T1.19. Integrand involving $\sqrt{a+b\,x+c\,x^2}$ and first- and second-degree polynomials.

Notation used: $X = a + b\,x + c\,x^2$:

1. $$\int \frac{dx}{(x+p)^n\sqrt{X}} = -\int \frac{t^{n-1}\,dt}{\sqrt{c+(b-2pc)t+(a-bp+cp^2)t^2}}, \quad \text{where } t = \frac{1}{x+p} > 0.$$

2. $$\int \frac{\sqrt{X}\,dx}{x+p} = c\int \frac{x\,dx}{\sqrt{X}} + (b-cp)\int \frac{dx}{\sqrt{X}} + (a-bp+cp^2)\int \frac{dx}{(x+p)\sqrt{X}}, \quad x+p>0.$$

3. $$\int \frac{dx}{(x+p)(x+q)\sqrt{X}} = \frac{1}{q-p}\int \frac{dx}{(x+p)\sqrt{X}} + \frac{1}{p-q}\int \frac{dx}{(x+q)\sqrt{X}}.$$

4. $$\int \frac{\sqrt{X}\,dx}{(x+p)(x+q)} = \frac{1}{q-p}\int \frac{\sqrt{X}\,dx}{x+p} + \frac{1}{p-q}\int \frac{\sqrt{X}\,dx}{x+q}.$$

5. $$\int \frac{(x+p)\sqrt{X}\,dx}{x+q} = \int \sqrt{X}\,dx + (p-q)\int \frac{\sqrt{X}\,dx}{x+q}.$$

6. $$\int \frac{(rx+s)\,dx}{(x+p)(x+q)\sqrt{X}} = \frac{s-pr}{q-p}\int \frac{dx}{(x+p)\sqrt{X}} + \frac{s-qr}{p-q}\int \frac{dx}{(x+q)\sqrt{X}}.$$

7. $$\int \frac{(Ax+B)\,dx}{(p+X)^n\sqrt{X}} = \frac{A}{c}\int \frac{du}{(p+u^2)^n} + \frac{2Bc-Ab}{2c}\int \frac{(1-c\,w^2)^{n-1}\,dw}{\left[p+a-\frac{b^2}{4c}-cp\,w^2\right]^n},$$
   where $u = \sqrt{X}$ and $w = \dfrac{b+2cx}{2c\sqrt{X}}$.

8. $$\int \frac{Ax+B}{(p+X)\sqrt{X}}\,dx = \frac{A}{c}\,I_1 + \frac{2Bc-Ab}{\sqrt{c^2p[b^2-4(a+p)c]}}\,I_2,$$

$$\text{where } I_1 = \begin{cases} \dfrac{1}{\sqrt{p}}\arctan\sqrt{\dfrac{X}{p}}, & p>0, \\ \dfrac{1}{2\sqrt{-p}}\ln\dfrac{\sqrt{-p}-\sqrt{X}}{\sqrt{-p}+\sqrt{X}}, & p<0, \end{cases}$$

and

$$I_2 = \begin{cases} \arctan\sqrt{\dfrac{p}{b^2-4(a+p)c}}\,\dfrac{b+2cx}{\sqrt{X}}, & p\{b^2-4(a+p)c\}>0, \quad p<0, \\ -\arctan\sqrt{\dfrac{p}{b^2-4(a+p)c}}\,\dfrac{b+2cx}{\sqrt{X}}, & p\{b^2-4(a+p)c\}>0, \quad p>0, \\ \dfrac{1}{2i}\ln\dfrac{\sqrt{4(a+p)c-b^2}\sqrt{X}+\sqrt{p}(b+2cx)}{\sqrt{4(a+p)c-b^2}\sqrt{X}-\sqrt{p}(b+2cx)}, & p\{b^2-4(a+p)c\}<0, \quad p>0, \\ \dfrac{1}{2i}\ln\dfrac{\sqrt{b^2-4(a+p)c}\sqrt{X}-\sqrt{-p}(b+2cx)}{\sqrt{b^2-4(a+p)c}\sqrt{X}+\sqrt{-p}(b+2cx)}, & p\{b^2-4(a+p)c\}<0, \quad p<0. \end{cases}$$
