T1.25. Integrand involving rational functions of $\sin x$, $\cos x$, and $\sqrt{1 \pm a^2 \sin^2 x}$.

Notation used: $\alpha = \arcsin(a \sin x)$, $\beta = \arcsin\left(\dfrac{\sqrt{1+a^2}\,\sin x}{\sqrt{1+a^2\,\sin^2 x}}\right)$, $\gamma = \arcsin\left(\dfrac{\alpha\,\cos x}{\sqrt{a^2-1}}\right)$.

1. $$\int \frac{dx}{\sqrt{1-a^2\,\sin^2 x}} = \frac{1}{a}F\left(\alpha,\,\frac{1}{a}\right), \quad a^2 > 1.$$

2. $$\int \sqrt{1-a^2\,\sin^2 x}\,dx = a\,E\left(\alpha,\,\frac{1}{a}\right) - \frac{a^2-1}{a}F\left(\alpha,\,\frac{1}{a}\right), \quad a^2 > 1.$$

3. $$\int \frac{dx}{(1-r^2\,\sin^2 x)\sqrt{1-a^2\,\sin^2 x}} = \frac{1}{a}\Pi\left(\alpha,\,\frac{r^2}{a^2},\,\frac{1}{a}\right), \quad a^2 > 1.$$

4. $$\begin{aligned}\int \frac{\sqrt{1-a^2\,\sin^2 x}}{\cos^2 x}\,dx &= \tan x\sqrt{1-a^2\,\sin^2 x} + \frac{1}{a}F\left(\alpha,\,\frac{1}{a}\right)\\ &= -\left[a\,E\left(\alpha,\,\frac{1}{a}\right) - \frac{a^2-1}{a}F\left(\alpha,\,\frac{1}{a}\right)\right]\\ &= a\left[F\left(\alpha,\,\frac{1}{a}\right) - E\left(\alpha,\,\frac{1}{a}\right)\right] + \tan x\sqrt{1-a^2\,\sin^2 x}, \quad a^2 > 1.\end{aligned}$$

5. $$\begin{aligned}\int \frac{dx}{\sqrt{(1-a^2\,\sin^2 x)^3}} &= \frac{-1}{a^2-1}\left[a\,E\left(\alpha,\,\frac{1}{a}\right) - \frac{a^2-1}{a}F\left(\alpha,\,\frac{1}{a}\right)\right] - \frac{a^2}{1-a^2}\,\frac{\sin x\,\cos x}{\sqrt{1-a^2\,\sin^2 x}}\\ &= \frac{a^2}{a^2-1}\,\frac{\sin x\,\cos x}{\sqrt{1-a^2\,\sin^2 x}} + \frac{1}{a}F\left(\alpha,\,\frac{1}{a}\right) - \frac{a}{a^2-1}E\left(\alpha,\,\frac{1}{a}\right), \quad a^2 > 1.\end{aligned}$$

6. $$\begin{aligned}\int \frac{\cos^4 x\,dx}{\sqrt{1-a^2\,\sin^2 x}} &= \frac{\sin x\,\cos x\sqrt{1-a^2\,\sin^2 x}}{3a^2} + \frac{4a^2-2}{3a^4}\left[a\,E\left(\alpha,\,\frac{1}{a}\right)\right.\\ &\qquad \left. -\frac{a^2-1}{a}F\left(\alpha,\,\frac{1}{a}\right)\right] + \frac{2-5a^2+3a^4}{3a^5}\,F\left(\alpha,\,\frac{1}{a}\right)\\ &= \frac{\sin x\,\cos x\sqrt{1-a^2\,\sin^2 x}}{3a^2} - \frac{a^2-1}{3a^3}F\left(\alpha,\,\frac{1}{a}\right) + \frac{4a^2-2}{3a^3}E\left(\alpha,\,\frac{1}{a}\right), \quad a^2 > 1.\end{aligned}$$

7. $$\int \frac{dx}{\sqrt{1+a^2\,\sin^2 x}} = \frac{1}{\sqrt{1+a^2}}\,F\left(\beta,\frac{a}{\sqrt{1+a^2}}\right).$$

8. $\displaystyle\int \sqrt{1+a^2\,\sin^2 x}\,dx = \sqrt{1+a^2}\,E\left(\beta,\,\frac{a}{\sqrt{1+a^2}}\right) - a^2\frac{\sin x\,\cos x}{\sqrt{1+a^2\,\sin^2 x}}.$

9. $\displaystyle\frac{\sqrt{1+a^2\,\sin^2 x}\,dx}{1+(a^2-r^2a^2-r^2)\,\sin^2 x} = \frac{1}{\sqrt{1+a^2}}\Pi\left(\beta,\,r^2,\,\frac{a}{\sqrt{1+a^2}}\right).$

10. $\displaystyle\int \frac{\sin x\,dx}{\sqrt{1+a^2\,\sin^2 x}} = -\frac{1}{a}\,\arcsin\left(\frac{a\,\cos x}{\sqrt{1+a^2}}\right).$

11. $\displaystyle\int \frac{\cos x\,dx}{\sqrt{1+a^2\,\sin^2 x}} = \frac{1}{a}\,\ln(a\,\sin x + \sqrt{1+a^2\,\sin^2 x}).$

12. $\displaystyle\int \frac{dx}{\sin x\sqrt{1+a^2\,\sin^2 x}} = \frac{1}{2}\,\ln\frac{\sqrt{1+a^2\,\sin^2 x}-\cos x}{\sqrt{1+a^2\,\sin^2 x}+\cos x}.$

13. $\displaystyle\int \frac{dx}{\cos x\sqrt{1+a^2\,\sin^2 x}} = \frac{1}{2\sqrt{1+a^2}}\,\ln\frac{\sqrt{1+a^2\,\sin^2 x}+\sqrt{1+a^2}\,\sin x}{\sqrt{1+a^2\,\sin^2 x}-\sqrt{1+a^2}\,\sin x}.$

14. $\displaystyle\int \frac{\tan x\,dx}{\sqrt{1+a^2\,\sin^2 x}} = \frac{1}{2\sqrt{1+a^2}}\,\ln\frac{\sqrt{1+a^2\,\sin^2 x}+\sqrt{1+a^2}}{\sqrt{1+a^2\,\sin^2 x}-\sqrt{1+a^2}}.$

15. $\displaystyle\int \frac{\cot x\,dx}{\sqrt{1+a^2\,\sin^2 x}} = \frac{1}{2}\,\ln\frac{1-\sqrt{1+a^2\,\sin^2 x}}{1+\sqrt{1+a^2\,\sin^2 x}}.$

16. $\displaystyle\int \frac{\tan^2 x\,dx}{\sqrt{1+a^2\,\sin^2 x}} = \frac{1}{(1+a^2)}\Bigg[\tan x\sqrt{1+a^2\,\sin^2 x}$
$$-\sqrt{1+a^2}E\left(\beta,\,\frac{a}{\sqrt{1+a^2}}\right)+a^2\frac{\sin x\,\cos x}{\sqrt{1+a^2\,\sin^2 x}}\Bigg]$$
$$= -\frac{1}{\sqrt{1+a^2}}E\left(\beta,\,\frac{a}{\sqrt{1+a^2}}\right)+\frac{\tan x}{\sqrt{1+a^2\,\sin^2 x}}.$$

17. $\displaystyle\int \frac{dx}{\sqrt{(1+a^2\,\sin^2 x)^3}} = \frac{1}{\sqrt{1+a^2}}E\left(\beta,\,\frac{a}{\sqrt{1+a^2}}\right).$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2\,\sin^2 x-1}} = -\frac{1}{a}F\left(\gamma,\,\frac{\sqrt{a^2-1}}{a}\right),\quad a^2>1.$

19. $\displaystyle\int \sqrt{a^2\,\sin^2 x-1}\,dx = \frac{1}{a}F\left(\gamma,\,\frac{\sqrt{a^2-1}}{a}\right) - a\,E\left(\gamma,\,\frac{\sqrt{a^2-1}}{a}\right),\quad a^2>1.$

20. $\displaystyle\int \frac{dx}{(1-r^2\,\sin^2 x)\sqrt{a^2\,\sin^2 x-1}} = \frac{1}{a(r^2-1)}\Pi\left(\gamma,\,\frac{r^2(a^2-1)}{a^2(r^2-1)},\,\frac{\sqrt{a^2-1}}{a}\right),\ a^2>1,\ r^2>1.$

21. $\displaystyle\int \frac{\sin x\,dx}{\sqrt{a^2\,\sin^2 x-1}} = -\frac{\gamma}{a},\quad a^2>1.$

22. $\displaystyle\int \frac{\cos x\,dx}{\sqrt{a^2\,\sin^2\,x-1}} = \frac{1}{a}\,\ln(a\,\sin x+\sqrt{a^2\,\sin^2\,x-1}), \quad a^2>1.$

23. $\displaystyle\int \frac{dx}{\sin x\sqrt{a^2\,\sin^2\,x-1}} = -\arctan\frac{\cos x}{\sqrt{a^2\,\sin^2\,x-1}}, \quad a^2>1.$

24. $\displaystyle\int \frac{dx}{\cos x\sqrt{a^2\,\sin^2\,x-1}} = \frac{1}{2\sqrt{a^2-2}}\,\ln\frac{\sqrt{a^2-1}\,\sin x+\sqrt{a^2\,\sin^2\,x-1}}{\sqrt{a^2-1}\,\sin x-\sqrt{a^2\,\sin^2\,x-1}}, \quad a^2>1.$

25. $\displaystyle\int \frac{\tan x\,dx}{\sqrt{a^2\,\sin^2\,x-1}} = \frac{1}{2\sqrt{a^2-1}}\,\ln\frac{\sqrt{a^2-1}+\sqrt{a^2\,\sin^2\,x-1}}{\sqrt{a^2-1}-\sqrt{a^2\,\sin^2\,x-1}}, \quad a^2>1.$

26. $\displaystyle\int \frac{\cot x\,dx}{\sqrt{a^2\,\sin^2\,x-1}} = -\arcsin\left(\frac{1}{a\,\sin x}\right), \quad a^2>1.$

27. $\displaystyle\int \sqrt{1+\sin ax}\,dx = \pm 2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right),$

where + sign is taken when $(8k-1)\pi/2 < x \le (8k+3)\pi/2$, $k \in \mathrm{N}$; otherwise − sign.

28. $\displaystyle\int \sqrt{1-\sin ax}\,dx = \pm 2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right),$

where + sign is taken when $(8k-3)\pi/2 < x \le (8k+1)\pi/2$, $k \in \mathrm{N}$; otherwise − sign.

29. $\displaystyle\int (\sin ax)\,\sqrt{1+b^2\,\sin^2 ax}\,dx = -\frac{\cos ax}{2a}\,\sqrt{1+b^2\,\sin^2 ax}\,-\frac{1+b^2}{2a\,b}\,\arcsin\frac{b\,\cos ax}{\sqrt{1+b^2}}.$

30. $\displaystyle\int (\sin ax)\,\sqrt{1-b^2\,\sin^2 ax}\,dx = -\frac{\cos ax}{2a}\,\sqrt{1-b^2\,\sin^2 ax}-\frac{1+b^2}{2a\,b}\,\ln\left(b\,\cos ax+\sqrt{1-b^2\sin^2 ax}\right).$

31. $\displaystyle\int (\cos ax)\,\sqrt{1+b^2\,\sin^2 ax}\,dx = \frac{\sin ax}{2a}\,\sqrt{1+b^2\,\sin^2 ax}+\frac{1}{2a\,b}\,\ln\left(b\,\sin ax+\sqrt{1+b^2\sin^2 ax}\right).$

32. $\displaystyle\int (\cos ax)\,\sqrt{1-b^2\,\sin^2 ax}\,dx = \frac{\sin ax}{2a}\,\sqrt{1-b^2\,\sin^2 ax}\,+\frac{1}{2a\,b}\,\arcsin\left(b\,\sin ax\right).$