

**T1.18.** Integrand involving  $\sqrt{a+cx^2}$  and integral powers of  $x$ .

Notation used:  $X = \sqrt{a+cx^2}$ ,  $I_1 = \begin{cases} \frac{1}{\sqrt{c}} \ln(x\sqrt{c} + X), & c > 0, \\ \frac{1}{\sqrt{-c}} \arcsin x\sqrt{-\frac{c}{a}}, & c < 0 \text{ and } a > 0, \end{cases}$

and  $I_2 = \begin{cases} \frac{1}{2\sqrt{a}} \ln \frac{X - \sqrt{a}}{X + \sqrt{a}}, & a > 0 \text{ and } c > 0, \\ \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a} - X}{\sqrt{a} + X} & a > 0 \text{ and } c < 0, \\ \frac{1}{\sqrt{-a}} \operatorname{arcsec} x\sqrt{-\frac{c}{a}} \\ = \frac{1}{\sqrt{-a}} \arccos \frac{1}{x}\sqrt{-\frac{a}{c}}, & a < 0 \text{ and } c > 0. \end{cases}$

1.  $\int X \, dx = \frac{1}{2}xX + \frac{1}{2}aI_1.$

2.  $\int X^3 \, dx = \frac{1}{4}xX^3 + \frac{3}{8}axX + \frac{3}{8}a^2I_1.$

3.  $\int X^5 \, dx = \frac{1}{6}xX^5 + \frac{5}{24}axX^3 + \frac{5}{16}a^2xX + \frac{5}{16}a^3I_1.$

4.  $\int \frac{dx}{X} = I_1.$

5.  $\int \frac{dx}{X^3} = \frac{1}{a} \frac{x}{X}.$

6.  $\int \frac{dx}{X^{2n+1}} = \frac{1}{a^n} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \frac{c^k x^{2k+1}}{X^{2k+1}}.$

7.  $\int \frac{x \, dx}{X^{2n+1}} = -\frac{1}{(2n-1)cX^{2n-1}}.$

8.  $\int x^2 X \, dx = \frac{1}{4} \frac{xX^3}{c} - \frac{1}{8} \frac{axX}{c} - \frac{1}{8} \frac{a^2}{c} I_1.$

9.  $\int x^2 X^3 \, dx = \frac{1}{6} \frac{xX^5}{c} - \frac{1}{24} \frac{axX^3}{c} - \frac{1}{16} \frac{a^2xX}{c} - \frac{1}{16} \frac{a^3}{c} I_1.$

$$10. \int \frac{x^2}{X} dx = \frac{1}{2} \frac{xX}{c} - \frac{1}{2} \frac{a}{c} I_1.$$

$$11. \int \frac{x^2}{X^3} dx = -\frac{x}{cX} + \frac{1}{c} I_1.$$

$$12. \int \frac{x^2}{X^5} dx = \frac{1}{3} \frac{x^3}{aX^3}.$$

$$13. \int \frac{x^2 dx}{X^{2n+1}} = \frac{1}{a^{n-1}} \sum_{k=0}^{n-2} \frac{(-1)^k}{2k+3} \binom{n-2}{k} \frac{c^k x^{2k+3}}{X^{2k+3}}.$$

$$14. \int \frac{x^3 dx}{X^{2n+1}} = -\frac{1}{(2n-3)c^2 X^{2n-3}} + \frac{a}{(2n-1)c^2 X^{2n-1}}.$$

$$15. \int x^4 X dx = \frac{x^3 X^3}{6c} - \frac{a x X^3}{8c^2} + \frac{a^2 x X}{16c^2} + \frac{a^3}{16c^2} I_1.$$

$$16. \int x^4 X^3 dx = \frac{x^3 X^5}{8c} - \frac{a x X^5}{16c^2} + \frac{a^2 x X^3}{64c^2} + \frac{3a^3 x X}{128c^2} + \frac{3a^4}{128c^2} I_1.$$

$$17. \int \frac{x^4}{X} dx = \frac{x^3 X}{4c} - \frac{3 a x X}{8 c^2} + \frac{3 a^2}{8 c^2} I_1.$$

$$18. \int \frac{x^4}{X^3} dx = \frac{1}{2} \frac{xX}{c^2} + \frac{ax}{c^2 X} - \frac{3}{2} \frac{a}{c^2} I_1.$$

$$19. \int \frac{x^4}{X^5} dx = -\frac{x}{c^2 X} - \frac{1}{3} \frac{x^3}{c X^3} + \frac{1}{c^2} I_1.$$

$$20. \int \frac{x^4}{X^7} dx = \frac{1}{5} \frac{x^5}{a X^5}.$$

$$21. \int \frac{x^4 dx}{X^{2n+1}} = \frac{1}{a^{n-2}} \sum_{k=0}^{n-3} \frac{(-1)^k}{2k+5} \binom{n-3}{k} \frac{c^k x^{2k+5}}{X^{2k+5}}.$$

$$22. \int \frac{x^5 dx}{X^{2n+1}} = -\frac{1}{(2n-5)c^3 X^{2n-5}} + \frac{2a}{(2n-3)c^3 X^{2n-3}} - \frac{a^2}{(2n-1)c^3 X^{2n-1}}.$$

$$23. \int x^6 X dx = \frac{1}{8} \frac{x^5 X^3}{c} - \frac{5}{48} \frac{a x^3 X^3}{c^2} + \frac{5a^2 x X^3}{64c^3} - \frac{5a^3 x X}{128c^3} - \frac{5}{128} \frac{a^4}{c^3} I_1.$$

$$24. \int x^6 X^3 dx = \frac{1}{10} \frac{x^5 X^5}{c} - \frac{a x^3 X^5}{16c^2} + \frac{a^2 x X^5}{32c^3} - \frac{a^3 x X^3}{128c^3} - \frac{3a^4 x X}{256c^3} - \frac{3}{256} \frac{a^5}{c^3} I_1.$$

$$25. \int \frac{x^6}{X} dx = \frac{1}{6} \frac{x^5 X}{c} - \frac{5}{24} \frac{a x^3 X}{c^2} + \frac{5}{16} \frac{a^2 x X}{c^3} - \frac{5}{16} \frac{a^3}{c^3} I_1.$$

$$26. \int \frac{x^6}{X^3} dx = \frac{1}{4} \frac{x^5}{cX} - \frac{5}{8} \frac{a x^3}{c^2 X} - \frac{15}{8} \frac{a^2 x}{c^3 X} + \frac{15}{8} \frac{a^2}{c^3} I_1.$$

$$27. \int \frac{x^6}{X^5} dx = \frac{1}{2} \frac{x^5}{cX^3} + \frac{10}{3} \frac{a x^3}{c^2 X^3} + \frac{5}{2} \frac{a^2 x}{c^3 X^3} - \frac{5}{2} \frac{a}{c^3} I_1.$$

$$28. \int \frac{x^6}{X^7} dx = -\frac{23}{15} \frac{x^5}{cX^5} - \frac{7}{3} \frac{ax^3}{c^2X^5} - \frac{a^2x}{c^3X^5} + \frac{1}{c^3} I_1.$$

$$29. \int \frac{x^6}{X^9} dx = \frac{1}{7} \frac{x^7}{aX^7}.$$

$$30. \int \frac{x^6 dx}{X^{2n+1}} = \frac{1}{a^{n-3}} \sum_{k=0}^{n-4} \frac{(-1)^k}{2k+7} \binom{n-4}{k} \frac{c^k x^{2k+7}}{X^{2k+7}}.$$

$$31. \int \frac{x^7 dx}{X^{2n+1}} = -\frac{1}{(2n-7)c^4 X^{2n-7}} + \frac{3a}{(2n-5)c^4 X^{2n-5}} - \frac{3a^2}{(2n-3)c^4 X^{2n-3}} + \frac{a^3}{(2n-1)c^4 X^{2n-1}}.$$

$$32. \int \frac{X}{x} dx = X + aI_2.$$

$$33. \int \frac{X^3}{x} dx = \frac{X^3}{3} + aX + a^2 I_2.$$

$$34. \int \frac{X^5}{x} dx = \frac{X^5}{5} + \frac{1}{3} aX^3 + a^2 X + a^3 I_2.$$

$$35. \int \frac{dx}{xX} = I_2.$$

$$36. \int \frac{dx}{xX^{2n+1}} = \frac{1}{a^n} I_2 + \sum_{k=0}^{n-1} \frac{1}{(2k+1)a^{n-k} X^{2k+1}}.$$

$$37. \int \frac{X}{x^2} dx = -\frac{X}{x} + c I_1.$$

$$38. \int \frac{X^3}{x^2} dx = -\frac{X^3}{x} + \frac{3}{2} c xX + \frac{3}{2} a I_1.$$

$$39. \int \frac{X^5}{x^2} dx = -\frac{X^5}{x} + \frac{5}{4} c xX^3 + \frac{15}{8} ac xX + \frac{15}{8} a^2 I_1.$$

$$40. \int \frac{dx}{x^2 X^{2n+1}} = -\frac{1}{a^{n+1}} \left\{ \frac{X}{x} + \sum_{k=1}^n \frac{(-1)^{k+1}}{2k-1} \binom{n}{k} c^k \left( \frac{x}{X} \right)^{2k-1} \right\}.$$

$$41. \int \frac{X}{x^3} dx = -\frac{X}{2x^2} + \frac{c}{2} I_2.$$

$$42. \int \frac{X^3}{x^3} dx = -\frac{X^3}{2x^2} + \frac{3}{2} cX + \frac{3}{2} ac I_2.$$

$$43. \int \frac{X^5}{x^3} dx = -\frac{X^5}{2x^2} + \frac{5}{6} cX^3 + \frac{5}{2} acX + \frac{5}{2} a^2 c I_2.$$

$$44. \int \frac{dx}{x^3 X} = -\frac{X}{2ax^2} - \frac{c}{2a} I_2.$$

$$45. \int \frac{dx}{x^3 X^3} = -\frac{1}{2ax^2 X} - \frac{3c}{2a^2 X} - \frac{3c}{2a^2} I_2.$$

$$46. \int \frac{dx}{x^3 X^5} = -\frac{1}{2ax^2 X^3} - \frac{5}{6} \frac{c}{a^2 X^3} - \frac{5}{2} \frac{c}{a^3 X} - \frac{5}{2} \frac{c}{a^3} I_2.$$

$$47. \int \frac{X}{x^4} dx = -\frac{X^3}{3ax^3}.$$

$$48. \int \frac{X^3}{x^4} dx = -\frac{X^3}{3x^3} - \frac{cX}{x} + c I_1.$$

$$49. \int \frac{X^5}{x^4} dx = -\frac{aX^3}{3x^3} - \frac{2acX}{x} + \frac{c^2 x X}{2} + \frac{5}{2} ac I_1.$$

$$50. \int \frac{dx}{x^4 X^{2n+1}} = \frac{1}{a^{n+2}} \left\{ -\frac{X^3}{3x^3} + (n+1) \frac{cX}{x} + \sum_{k=2}^{n+1} \frac{(-1)^k}{2k-3} \binom{n+1}{k} c^k \left( \frac{x}{X} \right)^{2k-3} \right\}.$$

$$51. \int \frac{X}{x^5} dx = -\frac{X}{4x^4} - \frac{1}{8} \frac{cX}{ax^2} - \frac{1}{8} \frac{c^2}{a} I_2.$$

$$52. \int \frac{X^3}{x^5} dx = -\frac{X^3}{4x^4} - \frac{3}{8} \frac{cX^3}{ax^2} + \frac{3}{8} \frac{c^2 X}{a} + \frac{3}{8} c^2 I_2.$$

$$53. \int \frac{dx}{x^5 X} = -\frac{X}{4ax^4} + \frac{3}{8} \frac{cX}{a^2 x^2} + \frac{3}{8} \frac{c^2}{a^2} I_2.$$

$$54. \int \frac{dx}{x^5 X^3} = -\frac{1}{4ax^4 X} + \frac{5}{8} \frac{c}{a^2 x^2 X} + \frac{15}{8} \frac{c^2}{a^3 X} + \frac{15}{8} \frac{c^2}{a^3} I_2.$$

$$55. \int \frac{X^3}{x^6} dx = -\frac{X^5}{5ax^5}.$$

$$56. \int \frac{X}{x^6} dx = -\frac{X^3}{5ax^5} + \frac{2}{15} \frac{cX^3}{a^2 x^3}.$$

$$57. \int \frac{dx}{x^6 X} = \frac{1}{a^3} \left( -\frac{X^5}{5x^5} + \frac{2}{3} \frac{cX^3}{x^3} - \frac{c^2 X}{x} \right).$$

$$58. \int \frac{dx}{x^6 X^{2n+1}} = \frac{1}{a^{n+3}} \left\{ -\frac{X^5}{5x^5} + \frac{1}{3} \binom{n+2}{1} \frac{cX^3}{x^3} - \binom{n+2}{2} \frac{c^2 X}{x} + \sum_{k=3}^{n+2} \frac{(-1)^k}{2k-5} \binom{n+2}{k} c^k \left( \frac{x}{X} \right)^{2k-5} \right\}.$$


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