T1.24. Integrand involving $\sqrt{a \pm b \sin x}$ and $\sqrt{a \pm b \cos x}$.

Notation used: $\alpha = \arcsin\sqrt{\dfrac{1-\sin x}{2}}$, $\beta = \arcsin\sqrt{\dfrac{b(1-\sin x)}{a+b}}$,

$\gamma = \arcsin\sqrt{\dfrac{b(1-\cos x)}{a+b}}$, $\delta = \sqrt{\dfrac{(a+b)(1-\cos x)}{2(a-b\cos x)}}$, $r = \sqrt{\dfrac{2b}{a+b}}$.

1. $$\int \frac{dx}{\sqrt{a+b\sin x}} = \begin{cases} \dfrac{-2}{\sqrt{a+b}}F(\alpha,\, r), & a>b>0,\ -\dfrac{\pi}{2} \le x < \dfrac{\pi}{2}, \\ -\sqrt{\dfrac{2}{b}}\,F\left(\beta,\, \dfrac{1}{r}\right) & 0<|a|<b,\ -\arcsin\dfrac{a}{b} < x < \dfrac{\pi}{2}. \end{cases}$$

2. $$\int \frac{\sin x\, dx}{\sqrt{a+b\sin x}} = \begin{cases} \dfrac{2a}{b\sqrt{a+b}}F(\alpha,\, r) - \dfrac{2\sqrt{a+b}}{b}\,E(\alpha,\, r) & a>b>0,\ -\dfrac{\pi}{2} \le x < \dfrac{\pi}{2}, \\ \sqrt{\dfrac{2}{b}}\left\{F\left(\beta,\, \dfrac{1}{r}\right) - 2E\left(\beta,\, \dfrac{1}{r}\right)\right\}, & 0<|a|<b,\ -\arcsin\dfrac{a}{b} < x < \dfrac{\pi}{2}. \end{cases}$$

3. $$\int \frac{\sin^2 x\, dx}{\sqrt{a+b\sin x}}$$

$$= \begin{cases} \dfrac{4a\sqrt{a+b}}{3b^2}E(\alpha,\, r) - \dfrac{2(2a^2+b^2)}{3b^2\sqrt{a+b}}F(\alpha,\, r) - \dfrac{2}{3b}\cos x\sqrt{a+b\sin x}, & a>b>0,\ -\dfrac{\pi}{2} \le x < \dfrac{\pi}{2}, \\ \sqrt{\dfrac{2}{b}}\left\{\dfrac{4a}{3b}E\left(\beta,\, \dfrac{1}{r}\right) - \dfrac{2a+b}{3b}F\left(\beta,\, \dfrac{1}{r}\right)\right\} - \dfrac{2}{3b}\cos x\,\sqrt{a+b\sin x}, & 0<|a|<b,\ -\arcsin\dfrac{a}{b} < x < \dfrac{\pi}{2}. \end{cases}$$

4. $$\int \frac{dx}{\sqrt{a+b\cos x}} = \begin{cases} \dfrac{2}{\sqrt{a+b}}\,F\left(\dfrac{x}{2},\, r\right) & a>b>0,\ 0 \le x \le \pi, \\ \sqrt{\dfrac{2}{b}}\,F\left(\gamma,\, \dfrac{1}{r}\right), & b \ge |a| \ge 0,\ 0 \le x < \arccos\left(-\dfrac{a}{b}\right). \end{cases}$$

5. $$\int \frac{dx}{\sqrt{a-b\cos x}} = \frac{2}{\sqrt{a+b}}F(\delta,\, r), \quad a>b>0,\ 0 \le x \le \pi.$$

6. $$\int \frac{\cos x\, dx}{\sqrt{a+b\cos x}} = \begin{cases} \dfrac{2}{b\sqrt{a+b}}\left\{(a+b)E\left(\dfrac{x}{2},\, r\right) - aF\left(\dfrac{x}{2},\, r\right)\right\}, & a>b>0,\ 0 \le x \le \pi, \\ \sqrt{\dfrac{2}{b}}\left\{2E\left(\gamma,\, \dfrac{1}{r}\right) - F\left(\gamma,\, \dfrac{1}{r}\right)\right\}, & b>|a|>0,\ 0 \le x < \arccos\left(-\dfrac{b}{a}\right). \end{cases}$$

7. $$\int \frac{\cos x\, dx}{\sqrt{a-b\cos x}} = \frac{2}{b\sqrt{a+b}}\{(b-a)\Pi(\delta,\, r^2,\, r) + a\,F(\delta,\, r)\}, \quad a>b>0,\ 0 \le x \le \pi.$$

8. $$\int \frac{\cos^2 x\,dx}{\sqrt{a+b\cos x}}$$
$$= \begin{cases} \dfrac{2}{3b^2\sqrt{a+b}}\left\{(2a^2+b^2)F\left(\dfrac{x}{2},r\right)-2a(a+b)E\left(\dfrac{x}{2},r\right)\right\}+\dfrac{2}{3b}\sin x\sqrt{a+b\cos x}, \\ \qquad\qquad a>b>0,\ 0\le x\le\pi, \\ \dfrac{1}{3b}\sqrt{\dfrac{2}{b}}\left\{(2a+b)F\left(\gamma,\dfrac{1}{r}\right)-4aE\left(\gamma,\dfrac{1}{r}\right)\right\}+\dfrac{2}{3b}\sin x\sqrt{a+b\cos x}, \\ \qquad\qquad b\ge|a|>0,\ 0\le x<\arccos\left(-\frac{a}{b}\right). \end{cases}$$

9. $$\int \frac{\cos^2 x\,dx}{\sqrt{a-b\cos x}} = \frac{2}{3b^2\sqrt{a+b}}\{(2a^2+b^2)F(\delta,r)-2a(a+b)E(\delta,r)\} + \frac{2}{3b}\sin x\frac{a+b\cos x}{\sqrt{a-b\cos x}}, \quad a>b>0,\ 0\le x<\pi.$$

10. $$\int \frac{\tan^2 x\,dx}{\sqrt{a+b\sin x}}$$
$$= \begin{cases} \dfrac{1}{\sqrt{a+b}}F(\alpha,r)+\dfrac{a}{(a-b)\sqrt{a+b}}E(\alpha,r)-\dfrac{b-a\sin x}{(a^2-b^2)\cos x}\sqrt{a+b\sin x}, \\ \qquad\qquad 0<b<a,\ -\frac{\pi}{2}<x<\frac{\pi}{2}, \\ \sqrt{\dfrac{2}{b}}\left\{\dfrac{2a+b}{2(a+b)}F\left(\beta,\dfrac{1}{r}\right)+\dfrac{ab}{a^2-b^2}E\left(\beta,\dfrac{1}{r}\right)\right\}-\dfrac{b-a\sin x}{(a^2-b^2)\cos x}\sqrt{a+b\sin x}, \\ \qquad\qquad 0<|a|<b,\ -\arcsin\frac{a}{b}<x<\frac{\pi}{2}. \end{cases}$$

11. $$\int \frac{1-\sin x}{1+\sin x}\cdot\frac{dx}{\sqrt{a+b\sin x}} = \frac{2}{a-b}\left\{\sqrt{a+b}E(\alpha,r)-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\sqrt{a+b\sin x}\right\}, \quad 0<b<a,\ -\frac{\pi}{2}\le x<\frac{\pi}{2}.$$

12. $$\int \frac{1-\cos x}{1+\cos x}\frac{dx}{\sqrt{a+b\cos x}} = \frac{2}{a-b}\tan\frac{x}{2}\sqrt{a+b\cos x}-\frac{2\sqrt{a+b}}{a-b}E\left(\frac{x}{2},r\right), \quad a>b>0,\ 0\le x<\pi.$$

13. $$\int \frac{dx}{(2-p^2+p^2\sin x)\sqrt{a+b\sin x}} = -\frac{1}{a+b}\Pi(\alpha,p^2,r), \quad 0<b<a,\ -\frac{\pi}{2}\le x<\frac{\pi}{2}.$$

14. $$\int \frac{dx}{(a+b-p^2b+p^2b\sin x)\sqrt{a+b\sin x}} = -\frac{1}{a+b}\sqrt{\frac{2}{b}}\Pi\left(\beta,p^2,\frac{1}{r}\right), \quad 0<|a|<b,\ -\arcsin\frac{a}{b}<x<\frac{\pi}{2}.$$

15. $$\int \frac{dx}{(2-p^2+p^2\cos x)\sqrt{a+b\cos x}} = \frac{1}{\sqrt{a+b}}\Pi\left(\frac{x}{2},p^2,r\right), \quad a>b>0,\ 0\le x<\pi.$$

16. $$\int \frac{dx}{(a+b-p^2b+p^2b\cos x)\sqrt{a+b\cos x}} = \frac{\sqrt{2}}{(a+b)\sqrt{b}}\Pi\left(\gamma,p^2,\frac{1}{r}\right), \quad b\ge|a|>0,\ 0\le x<\arccos\left(-\frac{a}{b}\right).$$

17. $\displaystyle\int \frac{dx}{\sqrt{(a+b\,\sin x)^3}}$

$$= \begin{cases} \dfrac{2b\,\cos x}{(a^2-b^2)\sqrt{a+b\,\sin x}} - \dfrac{2}{(a-b)\sqrt{a+b}}\,E(\alpha,\,r), \quad 0<b<a,\ -\dfrac{\pi}{2}\le x<\dfrac{\pi}{2}, \\ \sqrt{\dfrac{2}{b}}\left\{\dfrac{2b}{b^2-a^2}E\left(\beta,\,\dfrac{1}{r}\right) - \dfrac{1}{a+b}F\left(\beta,\,\dfrac{1}{r}\right)\right\} + \dfrac{2b}{b^2-a^2}\,\dfrac{\cos x}{\sqrt{a+b\,\sin x}}, \\ \qquad\qquad 0<|a|<b,\ -\arcsin\frac{a}{b}<x<\frac{\pi}{2}. \end{cases}$$

18. $\displaystyle\int \frac{dx}{\sqrt{(a+b\,\sin x)^5}}$

$$= \begin{cases} \dfrac{2}{3(a^2-b^2)^2\sqrt{a+b}}\{(a^2-b^2)F(\alpha,\,r) - 4a(a+b)E(\alpha,\,r)\} + \dfrac{2b(5a^2-b^2+4ab\,\sin x)}{3(a^2-b^2)^2\sqrt{(a+b\,\sin x)^3}}\,\cos x, \\ \qquad\qquad 0<b<a,\ -\frac{\pi}{2}\le x<\frac{\pi}{2}, \\ -\dfrac{1}{3(a^2-b^2)^2}\sqrt{\dfrac{2}{b}}\left\{(3a-b)(a-b)F\left(\beta,\,\dfrac{1}{r}\right) + 8abE\left(\beta,\,\dfrac{1}{r}\right)\right\} \\ \qquad + \dfrac{2b[a^2-b^2+4a(a+b\,\sin x)]}{3(a^2-b^2)^2\sqrt{(a+b\,\sin x)^3}}\,\cos x, \quad 0<|a|<b,\ -\arcsin\dfrac{a}{b}<x<\dfrac{\pi}{2}. \end{cases}$$

19. $\displaystyle\int \frac{dx}{\sqrt{(a+b\,\cos x)^3}}$

$$= \begin{cases} \dfrac{2}{(a-b)\sqrt{a+b}}E\left(\dfrac{x}{2},\,r\right) - \dfrac{2b}{a^2-b^2}\,\dfrac{\sin x}{\sqrt{a+b\,\cos x}}, \quad a>b>0,\ 0\le x\le\pi, \\ \dfrac{1}{a^2-b^2}\sqrt{\dfrac{2}{b}}\left\{(a-b)F\left(\gamma,\,\dfrac{1}{r}\right) + 2bE\left(\gamma,\,\dfrac{1}{r}\right)\right\} + \dfrac{2b}{b^2-a^2}\,\dfrac{\sin x}{\sqrt{a+b\,\cos x}}, \\ \qquad\qquad b\ge|a|>0,\ 0\le x<\arccos\left(-\frac{a}{b}\right). \end{cases}$$

20. $\displaystyle\int \frac{dx}{\sqrt{(a-b\,\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}E(\delta,\,r), \quad a>b>0,\ 0\le x\le\pi.$

21. $\displaystyle\int \frac{dx}{\sqrt{(a+b\,\cos x)^5}}$

$$= \begin{cases} \dfrac{2\sqrt{a+b}}{3(a^2-b^2)^2}\left\{4aE\left(\dfrac{x}{2},\,r\right) - (a-b)\,F\left(\dfrac{x}{2},\,r\right)\right\} - \dfrac{2b}{3(a^2-b^2)^2}\,\dfrac{5a^2-b^2+4ab\,\cos x}{\sqrt{(a+b\,\cos x)^3}}\,\sin x, \\ \qquad\qquad a>b>0,\ 0\le x\le\pi, \\ \dfrac{1}{3(a^2-b^2)^2}\sqrt{\dfrac{2}{b}}\left\{(a-b)(3a-b)F\left(\gamma,\,\dfrac{1}{r}\right) + 8abE\left(\gamma,\,\dfrac{1}{r}\right)\right\} + \dfrac{2b(5a^2-b^2+4ab\,\cos x)\,\sin x}{3(a^b-b^2)^2\sqrt{(a+b\,\cos x)^3}}, \\ \qquad\qquad b\ge|a|>0,\ 0\le x<\arccos\left(-\frac{a}{b}\right). \end{cases}$$

22. $\displaystyle\int \sqrt{1-\cos ax}\,dx = -\frac{2\sin ax}{a\sqrt{1-\cos ax}} = -\frac{2\sqrt{2}}{a}\,\cos\frac{ax}{2}.$

23. $\displaystyle\int \sqrt{1+\cos ax}\,dx = \frac{2\sin ax}{a\sqrt{1+\cos ax}} = \frac{2\sqrt{2}}{a}\,\sin\frac{ax}{2}.$

24. $$\int \sqrt{a+b\,\cos x}\,dx = \begin{cases} 2\sqrt{a+b}E\left(\frac{x}{2},\,r\right), & a>b>0,\ 0\le x\le\pi, \\ \sqrt{\frac{2}{b}}\left\{(a-b)F\left(\gamma,\,\frac{1}{r}\right)+2bE\left(\gamma,\,\frac{1}{r}\right)\right\}, & b\ge|a|>0,\ 0\le x<\arccos\left(-\frac{a}{b}\right). \end{cases}$$

25. $$\int \sqrt{a-b\,\cos x}\,dx = 2\sqrt{a+b}E(\delta,\,r)-\frac{2b\,\sin x}{\sqrt{a-b\,\cos x}}, \quad a>b>0,\ 0\le x\le\pi.$$

26. $$\int \sqrt{1+\sin x}\,dx = \begin{cases} 2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right), & (8k-1)\pi/2<x\le(8k+3)\pi/2, \\ \text{or} & \\ -2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right), & (8k+3)\pi/2<x\le(8k+7)\pi/2,\ k\in\mathrm{N}. \end{cases}$$

27. $$\int \sqrt{1-\sin x}\,dx = \begin{cases} 2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right), & (8k-3)\pi/2<x\le(8k+1)\pi/2, \\ \text{or} & \\ -2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right), & (8k+1)\pi/2<x\le(8k+6)\pi/2,\ k\in\mathrm{N}. \end{cases}$$

28. $$\int \frac{dx}{\sqrt{1-\sin x}} = \begin{cases} 2\ln\tan\left(\frac{x}{4}-\frac{\pi}{8}\right), & (8k+1)\pi/2<x\le(8k+5)\pi/2, \\ \text{or} & \\ -2\ln\tan\left(\frac{x}{4}-\frac{\pi}{8}\right), & (8k+5)\pi/2<x\le(8k+9)\pi/2,\ k\in\mathrm{N}. \end{cases}$$

29. $$\int \frac{dx}{\sqrt{1+\sin x}} = \begin{cases} 2\ln\tan\left(\frac{x}{4}+\frac{\pi}{8}\right), & (8k-1)\pi/2<x\le(8k+3)\pi/2, \\ \text{or} & \\ -2\ln\tan\left(\frac{x}{4}+\frac{\pi}{8}\right), & (8k+3)\pi/2<x\le(8k+7)\pi/2,\ k\in\mathrm{N}. \end{cases}$$

30. $$\int \frac{dx}{\sqrt{1-\cos x}} = \begin{cases} 2\ln\tan\frac{x}{4}, & 4k\pi<x\le(4k+2)\pi, \\ \text{or} & \\ -2\ln\tan\frac{x}{4}, & (4k+2)\pi<x\le(4k+4)\pi,\ k\in\mathrm{N}. \end{cases}$$

31. $$\int \frac{dx}{\sqrt{1+\cos x}} = \begin{cases} 2\ln\tan\left(\frac{x+\pi}{4}\right), & (4k-1)\pi<x\le(4k+1)\pi, \\ \text{or} & \\ -2\ln\tan\left(\frac{x+\pi}{4}\right), & (4k+1)\pi<x\le(4k+3)\pi,\ k\in\mathrm{N}. \end{cases}$$

32. $$\int \frac{\sqrt{a-b\,\cos x}}{1+p\,\cos x}\,dx = \frac{2(a-b)}{(1+p)\sqrt{a+b}}\,\Pi\left(\delta,\,\frac{2ap}{(a+b)(1+p)},\,r\right), \quad a>b>0,\ 0\le x\le\pi,\ p\ne-1.$$

33. $$\int \sqrt{\frac{a-b\,\cos x}{1+p\,\cos x}}\,dx = \frac{2(a-b)}{\sqrt{(1+p)(a+b)}}\Pi\left(\delta,\,-r^2,\,\sqrt{\frac{2(ap+b)}{(1+p)(a+b)}}\right), \quad a>b>0,\ 0\le x\le\pi,\ p\ne-1.$$

34. $$\int \frac{\tan x\,dx}{\sqrt{a+b\,\tan^2 x}} = \frac{1}{\sqrt{b-a}}\,\arccos\left(\frac{\sqrt{b-a}}{\sqrt{b}}\,\cos x\right), \quad b>a,\ b>0.$$