! For an efficient use of these tables, first read `HowTo.pdf`.

T2.45C. Integrands involving powers of linear trigonometric functions on the interval $(0, \pi)$.

1. $$\int_0^{\pi} (a + b\,\cos x)^n\,dx = \frac{1}{2}\int_0^{2\pi} (a + b\,\cos x)^n\,dx = \pi (a^2 - b^2)^{n/2} P_n\left(\frac{a}{\sqrt{a^2 - b^2}}\right)$$
$$= \frac{\pi}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n - 2k)!}{k!(n-k)!(n-2k)!} a^{n-2k} (a^2 - b^2)^k, \quad a^2 > b^2.$$

2. $$\int_0^{\pi} \frac{dx}{(a + b\,\cos x)^{n+1}} = \frac{1}{2}\int_0^{2\pi} \frac{dx}{(a + b\,\cos x)^{n+1}} = \frac{\pi}{(a^2 - b^2)^{(n+1)/2}}\, P_n\left(\frac{a}{\sqrt{a^2 - b^2}}\right)$$
$$= \frac{\pi}{2^n (a+b)^n \sqrt{a^2 - b^2}} \sum_{k=0}^{n} \frac{(2n - 2k - 1)!!(2k-1)!!}{(n-k)!k!} \left(\frac{a+b}{a-b}\right)^k, \quad a > |b|.$$

3. $$\int_0^{\pi} (z + \sqrt{z^2 - 1}\,\cos x)^q\,dx = \pi P_q(z),$$
$$\Re\{z\} > 0,\ \arg(z + \sqrt{z^2 - 1}\,\cos x) = \arg z \text{ for } x = \pi/2.$$

4. $$\int_0^{\pi} \frac{dx}{(z + \sqrt{z^2 - 1}\,\cos x)^q} = \pi P_{q-1}(z),$$
$$\Re\{z\} > 0,\ \arg\left(z + \sqrt{z^2 - 1}\,\cos x\right) = \arg z \text{ for } x = \pi/2.$$

5. $$\int_0^{\pi} (z + \sqrt{z^2 - 1}\,\cos x)^q\,\cos nx\,dx = \frac{\pi}{(q+1)(q+2)\ldots(q+n)} P_q^{(n)}(z),$$
$$\Re\{z\} > 0,\ \arg\left(z + \sqrt{z^2 - 1}\,\cos x\right) = \arg z \text{ for } x = \pi/2; \quad z \text{ lies outside the interval } (-1, 1).$$

6. $$\int_0^{\pi} \left(z + \sqrt{z^2 - 1}\,\cos x\right)^{\mu} \sin^{2\nu - 1}\, x\,dx \frac{2^{2\nu - 1}\Gamma(\mu + 1)[\Gamma(\nu)]^2}{\Gamma(2\nu + \mu)} C_{\mu}^{\nu}(z)$$
$$\frac{\sqrt{\pi}\Gamma(\nu)\Gamma(2\nu)\Gamma(\mu + 1)}{\Gamma(2\nu + \mu)\Gamma\left(\nu + \frac{1}{2}\right)} C_{\mu}^{\nu}(z) = 2^{\nu}\sqrt{\frac{\pi}{2}}\, \left(z^2 - 1\right)^{1/4 - \nu/2} \Gamma(\nu)\, P_{\mu + \nu - 1/2}^{(1/2 - \nu)}(z), \quad \Re\{\nu\} > 0.$$

7. $$\int_0^\pi \frac{\sin^{\mu-1}\, x\,dx}{(a+b\,\cos x)^\mu} = \frac{2^{\mu-1}}{\sqrt{(a^2-b^2)^\mu}}\,\mathrm{B}\left(\frac{\mu}{2},\,\frac{\mu}{2}\right), \quad \Re\{\mu\}>0,\ 0<b<a].$$

8. $$\int_0^\pi \frac{\sin^{2\mu-1}\, x\,dx}{(1+2a\,\cos x+a^2)^\nu} = \mathrm{B}\left(\mu,\,\frac{1}{2}\right) F\left(\nu,\,\nu-\mu+\frac{1}{2};\,\mu+\frac{1}{2};\,a^2\right), \quad \Re\{\mu\}>0,\ |a|<1.$$

9. $$\int_0^\pi (\beta+\cos x)^{\mu-\nu-1/2}\,\sin^{2\nu}\, x\,dx = \frac{2^{\nu+1/2}e^{-i\,\mu\pi}(\beta^2-1)^{\mu/2}\Gamma\left(\nu+\frac{1}{2}\right)Q^{(\mu)}_{\nu-1/2}(\beta)}{\Gamma\left(\nu+\mu+\frac{1}{2}\right)},$$
$$\Re\{\nu+\mu+\tfrac{1}{2}\}>0,\ \Re\{\nu\}>-\tfrac{1}{2}.$$

10. $$\int_0^\pi (\cosh\beta+\sinh\beta\,\cos x)^{\mu+\nu}\,\sin^{-2\nu}\, x\,dx = \frac{\sqrt{\pi}}{2^\nu}\,\sinh^\nu(\beta)\,\Gamma\left(\frac{1}{2}-\nu\right)\,P^{(\nu)}_\mu(\cosh\beta),$$
$$\Re\{\nu\}<\tfrac{1}{2}.$$

11. $$\int_0^\pi (\cos t+i\,\sin t\,\cos x)^\mu\,\sin^{2\nu-1}\, x\,dx = 2^{\nu-1/2}\sqrt{\pi}\,\sin^{1/2-\nu}\, t\,\Gamma(\nu)\,P^{(1/2-\nu)}_{\mu+\nu-1/2}(\cos t),$$
$$\Re\{\nu\}>0,\ t^2<\pi^2.$$

12. $$\int_0^\pi \sqrt{a\pm b\,\cos\theta}\,d\theta = \int_{-\pi/2}^{\pi/2}\sqrt{a\pm b\,\sin\theta}\,d\theta = 2\sqrt{a+b}\,K\left(\sqrt{\frac{2b}{a+b}}\right), \quad a>b>0.$$

13. $$\int_0^\pi \frac{d\theta}{\sqrt{a\pm b\,\cos\theta}} = \int_{-\pi/2}^{\pi/2}\frac{d\theta}{\sqrt{a\pm b\,\cos\theta}} = \frac{2}{\sqrt{a+b}}\,E\left(\sqrt{\frac{2b}{a+b}}\right), \quad a>b>0.$$
