T1.29. Integrand involving trigonometric functions and exponentials.

1. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$$

2. $$\int e^{ax}\sin^2 bx\,dx = \frac{e^{ax}\sin bx(a\sin bx - 2b\cos bx)}{4b^2+a^2} + \frac{2b^2e^{ax}}{(4b^2+a^2)a}$$
$$= \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bx + b\sin 2bx\right).$$

3. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}.$$

4. $$\int e^{ax}\cos^2 bx\,dx = \frac{e^{ax}\cos bx(a\cos bx + 2b\sin bx)}{4b^2+a^2} + \frac{2b^2e^{ax}}{(4b^2+a^2)a}$$
$$= \frac{e^{ax}}{2a} + \frac{e^{ax}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bx + b\sin 2bx\right).$$

5. $$\int e^{ax}\sin^n bx\,dx = \frac{1}{a^2+n^2b^2}\left[(a\sin bx - nb\cos bx)e^{ax}\sin^{n-1} bx + n(n-1)b^2\int e^{ax}\sin^{n-2} bx\,dx\right].$$

6. $$\int e^{ax}\cos^n bx\,dx = \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)e^{ax}\cos^{n-1} bx + n(n-1)b^2\int e^{ax}\cos^{n-2} bx\,dx\right].$$

7. $$\int e^{ax}\sin^{2m} bx\,dx$$
$$= \sum_{k=0}^{m-1}\frac{(2m)!b^{2k}e^{ax}\sin^{2m-2k-1} bx}{(2m-2k)![a^2+(2m)^2b^2][a^2+(2m-2)^2b^2]\dots[a^2+(2m-2k)^2b^2]}$$
$$\times[a\sin bx - (2m-2k)b\cos bx] + \frac{(2m)!b^{2m}e^{ax}}{[a^2+(2m)^2b^2][a^2+(2m-2)^2b^2]\dots[a^2+4b^2]a}$$
$$= \binom{2m}{m}\frac{e^{ax}}{2^{2m}a} + \frac{e^{ax}}{2^{2m-1}}\sum_{k=1}^{m}(-1)^k\binom{2m}{m-k}\frac{1}{a^2+4b^2k^2}(a\cos 2bkx + 2bk\sin 2bkx).$$

8. $\displaystyle\int e^{ax}\,\sin^{2m+1}\,bx\,dx$

$$= \sum_{k=0}^{m} \frac{(2m+1)!b^{2k}e^{ax}\,\sin^{2m-2k}\,bx[a\,\sin bx-(2m-2k+1)b\,\cos bx]}{(2m-2k+1)![a^2+(2m+1)^2b^2][a^2+(2m-1)^2b^2]\ldots[a^2+(2m-2k+1)^2b^2]}$$

$$= \frac{e^{ax}}{2^{2m}}\sum_{k=0}^{m}\frac{(-1)^k}{a^2+(2k+1)^2b^2}\binom{2m+1}{m-k}[a\,\sin(2k+1)bx-(2k+1)b\,\cos(2k+1)bx].$$

9. $$\int e^{ax}\,\cos^{2m}\,bx\,dx = \sum_{k=0}^{m-1}\frac{(2m)!b^{2k}e^{ax}\,\cos^{2m-2k-1}\,bx[a\,\cos bx+(2m-2k)b\,\sin bx]}{(2m-2k)![a^2+(2m)^2b^2][a^2+(2m-2)^2b^2]\ldots[a^2+(2m-2k)^2b^2]}$$

$$+\frac{(2m)!b^{2m}e^{ax}}{[a^2+(2m)^2b^2][a^2+(2m-2)^2b^2]\ldots[a^2+4b^2]a}$$

$$= \binom{2m}{m}\frac{e^{ax}}{2^{2m}a}+\frac{e^{ax}}{2^{2m-1}}\sum_{k=1}^{m}\binom{2m}{m-k}\frac{1}{a^2+4b^2k^2}[a\,\cos 2kbx+2kb\,\sin 2kbx].$$

10. $\displaystyle\int e^{ax}\,\cos^{2m+1}\,bx\,dx$

$$= \sum_{k=0}^{m}\frac{(2m+1)!b^{2k}e^{ax}\,\cos^{2m-2k}\,bx}{(2m-2k+1)![a^2+(2m-1)^2b^2]\ldots[a^2+(2m-2k+1)^2b^2]}$$

$$= \frac{e^{ax}}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{m-k}\frac{1}{a^2+(2k+1)^2b^2}[a\,\cos(2k+1)bx+(2k+1)b\,\sin(2k+1)\,bx].$$

$$
11.\ \int e^{ax}\sin^p x\cos^q x\,dx = \begin{cases}
\dfrac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^p x\cos^{q-1} x[a\cos x+(p+q)\sin x] \\
\quad -pa\displaystyle\int e^{ax}\sin^{p-1} x\cos^{q-1} x\,dx+(q-1)(p+q)\int e^{ax}\sin^p x\cos^{q-2} x\,dx\Big\} \\
\quad\text{or} \\
\dfrac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1} x\cos^q x[a\sin x-(p+q)\cos x] \\
\quad +qa\displaystyle\int e^{ax}\sin^{p-1} x\cos^{q-1} x\,dx+(p-1)(p+q)\int e^{ax}\sin^{p-2} x\cos^q x\,dx\Big\}, \\
\quad\text{or} \\
\dfrac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1} x\cos^{q-1} x[a\sin x\cos x+q\sin^2 x-p\cos^2 x] \\
k\quad +q(q-1)\displaystyle\int e^{ax}\sin^p x\cos^{q-2} x\,dx+p(p-1)\int e^{ax}\sin^{p-2} x\cos^q x\,dx\Big\}, \\
\quad\text{or} \\
\dfrac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1} x\cos^{q-1} x(a\sin x\cos x+q\sin^2 x-p\cos^2 x) \\
\quad +q(q-1)\displaystyle\int e^{ax}\sin^{p-2} x\cos^{q-2} x\,dx \\
\quad -(q-p)(p+q-1)\displaystyle\int e^{ax}\sin^{p-2} x\cos^q x\,dx\Big\}, \\
\quad\text{or} \\
\dfrac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1} x\cos^{q-1} x(a\sin x\cos x+q\sin^2 x-p\cos^2 x) \\
\quad +p(p-1)\displaystyle\int e^{ax}\sin^{p-2} x\cos^{q-2} x\,dx \\
\quad +(q-p)(p+q-1)\displaystyle\int e^{ax}\sin^p x\cos^{q-2} x\,dx\Big\}.
\end{cases}
$$

$$
12.\ \int e^{ax}\sin bx\cos cx\,dx = \frac{e^{ax}}{2}\left[\frac{a\sin(b+c)x-(b+c)\cos(b+c)x}{a^2+(b+c)^2}\right.
$$
$$
\left.+\frac{a\sin(b-c)x-(b-c)\cos(b-c)x}{a^2+(b-c)^2}\right].
$$

$$
13.\ \int e^{ax}\sin^2 bx\cos cx\,dx = \frac{e^{ax}}{4}\left[2\frac{a\cos cx+c\sin cx}{a^2+c^2}-\frac{a\cos(2b+c)x+(2b+c)\sin(2b+c)x}{a^2+(2b+c)^2}\right.
$$
$$
\left.-\frac{a\cos(2b-c)x+(2b-c)\sin(2b-c)x}{a^2+(2b-c)^2}\right].
$$

14. $$\int e^{ax}\,\sin bx\,\cos^2\,cx\,dx=\frac{e^{ax}}{4}\left[2\frac{a\,\sin bx-b\,\cos bx}{a^2+b^2}+\frac{a\,\sin(b+2c)x-(b+2c)\,\cos(b+2c)x}{a^2+(b+2c)^2}+\frac{a\,\sin(b-2c)x-(b-2c)\,\cos(b-2c)x}{a^2+(b-2c)^2}\right].$$

15. $$\int\frac{e^{ax}\,dx}{\sin^p\,bx}=-\frac{e^{ax}[a\,\sin bx+(p-2)b\,\cos bx]}{(p-1)(p-2)b^2\,\sin^{p-1}\,bx}+\frac{a^2+(p-2)^2b^2}{(p-1)(p-2)b^2}\int\frac{e^{ax}\,dx}{\sin^{p-2}\,bx}.$$

16. $$\int\frac{e^{ax}\,dx}{\cos^p\,bx}=-\frac{e^{ax}[a\,\cos bx-(p-2)b\,\sin bx]}{(p-1)(p-2)b^2\,\cos^{p-1}\,bx}+\frac{a^2+(p-2)^2b^2}{(p-1)(p-2)b^2}\int\frac{e^{ax}\,dx}{\cos^{p-2}\,bx}.$$

17. $$\int e^{ax}\,\tan x\,dx=\frac{e^{ax}\,\tan x}{a}-\frac{1}{a}\int\frac{e^{ax}\,dx}{\cos^2\,x}.$$

18. $$\int e^{ax}\,\tan^2\,x\,dx=\frac{e^{ax}}{a}(a\,\tan x-1)-a\int e^{ax}\,\tan x\,dx.$$

19. $$\int e^{ax}\,\cot x\,dx=\frac{e^{ax}\,\cot x}{a}+\frac{1}{a}\int\frac{e^{ax}\,dx}{\sin^2\,x}.$$

20. $$\int e^{ax}\,\cot^2\,x\,dx=-\frac{e^{ax}}{a}(a\,\cot x+1)+a\int e^{ax}\,\cot x\,dx.$$

21. $$\int e^{ax}\,\tan^p\,x\,dx=\frac{e^{ax}}{p-1}\,\tan^{p-1}\,x-\frac{a}{p-1}\int e^{ax}\,\tan^{p-1}\,x\,dx-\int e^{ax}\,\tan^{p-2}\,x\,dx.$$

22. $$\int e^{ax}\,\cot^p\,x\,dx=-\frac{e^{ax}\,\cot^{p-1}x}{p-1}+\frac{a}{p-1}\int e^{ax}\,\cot^{p-1}\,x\,dx-\int e^{ax}\,\cot^{p-2}\,x\,dx.$$

23. $$\int xe^{ax}\,\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[\left(ax-\frac{a^2-b^2}{a^2+b^2}\right)\sin bx-\left(bx-\frac{2ab}{a^2+b^2}\right)\cos bx\right].$$

24. $$\int xe^{ax}\,\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[\left(ax-\frac{a^2-b^2}{a^2+b^2}\right)\cos bx+\left(bx-\frac{2ab}{a^2+b^2}\right)\sin bx\right].$$

25. $$\int x^2e^{ax}\,\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left\{\left[ax^2-\frac{2(a^2-b^2)}{a^2+b^2}x+\frac{2a(a^2-3b^2)}{(a^2+b^2)^2}\right]\sin bx-\left[bx^2-\frac{4ab}{a^2+b^2}x+\frac{2b(3a^2-b^2)}{(a^2+b^2)^2}\right]\cos bx\right\}.$$

26. $$\int x^2e^{ax}\,\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left\{\left[ax^2-\frac{2(a^2-b^2)}{a^2+b^2}x+\frac{2a(a^2-3b^2)}{(a^2+b^2)^2}\right]\cos bx+\left[bx^2-\frac{4ab}{a^2+b^2}x+\frac{2b(3a^2-b^2)}{(a^2+b^2)^2}\right]\sin bx\right\}.$$

27. $$\int x^p e^{ax} \sin bx \, dx = \frac{x^p e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) - \frac{p}{a^2+b^2}\int x^{p-1} e^{ax}(a \sin bx - b \cos bx)\, dx$$
$$= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}} \sin(bx+t) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1} e^{ax} \sin(bx+t)\, dx.$$

28. $$\int x^p e^{ax} \cos bx \, dx = \frac{x^p e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) - \frac{p}{a^2+b^2}\int x^{p-1} e^{ax}(a \cos bx + b \sin bx)\, dx$$
$$= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}} \cos(bx+t) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1} e^{ax} \cos(bx+t)\, dx.$$

29. $$\int x^n e^{ax} \sin bx \, dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)!(a^2+b^2)^{k/2}} \sin(bx+kt).$$

30. $$\int x^n e^{ax} \cos bx \, dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)!(a^2+b^2)^{k/2}} \cos(bx+kt).$$

In the four formulas (27-30) we have used $\sin t = -\frac{b}{\sqrt{a^2+b^2}}$ and $\cos t = \frac{a}{\sqrt{a^2+b^2}}$.

31. $$\int x\, e^{ax} \sin(bx)\, dx = \frac{x\, e^{ax}}{a^2+b^2} \left[a \sin(bx) - b \cos(bx)\right]$$
$$- \frac{e^{ax}}{(a^2+b^2)^2} \left[\left(a^2-b^2\right) \sin(bx) - 2ab \cos(bx)\right].$$

32. $$\int x\, e^{ax} \cos(bx)\, dx = \frac{x\, e^{ax}}{a^2+b^2} \left[a \cos(bx) + b \sin(bx)\right]$$
$$- \frac{e^{ax}}{(a^2+b^2)^2} \left[\left(a^2-b^2\right) \cos(bx) + 2ab \sin(bx)\right].$$

33. $$\int \frac{e^{ax}}{\sin^n x}\, dx = -\frac{e^{ax}\left[a \sin x + (n-2) \cos x\right]}{(n-1)(n-2) \sin^{n-1} x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\sin^{n-2} x}\, dx.$$

34. $$\int \frac{e^{ax}}{\cos^n x}\, dx = -\frac{e^{ax}\left[a \cos x - (n-2) \cos x\right]}{(n-1)(n-2) \cos^{n-1} x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\cos^{n-2} x}\, dx.$$

35. $$\int e^{ax} \tan^n x\, dx = \frac{e^{ax}}{(n-1)} \tan^{n-1} x - \frac{a}{n-1} \int e^{ax} \tan^{n-1} x\, dx - \int e^{ax} \tan^{n-2} x\, dx.$$

36. $$\int x^n\, b^{ax}\, dx = \frac{x^n b^{ax}}{a \ln b} - \frac{n}{a \ln b} \int x^{n-1} b^{ax}\, dx, \quad n > 0.$$