T1.20. Integrand involving exponential functions.

Note that $a^x \stackrel{\text{def}}{=} e^{x \ln a}$.

1. $\displaystyle\int e^x \, dx = e^x.$

2. $\displaystyle\int e^{-x} \, dx = -\, e^{-x}.$

3. $\displaystyle\int e^{ax} \, dx = \frac{e^{ax}}{a}.$

4. $\displaystyle\int \frac{dx}{a + be^{mx}} = \frac{1}{am} \left[mx - \ln(a + be^{mx}) \right].$

5. $\displaystyle\int \frac{dx}{1 + e^x} = \ln \frac{e^x}{1 + e^x} = x - \ln(1 + e^x).$

6. $$\int \frac{dx}{ae^{mx} + be^{-mx}} = \begin{cases} \dfrac{1}{m\sqrt{ab}} \arctan\left(e^{mx} \sqrt{\dfrac{a}{b}} \right), & ab > 0, \\ \dfrac{1}{2m\sqrt{-ab}} \ln \dfrac{e^{mx}\sqrt{-ab} - |b|}{e^{mx}\sqrt{-ab} + |b|}, & ab < 0. \end{cases}$$

7. $$\int \frac{dx}{ae^{mx} - be^{-mx}} = \begin{cases} \dfrac{1}{2m\sqrt{ab}} \ln \left(\dfrac{\sqrt{a}\, e^{mx} - \sqrt{b}}{\sqrt{a}\, e^{mx} - \sqrt{b}} \right), & a > 0,\ b > 0, \\ \dfrac{-1}{2m\sqrt{-ab}} \operatorname{arctanh}\left(\sqrt{\dfrac{a}{b}}\, e^{mx} \right), & a > 0,\ b < 0. \end{cases}$$

8. $$\int \frac{dx}{\sqrt{a + be^{mx}}} = \begin{cases} \dfrac{1}{m\sqrt{a}} \ln \dfrac{\sqrt{a + be^{mx}} - \sqrt{a}}{\sqrt{a + be^{mx}} + \sqrt{a}}, & a > 0, \\ \dfrac{2}{m\sqrt{-a}} \arctan \dfrac{\sqrt{a + be^{mx}}}{\sqrt{-a}}, & a < 0. \end{cases}$$

9. $$\int x^m e^{ax} \, dx = \begin{cases} \dfrac{x^m e^{ax}}{a} - \dfrac{m}{a} \displaystyle\int x^{m-1} e^{ax} \, dx, \\ \text{or} \\ e^{ax} \displaystyle\sum_{k=0}^{m} (-1)^k \frac{m!\, x^{m-k}}{(m-1)!\, a^{k+1}}. \end{cases}$$

10. $$\int x^n e^{ax}\,dx = e^{ax}\left(\frac{x^n}{a} + \sum_{k=1}^{n}(-1)^k \frac{n(n-1)\dots(n-k+1)}{a^{k+1}}x^{n-k}\right)$$
$$= e^{ax}\left[\sum_{k=0}^{n}\frac{(-1)^k\, k!\binom{n}{k}}{a^{k+1}}x^{n-k}\right].$$

11. $$\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$$

12. $$\int x^2 e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right).$$

13. $$\int x^3 e^{ax}\,dx = e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right).$$

14. $$\int x^4 e^{ax}\,dx = e^{ax}\left(\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{24}{a^5}\right).$$

15. $$\int P_m(x)e^{ax}\,dx = \frac{e^{ax}}{a}\sum_{k=0}^{m}(-1)^k\frac{P^{(k)}(x)}{a^k},$$ where $P_m(x)$ is a polynomial in x of degree m, and

$P^{(k)}(x)$ is the k-th derivative of $P_m(x)$ with respect to x.

16. $$\int \frac{e^{ax}}{x}\,dx = \operatorname{Ei}(ax) = \ln x + \frac{ax}{1!} + \frac{a^2x^2}{2\cdot 2!} + \frac{a^3x^3}{3\cdot 3!} + \cdots.$$

17. $$\int \frac{e^{ax}}{x^2}\,dx = -\frac{e^{ax}}{x} + a\operatorname{Ei}(ax).$$

18. $$\int \frac{e^{ax}}{x^3}\,dx = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2}\operatorname{Ei}(ax).$$

19. $$\int \frac{e^{ax}\,dx}{x^m} = \frac{1}{m-1}\left[-\frac{e^{ax}}{x^{m-1}} + a\int\frac{e^{ax}\,dx}{x^{m-1}}\right].$$

20. $$\int \frac{e^{ax}}{x^n}\,dx = -e^{ax}\sum_{k=1}^{n-1}\frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!}\operatorname{Ei}(ax).$$

21. $$\int \frac{xe^{ax}\,dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}.$$

22. $$\int e^{-(ax^2+2bx+c)}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\exp\left(\frac{b^2-ac}{a}\right)\operatorname{erf}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right), \quad a \neq 0.$$

23. $\displaystyle\int \left(a^x - a^{-x}\right)\, dx = \frac{a^x + a^{-x}}{\ln\, a}.$

24. $\displaystyle\int x\, e^{-x^2}\, dx = -\frac{1}{2}\, e^{-x^2}.$