T1.31. Integrand involving rational functions of hyperbolic functions.

1. $$\int \frac{dx}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}} \ln \frac{a\tanh\frac{x}{2} - b + \sqrt{a^2+b^2}}{a\tanh\frac{x}{2} - b - \sqrt{a^2+b^2}}$$
$$= \frac{2}{\sqrt{a^2+b^2}} \operatorname{arctanh} \frac{a\tanh\frac{x}{2} - b}{\sqrt{a^2+b^2}}.$$

2. $$\int \frac{A+B\sinh x}{a+b\sinh x}\,dx = \frac{B}{b}x - \frac{aB-bA}{b}\int \frac{dx}{a+b\sinh x}.$$

3. $$\int \frac{A+B\sinh x}{(a+b\sinh x)^n}\,dx = \frac{aB-bA}{(n-1)(a^2+b^2)} \cdot \frac{\cosh x}{(a+b\sinh x)^{n-1}}$$
$$+\frac{1}{(n-1)(a^2+b^2)} \int \frac{(n-1)(aA+bB) + (n-2)(aB-bA)\sinh x}{(a+b\sinh x)^{n-1}}\,dx.$$

4. $$\int \frac{dx}{a+b\cosh x} = \begin{cases} \dfrac{1}{\sqrt{b^2-a^2}} \operatorname{arcsinh} \dfrac{b+a\cosh x}{a+b\cosh x}, & b^2 > a^2,\ x < 0, \\ -\dfrac{1}{\sqrt{b^2-a^2}} \operatorname{arcsinh} \dfrac{b+a\cosh x}{a+b\cosh x}, & b^2 > a^2,\ x > 0, \\ \dfrac{1}{\sqrt{a^2-b^2}} \ln \dfrac{a+b+\sqrt{a^2-b^2}\tanh\frac{x}{2}}{a+b-\sqrt{a^2-b^2}\tanh\frac{x}{2}}, & a^2 > b^2. \end{cases}$$

5. $$\int \frac{A+B\cosh x}{a+b\cosh x}\,dx = \frac{B}{b}x - \frac{aB-bA}{b}\int \frac{dx}{a+b\cosh x}.$$

6. $$\int \frac{A+B\cosh x}{(a+b\cosh x)^n}\,dx = \frac{aB-bA}{(n-1)(a^2-b^2)} \cdot \frac{\sinh x}{(a+b\cosh x)^{n-1}}$$
$$+\frac{1}{(n-1)(a^2-b^2)} \int \frac{(n-1)(aA-bB) + (n-2)(aB-bA)\cosh x}{(a+b\cosh x)^{n-1}}\,dx.$$

7. $$\int \frac{A+B\cosh x}{a+b\sinh x}\,dx = \frac{B}{b}\ln(a+b\sinh x) + A\int \frac{dx}{a+b\sinh x}.$$

8. $$\int \frac{A+B\sinh x}{a+b\cosh x}\,dx = \frac{B}{b}\ln(a+b\cosh x) + A\int \frac{dx}{a+b\cosh x}.$$

9. $$\int \frac{A+B\cosh x}{(a+b\sinh x)^n}\,dx = -\frac{B}{(n-1)b(a+b\sinh x)^{n-1}} + A\int \frac{dx}{(a+b\sinh x)^n}.$$

10. $$\int \frac{A+B\sinh x}{(a+b\cosh x)^n}\,dx = -\frac{B}{(n-1)b(a+b\cosh x)^{n-1}} + A\int \frac{dx}{(a+b\cosh x)^n}, \quad n \neq 1.$$

11. $$\int \frac{A + B\,\cosh\,x}{c + \cosh\,x}\,dx = Bx + (c\,A - B)\frac{\cosh\,x - c}{\sinh\,x}, \quad c = \pm 1.$$

12. $$\int \frac{A + B\,\cosh x}{(c + \cosh x)^n}\,dx$$
$$= \frac{B\,\sinh x}{(1-n)(c+\cosh x)^n} + \left(c\,A + \frac{n}{n-1}B\right)\frac{(n-1)!}{(2n-1)!!}\sinh\,x \sum_{k=0}^{n-1}\frac{(2n-2k-3)!!}{(n-k-1)!}\,\frac{c^n}{(c+\cosh\,x)^{n-k}},$$
$$c = \pm 1,\ n > 1.$$

13. $$\int \frac{dx}{\cosh a + \cosh x} = \operatorname{csch}\,a\left[\ln\cosh\frac{x+a}{2} - \ln\cosh\frac{x-a}{2}\right]$$
$$= 2\,\operatorname{csch}\,a\,\operatorname{arctanh}\left(\tanh\frac{x}{2}\tanh\frac{a}{2}\right).$$

14. $$\int \frac{dx}{\cos a + \cosh\,x} = 2\,\csc a\,\arctan\left(\tanh\frac{x}{2}\,\tan\frac{a}{2}\right).$$

15. $$\int \frac{\sinh x\,dx}{a\cosh x + b\sinh x} = \begin{cases} \dfrac{a\,\ln\cosh\left(x + \operatorname{arctanh}\frac{b}{a}\right) - bx}{a^2 - b^2}, & a > |b|, \\ \dfrac{bx - a\,\ln\sinh\left(x + \operatorname{arctanh}\frac{a}{b}\right)}{b^2 - a^2}, & b > |a|. \end{cases}$$

16. $$\int \frac{\sinh\,x\,dx}{\cosh\,x + \sinh\,x} = \frac{x}{2} + \frac{1}{4}e^{-2x}.$$

17. $$\int \frac{\sinh\,x\,dx}{\cosh\,x - \sinh\,x} = -\frac{x}{2} + \frac{1}{4}e^{2x}.$$

18. $$\int \frac{\cosh x\,dx}{a\,\cosh x + b\,\sinh x} = \begin{cases} \dfrac{ax - b\,\ln\cosh\left(x + \operatorname{arctanh}\frac{b}{a}\right)}{a^2 - b^2}, & a > |b|, \\ \dfrac{-ax + b\,\ln\sinh\left(x + \operatorname{arctanh}\frac{a}{b}\right)}{b^2 - a^2}, & b > |a|. \end{cases}$$

19. $$\int \frac{\cosh\,x\,dx}{\cosh\,x + \sinh\,x} = \frac{x}{2} - \frac{1}{4}e^{-2x}.$$

20. $$\int \frac{\cosh\,x\,dx}{\cosh\,x - \sinh\,x} = \frac{x}{2} + \frac{1}{4}e^{2x}.$$

21. $$\int \frac{dx}{(a\,\cosh x + b\,\sinh x)^n} = \begin{cases} \dfrac{1}{\sqrt{(a^2-b^2)^n}}\displaystyle\int \frac{dx}{\sinh^n\left(x + \operatorname{arctanh}\frac{b}{a}\right)}, & a > |b|, \\ \dfrac{1}{\sqrt{(b^2-a^2)^n}}\displaystyle\int \frac{dx}{\cosh^n\left(x + \operatorname{arctanh}\frac{a}{b}\right)}, & b > |a|. \end{cases}$$

22. $$\int \frac{dx}{a\,\cosh\,x + b\,\sinh\,x} = \begin{cases} \dfrac{1}{\sqrt{a^2-b^2}}\,\arctan\left|\sinh\left(x + \operatorname{arctanh}\dfrac{b}{a}\right)\right|, & a > |b|, \\ \dfrac{1}{\sqrt{b^2-a^2}}\,\ln\left|\tanh\dfrac{x + \operatorname{arctanh}\frac{a}{b}}{2}\right|, & b > |a|. \end{cases}$$

23. $$\int \frac{ax}{\cosh x + \sinh x} = -e^{-x} = \sinh x - \cosh x.$$

24. $$\int \frac{dx}{\cosh x - \sinh x} = e^{x} = \sinh x + \cosh x.$$

25. $$\int \frac{A + B\cosh x + C\sinh x}{(a + b\cosh x + c\sinh x)^n}\,dx$$

$$= \begin{cases} \dfrac{Bc - Cb + (Ac - Ca)\cosh x + (Ab - Ba)\sinh x}{(1-n)(a^2 - b^2 + c^2)(a + b\cosh x + c\sinh x)^{n-1}} + \dfrac{1}{(n-1)(a^2 - b^2 + c^2)} \\ \times \displaystyle\int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba)\cosh x - (n-2)(Ac - Ca)\sinh x}{(a + b\cosh x + c\sinh x)^{n-1}}\,dx, \quad a^2 + c^2 \neq b^2, \\ \dfrac{Bc - Cb - Ca\cosh x - Ba\sinh x}{(n-1)a(a + b\cosh x + c\sinh x)^n} + \left[\dfrac{A}{a} + \dfrac{n(Bb - Cc)}{(n-1)a^2}\right](c\cosh x + b\sinh x)\dfrac{(n-1)!}{(2n-1)!!} \\ \times \displaystyle\sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k} \frac{1}{(a + b\cosh x + c\sinh x)^{n-k}}, \qquad a^2 + c^2 = b^2. \end{cases}$$

26. $$\int \frac{A + B\cosh x + C\sinh x}{a + b\cosh x + c\sinh x}\,dx = \frac{Cb - Bc}{b^2 - c^2}\ln(a + b\cosh x + c\sinh x)$$
$$+ \frac{Bb - Cc}{b^2 - c^2}x + \left(A - a\frac{Bb - Cc}{b^2 - c^2}\right)\int \frac{dx}{a + b\cosh x + c\sinh x}, \quad b^2 \neq c^2.$$

27. $$\int \frac{A + B\cosh x + C\sinh x}{a + b\cosh x \pm b\sinh x}\,dx = \frac{C \mp B}{2a}(\cosh x \mp \sinh x) + \left[\frac{A}{a} - \frac{(B \mp C)b}{2a^2}\right]x$$
$$+ \left[\frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)b}{2a^2}\right]\ln(a + b\cosh x \pm b\sinh x), \quad ab \neq 0.$$

28. $$\int \frac{dx}{a + b\cosh x + c\sinh x}$$

$$= \begin{cases} \dfrac{2}{\sqrt{b^2 - a^2 - c^2}}\arctan\dfrac{(b-a)\tanh\frac{x}{2} + c}{\sqrt{b^2 - a^2 - c^2}}, & b^2 > a^2 + c^2 \text{ and } a \neq b, \\ \dfrac{1}{\sqrt{a^2 - b^2 + c^2}}\ln\dfrac{(a-b)\tanh\frac{x}{2} - c + \sqrt{a^2 - b^2 + c^2}}{(a-b)\tanh\frac{x}{2} - c - \sqrt{a^2 - b^2 + c^2}}, & b^2 < a^2 + c^2 \text{ and } a \neq b, \\ \dfrac{1}{c}\ln\left(a + c\tanh\dfrac{x}{2}\right), & a = b,\ c \neq 0, \\ \dfrac{2}{(a-b)\tanh\frac{x}{2} + c}, & b^2 = a^2 + c^2. \end{cases}$$

29. $$\int \frac{A+B\cosh x+C\sinh x}{(a_1+b_1\cosh x+c_1\sinh x)(a_2+b_2\cosh x+c_2\sinh x)}\,dx$$
$$= A_0\ln\frac{a_1+b_1\cosh x+c_1\sinh x}{a_2+b_2\cosh x+c_2\sinh x}+A_1\int\frac{dx}{a_1+b_1\cosh x+c_1\sinh x}$$
$$+A_2\int\frac{dx}{a_2+b_2\cosh x+c_2\sinh x},$$

where

$$A_0=\frac{\begin{vmatrix}a_1&b_1&c_1\\A&B&C\\a_2&b_2&c_2\end{vmatrix}}{\Delta},\ A_1=\frac{\begin{vmatrix}a_1&b_1&c_1\\\begin{vmatrix}b_1&c_1\\B&C\end{vmatrix}&\begin{vmatrix}c_1&a_1\\C&A\end{vmatrix}&\begin{vmatrix}a_1&b_1\\A&B\end{vmatrix}\\a_2&b_2&c_2\end{vmatrix}}{\Delta},\ A_2=\frac{\begin{vmatrix}a_1&b_1&c_1\\\begin{vmatrix}C&B\\c_2&b_2\end{vmatrix}&\begin{vmatrix}C&A\\c_2&a_2\end{vmatrix}&\begin{vmatrix}B&A\\b_2&a_2\end{vmatrix}\\a_2&b_2&c_2\end{vmatrix}}{\Delta},$$

$$\Delta=\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}^2+\begin{vmatrix}b_1&c_1\\b_2&c_2\end{vmatrix}^2-\begin{vmatrix}c_1&a_1\\c_2&a_2\end{vmatrix}^2\neq 0.$$

30. $$\int\frac{A\cosh^2x+2B\sinh x\cosh x+C\sinh^2x}{a\cosh^2x+2b\sinh x\cosh x+c\sinh^2x}\,dx$$
$$=\frac{1}{4b^2-(a+c)^2}\Big\{[4Bb-(A+C)(a+c)]\,x$$
$$+[(A+C)b-B(a+c)]\ln\left(a\cosh^2x+2b\sinh x\cosh x+c\sinh^2x\right)$$
$$+[2(A-C)\,b^2-2Bb(a-c)+(Ca-Ac)(a+c)]f(x)\Big\},$$

where

$$f(x)=\begin{cases}\dfrac{1}{2\sqrt{b^2-ac}}\ln\dfrac{c\,\tanh x+b-\sqrt{b^2-ac}}{c\,\tanh x+b+\sqrt{b^2-ac}}, & b^2>a\,c,\\[2ex]\dfrac{1}{\sqrt{ac-b^2}}\arctan\dfrac{c\,\tanh x+b}{\sqrt{ac-b^2}}, & b^2<a\,c,\\[2ex]=-\dfrac{1}{c\,\tanh x+b}, & b^2=a\,c.\end{cases}$$

31. $$\int\frac{(A+B\sinh x)\,dx}{\sinh x(a+b\sinh x)}=\frac{1}{a}\left[A\ln\left|\tanh\frac{x}{2}\right|+(aB-bA)\int\frac{dx}{a+b\sinh x}\right].$$

32. $$\int\frac{(A+B\sinh x)\,dx}{\sinh x(a+b\cosh x)}=\frac{A}{a^2-b^2}\left(a\ln\left|\tanh\frac{x}{2}\right|+b\ln\left|\frac{a+b\cosh x}{\sinh x}\right|\right)+B\int\frac{dx}{a+b\cosh x}.$$

33. $$\int\frac{(A+B\sinh x)\,dx}{\sinh x(1+\cosh x)}=\frac{A}{2}\left(\ln\left|\tanh\frac{x}{2}\right|-\frac{1}{2}\tanh^2\frac{x}{2}\right)+B\tanh\frac{x}{2}.$$

34. $$\int\frac{(A+B\sinh x)\,dx}{\sinh x(1-\cosh x)}=\frac{A}{2}\left(-\ln\left|\coth\frac{x}{2}\right|+\frac{1}{2}\coth^2\frac{x}{2}\right)+B\coth\frac{x}{2}.$$

35. $$\int\frac{(A+B\sinh x)\,dx}{\cosh x(a+b\sinh x)}=\frac{1}{a^2+b^2}\left[(Aa+Bb)\arctan(\sinh x)+(Ab-Ba)\ln\left|\frac{a+b\sinh x}{\cosh x}\right|\right].$$

36. $\int \frac{(A+B\,\cosh x)\,dx}{\sinh x(a+b\,\sinh x)} = \frac{1}{a}\left(A\,\ln\left|\tanh\frac{x}{2}\right| + B\,\ln\left|\frac{\sinh x}{a+b\,\sinh x}\right| - Ab\int\frac{dx}{a+b\,\sinh x}\right).$

37. $\int \frac{(A+B\,\cosh x)\,dx}{\sinh x(a+b\,\cosh x)} = \frac{1}{a^2-b^2}\left[(Aa+Bb)\ln\left|\tanh\frac{x}{2}\right| + (Ab-Ba)\ln\left|\frac{a+b\,\cosh x}{\sinh x}\right|\right].$

38. $\int \frac{(A+B\,\cosh x)\,dx}{\sinh x(1+\cosh x)} = \frac{A+B}{2}\ln\left|\tanh\frac{x}{2}\right| - \frac{A-B}{4}\tanh^2\frac{x}{2}.$

39. $\int \frac{(A+B\,\cosh x)\,dx}{\sinh x(1-\cosh x)} = \frac{A+B}{4}\coth^2\frac{x}{2} - \frac{A-B}{2}\ln\coth\frac{x}{2}.$

40. $\int \frac{(A+B\,\cosh x)\,dx}{\cosh x(a+b\,\sinh x)} = \frac{A}{a^2+b^2}\left[a\,\arctan(\sinh x) + b\,\ln\left|\frac{a+b\,\sinh x}{\cosh x}\right|\right] + B\int\frac{dx}{a+b\,\sinh x}.$

41. $\int \frac{(A+B\,\cosh x)\,dx}{\cosh x(a+b\,\cosh x)} = \frac{1}{a}\left[A\,\arctan\sinh x - (Ab-Ba)\int\frac{dx}{a+b\,\cosh x}\right].$

42. $\int \frac{dx}{1+\sinh^2 x} = \tanh x.$

43. $\int \frac{dx}{1-\sinh^2 x} = \begin{cases} \frac{1}{\sqrt{2}}\,\mathrm{arctanh}(\sqrt{2}\,\tanh x), & \sinh^2 x < 1, \\ \frac{1}{\sqrt{2}}\,\mathrm{arccoth}(\sqrt{2}\tanh x), & \sinh^2 x > 1. \end{cases}$

44. $\int \frac{dx}{1+\cosh^2 x} = \frac{1}{\sqrt{2}}\,\mathrm{arccoth}(\sqrt{2}\,\coth x).$

45. $\int \frac{dx}{1-\cosh^2 x} = \coth x.$

46. $\int \frac{dx}{a+b\,\sinh^2 x} = \begin{cases} \frac{1}{\sqrt{a(b-a)}}\arctan\left(\sqrt{\frac{b}{a}-1}\,\tanh x\right), & \frac{b}{a} > 1, \\ \frac{1}{\sqrt{a(a-b)}}\,\mathrm{arctanh}\left(\sqrt{1-\frac{b}{a}}\,\tanh x\right), & 0 < \frac{b}{a} < 1 \text{ or } \frac{b}{a} < 0 \text{ and } \sinh^2 x < -\frac{a}{b}, \\ \frac{1}{\sqrt{a(a-b)}}\,\mathrm{arccoth}\left(\sqrt{1-\frac{b}{a}}\,\tanh x\right), & \frac{b}{a} < 0 \text{ and } \sinh^2 x > -\frac{a}{b}. \end{cases}$

47. $\int \frac{dx}{a+b\,\cosh^2 x} = \begin{cases} \frac{1}{\sqrt{-a(a+b)}}\arctan\left(\sqrt{-\left(1+\frac{b}{a}\right)}\,\coth x\right), & \frac{b}{a} < -1, \\ \frac{1}{\sqrt{a(a+b)}}\,\mathrm{arctanh}\left(\sqrt{1+\frac{b}{a}}\,\coth x\right), & -1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x > -\frac{a}{b}, \\ \frac{1}{\sqrt{a(a+b)}}\,\mathrm{arccoth}\left(\sqrt{1+\frac{b}{a}}\,\coth x\right), & \frac{b}{a} > 0 \text{ or } -1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x < -\frac{a}{b}. \end{cases}$

48. $$\int \frac{dx}{(a+b\,\sinh^2 x)^2} = \frac{1}{2a(b-a)}\left[\frac{b\,\sinh x\,\cosh x}{a+b\,\sinh^2 x} + (b-2a)\int \frac{dx}{a+b\,\sinh^2 x}\right].$$

49. $$\int \frac{dx}{(a+b\,\cosh^2 x)^2} = \frac{1}{2a(a+b)}\left[-\frac{b\,\sinh x\,\cosh x}{a+b\,\cosh^2 x} + (2a+b)\int \frac{dx}{a+b\,\cosh^2 x}\right].$$

50. $$\int \frac{dx}{(a+b\,\sinh^2 x)^3}$$
$$= \begin{cases} \dfrac{1}{8pa^3}\left[\left(3-\dfrac{2}{p^2}+\dfrac{3}{p^4}\right)\arctan(p\,\tanh x) + \left(3-\dfrac{2}{p^2}-\dfrac{3}{p^4}\right)\dfrac{p\,\tanh x}{1+p^2\,\tanh^2 x} \right. \\ \qquad \left. + \left(1+\dfrac{2}{p^2}-\dfrac{1}{p^2}\tanh^2 x\right)\dfrac{2p\,\tanh x}{\left(1+p^2\,\tanh^2 x\right)^2}\right], \quad p^2 = \dfrac{b}{a}-1>0, \\ \dfrac{1}{8qa^3}\left[\left(3+\dfrac{2}{q^2}+\dfrac{3}{q^4}\right)\operatorname{arctanh}(q\,\tanh x) + \left(3+\dfrac{2}{q^2}-\dfrac{3}{q^4}\right)\dfrac{q\,\tanh x}{1-q^2\,\tanh^2 x} \right. \\ \qquad \left. + \left(1-\dfrac{2}{q^2}+\dfrac{1}{q^2}\tanh^2 x\right)\dfrac{2q\,\tanh x}{\left(1-q^2\,\tanh^2 x\right)^2}\right], \quad q^2 = 1-\dfrac{b}{a}>0. \end{cases}$$

51. $$\int \frac{dx}{(a+b\,\cosh^2 x)^3} = \begin{cases} \dfrac{1}{8pa^3}\left[\left(3-\dfrac{2}{p^2}+\dfrac{3}{p^4}\right)\arctan(p\,\coth x) + \left(3-\dfrac{2}{p^2}-\dfrac{3}{p^4}\right)\dfrac{p\,\coth x}{1+p^2\,\coth^2 x} \right. \\ \qquad \left. + \left(1+\dfrac{2}{p^2}-\dfrac{1}{p^2}\coth^2 x\right)\dfrac{2p\,\coth x}{\left(1+p^2\,\coth^2 x\right)^2}\right], \quad p^2 = -1-\dfrac{b}{a}>0, \\ \dfrac{1}{8qa^3}\left[\left(3+\dfrac{2}{q^2}+\dfrac{3}{q^4}\right)\varphi(x)^\dagger + \left(3+\dfrac{2}{q^2}-\dfrac{3}{q^4}\right)\dfrac{q\,\coth x}{1-q^2\,\coth^2 x} \right. \\ \qquad \left. + \left(1-\dfrac{2}{q^2}+\dfrac{1}{q^2}\coth^2 x\right)\dfrac{2q\,\coth x}{\left(1-q^2\,\coth^2 x\right)^2}\right], \quad q^2 = 1+\dfrac{b}{a}>0. \end{cases}$$

† If $\dfrac{b}{a}<0$ and $\cosh^2 x > -\dfrac{a}{b}$, then $\varphi(x) = \operatorname{arctanh}(q\coth x)$. If $\dfrac{b}{a}<0$ and $\cosh^2 x < -\dfrac{a}{b}$, or if $\dfrac{b}{a}>0$, then $\varphi(x) = \operatorname{arctanh}(q\coth x)$.