! For an efficient use of these tables, first read `HowTo.pdf`.

T3.56A. Integrands involving exponentials, logarithm functions and powers of $(a+b\,x)$ on the interval $(0,\infty)$.

1. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\ln\,x\,dx=\frac{1}{\mu^\nu}\Gamma(\nu)[\psi(\nu)-\ln\,\mu],\quad \Re\{\mu\}>0,\,\Re\{\nu\}>0.$$

2. $$\int_0^\infty x^n e^{-\mu x}\ln\,x\,dx=\frac{n!}{\mu^{n+1}}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\gamma_e-\ln\,\mu\right],\quad \Re\{\mu\}>0.$$

3. $$\int_0^\infty x^{n-1/2}e^{-\mu x}\ln\,x\,dx=\sqrt{\pi}\frac{(2n-1)!!}{2^n\mu^{n+1/2}}\left[2\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)-\gamma_e-\ln\,4\mu\right],$$ $$\Re\{\mu\}>0.$$

4. $$\int_0^\infty x^{\mu-1}e^{-x}\ln\,x\,dx=\Gamma'(\mu),\quad \Re\{\mu\}>0.$$

5. $$\int_0^\infty (x-\nu)x^{\nu-1}e^{-x}\ln\,x\,dx=\Gamma(\nu),\quad \Re\{\nu\}>0.$$

6. $$\int_0^\infty \left(\mu x-n-\frac{1}{2}\right)x^{n-1/2}e^{-\mu x}\ln\,x\,dx=\frac{(2n-1)!!}{(2\mu)^n}\sqrt{\frac{\pi}{\mu}},\quad \Re\{\mu\}>0.$$

7. $$\int_0^\infty \frac{x^{\nu-1}\ln\,x}{e^x+1}\,dx=\begin{cases}\Gamma(\nu)\displaystyle\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k^\nu}[\psi(\nu)-\ln\,k], & \Re\{\nu\}>0,\\ -\dfrac{1}{2}(\ln\,2)^2, & \nu=1.\end{cases}$$

8. $$\int_0^\infty \frac{x^{\nu-1}\ln\,x}{(e^x+1)^2}\,dx=\Gamma(\nu)\sum_{k=2}^\infty\frac{(-1)^k(k-1)}{k^\nu}[\psi(\nu)-\ln\,k],\quad \Re\{\nu\}>0.$$

9. $$\int_0^\infty \frac{(x-\nu)e^x-\nu}{(e^x+1)^2} x^{\nu-1}\ln x\,dx = \Gamma(\nu)\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^\nu}, \quad \Re\{\nu\}>0.$$

10. $$\int_0^\infty \frac{(x-2n)e^x-2n}{(e^x+1)^2} x^{2n-1}\ln x\,dx = \frac{2^{2n-1}-1}{2n}\pi^{2n}|B_{2n}|, \quad n=1,\,2,\ldots.$$

11. $$\int_0^\infty \frac{x^{\nu-1}\ln x}{(e^x+1)^n}\,dx = (-1)^n\frac{\Gamma(\nu)}{(n-1)!}\sum_{k=n}^\infty \frac{(-1)^k(k-1)!}{(k-n)!k^\nu}\,[\psi(\nu)-\ln k], \quad \Re\{\nu\}>0.$$

12. $$\int_0^\infty x^2 e^{-\mu x^2}\ln x\,dx = \frac{\sqrt{\pi}\,(2-\ln 4\mu-\gamma_e)}{\mu^{3/2}}, \quad \Re\{\mu\}>0.$$

13. $$\int_0^\infty x(\mu x^2-\nu x-1)e^{-\mu x^2+2\nu x}\ln x\,dx = \frac{1}{4\mu}+\frac{\nu}{4\mu}\sqrt{\frac{\pi}{\mu}}\exp\left(\frac{\nu^2}{\mu}\right)\left[1+\operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right], \quad \Re\{\mu\}>0.$$

14. $$\int_0^\infty (\mu x^2-n)x^{2n-1}e^{-\mu x^2}\ln x\,dx = \frac{(n-1)!}{4\mu^n}, \quad \Re\{\mu\}>0.$$

15. $$\int_0^\infty (2\mu x^2-2n-1)x^{2n}e^{-\mu x^2}\ln x\,dx = \frac{(2n-1)!!}{2(2\mu)^n}\sqrt{\frac{\pi}{\mu}}, \quad \Re\{\mu\}>0.$$

16. $$\int_0^\infty \exp\left[-\mu\left(\frac{x}{a}+\frac{a}{x}\right)\right]\ln x\frac{dx}{x} = 2\ln a\,K_0(2\mu), \quad a>0,\,\Re\{\mu\}>0.$$

17. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\cdot\left[2ax^2-(2n+1)x-2b\right]x^{n-1/2}\,dx$$
$$= 2\left(\frac{b}{a}\right)^{n/2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}\sum_{k=0}^\infty \frac{(n+k)!}{(n-k)!(2k)!!(2\sqrt{ab})^k}, \quad a>0,\,b>0.$$

18. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\cdot\left[2ax^2+(2n-1)x-2b\right]\frac{dx}{x^{n+3/2}}$$
$$= 2\left(\frac{a}{b}\right)^{n/2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}\sum_{k=0}^\infty \frac{(n+k-1)!}{(n-k-1)!\,(2k)!!\,(2\sqrt{ab}\,)^k}, \quad a>0,\,b>0.$$

19. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\,\frac{ax^2-b}{x^2}\,dx = 2K_0(2\sqrt{ab}), \quad a>0,\,b>0.$$

20. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\,\frac{2ax^2-x-2b}{x\sqrt{x}}\,dx = 2\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}. \quad a>0,\ b>0.$$

21. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\,\frac{2ax^2-3x-2b}{\sqrt{x}}\,dx = \frac{1+2\sqrt{ab}}{a}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}, \quad a>0,\ b>0.$$

22. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\,\left(a-\frac{b}{x^2}\right)dx = K_0(2\sqrt{ab}), \quad a>0,\ b>0.$$

23. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\,\left[2ax^2-(2n+1)x-2b\right]x^{n-3/2}\,dx$$
$$= \begin{cases} 4(b/a)^{(2n+1)/4}K_{n+1/2}(2\sqrt{ab}), \\ \qquad \text{or} \\ = 2(b/a)^{n/2}\sqrt{\dfrac{\pi}{a}}\,e^{-2\sqrt{ab}}\displaystyle\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!\,(2k)!!\,(2\sqrt{ab})^k}, \quad n=0,1,\ldots,;\ a>0,\ b>0. \end{cases}$$

24. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln\left[(ax^2-b)\cos(\alpha\ln x)+\alpha x\sin(\alpha\ln x)\right]\frac{dx}{x^2}$$
$$= 2\cos(\alpha\ \ln\ \sqrt{b/a})\,K_{i\,\alpha}(2\sqrt{ab}), \quad a>0,\ b>0,\ -\infty<\alpha<\infty.$$

25. $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\left[(ax^2-b)\sin(\alpha\ln x)-\alpha x\cos(\alpha\ln x)\right]\frac{dx}{x^2}$$
$$= 2\sin(\alpha\ \ln\ \sqrt{b/a})\,K_{i\,\alpha}(2\sqrt{ab}), \quad a>0,\ b>0,\ -\infty<\alpha<\infty.$$

26. $$\int_0^\infty x^\alpha\ln x\left[a-\frac{\alpha}{x}-\frac{b}{x^2}\right]\exp\left(-ax-\frac{b}{x}\right)dx = 2\left(\frac{b}{a}\right)^{\alpha/2}K_\alpha(2\sqrt{ab}),$$
$$a>0,\ b>0,\ -\infty<\alpha<\infty.$$

27. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right)\ln x\,\frac{1+ax^2-x^4}{x^2}\,dx = -\frac{\sqrt{2a^3\pi}}{2\sqrt[a]{e}}, \quad a>0.$$

28. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right)\ln x\,\frac{x^4+ax^2-1}{x^4}\,dx = \frac{\sqrt{2a^3\pi}}{2\sqrt[a]{e}}, \quad a>0.$$

29. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right)\ln x\,\frac{x^4+3ax-1}{x^6}\,dx = \frac{(1+a)\sqrt{2a^3\pi}}{2\sqrt[a]{e}}, \quad a>0.$$

30. $$\int_0^\infty x^{\nu-1}e^{-\mu x}(\ln\, x)^2\,dx = \frac{\Gamma(\nu)}{\mu^\nu}\{[\psi(\nu)-\ln\,\mu]^2+\zeta(2,\,\nu)\},\quad \Re\{\mu\}>0,\ \Re\{\nu\}>0.$$

31. $$\int_0^\infty x^{\nu-1}e^{-\mu x}(\ln\, x)^3\,dx = \frac{\Gamma(\nu)}{\mu^\nu}\{[\psi(\nu)-\ln\,\mu]^3+3\,\zeta(2,\nu)[\psi(\nu)-\ln\,\mu]-2\,\zeta(3,\,\nu)\},\quad \Re\{\mu\}>0,\ \Re\{\nu\}>0.$$

32. $$\int_0^\infty x^{\nu-1}e^{-\mu x}(\ln\, x)^4\,dx = \frac{\Gamma(\nu)}{\mu^\nu}\{[\psi(\nu)-\ln\,\mu]^4+6\,\zeta(2,\,\nu)[\psi(\nu)-\ln\,\mu]^2 -8\,\zeta(3,\,\nu)[\psi(\nu)-\ln\,\mu]+3[\zeta(2,\nu)]^2+6\,\zeta(4,\,\nu)\},\quad \Re\{\mu\}>0,\ \Re\{\nu\}>0.$$

33. $$\int_0^\infty x^{\nu-1}e^{-\mu x}(\ln\, x)^n\,dx = \frac{\partial^n}{\partial\nu^n}\{\mu^{-\nu}\Gamma(\nu)\},\quad n=0,1,2,\ldots.$$

34. $$\int_0^\infty e^{-\mu x}\frac{x^{p-1}-x^{q-1}}{\ln\,x}\,dx = \frac{1}{\mu}[\lambda(\mu,\,p-1)-\lambda(\mu,\,q-1)],\quad \Re\{\mu\}>0,\ p>0,\ q>0.$$

35. $$\int_0^\infty \frac{(x+1)e^{-\mu x}}{\pi^2+(\ln\,x)^2}\,dx = \nu'(\mu)-\nu''(\mu),\quad \Re\{\mu\}>0.$$

36. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{x[\pi^2+(\ln\,x)^2]} = e^\mu-\nu(\mu),\quad \Re\{\mu\}>0.$$

37. $$\int_0^\infty e^{-\mu x}\ln\,(a+x)\frac{\mu(x+a)\ln\,(x+a)-2}{x+a}\,dx = \frac{1}{4}\int_0^\infty e^{-\mu x}\ln\,(a-x)^2\frac{\mu(x-a)\,\ln\,(x-a)^2-4}{x-a}\,dx = (\ln\,a)^2,\quad \Re\{\mu\}>0,\ a>0.$$

38. $$\int_0^\infty e^{-\mu x}\ln\,[(x+a)(x+b)]\frac{dx}{x+a+b} = e^{(a+b)\mu}\{\mathrm{Ei}(-a\mu)\mathrm{Ei}(-b\mu)-\ln\,(ab)\mathrm{Ei}[-(a+b)\mu]\},\quad a>0,\ b>0,\ \Re\{\mu\}>0.$$

39. $$\int_0^\infty e^{-\mu x}\ln\,(x+a+b)\left(\frac{1}{x+a}+\frac{1}{x+b}\right)dx = (1+\ln\,a\ln\,b)\ln\,(a+b)+e^{-(a+b)\mu}\{\mathrm{Ei}(-\alpha\mu)\mathrm{Ei}(-b\mu) +(1-\ln\,(ab))\,\mathrm{Ei}[-(a+b)\mu]\},\quad a>0,\ b>0,\ \Re\{\mu\}>0.$$

40. $$\int_0^\infty \left[e^{-x}-\frac{x}{(1+x)^{p+1}\,\ln\,(1+x)}\right]\frac{dx}{x} = \ln\,p,\quad p>0.$$

41. $$\int_0^\infty e^{-\mu x} \ln\left(1+\frac{x^2}{a^2}\right)\frac{dx}{x} = [\mathrm{Ci}(a\mu)]^2 + [\mathrm{Si}(a\mu)]^2, \quad \Re\{\mu\} > 0.$$

42. $$\int_0^\infty e^{-\mu x} \ln\left|1-\frac{x^2}{a^2}\right|\frac{dx}{x} = \mathrm{Ei}(a\mu)\,\mathrm{Ei}(-a\mu), \quad \Re\{\mu\} > 0.$$

43. $$\int_0^\infty x e^{-\mu x^2} \ln\left|\frac{1+x^2}{1-x^2}\right| dx = \frac{1}{\mu}[\cosh\mu\,\sinh i(\mu) - \sinh\mu\,\cosh i(\mu)], \quad \Re\{\mu\} > 0.$$ 4.367

44. $$\int_0^\infty x e^{-\mu x^2} \ln\frac{x+\sqrt{x^2+2\beta}}{\sqrt{2\beta}}\,dx = \frac{e^{\beta\mu}}{4\mu}K_0(\beta\mu), \quad |\arg\beta| < \pi,\ \Re\{\mu\} > 0.$$

45. $$\int_0^\infty x^{\nu-1} e^{-\mu x}[\psi(\nu) - \ln x]\,dx = \frac{\Gamma(\nu)\ln\mu}{\mu^\nu}, \quad \Re\{\nu\} > 0.$$

46. $$\int_0^\infty x^n e^{-\mu x}\left\{\left[\ln x - \frac{1}{2}\psi(n+1)\right]^2 - \frac{1}{2}\psi'(n+1)\right\}dx$$
$$= \frac{n!}{\mu^{n+1}}\left\{\left[\ln\mu - \frac{1}{2}\psi(n+1)\right]^2 + \frac{1}{2}\psi'(n+1)\right\}, \quad \Re\{\mu\} > 0.$$