! For an efficient use of these tables, first read `HowTo.pdf`.

T2.63D. Integrands involving logarithm functions of complicated arguments, like $\big(1+a^2/x^2\big)$, $(1 \pm e^{-x})$, $\big(1+2e^{-x}\cos t+e^{-2x}\big)$ and others, on the intervals $(0,\pi/2)$ and $(\pi/4,\pi/2)$.

1. $$\int_0^{\pi/2} \ln \sin x\,dx = \frac{1}{2}\int_0^{\pi} \ln \sin x\,dx = -\frac{\pi}{2}\ln 2..$$

2. $$\int_0^{\pi/2} \ln \cos x\,dx = -\frac{\pi}{2}\ln 2.$$

3. $$\int_0^{\pi/2} (\ln \sin x)^2\,dx = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right].$$

4. $$\int_0^{\pi/2} (\ln \cos x)^2\,dx = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right].$$

5. $$\int_0^{\pi/2} \ln\,(1+a\sin x)^2\,dx = \begin{cases} \pi\,\ln(a/2) + 4\mathbf{G} + 4\displaystyle\sum_{k=1}^{\infty}\frac{bk}{k}\sum_{n=1}^{k}\frac{(-1)^{n+1}}{2n-1}, & a>0, \quad b=\dfrac{(1-a)}{(1+a)}, \\ = -\pi\,\ln 2 - 4\mathbf{G}, & a=-1. \end{cases}$$

6. $$\int_0^{\pi/2} \ln\,(1+2a\sin x + a^2)\,dx = \sum_{k=0}^{\infty}\frac{2^{2k}(k!)^2}{(2k+1)\cdot(2k+1)!!}\left(\frac{2a}{1+a^2}\right)^{2k+1}, \quad a^2 \le 1.$$

7. $$\int_0^{\pi/2} \ln\,(a^2-\sin^2 x)^2 dx = \begin{cases} -2\pi\ln 2, & a^2\le 1, \\ 2\pi\ln\,\dfrac{a+\sqrt{a^2-1}}{2} = 2\pi(\operatorname{arccosh}\ a - \ln 2), & a>1. \end{cases}$$

8. $$\int_0^{\pi/2} \ln\,(1+a\sin^2 x)\,dx = \frac{1}{2}\int_0^{\pi} \ln\,(1+a\sin^2 x)\,dx = \int_0^{\pi/2} \ln\,(1+a\cos^2 x)\,dx$$
$$= \frac{1}{2}\int_0^{\pi} \ln\,(1+a\cos^2 x)\,dx = \pi\ln\,\frac{1+\sqrt{1+a}}{2}, \quad a \ge -1.$$

9. $$\int_0^{\pi/2} \ln\left[1-\cos^2 x(\sin^2\alpha-\sin^2\beta\sin^2 x)\right] dx = \pi \ln \left[\frac{1}{2}\left(\cos^2\frac{\alpha}{2} + \sqrt{\cos^4\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2}}\right)\right],$$
$\alpha > \beta > 0.$

10. $$\int_0^{\pi/2} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right) dx = \frac{1}{2}\int_0^{\pi} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right) dx = \pi\ln\frac{a+b}{2},$$
$a > 0,\ b > 0.$

11. $$\int_0^{\pi/2} \ln\frac{1+\sin t\cos^2 x}{1-\sin t\cos^2 x}\, dx = \pi\ln\frac{1+\sin(t/2)}{\cos(t/2)} = \pi\ln\cot\frac{\pi-t}{4}, \quad |t| < \frac{\pi}{2}.$$

12. $$\int_0^{\pi/2} \ln\left(a\tan x\right) dx = \frac{\pi}{2}\ln a, \quad a > 0.$$

13. $$\int_0^{\pi/2} (\ln\tan x)^{2n}\, dx = 2(2n)!\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}\left(\frac{\pi}{2}^{2n+1}|E_{2n}|\right).$$

14. $$\int_0^{\pi/2} (\ln\tan x)^{2n+1}\, dx = 0.$$

15. $$\int_0^{\pi/2} \ln\left(1+\tan x\right) dx = \frac{\pi}{4}\ln 2 + \mathbf{G}.$$

16. $$\int_0^{\pi/2} \ln\left(1-\tan x\right) dx = \frac{\pi}{8}\ln 2 - \mathbf{G}.$$

17. $$\int_0^{\pi/2} \ln\left(a^2 + b^2\tan^2 x\right) dx = \pi\ln(a+b), \quad a > 0,\ b > 0.$$

18. $$\int_0^{\pi/2} \ln\left(\sin t\sin x + \sqrt{1-\cos^2 t\sin^2 t}\right) dx = \frac{\pi}{2}\ln 2 - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi-t}{2}\right).$$

19. $$\int_0^{\pi/2} \sqrt{\ln\cot x}\, dx = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{(2k+1)^3}}.$$

20. $$\int_0^{\pi/2} \ln\left(\cos x + \sin x\right) dx = -\frac{\pi}{4}\ln 2 + \mathbf{G}.$$

21. $\displaystyle\int_0^{\pi/2} \ln(\tan x + \cot x)\, dx = \pi \ \ln\ 2.$

22. $\displaystyle\int_0^{\pi/4} \ln(\cot x - \tan x)\, dx = \frac{1}{2} \int_0^{\pi/2} \ln^2(\cot x - \tan x)\, dx = \frac{\pi}{2}\ \ln\ 2.$

23. $\displaystyle\int_0^{\pi/2} \ln\left(\sqrt{\tan x} + \sqrt{\cot x}\right)\, dx = \frac{\pi}{4}\ \ln\ 2 + \mathbf{G}.$

24. $\displaystyle\int_0^{\pi/2} \ln^2\left(\sqrt{\tan x} - \sqrt{\cot x}\right)\, dx = \frac{\pi}{2}\ \ln\ 2 - 2\mathbf{G}.$

25. $\displaystyle\int_{\pi/4}^{\pi/2} \ln\ \tan x\, dx = \mathbf{G}.$

26. $\displaystyle\int_{\pi/4}^{\pi/2} \ln\ \ln\ \tan x\, dx = \frac{\pi}{2}\ \ln\left\{\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{2\pi}\right\}.$
