! For an efficient use of these tables, first read `HowTo.pdf`.

T3.54A. Integrands involving logarithm functions and powers of logarithm functions and rational functions on the interval $(0,\infty)$.

1. $$\int_0^\infty \frac{\ln(1+ax)}{1+x}\,dx = \frac{\pi}{4}\ln(1+a^2) - \int_0^a \frac{\ln u\,du}{1+u^2}, \quad a>0.$$

2. $$\int_0^\infty \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{4}\ln 2 + \mathbf{G}.$$

3. $$\int_0^\infty \frac{\ln(1+x)}{x(1+x)}\,dx = \frac{\pi^2}{6}.$$

4. $$\int_0^\infty \frac{\ln(1+x)}{(ax+b)^2}\,dx = \frac{\ln\frac{a}{b}}{a(a-b)}, \quad ab>0.$$

5. $$\int_0^\infty \ln(a+x)\frac{dx}{(b+x)^2} = \frac{a\ln a - b\ln b}{b(a-b)}, \quad a>0,\ b>0,\ a\neq b.$$

6. $$\int_0^\infty \frac{\ln(ax+b)}{(1+x)^2}\,dx = \frac{1}{a-b}[a\ln a - b\ln b], \quad a>0,\ b>0.$$

7. $$\int_0^\infty \ln(a+x)\frac{x\,dx}{(b^2+x^2)^2} = \frac{1}{2(a^2+b^2)}\left(\ln b + \frac{a\pi}{2b} + \frac{a^2}{b^2}\ln a\right), \quad a>0,\ b>0.$$

8. $$\int_0^\infty \ln(1+x)\frac{1-x^2}{(ax+b)^2}\,\frac{dx}{(bx+a)^2} = \frac{1}{ab(a^2-b^2)}\ln\frac{b}{a}, \quad a>0,\ b>0.$$

9. $$\int_0^\infty \ln(a+x)\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \frac{1}{a^2+b^2}\left(a\ln\frac{b}{a} - \frac{b\pi}{2}\right), \quad a>0,\ b>0.$$

10. $$\int_0^\infty \ln(a-x)^2\frac{b^2-x^2}{(b^2+x^2)^2}\,dx = \frac{2}{a^2+b^2}\left(a\ln\frac{a}{b} - \frac{b\pi}{2}\right), \quad a>0,\ b>0.$$

11. $$\int_0^\infty \ln{(a-x)^2}\frac{x\,dx}{(b^2+x^2)^2} = \frac{1}{a^2+b^2}\left(\ln\,b - \frac{a\pi}{2b} + \frac{a^2}{b^2}\ln\,a\right), \quad a>0,\, b>0.$$

12. $$\int_0^\infty x^{\mu-1}\ln{(1+x)}\,dx = \frac{\pi}{\mu\sin\mu\pi}, \quad -1<\Re\{\mu\}<0.$$

13. $$\int_0^\infty x^{\mu-1}\ln\,|1-x|\,dx = \frac{\pi}{\mu}\cot(\mu\pi), \quad -1<\Re\{\mu\}<0.$$

14. $$\int_0^\infty x^{\mu-1}\ln{(1+\gamma x)}\,dx = \frac{\pi}{\mu\gamma^\mu\sin\mu\pi}, \quad -1<\Re\{\mu\}<0,\, |\arg\gamma|<\pi.$$

15. $$\int_0^\infty \frac{x^{\mu-1}\,\ln{(1+x)}}{1+x}\,dx = \frac{\pi}{\sin\mu\pi}[\gamma_e+\psi(1-\mu)], \quad -1<\Re\{\mu\}<1.$$

16. $$\int_0^\infty \frac{x^{\mu-1}\,\ln{(\gamma+x)}}{(\gamma+x)^\nu}\,dx = \gamma^{\mu-\nu}\mathrm{B}(\mu,\,\nu-\mu)[\psi(\nu)-\psi(\nu-\mu)+\ln\,\gamma], \quad 0<\Re\{\mu\}<\Re\{\nu\}.$$

17. $$\int_0^\infty \ln{(1-x)^2}x^p\,dx = \frac{2\pi}{p+1}\cot p\pi, \quad -2<p<-1.$$

18. $$\int_0^\infty \ln{(\mu x^2+\beta)}\,\frac{dx}{\gamma+x^2} = \frac{\pi}{\sqrt{\gamma}}\,\ln\left(\sqrt{\mu\gamma}+\sqrt{\beta}\right), \quad \Re\{\beta\}>0,\, \Re\{\mu\}>0,\, |\arg\gamma|<\pi.$$

19. $$\int_0^\infty \ln{(1+x^2)}\frac{dx}{x^2} = \pi.$$

20. $$\int_0^\infty \ln{(1+x^2)}\frac{dx}{(a+x)^2} = \frac{2a}{1+a^2}\left(\frac{\pi}{2a}+\ln\,a\right), \quad a>0.$$

21. $$\int_0^\infty \ln{(a^2+b^2x^2)}\frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg}\ln\,\frac{ag+bc}{g}, \quad a>0,\, b>0,\, c>0,\, g>0.$$

22. $$\int_0^\infty \ln{(a^2+b^2x^2)}\frac{dx}{c^2-g^2x^2} = -\frac{\pi}{cg}\arctan\frac{bc}{ag}, \quad a>0,\, b>0,\, c>0,\, g>0.$$

23. $$\int_0^\infty \frac{\ln{(1+p^2x^2)}-\ln{(1+q^2x^2)}}{x^2}\,dx = \pi(p-q), \quad p>0,\, q>0.$$

24. $$\int_0^\infty \ln{(1-x^2)^2}\frac{dx}{x^2} = 0.$$

25. $\displaystyle\int_0^\infty \ln(a^2 - x^2)^2 \frac{dx}{b^2 + x^2} = \frac{\pi}{b} \ln(a^2 + b^2), \quad b > 0.$

26. $\displaystyle\int_0^\infty \ln(a^2 - x^2)^2 \frac{b^2 - x^2}{(b^2 + x^2)^2}\, dx = -\frac{2b\pi}{a^2 + b^2}, \quad b > 0.$

27. $\displaystyle\int_0^\infty \ln(1 + x^2) \frac{dx}{x(1 + x^2)} = \frac{\pi^2}{12}.$

28. $$\int_0^\infty \ln(a^2 + b^2x^2) \frac{dx}{(c + gx)^2} = \frac{2\ln b}{cg} + \frac{b^2}{a^2g^2 + b^2c^2} \left(\frac{a}{b}\pi + 2\frac{c}{g} \ln \frac{c}{g} + 2\frac{a^2g}{b^2c} \ln \frac{a}{b}\right),$$
$$a > 0,\ b > 0,\ c > 0,\ g > 0.$$

29. $\displaystyle\int_0^\infty \frac{\ln(1 + p^2x^2)}{r^2 + q^2x^2}\, dx = \int_0^\infty \frac{\ln(p^2 + x^2)}{q^2 + r^2x^2}\, dx = \frac{\pi}{qr} \ln \frac{q + pr}{q}, \quad qr > 0,\ p > 0.$

30. $$\int_0^\infty \frac{\ln(1 + a^2x^2)}{b^2 + c^2x^2} \frac{dx}{d^2 + g^2x^2} = \frac{\pi}{b^2g^2 - c^2d^2} \left[\frac{g}{d} \ln\left(1 + \frac{ad}{g}\right) - \frac{c}{b} \ln\left(1 + \frac{ab}{c}\right)\right],$$
$$a > 0,\ b > 0,\ c > 0,\ d > 0,\ g > 0,\ b^2g^2 \neq c^2d^2.$$

31. $$\int_0^\infty \frac{\ln(1 + a^2x^2)}{b^2 + c^2x^2} \frac{x^2\, dx}{d^2 + g^2x^2} = \frac{\pi}{b^2g^2 - c^2d^2} \left[\frac{b}{c} \ln\left(1 + \frac{ab}{c}\right) - \frac{d}{g} \ln\left(1 + \frac{ad}{g}\right)\right],$$
$$a > 0,\ b > 0,\ c > 0,\ d > 0,\ g > 0,\ b^2g^2 \neq c^2d^2.$$

32. $$\int_0^\infty \ln(a^2 + b^2x^2) \frac{dx}{(c^2 + g^2x^2)^2} = \frac{\pi}{2c^3g} \left(\ln \frac{ag + bc}{g} - \frac{bc}{ag + bc}\right),$$
$$a > 0,\ b > 0,\ c > 0,\ g > 0.$$

33. $$\int_0^\infty \ln(a^2 + b^2x^2) \frac{x^2\, dx}{(c^2 + g^2x^2)^2} = \frac{\pi}{2cg^3} \left(\ln \frac{ag + bc}{g} + \frac{bc}{ag + bc}\right),$$
$$a > 0,\ b > 0,\ c > 0,\ g > 0.$$

34. $\displaystyle\int_0^\infty \ln(1 + x^2) x^{\mu - 1}\, dx = \frac{\pi}{\mu \sin \frac{\mu\pi}{2}} \quad -2 < \Re\{\mu\} < 0.$

35. $$\int_0^\infty \ln(1 + x^2) \frac{x^{\mu - 1} dx}{1 + x}$$
$$= \frac{\pi}{\sin \mu\pi} \left\{\ln 2 - (1 - \mu) \sin \frac{\mu\pi}{2} \beta\left(\frac{1 - \mu}{2}\right) - (2 - \mu) \cos \frac{\mu\pi}{2} \beta\left(\frac{2 - \mu}{2}\right)\right\}, \quad -2 < \Re\{\mu\} < 1.$$

36. $\displaystyle\int_0^\infty \ln(1+2x\cos t + x^2)x^{\mu-1}\,dx = \frac{2\pi}{\mu}\,\frac{\cos\mu t}{\sin\mu\pi}, \quad |t|<\pi,\ -1<\Re\{\mu\}<0.$

37. $\displaystyle\int_0^\infty \ln\left(\frac{x^2+2ax\cos t+a^2}{x^2-2ax\cos t+a^2}\right)\frac{x\,dx}{x^2+b^2} = \frac{1}{2}\pi^2-\pi t+\pi\arctan\frac{(a^2-b^2)\cos t}{(a^2+b^2)\sin t+2ab},$
$a>0,\ b>0,\ 0<t<\pi.$

38. $\displaystyle\int_0^\infty \ln\frac{ax+b}{bx+a}\,\frac{dx}{(1+x)^2} = 0, \quad ab>0.$

39. $\displaystyle\int_0^\infty \ln\left(\frac{1+x}{1-x}\right)^2\,\frac{dx}{x(1+x^2)} = \frac{\pi^2}{2}.$

40. $\displaystyle\int_0^\infty \frac{b\ln(1+ax)-a\ln(1+bx)}{x^2}\,dx = ab\ln\frac{b}{a}, \quad a>0,\ b>0.$

41. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x}\,\frac{x^{2n-1}}{1+x}\,dx = \frac{\ln 2}{2n}+\frac{1}{4n^2}-\frac{1}{2n}\beta(2n+1).$

42. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x}\,\frac{x^{2n}}{1+x}\,dx = \frac{\ln 2}{2n}+\frac{1}{4n^2}-\frac{1}{2n}\beta(2n+1).$

43. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x}\,\frac{x^{2n-1}}{1-x}\,dx = \frac{\ln 2}{2n}+\frac{1}{4n^2}-\frac{1}{2n}\beta(2n+1).$

44. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x}\,\frac{x^{2n}}{1-x}\,dx = -\frac{\ln 2}{2n}-\frac{1}{4n^2}+\frac{1}{2n}\beta(2n+1).$

45. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x}\,\frac{x^{2n-1}}{1+x^2}\,dx = \frac{\ln 2}{2n}+\frac{1}{4n^2}-\frac{1}{2n}\beta(2n+1).$

46. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x}\,\frac{dx}{1+x^2} = \pi\ln 2.$

47. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x}\,\frac{dx}{1-x^2} = 0.$

48. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x^2}\,\frac{x\,dx}{1+x^2} = \frac{\pi^2}{12}.$

49. $\displaystyle\int_0^\infty \ln\,\frac{a^2+b^2x^2}{x^2}\,\frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg}\ln\,\frac{ag+bc}{c}, \quad a>0,\ b>0,\ c>0,\ g>0.$

50. $\displaystyle\int_0^\infty \ln\,\frac{a^2+b^2x^2}{x^2}\,\frac{dx}{c^2-g^2x^2} = \frac{1}{cg}\arctan\frac{ag}{bc}, \quad a>0,\ b>0,\ c>0,\ g>0.$

51. $\displaystyle\int_0^\infty \ln\,\frac{1+x^2}{x^2}\,\frac{x^2\,dx}{(1+x^2)^2} = \frac{\pi}{4}(\ln\,4-1).$

52. $\displaystyle\int_0^\infty \ln\,\frac{1+2x\cos t+x^2}{(1+x)^2}\,x^{p-1}\,dx = -\frac{2\pi(1-\cos pt)}{p\,\sin p\pi}, \quad 0<|p|<1,\ |t|<\pi.$

53. $\displaystyle\int_0^\infty \ln\,\frac{(x+1)(x+a^2)}{(x+a)^2}\,\frac{dx}{x} = (\ln\,a)^2, \quad a>0.$

54. $\displaystyle\int_0^\infty \ln\,(a^3-x^3)\frac{dx}{x^3} = \infty.$

55. $\displaystyle\int_0^\infty \ln\,(1+x^3)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\ln\,3.$

56. $\displaystyle\int_0^\infty \ln\,(1+x^3)\frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln\,3-\frac{\pi^2}{9}.$

57. $\displaystyle\int_0^\infty \ln\,(1+x^3)\frac{x\,dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln\,3+\frac{\pi^2}{9}.$

58. $\displaystyle\int_0^\infty \ln\,(1+x^3)\frac{1-x}{1+x^3}\,dx = -\frac{2}{9}\,\pi^2.$

59. $\displaystyle\int_0^\infty \ln\,\left|1-\frac{n^3}{a^3}\right|\frac{dx}{x^3} = -\frac{\pi\sqrt{3}}{6a^2}.$

60. $\displaystyle\int_0^\infty \ln\,\frac{1+x^3}{x^3}\,\frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln\,3+\frac{\pi^2}{9}.$

61. $\displaystyle\int_0^\infty \ln\,\frac{1+x^3}{x^3}\,\frac{x\,dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln\,3-\frac{\pi^2}{9}.$

62. $$\int_0^\infty \ln x \, \ln(1+a^2x^2)\frac{dx}{x^2} = \pi a(1-\ln a), \quad a>0.$$

63. $$\int_0^\infty \ln(1+c^2x^2)\ln(a^2+b^2x^2)\frac{dx}{x^2} = 2\pi\left[\left(c+\frac{b}{a}\right)\ln(b+ac) - \frac{b}{a}\ln b - c\ln c\right],$$ $$a>0,\ b>0,\ c>0.$$

64. $$\int_0^\infty \ln(1+c^2x^2)\ln\left(a^2+\frac{b^2}{x^2}\right)\frac{dx}{x^2} = 2\pi\left[\frac{a+bc}{b}\ln(a+bc) - \frac{a}{b}\ln a - c\right],$$ $$a>0,\ a+bc>0.$$

65. $$\int_0^\infty \ln x \ln\frac{1+a^2x^2}{1+b^2x^2}\,\frac{dx}{x^2} = \pi(a-b) + \pi\ln\frac{b^b}{a^a}, \quad a>0,\ b>0.$$

66. $$\int_0^\infty \ln x \ln\frac{a^2+2bx+x^2}{a^2-2bx+x^2}\,\frac{dx}{x} = 2\pi\ln a\,\arcsin\frac{b}{a}, \quad a\ge |b|.$$

67. $$\int_0^\infty \ln(1+x)\,\frac{x\ln x - x - a}{(x+a)^2}\,\frac{dx}{x} = \frac{(\ln a)^2}{2(a-1)}, \quad a>0.$$

68. $$\int_0^\infty \ln(1-x)^2\,\frac{x\ln x - x - a}{(x+a)^2}\,\frac{dx}{x} = \frac{\pi^2+(\ln a)^2}{1+a}, \quad a>0.$$

69. $$\int_0^\infty \left[\frac{(q-1)x}{(1+x)^2} - \frac{1}{x+1} + \frac{1}{(1+x)^q}\right]\frac{dx}{x\ln(1+x)} = \ln\Gamma(q), \quad q>0.$$

70. $$\int_0^\infty \ln\frac{\sqrt{1+x^2}+a}{\sqrt{1+x^2}-a}\,\frac{dx}{\sqrt{1+x^2}} = \pi\,\arcsin a, \quad |a|<1].$$

71. $$\int_0^\infty \ln(1+x^s)\left[\frac{(p-s)x^p-(q-s)x^q}{\ln x} + \frac{x^q-x^p}{(\ln x)^2}\right]\frac{dx}{x^{s+1}} = s\,\ln\left(\tan\frac{q\pi}{2s}\cot\frac{p\pi}{2s}\right),$$ $$p<s,\ q<s.$$

72. $$\int_0^\infty x^{p-1}\ln(1+x^s)\,dx = \frac{1}{s}\int_0^\infty t^{p/s-1}\ln(1+t)\,dt = \frac{\pi}{p\sin(p\pi/s)}.$$

73. $$\int_0^\infty \ln(1-e^{-2a\pi x})\frac{dx}{1+x^2} = -\pi\left[\frac{1}{2}\ln 2a\pi + a(\ln a - 1) - \ln\Gamma(a+1)\right], \quad a>0.$$

74. $\displaystyle\int_0^\infty \ln\left(1+e^{-2a\pi x}\right)\frac{dx}{1+x^2} = \pi\left[\ln\Gamma(2a) - \ln\Gamma(a) + a(1-\ln a) - \left(2a-\frac{1}{2}\right)\ln 2\right], \quad a>0.$

75. $\displaystyle\int_0^\infty \ln\frac{a+be^{-px}}{a+be^{-qx}}\,\frac{dx}{x} = \ln\frac{a}{a+b}\ln\frac{p}{q}, \quad \frac{b}{a} > -1,\, pq>0.$

76. $\displaystyle\int_0^\infty \ln\cosh x\,\frac{dx}{1-x^2} = 0.$

77. $\displaystyle\int_0^\infty \frac{\ln\sin^2 ax}{b^2+x^2}\,dx = \frac{\pi}{b}\ln\frac{1-e^{-2ab}}{2}, \quad a>0,\, b>0.$

78. $\displaystyle\int_0^\infty \frac{\ln\cos^2 ax}{b^2+x^2}\,dx = \frac{\pi}{b}\ln\frac{1+e^{-2ab}}{2}, \quad a>0,\, b>0.$

79. $\displaystyle\int_0^\infty \frac{\ln\sin^2 ax}{b^2-x^2}\,dx = -\frac{\pi^2}{2b} + a\pi, \quad a>0,\, b>0.$

80. $\displaystyle\int_0^\infty \frac{\ln\cos^2 ax}{b^2-x^2}\,dx = a\pi, \quad a>0.$

81. $\displaystyle\int_0^\infty \frac{\ln\cos^2 x}{x^2}\,dx = -\pi.$

82. $\displaystyle\int_0^\infty \ln\left(1\pm 2p\cos\beta x + p^2\right)\frac{dx}{q^2+x^2} = \begin{cases} \dfrac{\pi}{q}\ln\left(1\pm pe^{-\beta q}\right), & p^2<1, \\ \dfrac{\pi}{q}\ln\left(p\pm e^{-\beta q}\right), & p^2>1. \end{cases}$

83. $\displaystyle\int_0^\infty \frac{\ln\tan^2 ax}{b^2+x^2}\,dx = \frac{\pi}{b}\ln\tanh ab, \quad a>0,\, b>0.$

84. $\displaystyle\int_0^\infty \ln\left(\frac{1+\tan x}{1-\tan x}\right)^2\frac{dx}{x} = \frac{\pi^2}{2}.$

85. $\displaystyle\int_0^\infty \ln\left(\frac{1+\sin x}{1-\sin x}\right)^2\frac{dx}{x} = \pi^2.$

86. $$\int_0^\infty \ln \frac{1+2a\cos px+a^2}{1+2a\cos qx+a^2}\,\frac{dx}{x} = \begin{cases} \ln(1+a)\ln\dfrac{q^2}{p^2}, & -1<a\le 1, \\ \ln\left(1+\dfrac{1}{a}\right)\ln\dfrac{q^2}{p^2}, & a<-1 \text{ or } a\ge 1. \end{cases}$$

87. $$\int_0^\infty \ln\left(a^2\sin^2 px+b^2\cos^2 px\right)\frac{dx}{c^2+x^2} = \frac{\pi}{c}\left[\ln\left(a\sinh cp+b\cosh cp\right)-cp\right],$$
$$a>0,\ b>0,\ c>0,\ p>0.$$

88. $$\int_1^\infty \ln\ln x\,\frac{x^{n-2}\,dx}{1+x^2+x^4+\cdots+x^{2n-2}}$$
$$= \begin{cases} \dfrac{\pi}{2n}\tan\dfrac{\pi}{2n}\ln 2\pi+\dfrac{\pi}{n}\displaystyle\sum_{k=1}^{n-1}(-1)^{k-1}\sin\dfrac{k\pi}{n}\ln\dfrac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)}, & n \text{ is even}, \\ \dfrac{\pi}{2n}\tan\dfrac{\pi}{2n}\ln\pi+\dfrac{\pi}{n}\displaystyle\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\dfrac{k\pi}{n}\ln\dfrac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)}, & n \text{ is odd}. \end{cases}$$

89. $$\int_0^\infty \ln\left[a^2+(\ln x)^2\right]x^{\mu-1}\,dx = \frac{2}{\mu}\left[-\cos a\mu\,\mathrm{Ci}(a\mu)-\sin a\mu\,\mathrm{Si}(a\mu)+\ln a\right], \quad a>0,\ \Re\mu\}>0.$$