! For an efficient use of these tables, first read `HowTo.pdf`.

T2.72B. Integrands involving logarithms and exponentials on the interval $(0, \pi/4)$.

1. $$\int_0^{\pi/4} \ln\sin x \frac{\sin^{2n} x}{\cos^{2n+2} x}\,dx = \frac{1}{2n+1}\left[\frac{1}{2}\ln 2 + (-1)^n \frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1}\right].$$

2. $$\int_0^{\pi/4} \ln\sin x \frac{\sin^{2n-1} x}{\cos^{2n+1} x}\,dx = \frac{1}{4n}\left[-\ln 2 + (-1)^n \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^k}{n-k}\right].$$

3. $$\int_0^{\pi/4} \ln\cos x \frac{\sin^{2n} x}{\cos^{2n+2} x}\,dx = \frac{1}{2n+1}\left[-\frac{1}{2}\ln 2 + (-1)^{n+1} \frac{\pi}{4} + \sum_{k=0}^{n} \frac{(-1)^{k-1}}{2n-2k+1}\right].$$

4. $$\int_0^{\pi/4} \ln\cos x \frac{\sin^{2n-1} x}{\cos^{2n+1} x}\,dx = \frac{1}{4n}\left[-\ln 2 + (-1)^n \ln 2 + \sum_{k=0}^{n-1} \frac{(-1)^k}{n-k}\right].$$

5. $$\int_0^{\pi/4} \ln\sin x\,\cos^n 2x \sin 2x\,dx = -\frac{1}{4(n+1)}\left[\gamma_e + \psi(n+2) + \ln 2\right].$$

6. $$\int_0^{\pi/4} \ln\cos x\,\cos^{\mu-1} 2x\,\tan 2x\,dx = \frac{1}{4(1-\mu)}\beta(\mu), \quad \Re\{\mu\} > 0.$$

7. $$\int_0^{\pi/4} (\ln\cos 2x)^n \cos^{p-1} 2x\,\tan x\,dx = \int_0^{\pi/4} (\ln\sin 2x)^n \sin^{p-1} 2x\,\tan\left(\frac{\pi}{4} - x\right)dx$$
$$= \frac{1}{2}\beta^{(n)}(p), \quad p > 0.$$

8. $$\int_0^{\pi/4} (\ln\sin 2x)^n \sin^{p-1} 2x\,\tan\left(\frac{\pi}{4} + x\right)dx = \frac{(-1)^n\,n!}{2}\,\zeta(n+1,\,p).$$

9. $$\int_0^{\pi/4} (\ln\cos 2x)^{2n-1}\,\tan x\,dx = \frac{1-2^{2n-1}}{4n}\pi^{2n}|B_{2n}|, \quad n = 1, 2, \ldots.$$

10. $\displaystyle\int_0^{\pi/4} (\ln\cos\, 2x)^{2n}\ \tan x\, dx = \frac{2^{2n}-1}{2^{2n+1}}(2n)!\,\zeta(2n+1).$

11. $\displaystyle\int_0^{\pi/4} \ln(\sin x\cos x)\frac{\sin^{2n} x}{\cos^{2n+2}\, x}\, dx = \frac{1}{2n+1}\left[(-1)^{n+1}\frac{\pi}{2} - \ln 2 + \frac{1}{2n+1} + 2\sum_{k=0}^{n-1}\frac{(-1)^{k-1}}{2n-2k-1}\right].$

12. $\displaystyle\int_0^{\pi/4} \ln(\sin x\cos x)\frac{\sin^{2n-1} x}{\cos^{2n+1}\, x}\, dx = \frac{1}{2n}\left[(-1)^n\ \ln 2 - \ln 2 + \frac{1}{2n} + (-1)^n\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right].$

13. $\displaystyle\int_0^{\pi/4} \frac{\ln\tan\, x}{\cos 2x}\, dx = -\frac{\pi^2}{8}.$

14. $\displaystyle\int_0^{\pi/4} \ln\tan\, x\frac{\cos 2x\, dx}{1 - a\sin 2x} = -\frac{\arcsin a}{4a}(\pi + \arcsin a), \quad a^2 \le 1.$

15. $\displaystyle\int_0^{\pi/4} \ln\tan\, x\frac{\cos 2x\, dx}{1 - a^2\sin^2 2x} = -\frac{\pi}{4a}\ \arcsin a, \quad a^2 < 1.$

16. $\displaystyle\int_0^{\pi/4} \ln\tan\, x\frac{\cos 2x\, dx}{1 + a^2\sin^2 x} = -\frac{\pi}{4a}\ \operatorname{arcsinh}\, a = -\frac{\pi}{4a}\ln\left(a + \sqrt{1+a^2}\right), \quad a^2 < 1.$

17. $\displaystyle\int_0^{\pi/4} \frac{\ln\tan\, x\sin 2x\, dx}{1 - \cos^2 t\sin^2\, 2x} = \csc 2t\left[L\left(\frac{\pi}{2} - t\right) - \left(\frac{\pi}{2} - t\right)\ln 2\right].$

18. $\displaystyle\int_u^{\pi/4} \frac{\ln\tan\, x\sin 4x\, dx}{(\sin^2 u + \tan^2\, v\sin^2 2x)\sqrt{\sin^2 2x - \sin^2 u}} = -\frac{\pi}{2}\frac{\cos^2 v}{\sin u\sin v}\ln\frac{\sin v + \sqrt{1 - \cos^2 u\cos^2 v}}{\sin u(1+\sin v)},$

$$0 < u < \frac{\pi}{2},\ 0 < v < \frac{\pi}{2}.$$

19. $\displaystyle\int_0^{\pi/4} (\ln\tan\, x)^n\ \tan^p x\, dx = \frac{1}{2^{n+1}}B^{(n)}\left(\frac{p+1}{2}\right), \quad p > -1.$

20. $\displaystyle\int_0^{\pi/4} \ln\tan\, x\ \tan^{2n+1} x\, dx = \frac{(-1)^{n+1}}{4}\left[\frac{\pi^2}{12} + \sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right].$

21. $\displaystyle\int_0^{\pi/4} \ln\ \tan\left(\frac{\pi}{4} \pm x\right)\frac{dx}{\sin 2x} = \pm\frac{\pi^2}{8}.$

22. $\displaystyle\int_0^{\pi/4} \ln\,\tan\left(\frac{\pi}{4} \pm x\right) \frac{dx}{\tan 2x} = \pm\frac{\pi^2}{16}.$

23. $\displaystyle\int_0^{\pi/4} \ln\,\tan\left(\frac{\pi}{4} \pm x\right) (\ln\,\tan\,x)^{2n} \frac{dx}{\sin 2x} = \pm\frac{2^{2n+2}-1}{4(n+1)(2n+1)}\pi^{2n+2}|B_{2n+2}|.$

24. $\displaystyle\int_0^{\pi/4} \ln\,\tan\left(\frac{\pi}{4} \pm x\right) (\ln\,\tan\,x)^{2n-1} \frac{dx}{\sin 2x} = \pm\frac{1-2^{2n+1}}{2^{2n+2}n}(2n)!\,\zeta(2n+1).$

25. $\displaystyle\int_0^{\pi/4} \ln\,\tan\left(\frac{\pi}{4} \pm x\right) (\ln\,\sin\,2x)^{n-1} \frac{dx}{\tan 2x} = \frac{(-1)^{n-1}}{2}(n-1)!\,\zeta(n+1).$

26. $\displaystyle\int_0^{\pi/4} \ln\,\tan\,x(\ln\,\cos\,2x)^{n-1}\,\tan 2x\,dx = \frac{1}{2}(-1)^n(n-1)!(1-2^{-(n+1)})\,\zeta(n+1).$
