How the VSRT Simulates a Cross-Correlation Interferometer

This document provides a detailed discussion of the mathematical theory behind how the VSRT Interferometer works, and how it simulates a modern cross-correlation interferometer that professional radio astronomers use. This discussion, in general, is not intended for students. Instructors are advised to read this and decide for themselves whether their students would benefit from such a discussion. The labs discussed in "VSRTlabs.pdf" can be conducted without a discussion of the mathematical theory and the students will still attain an intuitive understanding of how aperture synthesis data leads to images of radio sources.

This discussion, here, in fact, is designed primarily for instructors who have some familiarity with aperture synthesis, although we do provide a very quick synopsis of the basic idea. The VSRT Interferometer is different, and simpler, than a professional radio astronomy interferometer, and so the purpose of this document is to provide an explanation of how the VSRT interferometer produces (mostly) the same kind of data.

I. SOME BASICS OF APERTURE SYNTHESIS

In most professional radio astronomy aperture synthesis instruments, a number of antennas arranged in a designed array, with assorted "baselines" (separation distances and directions between antennas), receive the radiation from a given source simultaneously. Each pair of antennas, with a specific baseline, act as an interferometer in which the output signals are cross-correlated. Correcting for a few systematics, the cross-correlations lead to what is known as the "Visibility function," which is a complex-valued function of the baseline vector, $V(\vec{b})$, where $\vec{b}$ is the projection of the baseline onto the sky plane. An image of the source is then obtained via the Fourier transform of the Visibility function,

$$I(x, y) = FT[V(\vec{b})],$$

where x and y are angles on the sky and FT indicates the Fourier transform of the argument inside the square brackets, [ ]. In the observation of some source, the Visibility for any given baseline vector contains an amplitude and a phase and so is represented by

$$V(\vec{b}) = Ae^{-i\phi}, \tag{1}$$

where the real part is given by Acos(ϕ) and the imaginary part is Asin(ϕ).

For these labs, we simplify the discussion and consider just the two-dimensional situation in which the source structure and the antennas are located in a single plane, in which case the baselines and the source structure are each one-dimensional.

For a single point source located an angular distance θ (in radians) from the center of the map (called the "phase center"), $A = E_0^2$ and $\phi = \frac{2\pi b \sin(\theta)}{\lambda}$, where b is the baseline distance. For a number of point sources the visibilities due to all the individual sources simply add. For N sources, the total Visibility for baseline b is

$$V_T(b) = Ae^{i\phi} = \sum_{k=1}^{N} A_k e^{i\phi_k} = \sum_{k=1}^{N} E_k^2 \exp(-i\frac{2\pi b \sin\theta_k}{\lambda}) \tag{2}$$

The relation between the magnitude of any particular Visibility and the Visibilities due to each source is given as the square root of the sum of the squares of the total real part and the total imaginary part, i.e.

$$A = \sqrt{\left(\sum_{k=1}^{N} A_k \cos\phi_k\right)^2 + \left(\sum_{k=1}^{N} A_k \sin\phi_k\right)^2}, \tag{3}$$

which is equivalent to

$$A = \sqrt{\sum_{k=1}^{N} A_k^2 \cos^2\phi_k + \sum_{k=1}^{N}\sum_{l>k}^{N} 2A_k A_l \cos\phi_k \cos\phi_l + \sum_{k=1}^{N} A_k^2 \sin^2\phi_k + \sum_{k=1}^{N}\sum_{l>k}^{N} 2A_k A_l \sin\phi_k \sin\phi_l}. \tag{4}$$

Using the trig identities

$$\cos^2 A + \sin^2 A = 1 \tag{5}$$

and

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \tag{6}$$

we can be simplify Equation 4 to

$$A = \sqrt{\sum_{k=1}^{N} A_k^2 + \sum_{k=1}^{N}\sum_{l>k}^{N} 2A_k A_l \cos(\phi_k - \phi_l)}. \tag{7}$$

The phase of the total Visibility can be obtained from the arctan of the total imaginary part divided by the total real part. The VSRT interferometer does not measure the phase, though, and so we do not need to derive the phase equation here.

Since the Visibility function is the inverse Fourier transform of the image, the shape of the Visibility function is determined by the structure of the source. Even without knowledge of the phases, the magnitudes of the Visibility function contain much significant information about the structural aspects of the source. For example, if observing a pair of equally bright point sources, the magnitudes will oscillate with a wavelength that is inversely proportional to the angular distance between the point sources. More specifically,

$$\frac{\bar{b}}{\lambda} = \frac{1}{\Delta\theta} \tag{8}$$

where $\bar{b}$ is the wavelength of the Visibility oscillations and $\Delta\theta$ is the angular separation of the point sources.

And, if observing a single but resolved source, the magnitudes will be a decreasing function with baseline length and the rate of decrease will be inversely proportional to the angular size of the source. For a source with a Gaussian brightness profile, for example, the magnitudes will also decrease with a Gaussian profile, and the half-maximum width of the magnitudes as a function of baseline will be inversely proportional to the half-maximum width of the source brightness distribution.

II. THE VSRT INTERFEROMETER

Information about the components and assembly of the VSRTs can be found at http://www.haystack.mit.edu/edu/undergrad/VSRT/index.html.

The VSRTs come with satellite TV dishes, which are needed for observations of the Sun, but for the labs in the classroom the dishes are not needed and so in these labs the feeds act as the antennas. In the following, then, the words "feeds" and "antennas" are interchangeable.

A. How the VSRT Interferometer Works

The VSRT interferometer differs from modern interferometers used by radio astronomers today in two significant ways. First, the signals are added instead of cross correlated (i.e., it is an additive interferometer) and, secondly, the two feeds involve different mix-down (or LO) frequencies. Therefore, when the signals are combined a beat signal (with frequency about

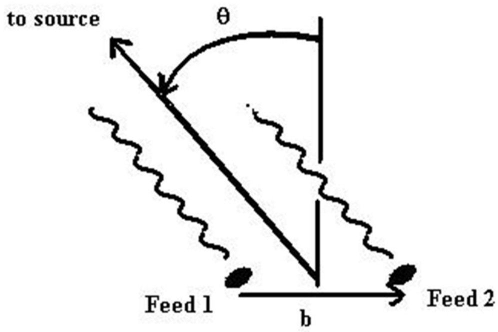

FIG. 1: The feeds (acting as antennas) separated by a distance b recieve tadiation from the source at position angle θ relative to the midplane position.

500 kHz) results. The end result, however, is that the response of the VSRT interferometer mimics, in many ways, a standard cross-correlation interferometer, as we show below.

First consider the radiation from a single source entering the two feeds and the detected power that exits the square-law detector. We'll denote the baseline distance between the feeds as 'b,' and, assuming that the distance of the source is much larger than b, we assign the position of the source by its direction angle, θ, relative to the mid-plane between the feeds. Figure 1 shows the arrangement.

The electric field entering each feed is

$$E_a = E_0 \ \cos(2\pi\nu t) \text{ and } E_b = E_0 \ \cos(2\pi\nu t - \phi), \tag{9}$$

where the phase difference, ϕ, is due to the extra path length to the second antenna and is given by

$$\phi = 2\pi\frac{b\sin\theta}{\lambda}. \tag{10}$$

The induced voltages (which are proportional to the electric fields of the incident waves) are then mixed down with frequencies ν_a and ν_b, added, and squared, yielding

$$V_T^2 = V_0^2(\cos^2[2\pi(\nu-\nu_a)t]+2\cos[2\pi(\nu-\nu_a)t]\cos[2\pi(\nu-\nu_b)t-\phi]+\cos^2[2\pi(\nu-\nu_b)t-\phi]). \tag{11}$$

The output signal we detect is the beat signal, which occurs because of the middle term. So, ignoring the first and third terms, and using 2cosAcosB = cos(A+B) + cos(A-B), we get

$$V_T^2 = V_0^2(\cos[2\pi(2\nu - (\nu_a + \nu_b)t - \phi] + \cos[2\pi(\nu_b - \nu_a)t + \phi]) \tag{12}$$

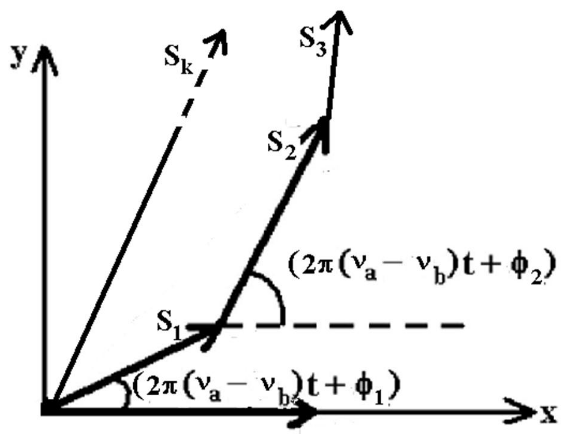

FIG. 2: Vectors of magnitudes S_k and phase angles relative to the x-axis of $(2\pi(\nu_b - \nu_a)t + \phi_k)$ are added to make a total vector of magnitude S_T.

Only the second term passes through the low-pass filter, and so we find that the detected signal is

$$V_T^2 = V_0^2 \cos[2\pi(\nu_b - \nu_a)t + \phi]. \tag{13}$$

This, now is, simply, the beat signal. For simplicity of notation, we will write the amplitude of the beat signal as 'S', and so the last equation becomes

$$S = S_0 \cos[2\pi(\nu_b - \nu_a)t + \phi]. \tag{14}$$

Now consider the total power in the beat signal when there are N sources. In general, the sources are incoherent and so we must add the powers (which are proportional to V^2). The total power in the beat signals of N sources, then, is given by

$$S_T = \sum_{k=1}^{N} S_k \cos[2\pi(\nu_b - \nu_a)t + \phi_k], \tag{15}$$

where S_k is the power in the beat signal from the kth source and ϕ_k is the phase delay to the second feed from the kth source. Note that we now have the sum of many numbers each multiplied by a cosine term. This is mathematically identical to summing the x-components of a series of two-dimensional vectors, with magnitudes of S_k and angles relative to the x-axis of $(2\pi(\nu_b - \nu_a)t + \phi_k)$, as represented by Figure 2.

And, this sum, then, must equal the x-component of the resultant vector. Now, since the angles of all these vectors depend on time identically, they all rotate in the x-y plane together at angular frequency $2\pi(\nu_b - \nu_a)$, and so the resultant vector also rotates at this frequency. Therefore, by a suitable choice of our t = 0 time, we can define the phase of the resultant vector to be zero and so that

$$S_T = S_0 \cos[2\pi(\nu_b - \nu_a)t], \tag{16}$$

and we can find S_0 by calculating the magnitude of the resultant vector, i.e.

$$\begin{aligned} S_0 &= \sqrt{\left(\sum \text{x} - \text{components}\right)^2 + \left(\sum \text{y} - \text{components}\right)^2} \\ &= \sqrt{\left(\sum_{k=1}^{N} S_k \cos[2\pi(\nu_b - \nu_a)t + \phi_k]\right)^2 + \left(\sum_{k=1}^{N} S_k \sin[2\pi(\nu_b - \nu_a)t + \phi_k]\right)^2}. \end{aligned} \tag{17}$$

Expanding, we get

$$\begin{aligned} S_0 = \Bigg(&\sum_{k=1}^{N} S_k^2 \cos^2[2\pi(\nu_b - \nu_a)t + \phi_k] + \sum_{k=1}^{N} S_k^2 \sin^2[2\pi(\nu_b - \nu_a)t + \phi_k] \\ &+ \sum_{k=1}^{N}\sum_{l>k}^{N} 2S_k S_l \cos[2\pi(\nu_b - \nu_a)t + \phi_k] \cos[2\pi(\nu_b - \nu_a)t + \phi_l] \\ &+ \sum_{l>k} 2S_k S_l \sin[2\pi(\nu_b - \nu_a)t + \phi_k] \sin[2\pi(\nu_b - \nu_a)t + \phi_l]\Bigg)^{1/2}. \end{aligned} \tag{18}$$

Again using the trig identities in Equations 5 and 6 this can be simplified to

$$S_0 = \sqrt{\sum_{k=1}^{N} S_k^2 + \sum_{k=1}^{N}\sum_{l>k}^{N} 2S_k S_l \cos(\phi_k - \phi_l)}. \tag{19}$$

Substituting this back into Equaton 16 we find that the total output signal is given by

$$S_T = \left(\sqrt{\sum_{k=1}^{N} S_k^2 + \sum_{k=1}^{N}\sum_{l>k}^{N} 2S_k S_l \cos(\phi_k - \phi_l)}\right) \cos(2\pi(\nu_b - \nu_a)t). \tag{20}$$

Note that this, again, is the beat signal, but with an amplitude that is modified by the factor with the radical. Note also that the amplitude in Equation 20 is identical to Equation 7. We see, therefore, that the power of the beat signal with the VSRT interferometer is identical to the magnitude of the complex visibility for that baseline for any arrangement of sources.

With a professional radio astronomy antenna array, the magnitudes and phases of the Visibilities are obtained and so a two-dimensional Fourier transform is applied to yield an image of the source. With the VSRT interferometer, though, the phases are not measured, and so one must analyze the shape of the magnitudes as a function of baseline.