T1.36. Integrand involving logarithm functions.

1. $\int \ln ax\,dx = x \ln ax - x.$
2. $\int x \ln ax\,dx = \frac{x^2}{2} \ln ax - \frac{x^2}{4}.$
3. $\int x^2 \ln ax\,dx = \frac{x^3}{3} \ln ax - \frac{x^3}{9}.$
4. $\int x^n \ln x\,dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right].$
5. $\int x^n (\ln x)^2\,dx = x^{n+1}\left[\frac{\ln^2 x}{n+1} - \frac{2\ln x}{(n+1)^2} + \frac{2}{(n+1)^3}\right].$
6. $\int x^n (\ln x)^3\,dx = x^{n+1}\left[\frac{\ln^3 x}{n+1} - \frac{3\ln^2 x}{(n+1)^2} + \frac{6\ln x}{(n+1)^3} - \frac{6}{(n+1)^4}\right].$
7. $\int x^n (\ln x)^m\,dx = \frac{x^{n+1}(\ln x)^m}{n+1} - \frac{m}{n+1}\int x^n (\ln x)^{m-1}\,dx.$
8. $\int \frac{(\ln x)^m\,dx}{x} = \frac{(\ln x)^{m+1}}{m+1}.$
9. $\int \frac{dx}{x \ln x} = \ln(\ln x).$
10. $\int x^n (\ln x)^m\,dx = \frac{x^{n+1}}{m+1}\sum_{k=0}^{m}(-1)^k (m+1)m(m-1)\ldots(m-k+1)\frac{(\ln x)^{m-k}}{(n+1)^{k+1}}.$
11. $\int (\ln x)^m\,dx = x(\ln x)^m - m\int (\ln x)^{m-1}\,dx$
$$= \frac{x}{m+1}\sum_{k=0}^{m}(-1)^k (m+1)m(m+1)\ldots(m-k+1)(\ln x)^{m-k}, \quad m > 0.$$
12. $\int (a+bx) \ln x\,dx = \left[\frac{(a+bx)^2}{2b} - \frac{a^2}{2b}\right] \ln x - \left(ax + \frac{1}{4}bx^2\right).$
13. $\int (a+bx)^2 \ln x\,dx = \frac{1}{3b}[(a+bx)^3 - a^3] \ln x - \left(a^2 x + \frac{abx^2}{2} + \frac{b^2 x^3}{9}\right).$

14. $\int (a+bx)^3 \ln x\,dx = \frac{1}{4b}[(a+bx)^4 - a^4] \ln x - \left(a^3x + \frac{3}{4}a^2bx^2 + \frac{1}{3}ab^2x^3 + \frac{1}{16}b^3x^4\right).$

15. $\int (a+bx)^m \ln x\,dx = \frac{1}{(m+1)b}\left[(a+bx)^{m+1} \ln x - \int \frac{(a+bx)^{m+1}\,dx}{x}\right].$

16. $\int (a+bx)^m \ln x\,dx = \frac{1}{(m+1)b}[(a+bx)^{m+1} - a^{m+1}] \ln x - \sum_{k=0}^{m} \frac{\binom{m}{k} a^{m-k} b^k x^{k+1}}{(k+1)^2}.$

17. $\int \frac{x^n\,dx}{(\ln x)^m} = -\frac{x^{n+1}}{(m-1)(\ln x)^{m-1}} + \frac{n+1}{m-1}\int \frac{x^n\,dx}{(\ln x)^{m-1}}.$

18. $\int \frac{x^n\,dx}{\ln x} = \operatorname{li}(x^{n+1}).$

19. $\int \frac{\ln x\,dx}{(a+bx)^m} = \frac{1}{b(m-1)}\left[-\frac{\ln x}{(a+bx)^{m-1}} + \int \frac{dx}{x(a+bx)^{m-1}}\right].$

20. $\int \frac{\ln x\,dx}{a+bx} = \frac{1}{b} \ln x \ln(a+bx) - \frac{1}{b}\int \frac{\ln(a+bx)\,dx}{x}.$

21. $\int \frac{\ln x\,dx}{(a+bx)^2} = -\frac{\ln x}{b(a+bx)} + \frac{1}{ab} \ln \frac{x}{a+bx}.$

22. $\int \frac{\ln x\,dx}{(a+bx)^3} = -\frac{\ln x}{2b(a+bx)^2} + \frac{1}{2ab(a+bx)} + \frac{1}{2a^2b} \ln \frac{x}{a+bx}.$

23. $\int \frac{\ln x\,dx}{\sqrt{a+bx}} = \begin{cases} \frac{2}{b}\left\{(\ln x - 2)\sqrt{a+bx} + \sqrt{a} \ln \frac{\sqrt{a+bx}+\sqrt{a}}{\sqrt{a+bx}-\sqrt{a}}\right\}, & a > 0, \\ \frac{2}{b}\left\{(\ln x - 2)\sqrt{a+bx} + 2\sqrt{-a} \arctan\sqrt{\frac{a+bx}{-a}}\right\}, & a < 0. \end{cases}$

24. $\int x^m \ln(a+bx)\,dx = \frac{1}{m+1}\left[x^{m+1} \ln(a+bx) - b\int \frac{x^{m+1}\,dx}{a+bx}\right].$

25. $\int \frac{\ln(a+bx)}{x} = \ln a + \ln x + \frac{bx}{a}\Phi\left(-\frac{bx}{a}, 2, 1\right), \quad a > 0.$

26. $\int x \ln(a+bx)\,dx = \frac{1}{2}\left[x^2 - \frac{a^2}{b^2}\right] \ln(a+bx) - \frac{1}{2}\left[\frac{x^2}{2} - \frac{ax}{b}\right].$

27. $\int x^2 \ln(a+bx)\,dx = \frac{1}{3}\left[x^3 + \frac{a^3}{b^3}\right] \ln(a+bx) - \frac{1}{3}\left[\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2}\right].$

28. $\int x^3 \ln(a+bx)\,dx = \frac{1}{4}\left[x^4 - \frac{a^4}{b^4}\right] \ln(a+bx) - \frac{1}{4}\left[\frac{x^4}{4} - \frac{ax^3}{3b} + \frac{a^2x^2}{2b^2} - \frac{a^3x}{b^3}\right].$

29. $\int x^m \ln(a+bx)\,dx = \frac{1}{m+1}\left[x^{m+1} - \frac{(-a)^{m+1}}{b^{m+1}}\right] \ln(a+bx) + \frac{1}{m+1}\sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} a^{k-1}}{(m-k+2)b^{k-1}}.$

30. $$\int x^{2n} \ln(x^2+a^2)\,dx = \frac{1}{2n+1}\left\{x^{2n+1}\ln(x^2+a^2) + (-1)^n 2a^{2n+1}\arctan\frac{x}{a} - 2\sum_{k=0}^{n}\frac{(-1)^{n-k}}{2k+1}a^{2n-2k}x^{2k+1}\right\}.$$

31. $$\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\arctan\frac{x}{a}.$$

32. $$\int x\ln(x^2+a^2)\,dx = \frac{1}{2}[(x^2+a^2)\ln(x^2+a^2) - x^2].$$

33. $$\int x^2\ln(x^2+a^2)\,dx = \frac{1}{3}\left[x^3\ln(x^2+a^2) - \frac{2}{3}x^3 + 2a^2x - 2a^3\arctan\frac{x}{a}\right].$$

34. $$\int x^3\ln(x^2+a^2)\,dx = \frac{1}{4}\left[(x^4-a^4)\ln(x^2+a^2) - \frac{x^4}{2} + a^2x^2\right].$$

35. $$\int x^4\ln(x^2+a^2)\,dx = \frac{1}{5}\left[x^5\ln(x^2+a^2) - \frac{2}{5}x^5 + \frac{2}{3}a^2x^3 - 2a^4x + 2a^5\arctan\frac{x}{a}\right].$$

36. $$\int x^{2n+1}\ln(x^2+a^2)\,dx = \frac{1}{2n+1}\left\{(x^{2n+2} + (-1)^n a^{2n+2})\ln(x^2+a^2) + \sum_{k=1}^{n+1}\frac{(-1)^{n-k}}{k}a^{2n-2k+2}x^{2k}\right\}.$$

37. $$\int \ln|x^2-a^2|\,dx = x\ln|x^2-a^2| - 2x + a\ln\left|\frac{x+a}{x-a}\right|.$$

38. $$\int x\ln|x^2-a^2|\,dx = \frac{1}{2}\{(x^2-a^2)\ln|x^2-a^2| - x^2\}.$$

39. $$\int x^2\ln|x^2-a^2|\,dx = \frac{1}{3}\left\{x^3\ln|x^2-a^2| - \frac{2}{3}x^3 - 2a^2x + a^3\ln\left|\frac{x+a}{x-a}\right|\right\}.$$

40. $$\int x^3\ln|x^2-a^2|\,dx = \frac{1}{4}\left\{(x^4-a^4)\ln|x^2-a^2| - \frac{x^4}{2} - a^2x^2\right\}.$$

41. $$\int x^4\ln|x^2-a^2|\,dx = \frac{1}{5}\left\{x^5\ln|x^2-a^2| - \frac{2}{5}x^5 - \frac{2}{3}a^2x^3 - 2a^4x + a^5\ln\left|\frac{x+a}{x-a}\right|\right\}.$$

42. $$\int x^{2n}\ln|x^2-a^2|\,dx = \frac{1}{2n+1}\left\{x^{2n+1}\ln|x^2-a^2| + a^{2n+1}\ln\left|\frac{x+a}{x-a}\right| - 2\sum_{k=0}^{n}\frac{1}{2k+1}a^{2n-2k}x^{2k+1}\right\}.$$

43. $$\int x^{2n+1} \ln|x^2 - a^2|\, dx = \frac{1}{2n+2}\left\{(x^{2n+2} - a^{2n+2}) \ln|x^2 - a^2| - \sum_{k=1}^{n+1} \frac{1}{k} a^{2n-2k+2} x^{2k}\right\}.$$