T1.35. Integrand involving trigonometric and hyperbolic functions.

1. $$\int \sinh(ax+b)\,\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\,\sin(cx+d) - \frac{c}{a^2+c^2}\sinh(ax+b)\,\cos(cx+d).$$

2. $$\int \sinh(ax+b)\,\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\,\cos(cx+d) + \frac{c}{a^2+c^2}\sinh(ax+b)\,\sin(cx+d).$$

3. $$\int \cosh(ax+b)\,\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\sinh(ax+b)\,\sin(cx+d) - \frac{c}{a^2+c^2}\cosh(ax+b)\,\cos(cx+d).$$

4. $$\int \cosh(ax+b)\,\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\sinh(ax+b)\,\cos(cx+d) + \frac{c}{a^2+c^2}\cosh(ax+b)\,\sin(cx+d).$$

5. $$\int \sinh x\,\sin x\,dx = \frac{1}{2}(\cosh x\,\sin x - \sinh x\,\cos x).$$

6. $$\int \sinh x\,\cos x\,dx = \frac{1}{2}(\cosh x\,\cos x + \sinh x\,\sin x).$$

7. $$\int \cosh x\,\sin x\,dx = \frac{1}{2}(\sinh x\,\sin x - \cosh x\,\cos x).$$

8. $$\int \cosh x\,\cos x\,dx = \frac{1}{2}(\sinh x\,\cos x + \cosh x\,\sin x).$$

9. $$\int \sinh^{2m}(ax+b)\,\sin^{2n}(cx+d)\,dx$$

$$= \frac{(-1)^m}{2^{2m+2n}}\binom{2m}{m}\binom{2n}{n}x + \frac{(-1)^{m+n}}{2^{2m+2n-1}}\binom{2m}{m}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k)c}\binom{2n}{k}\,\sin[(2n-2k)(cx+d)]$$

$$+\frac{(-1)^n}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$

$$\times\{(2m-2j)a\,\sinh[(2m-2j)(ax+b)]\,\cos[(2n-2k)(cx+d)]$$

$$+(2-2k)c\,\cosh[(2m-2j)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

10. $$\int \sinh^{2m}(ax+b)\,\sin^{2n-1}(cx+d)\,dx$$

$$= \frac{(-1)^{m+n}}{2^{2m+2n-2}}\binom{2m}{m}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k-1)c}\binom{2n-1}{k}\,\cos[(2n-2k-1)(cx+d)]$$

$$+\frac{(-1)^{n-1}}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$

$$\times\{(2m-2j)a\,\sinh[(2m-2j)(ax+b)]\,\sin[(2n-2k-1)(cx+d)]$$

$$-(2n-2k-1)c\,\cosh[(2m-2j)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]\}.$$

11. $$\int \sinh^{2m-1}(ax+b)\,\sin^{2n}(cx+d)\,dx$$

$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m-1}{j}}{(2m-2j-1)a}\,\cosh[(2m-2j-1)(ax+d)]$$

$$+\frac{(-1)^n}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$

$$\times\{(2m-2j-1)a\,\cosh[(2m-2j-1)(ax+b)]\,\cos[(2n-2k)(cx+d)]$$

$$+(2n-2k)c\,\sinh[(2m-2j-1)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

12. $$\int \sinh^{2m-1}(ax+b)\,\sin^{2n-1}(cx+d)\,dx$$

$$= \frac{(-1)^{n-1}}{2^{2m-2n-4}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$

$$\times\{(2m-2j-1)a\,\cosh[(2m-2j-1)(ax+b)]\,\sin[(2n-2k-1)(cx+d)]$$

$$-(2n-2k-1)c\,\sinh[(2m-2j-1)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]\}.$$

13. $\displaystyle\int \sinh^{2m}(ax+b)\,\cos^{2n}(cx+d)\,dx$

$$= \frac{(-1)^m}{2^{2m+2n}}\binom{2m}{m}\binom{2n}{n}x + \frac{\binom{2n}{n}}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m}{j}}{(2m-2j)a}\,\sinh[(2m-2j)(ax+b)]$$

$$+\frac{(-1)^m\binom{2m}{m}}{2^{2m+2n-1}}\sum_{k=0}^{n-1}\frac{\binom{2n}{k}}{(2n-2k)c}\,\sin[(2n-2k)(cx+d)]$$

$$+\frac{1}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^j\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$

$$\times\{(2m-2j)a\,\sinh[(2m-2j)(ax+b)]\,\cos[(2n-2k)(cx+d)]$$

$$+(2-2k)c\,\cosh[(2m-2j)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

14. $\displaystyle\int \sinh^{2m}(ax+b)\,\cos^{2n-1}(cx+d)\,dx$

$$= \frac{(-1)^m\binom{2m}{m}}{2^{2m+2n-2}}\sum_{k=0}^{n-1}\frac{\binom{2n-1}{k}}{(2n-2k-1)c}\,\sin[(2n-2k-2)(cx+d)]$$

$$+\frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^j\binom{2m}{j}\binom{2-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$

$$\times\{(2m-2j)a\,\sinh[(2m-2j)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]$$

$$+(2n-2k-1)c\,\cosh[(2m-2j)(ax+b)]\,\sin[(2n-2k-1)(cx+d)]\}.$$

15. $\displaystyle\int \sinh^{2m-1}(ax+b)\,\cos^{2n}(cx+d)\,dx$

$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m-1}{j}}{(2m-2j-1)a}\,\cosh[(2m-2j-1)(ax+d)]$$

$$+\frac{1}{2^{2m-2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^j\binom{2m}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$

$$\times\{(2m-2j-1)a\,\cosh[(2m-2j-1)(ax+b)]\,\cos[(2n-2k)(cx+d)]$$

$$+(2n-2k)c\,\sinh[(2m-2j-1)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

16. $\int \sinh^{2m-1}(ax+b)\,\cos^{2n-1}(cx+d)\,dx$

$$= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j-1)a\,\cosh[(2m-2j-1)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]$$
$$+(2n-2k-1)c\,\sinh[(2m-2j-1)(ax+b)]\,\sin[(2n-2k-1)(cx+d)]\}.$$

17. $\int \cosh^{2m}(ax+b)\,\sin^{2n}(cx+d)\,dx$

$$= \frac{\binom{2m}{m}\binom{2n}{n}}{2^{2m+2n}}x + \frac{(-1)^n\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k\binom{2n}{k}}{(2n-2k)c}\,\sin[(2n-2k)(cx+d)]$$
$$+\frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a}\,\sinh[(2m-2j)(ax+b)]$$
$$+\frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^k\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j)a\,\sinh[(2m-2j)(ax+b)]\,\cos[(2n-2k)(cx+d)]$$
$$+(2n-2k)c\,\cosh[(2m-2j)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

18. $\int \cosh^{2m-1}(ax+b)\,\sin^{2n}\,(cx+d)\,dx$

$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a}\,\sinh[(2m-2j-1)(ax+b)]$$
$$+\frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^k\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j-1)a\,\sinh[(2m-2j-1)(ax+b)]\,\cos[(2n-2k)(cx+d)]$$
$$+(2n-2k)c\,\cosh[(2m-2j-1)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

19. $\displaystyle\int \cosh^{2m}(ax+b)\,\sin^{2n-1}(cx+d)\,dx$

$$= \frac{(-1)^{n-1}\binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{k+1}\binom{2n-1}{k}}{(2n-2k-1)c} \cos[(2n-2k-1)(cx+d)]$$

$$+\frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^k\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$

$$\times\{(2m-2j)a\,\sinh[(2m-2j)(ax+b)]\,\sin[(2n-2k-1)(cx+d)]$$

$$-(2n-2k-1)c\,\cosh[(2m-2j)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]\}.$$

20. $\displaystyle\int \cosh^{2m-1}(ax+b)\,\sin^{2n-1}(cx+d)\,dx$

$$= \frac{(-1)^{n-1}}{2^{2m+2n-4}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^k\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$

$$\times\{(2m-2j-1)a\,\sinh[(2m-2j-1)(ax+b)]\,\sin[(2n-2k-1)(cx+d)]$$

$$-(2n-2k-1)c\,\cosh[(2m-2j-1)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]\}.$$

21. $\displaystyle\int \cosh^{2m}(ax+b)\,\cos^{2n}(cx+d)\,dx$

$$= \frac{\binom{2m}{m}\binom{2n}{n}}{2^{2m+2n}}x + \frac{\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2}{k}}{(2n-2k)c}\, \sin[(2n-2k)(cx+d)]$$

$$+\frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a}\, \sinh[(2m-2j)(ax+b)]$$

$$+\frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$

$$\times\{(2m-2j)a\,\sinh[(2m-2j)(ax+b)]\,\cos[(2n-2k)(cx+d)]$$

$$+(2n-2k)c\,\cosh[(2m-2j)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

22. $\displaystyle\int \cosh^{2m-1}(ax+b)\,\cos^{2n}(cx+d)\,dx$

$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{\binom{2m-1}{j}}{(2m-2j-1)a}\,\sinh[(2m-2j-1)(ax+b)]$$

$$+\frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$

$$\times\{(2m-2j-1)a\,\sinh[(2m-2j-1)(ax+b)]\,\cos[(2n-2k)(cx+d)]$$

$$+(2n-2k)c\,\cosh[(2m-2j-1)(ax+b)]\,\sin[(2n-2k)(cx+d)]\}.$$

23. $\displaystyle\int \cosh^{2m}(ax+b)\,\cos^{2n-1}(cx+d)\,dx$

$$= \frac{\binom{2m}{m}}{2^{2m+2n-2}}\sum_{k=0}^{n-1}\frac{\binom{2n-1}{k}}{(2n-2k-1)c}\,\sin[(2n-2k-1)(cx+d)]$$

$$+\frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$

$$\times\{(2m-2j)a\,\sinh[(2m-2j)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]$$

$$+(2n-2k-1)c\,\cosh[(2m-2j)(ax+b)]\,\sin[(2n-2k-1)(cx+d)]\}.$$

24. $\displaystyle\int \cosh^{2m-1}(ax+b)\,\cos^{2n-1}(cx+d)\,dx$

$$= \frac{1}{2^{2m+2n-4}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$

$$\times\{(2m-2j-1)a\,\sinh[(2m-2j-1)(ax+b)]\,\cos[(2n-2k-1)(cx+d)]$$

$$+(2n-2k-1)c\,\cosh[(2m-2j-1)(ax+b)]\,\sin[(2n-2k-1)(cx+d)]\}.$$

25. $$\int e^{ax}\,\sinh bx\,\sin cx\,dx = \frac{e^{(a+b)x}}{2[(a+b)^2+c^2]}[(a+b)\,\sin cx - c\,\cos cx]$$

$$-\frac{e^{(a-b)x}}{2[(a-b)^2+c^2]}[(a-b)\,\sin cx - c\,\cos cx].$$

26. $$\int e^{ax}\,\sinh bx\,\cos cx\,dx = \frac{e^{(a+b)x}}{2[(a+b)^2+c^2]}[(a+b)\,\cos cx + c\,\sin cx]$$

$$-\frac{e^{(a-b)x}}{2[(a-b)^2+c^2]}[(a-b)\,\cos cx + c\,\sin cx].$$

27. $$\int e^{ax}\cosh bx\,\sin cx\,dx = \frac{e^{(a+b)x}}{2[(a+b)^2+c^2]}[(a+b)\sin cx - c\cos cx] + \frac{e^{(a-b)x}}{2[(a-b)^2+c^2]}[(a-b)\sin cx - c\cos cx].$$

28. $$\int e^{ax}\cosh bx\,\cos cx\,dx = \frac{e^{(a+b)x}}{2[(a+b)^2+c^2]}[(a+b)\cos cx + c\sin cx] + \frac{e^{(a-b)x}}{2[(a-b)^2+c^2]}[(a-b)\cos cx + c\sin cx].$$