


Figure G.26: The hydrogen wave function and its probability for $n = 3$. The horizontal axis gives the distance of the electron from the nucleus in units of a_0.

Exercise G.22.4. A precision-tools company claims that it has invented a device that can measure energy of an atom with an uncertainty of only 0.01 eV in a time interval of 10^{-14} second. Is the company's claim valid?

G.26 Numerical Exercises for Chapter 26

Exercise G.26.1. The crew of Apollo 23 goes to the Moon with a speed of 10 km/s. It spends 20 hours exploring the Moon, and comes back with the same speed. The captain of

the spaceship has just had a baby when he leaves on the mission. The whole trip takes 42 hours for the crew.
(a) To feel the enormity of Apollo's speed, estimate its travel time from New York to Los Angeles (a distance of about 5000 km).
(b) How many hours does the captain spend on the way from Earth to the Moon?
(c) How many more seconds has the baby aged than the captain?
(d) How far is the Moon from the Earth?
(e) What is the Earth-Moon distance according to the crew?

Exercise G.26.2. Epsilon Eridani is a star that is situated 10.8 light years away from us. The crew of the Spaceship Enterprise set out on their journey to Epsilon Eridani with 95% light speed.
(a) What is the distance between Earth and Epsilon Eridani according to the crew of Enterprise?
(b) How long does it take the crew to get there according to the crew?
(c) What is the round-trip travel time to Epsilon Eridani and back for the crew?
(d) How long does it take the crew to get there according to the Earth observers?
(e) What is the round-trip travel time of the journey for the people on Earth?
(f) Verify that (b) and (c) are related via time dilation formula.

Exercise G.26.3. The crew of a spaceship goes to a distant planet 30 l.y. away with a speed of $0.87c$. It spends a year exploring the planet, and comes back with the same speed. The captain of the spaceship is 35 years old and has just had a baby when he leaves on the mission.
(a) What is the Earth-planet distance according to the crew?
(b) How many years does the captain spend on the way from Earth to the planet?
(c) How long does it take the "baby" for her father to land on the planet?
(d) How old is the "baby" when the captain gets back? How old is the captain?
(e) How old would a classmate of the captain be if (s)he were alive?

Exercise G.26.4. In the table below, you are to calculate $1/\sqrt{1-x}$ for small values of x and compare the result with $1 + \frac{1}{2}x$.

Exercise G.26.5. To compare the results you obtain below, note that the length of a proton (the smallest *measurable* length in the universe) is about 10^{-15} m, the length of an atom is about 10^{-10} m, the length of a molecule is about 10^{-9} m, and the length of a bacterium is about 10^{-6} m.
(a) How much does a car shrink as it moves on a highway with a speed of 65 mph? Length of a typical car is about 4 meters.
(b)How much does a race car shrink as it moves on a speedway with a speed of 230 mph? Length of a typical race car is about 3 meters.
(c) How much does a jet plane shrink as it moves with a speed of 500 mph? Length of a typical jet plane is about 30 meters.
(d)How much does a space shuttle shrink as it moves with a speed of 8 km/s? Length of a space shuttle is about 50 meters.

G.27 Numerical Exercises for Chapter 27

Exercise G.27.1. Figure G.27 shows some events as seen by Emmy.
(a) Which pair of events are causally disconnected?
(b) For all pairs that are causally connected, find the speed relative to Emmy of Karl who, moving at constant speed, is present at both events.

x	$1-x$	$\sqrt{1-x}$	$1-\frac{1}{2}x$	$1/\sqrt{1-x}$	$1+\frac{1}{2}x$
0.5					
0.2					
0.1					
0.05					
0.005					
0.0002					
0.00001					

(c) Find the time interval between the pair of events of the previous part as measured by Karl by using the formula that connect proper time to nonproper time.
(d) Find the time interval between the pair of events of the previous part as measured by Karl by calculating their spacetime distances.

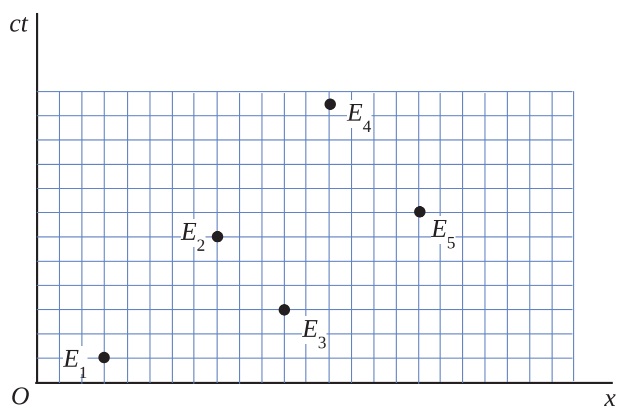


Figure G.27: Some events in a spacetime plane of observer O.

Exercise G.27.2. Figure G.28 shows two world-line diagrams. (a) shows two trips to the Moon (distance of 400,000 km from Earth) by two inertial observers, with the initial point of the trip being half way from Earth. (b) shows a trip to Alpha Centauri which is 4 l.y. away from Earth.
(a) Estimate the travel time of observer 1 to the Moon. What is her speed?
(b) Estimate the travel time of observer 2 to the Moon. What is his speed?
(c) Estimate the travel time of observer in (b). What is his speed?

Exercise G.27.3. In Figure G.29 Emmy is the observer O. Karl is moving relative to Emmy and has a coordinate system, whose axes are also shown in the figure. All units are

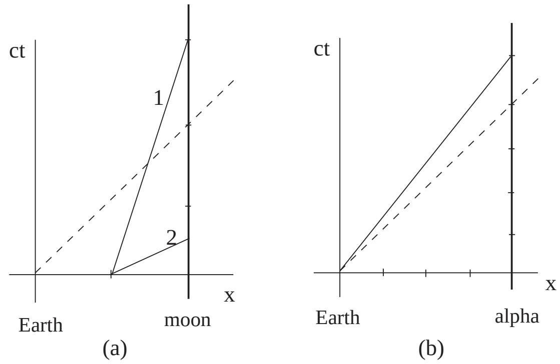


Figure G.28: Spacetime diagrams, world lines, and space travels.

in light years. Assume that at $t = 0$ the origin of both coordinates coincide.
(a) How fast is Karl moving relative to Emmy? In which direction?
(b) How many years after Karl's take off did event E_1 occur according to Emmy?
(c) When did event E_1 occur according to Karl?
(d) Karl asks a friend in his RF to send a light signal back to Emmy. Which event corresponds to this process? When does the friend send the signal?
(e) How far from Emmy is Karl's friend at the moment that he sends the signal?
(f) How far from Karl is his friend?
(g) How many years after E_2 did E_3 occur according to Karl?According to Emmy?
(h) When does Emmy receive the light signal from Karl's friend?

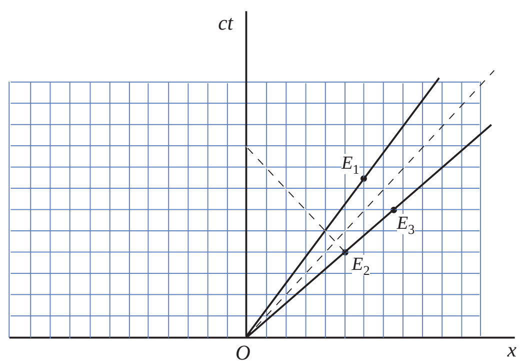


Figure G.29: Emmy is O. Karl's axes are as shown.

Exercise G.27.4. In Figure G.30 Emmy is the observer O. Karl is moving relative to Emmy and has a coordinate system, whose axes are also shown in the figure. All units are in light years. Assume that at $t = 0$ the origin of both coordinates coincide.
(a) How fast is Karl moving relative to Emmy? In which direction?
(b) How many years after Karl's take off did Emmy send Karl a light signal? How many years later (according to Emmy) did Karl receive the signal? How many years after take off did Karl receive Emmy's signal?
(c) How many years after take off did Karl send a light signal to Emmy according to Emmy? According to Karl? When did Emmy receive this signal?
(d) Karl asks a friend in his RF to send a light signal back to Emmy. Which event corre-

sponds to this process? When does the friend send the signal?
(e) How far from Emmy is Karl's friend at the moment that he sends the signal?
(f) How far from Karl is his friend?
(g) When does Emmy receive the light signal from Karl's friend?
(h) When according to Karl does event E_2 occur? Hint: To find coordinates of an event, draw parallel lines!
(i) One year after the take off, Emmy sends Karl a light signal. Immediately after receiving the signal, Karl responds by sending a signal to Emmy. How long after she sends her signal does she receive Karl's response?

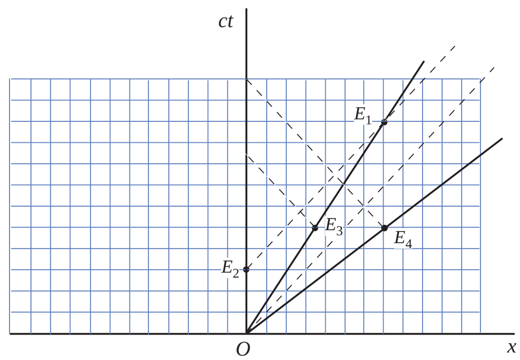


Figure G.30: Emmy is O. Karl's axes are as shown.

Exercise G.27.5. Emmy (observer O) is in the middle of a 100-meter train car moving at 99.5% the speed of light. Karl (observer O') is standing on a platform seeing Emmy pass by. Emmy passes Karl at time zero for both observers. At the moment that she passes him, she observes that firecrackers A (at the rear end of the train) and B (at the front end of the train) explode simultaneously. With Karl's spacetime axes as perpendicular, sketch the following in the space provided below:
(a) Emmy's axes, and the location of the firecrackers (label the points A and B).
(b) The space coordinates for the two explosions as seen by Karl (label these x'_A and x'_B).
(c) The actual value of x'_A and x'_B in meters.
(d) The time coordinates for the two explosions as seen by Karl (label these ct'_A and ct'_B).
(e) The actual value of ct'_A and ct'_B in meters.
(f) The actual value of t'_A and t'_B in μs.
(g) The time coordinate at which Emmy receives the two light signals from A and B (label it cT_{AB}).
(h) The actual value of cT_{AB} in meters.
(i) The actual value of T_{AB} in μs.
(j) The time coordinates at which Karl receives the two light signals from A and B (label them cT'_A, cT'_B).
(k) The actual value of cT'_A and cT'_B in meters.
(l) The actual value of T'_A and T'_B in μs.

Exercise G.27.6. Emmy (observer O) is in a spaceship moving at 86.6% light speed. Karl (observer O') is standing on Earth seeing Emmy pass by. Emmy passes Karl at time zero for both observers. At the moment that she passes him, she observes that supernovae A (10 ly away on her positive direction) and B (10 ly away on her negative direction) explode simultaneously. With Karl's spacetime axes as perpendicular, sketch the following in the space provided below:

(a) Emmy's axes, and the location of the supernovae (label the points A and B).
(b) The space coordinates for the two explosions as seen by Karl (label these x'_A and x'_B).
(c) The actual value of x'_A and x'_B in light years.
(d) The time coordinates for the two explosions as seen by Karl (label these ct'_A and ct'_B).
(e) The actual value of t'_A and t'_B in years.
(f) The time coordinate at which Emmy receives the two light signals from A and B (label it cT_{AB}).
(g) The actual value of T_{AB} in years.
(h) The time coordinates at which Karl receives the two light signals from A and B (label them cT'_A, cT'_B).
(i) The actual value of T'_A and T'_B in years.

Exercise G.27.7. Emmy (observer O) is in a spaceship moving at 99.5% light speed. Karl (observer O') is standing on Earth seeing Emmy pass by. Emmy passes Karl at time zero for both observers. At the moment that she passes him, Karl observes that supernovae A (10 ly away on his positive direction) and B (10 ly away on his negative direction) explode simultaneously. With Emmy's spacetime axes as perpendicular, sketch the following in the space provided below:
(a) Karl's axes, and the location of the supernovae (label the points A and B).
(b) The space coordinates for the two explosions as seen by Emmy (label these x_A and x_B).
(c) The actual value of x_A and x_B in light years.
(d) The time coordinates for the two explosions as seen by Emmy (label these ct_A and ct_B).
(e) The actual value of t_A and t_B in years.
(f) The time coordinate at which Karl receives the two light signals from A and B (label it cT'_{AB}).
(g) The actual value of T'_{AB} in years.
(h) The time coordinates at which Emmy receives the two light signals from A and B (label them cT_A, cT_B).
(i) The actual value of T_A and T_B in years.

Exercise G.27.8. In Figure G.31 are shown the worldlines of some observers as seen by Emmy (observer O).
(a) Specify which worldlines are correct.
(b) Which observers change direction of their motion?
(c) Which observer leaves Emmy and then comes back to her?
(d) Which observers leave each other and then comes back together?
(e) Mark, when appropriate, the event on the world lines when the observers are temporarily not moving relative to Emmy.

Exercise G.27.9. The year is 2209 and the Intergalactic Space Federation (ISF) is trying to go back to the year 1999 to calm down the world's fear of the "millennium disaster" by going to the radio station that is announcing the "doom's day" news. It finds the spaceship Diracus, which is 212 ly away and for which the event of 1999 is NOW. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.
(a) Draw the Earth's coordinate axes with origin O' and axes x' and ct' and place both the origin of Diracus (event O) and the event of the announcement of the radio station in 1999 (event E) in the Earth's spacetime plane.
(b) What is the space separation between the two events in the Earth's RF?
(c) What is the time separation between the two events in the Earth's RF?
(d) Draw the Diracus axes x and ct?
(e) How fast is Diracus moving (i.e., what is β)?
(f) What is the time separation between the two events in the Diracus RF?
(g) What is γ?

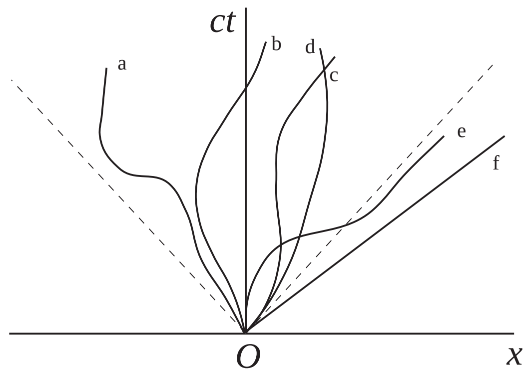


Figure G.31: Some worldlines that may or may not be right.

(h) How far is the radio station from Diracus according to the Diracus crew?
(i) Can the crew prevent the announcement of the doomsday?

Exercise G.27.10. The year is 2139 and the Intergalactic Space Federation (ISF) is trying to go back to the year 1939 to stop the World War II by capturing Hitler and taking him to outer space. It finds the spaceship Diracus, which is 201 ly away and for which the event of 1939 is NOW. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.
(a) Draw the Earth's coordinate axes with origin O' and axes x' and ct' and place both the origin of Diracus (event O) and the event of Hitler's capture (event E) in the Earth's spacetime plane.
(b) What is the space separation between the two events in the Earth's RF?
(c) What is the time separation between the two events in the Earth's RF?
(d) Draw the Diracus axes x and ct?
(e) How fast is Diracus moving (i.e., what is β)?
(f) What is the time separation between the two events in the Diracus RF?
(g) What is γ?
(h) How far is Hitler's residence from Diracus according to the Diracus crew?
(i) Can the crew prevent WWII?

Exercise G.27.11. The year is 2788 and the Intergalactic Space Federation (ISF) is trying to go back to the year 212 BC to stop the slaying of Archimedes by a Roman soldier by stunning the soldier with a laser gun. It finds the spaceship Diracus, which is 3001 ly away and for which the event of 212 BC is NOW. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.
(a) Draw the Earth's coordinate axes with origin O' and axes x' and ct' and place both the origin of Diracus (event O) and the stunning of the Roman soldier (event E) in the Earth's spacetime plane.
(b) What is the space separation between the two events in the Earth's RF?
(c) What is the time separation between the two events in the Earth's RF?
(d) Draw the Diracus axes x and ct?
(e) What is the time separation between the two events in the Diracus RF?
(f) How fast is Diracus moving (i.e., what is β)?
(g) What is γ?
(h) How far is Archimedes' residence from Diracus according to the Diracus crew?
(i) Can the crew prevent the slaying of Archimedes?

Exercise G.27.12. The year is 2788 and the Intergalactic Space Federation (ISF) is trying

to go back far enough in the past so the slaying of Archimedes by a Roman soldier in 212 BC can be stopped. It decides to go to the year 1212 BC and finds the spaceship Diracus, which is 4001 ly away and for which 1212 BC is NOW. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.
(a) Draw the Earth's coordinate axes with origin O' and axes x' and ct' and place the origin of Diracus (event O), the slaying of Archimedes (event E), and the year 1212 BC (event B) in the Earth's spacetime plane.
(b) Draw the Diracus axes x and ct?
(c) How fast is Diracus moving (i.e., what is β)?
(d) What is γ?
(e) What is the space separation between O and B in the Earth's RF?
(f) What is the time separation between the same two events in the Earth's RF?
(g) What is the time separation between O and B in the Diracus RF?
(h) How far is the site of the event B from Diracus according to the Diracus crew?
(i) When is E happening according to the Diracus crew?
(j) What is the x-coordinate of E according to the Diracus crew? To answer this question, do the following: From E, draw a line parallel to the x-axis to cut the ct-axis at A. Convince yourself that $\overline{EA}$ is the x-coordinate of E, and that $\overline{OA}$ is the time of the occurrence of E according to the Diracus crew (which you found earlier). From A draw a line parallel to the x'-axis to cut the ct'-axis at A'. Use Rule 4 to relate $\overline{EA}$ to $\overline{AA'}$. Use Rule 2 to relate $\overline{AA'}$ to $\overline{EA'}$. So, once you find $\overline{EA'}$, you are done. To find $\overline{EA'}$, you need $\overline{O'A'}$; but Rule 4 gives you $\overline{O'A'}$ in terms of $\overline{OA}$. Now put everything together to find $\overline{EA}$.
(k) Draw E and B in the rest frame of O, i.e., in a coordinate system in which ct-axis is perpendicular to the x-axis.
(l) Is it possible to save Archimedes?

Exercise G.27.13. Suppose we are in the distant future when speeds have reached close to light speed. On the 20th anniversary of her mother's tragic death in a car crash Karl tries to prevent the event from happening. So he plans to find a spaceship, for which 10 years earlier than the accident is NOW. That way, he would have 10 years to prepare for the prevention of the accident. He finds the spaceship Diracus, which is 31 ly away. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.
(a) Draw the Earth's coordinate axes with origin O' and axes x' and ct' and place the origin of Diracus (event O), Karl's mother fatal crash (event E), and 10 years earlier (event B) in the Earth's spacetime plane.
(b) Draw the Diracus axes x and ct?
(c) How fast is Diracus moving (i.e., what is β)?
(d) What is γ?
(e) What is the space separation between O and B in the Earth's RF?
(f) What is the time separation between the same two events in the Earth's RF?
(g) What is the time separation between O and B in the Diracus RF?
(h) How far is the site of the event B from Diracus according to the Diracus crew?
(i) When is E happening according to the Diracus crew?
(j) What is the x-coordinate of E according to the Diracus crew? To answer this question, do the following: From E, draw a line parallel to the x-axis to cut the ct-axis at A. Convince yourself that $\overline{EA}$ is the x-coordinate of E, and that $\overline{OA}$ is the time of the occurrence of E according to the Diracus crew (which you found earlier). From A draw a line parallel to the x'-axis to cut the ct'-axis at A'. Use Rule 4 to relate $\overline{EA}$ to $\overline{AA'}$. Use Rule 2 to relate $\overline{AA'}$ to $\overline{EA'}$. So, once you find $\overline{EA'}$, you are done. To find $\overline{EA'}$, you need $\overline{O'A'}$; but Rule 4 gives you $\overline{O'A'}$ in terms of $\overline{OA}$. Now put everything together to find $\overline{EA}$.
(k) Draw E and B in the rest frame of O, i.e., in a coordinate system in which ct-axis is perpendicular to the x-axis.

(l) Is it possible to save Karl's mother?

Exercise G.27.14. Jack is on a spaceship that travels to a planet of a star system 10 l.y. away on a world line shown in Figure G.32 as seen by observer O, Jill. All units are in light years.
(a) How long is the time interval between take-off from Earth (E_1) and landing on the planet (E_2) according to Jill?
(b) How long is the time interval between landing (E_2) and departure (E_3) from the planet according to Jill?
(c) How long is the time interval between departure (E_3) and landing on Earth (E_4) according to Jill?
(d) How long does the entire trip take according to Jill?
(e) From the figure determine what Δs is for the two events E_1 and E_2.
(f) From the figure determine what Δs is for the two events E_2 and E_3.
(g) From the figure determine what Δs is for the two events E_3 and E_4.
(h) What is Δs for the entire trip? How long does this trip take according to Jack?
(i) Who measures the proper time interval between E_1 and E_4, Jack or Jill (or both)?
(j) What is the speed of the spaceship in m/s between E_1 and E_2? Between E_2 and E_3? Between E_3 and E_4?

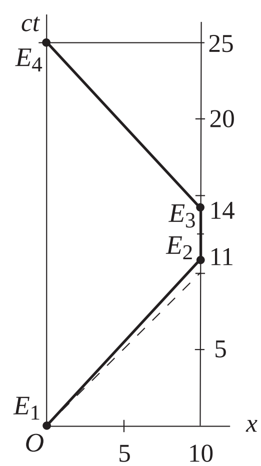


Figure G.32: The heavy path is the world line of the spaceship. All units are in light years.

Exercise G.27.15. Jill is on a spaceship that travels to a planet of a star system 50 l.y. away on a world line shown in Figure G.33 as seen by observer O, Jack. All units are in light years.
(a) How long is the time interval between take-off from Earth (E_1) and landing on the planet (E_2) according to Jack?
(b) How long is the time interval between landing (E_2) and departure (E_3) from the planet according to Jack?
(c) How long is the time interval between departure (E_3) and landing on Earth (E_4) according to Jack?
(d) How long does the entire trip take according to Jack?
(e) From the figure determine what Δs is for the two events E_1 and E_2.
(f) From the figure determine what Δs is for the two events E_2 and E_3.
(g) From the figure determine what Δs is for the two events E_3 and E_4.
(h) What is Δs for the entire trip? How long does the entire trip take according to Jill?

(i) Who measures the proper time interval between E_1 and E_4, Jack or Jill (or both)?
(j) What is the speed of the spaceship in m/s between E_1 and E_2? Between E_2 and E_3? Between E_3 and E_4?

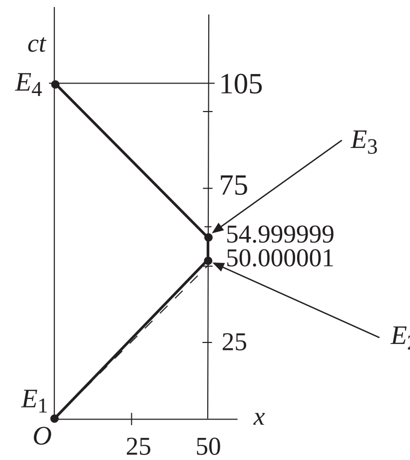


Figure G.33: The heavy path is the world line of the spaceship. All units are in light years.

Exercise G.27.16. Jack is on a spaceship that travels to a planet of a star system 10 ly away on a world line shown in Figure G.34 as seen by observer O, Jill. All units are in light years. (a) How long is the time interval between take-off from Earth (E_1) and landing on the planet (E_2) according to Jill?
(b) How long is the time interval between landing (E_2) and departure (E_3) from the planet according to Jill.
(c) How long is the time interval between departure (E_3) and landing on Earth (E_4) according to Jill?
(d) How long does the entire trip take according to Jill?
(e) From the figure determine what Δs is for the two events E_1 and E_2.
(f) From the figure determine what Δs is for the two events E_2 and E_3.
(g) From the figure determine what Δs is for the two events E_3 and E_4.
(h) What is Δs for the entire trip? How long does this trip take according to Jack?
(i) Who measures the proper time interval between E_1 and E_4, Jack or Jill (or both)?
(j) What is the speed of the spaceship between E_1 and E_2? Between E_2 and E_3? Between E_3 and E_4?

Exercise G.27.17. Jill is on a spaceship that travels to a planet of a star system 30 l.y. away on a world line shown in Figure G.35 as seen by observer O, Jack. All units are in light years.
(a) How long is the time interval between take-off from Earth (E_1) and landing on the planet (E_2) according to Jack?
(b) How long is the time interval between landing (E_2) and departure (E_3) from the planet according to Jack?
(c) How long is the time interval between departure (E_3) and landing on Earth (E_4) according to Jack?
(d) How long does the entire trip take according to Jack?
(e) From the figure determine what Δs is for the two events E_1 and E_2.
(f) From the figure determine what Δs is for the two events E_2 and E_3.

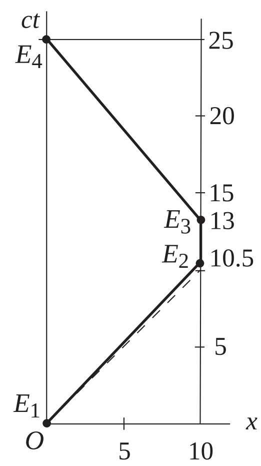


Figure G.34: The heavy path is the world line of the spaceship. All units are in light years.

(g) From the figure determine what Δs is for the two events E_3 and E_4.
(h) What is Δs for the entire trip? How long does this trip take according to Jill?
(i) Who measures the proper time interval between E_1 and E_4, Jack or Jill (or both)?
(j) What is the speed of the spaceship between E_1 and E_2? Between E_2 and E_3? Between E_3 and E_4?

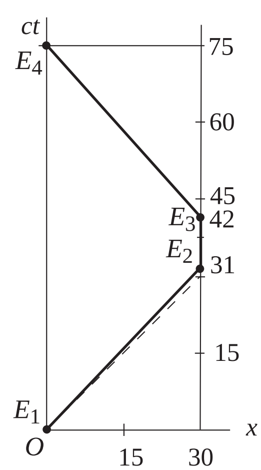


Figure G.35: The heavy path is the world line of the spaceship. All units are in light years.

Exercise G.27.18. Observer O is in a rocketship which is moving relative to observer O' with the speed of 270,000 km/s. Suppose that at $t = 0$ the origins of the two RFs coincide. Observer O fires an electron in the forward direction (E_1) with a speed of 150,000 km/s. This electron is detected one microsecond (10^{-6} sec) later (E_2) at a detector. In the following calculations, keep as many decimals as possible.
(a) What is γ for this relative motion?
(b) What is Δx and $c\Delta t$ (according to O)?
(c) Find Δs (according to O) and from that find $\Delta \tau$ (according to O)?
(d) What is $\Delta x'$ and $c\Delta t'$ (according to O')?

(e) Find $\Delta s'$ (according to O') and from that find $\Delta\tau'$ (according to O')?
(f) Compare $\Delta s'$ with Δs and $\Delta\tau'$ with $\Delta\tau$?

Exercise G.27.19. Observer O is in a rocketship which is moving relative to observer O' with the speed of 240,000 km/s. Suppose that at $t = 0$ the origins of the two RFs coincide. Observer O fires a light beam in the forward direction (E_1). This beam is detected one microsecond (10^{-6} sec) later (E_2) at a detector. In the following calculations, keep as many decimals as possible.
(a) What is γ for this relative motion?
(b) What is Δx and $c\Delta t$ (according to O)?
(c) Find Δs (according to O) and from that find $\Delta\tau$ (according to O)?
(d) What is $\Delta x'$ and $c\Delta t'$ (according to O')?
(e) Find $\Delta s'$ (according to O') and from that find $\Delta\tau'$ (according to O')?
(f) Compare $\Delta s'$ with Δs and $\Delta\tau'$ with $\Delta\tau$?

Exercise G.27.20. Consider the explosion of the two firecrackers at the two ends of a train moving with speed v. Label these two events as (x'_1, t'_1) and (x'_2, t'_2) for the ground observer, and as (x_1, t_1) and (x_2, t_2) for the train observer.
(a) Write the (four) Lorentz transformation for the two events.
(b) Subtract the equations to find a relation between $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$ (the length and time interval in the RF of the train observer) and $\Delta x' = x'_2 - x'_1$ and $\Delta t' = t'_2 - t'_1$ (the length and time interval in the RF of the ground observer).

Exercise G.27.21. Assume that the ground observer O' sees the explosion of two firecrackers at the two ends of a train, with observer O inside it, as simultaneous. The speed of the train is v.
(a) What is $\Delta t'$?
(b) Solve the equation involving $\Delta t'$ and find a relation between Δx and Δt.
(c) Substitute for Δt in the equation for $\Delta x'$ and find a relation connecting Δx and $\Delta x'$. Can you interpret this as length contraction?

Exercise G.27.22. Emmy (observer O) is riding on a train whose fractional speed is β relative to Karl (observer O'), who is at the platform watching the train go by as shown in Figure G.36. The moment that Emmy passes Karl is the origin of time for both Karl and Emmy, and that is also the time that both firecrackers pop simultaneously *according to Karl*. The location of each observer is the origin of his/her space coordinate. The length of the train is L according to Karl. Find the answer to all the following questions in terms of L, β, and γ.
(a) What are the spacetime coordinates of A and B according to Karl?
(b) What are the spacetime coordinates of A and B according to Emmy? What is the significance of the sign of t_B? What is the length of the train according to Emmy?
(c) In the middle diagram of Figure G.36, Emmy receives the signal from B. What are the spacetime coordinates of this event according to Emmy? Hint: Add the time of the occurrence of B to the time that it takes for light to travel from B to Emmy.
(d) According to Karl, when and where does Emmy receive the signal from B?
(e) In the last diagram of Figure G.36, Karl receives the signals from A and B. What are the spacetime coordinates of this event according to Karl?
(f) According to Emmy, when and where does Karl receive the signals from A and B? Has she seen the rear end of the train pass Karl when this happens?
(g) Assume that $L = 100$ m and $\beta = 0.995$. Now find numertical values for all the above questions.

Exercise G.27.23. The year is 2209 and the Intergalactic Space Federation (ISF) is trying to go back to the year 1999 to calm down the world's fear of the "millennium disaster" by

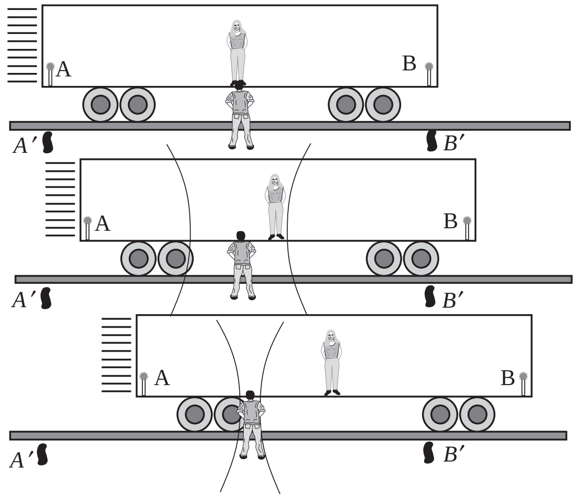


Figure G.36: At the moment shown on top of the page, the firecrackers explode *according to Karl*.

going to the radio station that is announcing the "doom's day" news. It finds the spaceship Diracus, which is 212 ly away and for which the event of 1999 is NOW. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.
(a) Draw the Earth's coordinate axes and place both the origin of Diracus (event E_1) and the event of the announcement of the radio station in 1999 (event E_2) in the Earth's spacetime plane.
(b) What is the space separation between the two events in the Earth's RF?
(c) What is the time separation between the two events in the Earth's RF?
(d) What is the time separation between the two events in the Diracus RF?
(e) How fast is Diracus moving (i.e., what is β)?
(f) What is γ?
(g) How far is the radio station from Diracus according to the Diracus crew?
(h) Can the crew prevent the announcement of the doomsday?

Exercise G.27.24. The year is 2788 and the Intergalactic Space Federation (ISF) is trying to go back to the year 212 BC to stop the slaying of Archimedes by a Roman soldier by stunning the soldier with a laser gun. It finds the spaceship Diracus, which is 3001 ly away and for which the event of 212 BC is NOW. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.
(a) Draw the Earth's coordinate axes and place both the origin of Diracus (event E_1) and the stunning of the Roman soldier (event E_2) in the Earth's spacetime plane.
(b) What is the space separation between the two events in the Earth's RF?
(c) What is the time separation between the two events in the Earth's RF?
(d) What is the time separation between the two events in the Diracus RF?
(e) How fast is Diracus moving (i.e., what is β)?
(f) What is γ?
(g) How far is Archimedes' residence from Diracus according to the Diracus crew?
(h) Can the crew prevent the slaying of Archimedes?

Exercise G.27.25. The year is 2788 and the Intergalactic Space Federation (ISF) is trying to go back far enough in the past so the slaying of Archimedes by a Roman soldier in 212 BC can be stopped. It decides to go to the year 1212 BC and finds the spaceship Diracus, which is 4001 ly away and for which 1212 BC is NOW. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.

(a) Draw the Earth's coordinate axes and place both the origin of Diracus (event E_1), the slaying of Archimedes (event E_2), and the year 1212 BC (event E_3) in the Earth's spacetime plane.
(b) What is the space separation between E_1 and E_3 in the Earth's RF?
(c) What is the time separation between the same two events in the Earth's RF?
(d) What is the time separation between the same two events in the Diracus RF?
(e) How fast is Diracus moving (i.e., what is β)?
(f) What is γ?
(g) How far is the site of the event E_3 from Diracus according to the Diracus crew?
(h) How much farther is the site of the slaying from E_3 according to the Diracus crew? In other words, what is Δx_{23}?
(i) How much later does the slaying occur than E_3 according to the Diracus crew? In other words, what is Δt_{23}?
(j) How far is the slaughter site from Diracus according to the Diracus crew?
(k) How much time does the Diracus crew have to get to the site?
(l) Now Diracus commander sends a manned probe to the site to divert the slaying. As a fraction of light speed, what should the probe speed be for it to get there at the time of the slaying?
(m) Can the probe save Archimedes?

Exercise G.27.26. Suppose we are in the distant future when speeds have reached close to light speed. On the 10th anniversary of her mother's tragic death in a car crash Karl tries to prevent the event from happening. So he plans to find a spaceship, for which 5 years earlier than the accident is NOW. That way, he would have 5 years to prepare for the prevention of the accident. He finds the spaceship Diracus, which is 16 ly away. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.
(a) Draw the Earth's coordinate axes and place the origin of Diracus (event E_1), Karl's mother fatal crash (event E_2), and 5 years earlier (event E_3) in the Earth's spacetime plane.
(b) What is the space separation between E_1 and E_3 in the Earth's RF?
(c) What is the time separation between the same two events in the Earth's RF?
(d) What is the time separation between the same two events in the Diracus RF?
(e) How fast is Diracus moving (i.e., what is β)?
(f) What is γ?
(g) How far is the site of the event E_3 from Diracus according to the Diracus crew?
(h) How much farther is the site of the crash from E_3 according to the Diracus crew? In other words, what is Δx_{23}?
(i) How much later does the crash occur than E_3 according to the Diracus crew? In other words, what is Δt_{23}?
(j) How far is the crash site from Diracus according to the Diracus crew?
(k) How much time does the Diracus crew have to get to the crash site?
(l) Now Diracus commander sends a manned probe to the crash site to divert the crash. As a fraction of light speed, what should the probe speed be for it to get there at the time of the crash?
(m) Can the probe make it to the accident?

Exercise G.27.27. Suppose we are in the distant future when speeds have reached close to light speed. On the 20th anniversary of her mother's tragic death in a car crash Karl tries to prevent the event from happening. So he plans to find a spaceship, for which 10 years earlier than the accident is NOW. That way, he would have 10 years to prepare for the prevention of the accident. He finds the spaceship Diracus, which is 31 ly away. Diracus happens to be just passing an outpost there, so the plan can be immediately communicated to Diracus.

(a) Draw the Earth's coordinate axes and place the origin of Diracus (event E_1), Karl's mother fatal crash (event E_2), and 10 years earlier (event E_3) in the Earth's spacetime plane.
(b) What is the space separation between E_1 and E_3 in the Earth's RF?
(c) What is the time separation between the same two events in the Earth's RF?
(d) What is the time separation between the same two events in the Diracus RF?
(e) How fast is Diracus moving (i.e., what is β)?
(f) What is γ?
(g) How far is the site of the event E_3 from Diracus according to the Diracus crew?
(h) How much farther is the site of the crash from E_3 according to the Diracus crew? In other words, what is Δx_{23}?
(i) How much later does the crash occur than E_3 according to the Diracus crew? In other words, what is Δt_{23}?
(j) How far is the crash site from Diracus according to the Diracus crew?
(k) How much time does the Diracus crew have to get to the crash site?
(l) Now Diracus commander sends a manned probe to the crash site to divert the crash. As a fraction of light speed, what should the probe speed be for it to get there at the time of the crash?
(m) Can the probe make it to the accident?

G.28 Numerical Exercises for Chapter 28

Exercise G.28.1. A very compact car has a mass of 200 kg. It is desired to accelerate this car to speeds very close to the speed of light. Use Newtonian nonrelativistic theory (not a valid theory at high speeds).
(a) Calculate the energy needed to accelerate the car from rest to $0.99c$.
(b) How much energy does it take to accelerate the car further from $0.99c$ to $0.9999c$?
(c) How much energy does it take to accelerate the car further from $0.9999c$ to $0.999999c$?
(d) How much energy does it take to accelerate the car further from $0.999999c$ to $0.99999999c$?
(e) How much energy does it take to accelerate the car further from $0.9999999999c$ to $0.999999999999c$? Each gallon of gasoline stores approximately 10^9 J of energy. How many gallons are needed to accomplish this acceleration?

Exercise G.28.2. A car has a mass of 1200 kg. It is desired to accelerate this car to speeds very close to the speed of light. Use Newtonian nonrelativistic theory (not a valid theory at high speeds).
(a) Calculate the energy needed to accelerate the car from rest to $0.99c$.
(b) How much energy does it take to accelerate the car further from $0.99c$ to $0.9999c$?
(c) How much energy does it take to accelerate the car further from $0.9999c$ to $0.999999c$?
(d) How much energy does it take to accelerate the car further from $0.999999c$ to $0.99999999c$?
(e) How much energy does it take to accelerate the car further from $0.9999999999c$ to $0.999999999999c$? Each gallon of gasoline stores approximately 10^9 J of energy. How many gallons are needed to accomplish this acceleration?

Exercise G.28.3. A car has a mass of 1200 kg. It is desired to accelerate this car to speeds very close to the speed of light.
(a) Calculate the energy needed to accelerate the car from rest to a speed of $0.99c$.
(b) How much energy does it take to accelerate the car further from $0.99c$ to $0.9999c$?
(c) How much energy does it take to accelerate the car further from $0.9999c$ to $0.999999c$?
(d) How much energy does it take to accelerate the car further from $0.999999c$ to $0.99999999c$?
(e) How much energy does it take to accelerate the car further from $0.9999999999c$ to $0.999999999999c$? Each gallon of gasoline stores approximately 10^9 J of energy.How many

gallons of gasoline are needed to accomplish this acceleration? Compare this with the nonrelativistic case.

Exercise G.28.4. It is desired to accelerate a penny (about 1 gram) to $0.9999999c$.
(a) How many Joules of energy is required?
(b) A medium city uses about one giga (10^9) Watts (Joules per second) of power. How many seconds should all power plants of such a city be devoted to the acceleration of the penny? How many years?

Exercise G.28.5. A medium city uses about one giga (10^9) Watts of power.
(a) How many Joules of energy is used in such a city in a year?
(b) How many kilograms of matter and antimatter are to annihilate each other to provide energy for one year of such a city?

G.29 Numerical Exercises for Chapter 29

Exercise G.29.1. A beam of light is sent across a distance of 10 km on a planet whose gravitational acceleration is 200 m/s^2.
(a) How long does it take light to go from one end of the field to the other?
(b) How far does this light beam falls during this time?
(c) What is the deflection angle of light, i.e., the angle between its original direction of motion and its direction at the other end of the field?

Exercise G.29.2. A beam of starlight grazes a star of mass 8×10^{32} kg and radius 600,000 km.
(a) What is the deflection angle for this light beam in radians?
(b) What is the deflection angle in degrees? In arcseconds?

Exercise G.29.3. A beam of starlight grazes a white dwarf of mass 10^{30} kg and radius 6,000 km.
(a) What is the deflection angle for this light beam in radians?
(b) What is the deflection angle in degrees? In arcseconds?

Exercise G.29.4. A flashlight emits green light of wavelength 0.55 μm. It is located at the top of a mountain peak 6000 m high. It shines a light beam downward, which is detected at the bottom.
(a) Is the wavelength of light at the bottom shorter or longer than 0.55 μm?
(b) What is the change in the wavelength of the light?

Exercise G.29.5. Yellow light of wavelength 0.6 μm is emitted by the Sun whose surface gravitational acceleration is 270 m/s^2. This light is detected 100 km away from the Sun.
(a) Is the wavelength of light at the bottom shorter or longer than 0.6 μm?
(b) What is the change in the wavelength of the yellow light?

Exercise G.29.6. A satellite 20 km above the surface of the Earth carries an atomic clock synchronized with a similar clock on Earth. The satellite is kept in orbit for one year. Consider only the GTR effects.
(a) Which clock will be running faster?
(b) How many seconds will the faster clock be ahead of the slower one after one year?

Exercise G.29.7. A satellite 50 km above the surface of Jupiter carries an atomic clock synchronized with a similar clock on Jupiter where the gravitational acceleration is 25 m/s^2. The satellite is kept in orbit for one year. Consider only the GTR effects.
(a) Which clock will be running faster?
(b) How many seconds will the faster clock be ahead of the slower one after one year?

Exercise G.29.8. A satellite 5000 km above the surface of a planet carries an atomic clock synchronized with a similar clock on the planet. The mass of the planet is 4×10^{25} kg and its radius in 3000 km. The satellite is kept in orbit for one year. Consider only the GTR effects.
(a) Which clock will be running faster?
(b) How many seconds will the faster clock be ahead of the slower one after one year?

Exercise G.29.9. A distant galaxy is 250 Mly away. Assume that the Hubble parameter is 22 km/s per Mly.
(a) What is the speed of the galaxy in m/s?
(b) How long (in seconds) did it take this galaxy to move this distance?
(c) How many *years* ago was this galaxy on top of the Milky Way?

Exercise G.29.10. A distant galaxy is 800 Mly away. Assume that the Hubble parameter is 20 km/s per Mly.
(a) What is the speed of the galaxy in m/s?
(b) How long (in seconds) did it take this galaxy to move this distance?
(c) How many *years* ago was this galaxy on top of the Milky Way?

Exercise G.29.11. At earlier times, the Hubble parameter was larger than its present value. When the Hubble parameter was 500 km/s per Mly,
(a) what was the speed of a galaxy 150 Mly away from the Milky Way?
(b) How long (in seconds) did it take the galaxy to move that distance?
(c) How many *years* earlier was this galaxy on top of the Milky Way?

Exercise G.29.12. At earlier times, the Hubble parameter was larger than its present value. When the Hubble parameter was 2000 km/s per Mly,
(a) what was the speed of a galaxy 120 Mly away from the Milky Way?
(b) How long (in seconds) did it take the galaxy to move that distance?
(c) How many *years* earlier was this galaxy on top of the Milky Way?

Exercise G.29.13. At earlier times, the Hubble parameter was larger than its present value. When the Hubble parameter was 70,000 km/s per Mly,
(a) what was the speed of a galaxy 4 Mly away from the Milky Way?
(b) How long (in seconds) did it take the galaxy to move that distance?
(c) How many *years* earlier was this galaxy on top of the Milky Way?

G.31 Numerical Exercises for Chapter 31

Exercise G.31.1. ${}^{12}_{6}\mathrm{C}$ has an atomic mass of exactly 12 u.
(a) What is the total binding energy of ${}^{12}_{6}\mathrm{C}$ in MeV?
(b) What is its binding energy per nucleon?

Exercise G.31.2. ${}^{14}_{7}\mathrm{N}$ has an atomic mass of 14.003074 u.
(a) What is the total binding energy of ${}^{14}_{7}\mathrm{N}$ in MeV?
(b) What is its binding energy per nucleon?

Exercise G.31.3. The binding energy per nucleon of ${}^{63}_{29}\mathrm{Cu}$ is 8.523 MeV.
(a) What is the atomic mass of ${}^{63}_{29}\mathrm{Cu}$ in MeV?
(b) What is its mass in unified atomic mass unit?

Exercise G.31.4. The mass excess for ${}^{40}_{20}\mathrm{Ca}$ is -34.85 MeV.
(a) What is the total binding energy of ${}^{40}_{20}\mathrm{Ca}$ in MeV?
(b) What is its binding energy per nucleon? What is its mass in MeV? What is its mass in unified atomic mass unit?

Exercise G.31.5. An isotope of Na has a half-life of 15 hours. What fraction of a sample of this isotope remains after one day? If you start with a million atoms, how many atoms are left after one week?

Exercise G.31.6. An isotope of Ca has a half-life of 1.5 million years.
(a) What fraction of a sample of this isotope remains after 1000 years?
(b) How long do we have to wait for 1% of a sample of this isotope to disappear?

Exercise G.31.7. A fission reaction is given by

$$\mathrm{n} + {}^{235}_{92}\,\mathrm{U} \rightarrow {}^{93}_{37}\mathrm{Rb} + {}^{141}_{55}\,\mathrm{Cs} + 2\mathrm{n}$$

The mass excesses of ${}^{235}_{92}\mathrm{U}$, ${}^{93}_{37}\mathrm{Rb}$, and ${}^{141}_{55}\mathrm{Cs}$ are 40.92 MeV, −72.62 MeV, and −74.48 MeV, respectively.
(a) What is the binding energy per nucleon for ${}^{235}_{92}\mathrm{U}$?
(b) What is the binding energy per nucleon for ${}^{93}_{37}\mathrm{Rb}$?
(c) What is the binding energy per nucleon for ${}^{141}_{55}\mathrm{Cs}$?
(d) What is the energy released in the reaction?

Exercise G.31.8. A hypothetical fission reaction is given by

$$\mathrm{n} + {}^{250}_{94}\,\mathrm{X} \rightarrow {}^{140}_{50}\mathrm{Y} + {}^{109}_{?}\,\mathrm{Z} + ?\mathrm{n}$$

The binding energy per nucleon of X, Y, and Z are 7.4 MeV, 8 MeV, and 8.2 MeV, respectively.
(a) What is the number of protons in Z?
(b) How many neutrons are produced?
(c) What is the energy released in the reaction?

Exercise G.31.9. A hypothetical fission reaction is given by

$$\mathrm{n} + {}^{242}_{93}\,\mathrm{X} \rightarrow {}^{141}_{54}\mathrm{Y} + {}^{98}_{?}\,\mathrm{Z} + ?\mathrm{n}$$

The binding energy per nucleon of X, Y, and Z are 7.2 MeV, 8.1 MeV, and 7.8 MeV, respectively.
(a) What is the number of protons in Z?
(b) How many neutrons are produced?
(c) What is the energy released in the reaction?

Exercise G.31.10. The first stage of the proton-proton cycle is

$$\mathrm{p} + \mathrm{p} \rightarrow {}^{2}_{1}\mathrm{D} + \mathrm{e} + \nu$$

where ν is a (neutral) neutrino. The mass of the proton, deuteron, and e are, respectively, 938.272 MeV, 1875.6 MeV, and 0.51 MeV.
(a) What is the sign of the electric charge of e?
(b) What is the energy released in the reaction?

G.38 Numerical Exercises for Chapter 38

Exercise G.38.1. The temperature of the universe is now 2.725 °K, and its scale size is 500 Mly.
(a) How hot was the universe when its length scale R was the size of the Sun (1,400,000 km)?
(b) What was $\lambda_{\max}$ then?

Exercise G.38.2. When the temperature of the universe was 100 times the present temperature,
(a) how much smaller was the length scale of the universe than present?
(b) How much larger was the photon number density compared to now?
(c) How much larger was the photon average energy compared to now?
(d) How much larger was the photon equivalent mass density compared to now?
(e) How much larger was the matter mass density compared to now?

Exercise G.38.3. When the length scale of the universe was 10000 times smaller than the present length scale,
(a) how much hotter was the universe than the present?
(b) How much larger was the photon number density compared to now?
(c) How much larger was the photon average energy compared to now?
(d) How much larger was the photon equivalent mass density compared to now?
(e) How much larger was the matter mass density compared to now?

Exercise G.38.4. The temperature of the universe is now 2.725 °K, and its scale size is 500 Mly. (a) How hot was the universe when its length scale R was the size of the local group of galaxies (3,000,000 ly)?
(b) What was the photon number density?
(c) What was the photon average energy?
(d) What was the photon equivalent mass density?
(e) What was the matter mass density?

Exercise G.38.5. The temperature of the universe is now 2.725 °K, and its scale size is 500 Mly.
(a) How hot was the universe when its length scale R was the size of the Earth (10,000 km)?
(b) What was the photon number density?
(c) What was the photon average energy?
(d) What was the photon equivalent mass density?
(e) What was the matter mass density?

Exercise G.38.6. Consider the universe one millisecond after the big bang, when $\alpha = 7.125$.
(a) What was the temperature then?
(b) What was the density then?
(c) Assuming that the current temperature of the universe is 2.725 °K and its scale size is 500 Mly, what was the scale size of the universe one millisecond after the big bang?
(d) How do you describe the rate of expansion of the universe during the first millisecond of its creation?

G.39 Numerical Exercises for Chapter 39

Exercise G.39.1. The luminosity of the Sun is the result of the conversion of hydrogen into helium and release of energy. The energy released is only 0.7% of the energy equivalent of the mass of the hydrogen converted.
(a) Given that the luminosity of the Sun is 4×10^{26} Watts, how much mass-energy of hydrogen is turned into helium per second?
(b) How much mass is the Sun losing per second?

Exercise G.39.2. The radius of the Sun is 700,000 km and its surface temperature is 6000 K.
(a) What is the intensity (brightness, energy flux) at the surface of the Sun?

(b) What is the surface area of the Sun?
(c) What is the luminosity of the Sun?
(d) Given that the Earth-Sun distance is 150 million km, find the brightness (intensity, energy flux) of the Sun on Earth.
(e) A star that is identical to the Sun has a brightness of 0.4 μW/m^2. How far is this star?