Special Functions

1. Gamma Function. $\Gamma(z)=\int_0^\infty e^{-t}\,t^{z-1}\,dt$, $\Re\{z\}>0$. The recursion formula is:

$$\Gamma(z+1)=z\,\Gamma(z).$$

Some special values are:

1. $\Gamma(n+1)=n!\quad$ if $n=0,1,2,\ldots$, where $0!=1$,
2. $\Gamma(1)=1,\quad \Gamma(2)=1,\quad \Gamma(2)=2,\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},\quad \Gamma\left(\frac{3}{2}\right)=\sqrt{\pi}/2,$
3. $\Gamma\left(n+\frac{1}{2}\right)=\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}=\dfrac{\sqrt{\pi}}{2^n}\,(2n-1)!!,\quad n=1,2,\ldots\,,$
4. $\Gamma\left(-n+\frac{1}{2}\right)=\dfrac{(-1)^n\,2^n}{1\cdot 3\cdot 5\cdots(n-1)}\sqrt{\pi}=(-1)^n\,\dfrac{2^n\sqrt{\pi}}{(2n-1)!!},\quad n=1,2,\ldots\,,$
5. $\dfrac{\Gamma\left(p+n+\frac{1}{2}\right)}{\Gamma\left(p-n+\frac{1}{2}\right)}=\dfrac{\left(4p^2-1^2\right)\left(4p^2-3^2\right)\ldots\left[4p^2-(2n-1)^2\right]}{2^{2n}}.$

2. Incomplete Gamma Functions. Some definitions are:

1. $P(a,x)=\dfrac{1}{\Gamma(a)}\displaystyle\int_0^x e^{-t}t^{a-1}\,dt,\quad \Re\{a\}>0,$
2. $\gamma(a,x)=P(a,x)\,\Gamma(a)=\displaystyle\int_0^x e^{-t}t^{a-1}\,dt,\quad \Re\{a\}>0,$
3. $\Gamma(a,x)=\Gamma(a)-\gamma(a,x)=\displaystyle\int_x^\infty e^{-t}t^{a-1}\,dt,\quad \Re\{a\}>0.$

Special cases:

1. $\gamma(1+n,x) = n!\left[1 - e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}\right], \quad n = 0, 1, \ldots\,,$

2. $\Gamma(1+n,x) = n!\, e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}, \quad n = 0, 1, \ldots\,,$

3. $\Gamma(-n,x) = \frac{(-1)^n}{n!}\left[\mathrm{Ei}(-x) - e^{-x}\sum_{k=0}^{n-1}(-1)^k\frac{k!}{x^{k+1}}\right], \quad n = 1, 2, \ldots.$

Relationship with other functions:

1. $\Gamma(0,x) = -\mathrm{Ei}(-x),$
2. $\Gamma\left(0, \ln\frac{1}{x}\right) = -\mathrm{li}(x),$
3. $\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi} - \sqrt{\pi}\,\Phi(x),$
4. $\gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi}\,\Phi(x).$

3. Logarithmic Derivative of the Gamma Function. The Euler psi function is defined as

$$\psi(z) = \frac{d}{dz}\ln\,\Gamma(z) = -\gamma_e + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{z+n}\right), \quad z \neq 0, -1, -2, \ldots\,,$$

where γ_e is the Euler constant (defined below in §4). The integral representations are:

1. $\psi(z) = \frac{d\,\ln\Gamma(z)}{dz} = \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt, \quad \Re\{z\} > 0.$

2. $\psi(z) = \int_0^\infty\left\{e^{-t} - \frac{1}{(1+t)^z}\right\}\frac{dt}{t}, \quad \Re\{z\} > 0.$

3. $\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty\frac{t\,dt}{(t^2+z^2)(e^{2\pi t}-1)}, \quad \Re\{z\} > 0.$

4. $\psi(z) = \int_0^1\left(\frac{1}{-\ln t} - \frac{t^{z-1}}{1-t}\right)dt, \quad \Re\{z\} > 0.$

5. $\psi(z) = \int_0^\infty\frac{e^{-t} - e^{-zt}}{1-e^{-t}}\,dt - \gamma_e, \quad \Re\{z\} > 0.$

6. $\psi(z) = \int_0^\infty\{(1+t)^{-1} - (1+t)^{-z}\}\frac{dt}{t} - \gamma_e, \quad \Re\{z\} > 0.$

7. $\psi(z) = \int_0^1\frac{t^{z-1}-1}{t-1}\,dt - \gamma_e, \quad \Re\{z\} > 0.$

8. $\psi(z) = \ln z + \int_0^{\infty} e^{-tz} \left[\frac{1}{t} - \frac{1}{1-e^{-t}} \right] dt, \quad \Re\{z\} > 0.$

Series representations are:

1. $\psi(x) = -\gamma_e - \sum_{k=0}^{\infty} \left(\frac{1}{x+k} - \frac{1}{k+1} \right) = -\gamma_e - \frac{1}{x} + x \sum_{k=1}^{\infty} \frac{1}{k(x+k)}.$

2. $\psi(x) = \ln x - \sum_{k=0}^{\infty} \left[\frac{1}{x+k} - \ln\left(1 + \frac{1}{x+k}\right) \right].$

3. $\psi(x) = -\gamma_e + \frac{\pi^2}{6}(x-1) - (x-1) \sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{x+k} \right) \sum_{n=0}^{k-1} \frac{1}{x+n}.$

Some particular values are

1. $\psi(1) = -\gamma_e.$
2. $\psi\left(\frac{1}{2}\right) = -\gamma_e - 2 \ln 2 \approx -1.963510026.$
3. $\psi\left(\frac{1}{2} \pm n\right) = -\gamma_e + 2\left[\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2 \right].$
4. $\psi\left(\frac{1}{4}\right) = -\gamma_e - \frac{\pi}{2} - 3 \ln 2.$
5. $\psi\left(\frac{3}{4}\right) = -\gamma_e + \frac{\pi}{2} - 3 \ln 2.$
6. $\psi\left(\frac{1}{3}\right) = -\gamma_e - \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3.$
7. $\psi\left(\frac{2}{3}\right) = -\gamma_e + \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3.$
8. $\psi'(1) = \frac{\pi^2}{6} \approx 1.644934066848.$
9. $\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} \approx 4.9348022005.$
10. $\psi'(-n) = \infty.$
11. $\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}.$
12. $\psi'\left(\frac{1}{2} + n\right) = \frac{\pi^2}{2} - 4 \sum_{k=1}^{n} \frac{1}{(2k-1)^2}.$

13. $\psi'\left(\frac{1}{2}-n\right)=\frac{\pi^2}{2}+4\sum_{k=1}^{n}\frac{1}{(2k-1)^2}$.

4. Euler's constant γ_e (usually denoted by γ):

1. $\gamma_e \approx -\psi(1)=0.57721566490$.
2. $\gamma_e=\lim_{n\to\infty}\left[\sum_{k=1}^{n-1}\frac{1}{k}-\ln\, n\right]$.
3. $\gamma_e=\lim_{x\to 1+0}\left[\zeta(x)-\frac{1}{x-1}\right]$.

Some integral representations are:

1. $\gamma_e=-\int_0^\infty e^{-t}\,\ln\, t\,dt$.
2. $\gamma_e=-\int_0^1 \ln\,\left(\ln\,\frac{1}{t}\right)\,dt$.
3. $\gamma_e=\int_0^1\left[\frac{1}{\ln\, t}+\frac{1}{1-t}\right]\,dt$.
4. $\gamma_e=-\int_0^\infty\left[\cos t-\frac{1}{1+t}\right]\frac{dt}{t}$.
5. $\gamma_e=1-\int_0^\infty\left[\frac{\sin t}{t}-\frac{1}{1+t}\right]\frac{dt}{t}$.
6. $\gamma_e=-\int_0^\infty\left[e^{-t}-\frac{1}{1+t}\right]\frac{dt}{t}$.
7. $\gamma_e=-\int_0^\infty\left[e^{-t}-\frac{1}{1+t^2}\right]\frac{dt}{t}$.
8. $\gamma_e=\int_0^\infty\left[\frac{1}{e^t-1}-\frac{1}{te^t}\right]\,dt$.
9. $\gamma_e=\int_0^1(1-e^{-t})\frac{dt}{t}-\int_1^\infty\frac{e^{-t}}{t}\,dt$.

It has the asymptotic expansion

$$\gamma_e\;=\sum_{k=1}^{n-1}\frac{1}{k}-\ln\,n+\frac{1}{2n}+\frac{1}{12n^2}-\frac{1}{120n^4}+\frac{1}{252n^6}-\frac{1}{240n^8}+\cdots+\frac{B_{2r}}{2r}\frac{1}{n^{2r}}$$
$$+\frac{B_{2r+2}}{2(r+1)}\frac{\theta}{n^{2r+2}},\quad 0<\theta<1.$$

5. The Function $\beta(x)$ is defined by

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right].$$

Integral representations are:

1. $\beta(x) = \displaystyle\int_0^1 \frac{t^{x-1}}{1+t}\,dt, \quad \Re\{x\} > 0.$
2. $\beta(x) = \displaystyle\int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt, \quad \Re\{x\} > 0.$
3. $\beta\left(\dfrac{x+1}{2}\right) = \displaystyle\int_0^\infty \frac{e^{-xt}}{\cosh t}\,dt, \quad \Re\{x\} > -1.$

Series representations are:

1. $\beta(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{x+k}, \quad -x \notin \mathrm{N}.$
2. $\beta(x) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+2k+1)}, \quad -x \notin \mathrm{N}.$
3. $\beta(x) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty} \frac{k!}{x(x+1)\dots(x+k)}\,\frac{1}{2^k}, \quad -x \notin \mathrm{N}.$
4. $\beta(x+1) = \ln 2 + \displaystyle\sum_{k=1}^{\infty} (-1)^k (1-2^{-k})\zeta(k+1)x^k, \quad |x| < 1.$
5. $\beta(x+1) = \ln 2 - 1 + \dfrac{1}{2x} - \dfrac{\pi}{2\,\sin \pi x} + \dfrac{1}{1-x^2} - \displaystyle\sum_{k=1}^{\infty} \left[1 - (1-2^{-2k})\zeta(2k+1)\right]\, x^{2k},$

   $0 < |x| < 2;\ x \neq p \pm 1.$
6. $\dfrac{d^n}{dx^n}\,\beta^{(n)}(x) = (-1)^n n! \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)^{n+1}}, \quad -x \notin \mathrm{N}.$

Representation in the form of a finite sum are:

1. $\beta\left(\dfrac{m}{n}\right) = \dfrac{\pi}{2\,\sin \frac{m\pi}{n}} - \displaystyle\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \cos \frac{m(2k+1)\pi}{n}\ \ln \sin \frac{(2k+1)\pi}{2n},$

   $n = 2, 3, \dots\ ;\ m = 1, 2, \dots, n-1.$
2. $\beta(n) = (-1)^{n+1} \ln 2 + \displaystyle\sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k}.$

6. Beta Function $\mathrm{B}(x, y)$ is defined by the following integral representation:

$$\begin{aligned}\mathrm{B}\,(x,\,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \\ &= 2\int_0^1 t^{2x-1}(1-t^2)^{y-1}\,dt, \quad \Re\{x\}\} > 0,\, \Re\{y\}\} > 0.\end{aligned}$$

Other definitions are:

1. $\mathrm{B}\,(x,\,y) = 2\displaystyle\int_0^{\pi/2} \sin^{2x-1}\,\theta\,\cos^{2y-1}\,\theta\,d\theta, \quad \Re\{x\} > 0,\, \Re\{y\} > 0.$

2. $\mathrm{B}\,(x,\,y) = \displaystyle\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt = 2\int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}}\,dt, \quad \Re\{x\} > 0,\, \Re\{y\} > 0.$

3. $\mathrm{B}\,(x,\,y) = 2^{2-y-x}\displaystyle\int_{-1}^1 \frac{(1+t)^{2x-1}(1-t)^{2y-1}}{(1+t^2)^{x+y}}\,dt, \quad \Re\{x\} > 0,\, \Re\{y\} > 0.$

4. $\mathrm{B}\,(x,\,y) = \displaystyle\int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt = \int_1^\infty \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt, \quad \Re\{x\} > 0,\, \Re\{y\} > 0.$

5. $\mathrm{B}\,(x,\,y) = \displaystyle\frac{1}{2^x+y-1}\int_0^1 \left[(1+t)^{x-1}(1-t)^{y-1} + (1+t)^{y-1}(1-t)^{x-1}\right]\,dt,$
$\Re\{x\} > 0,\, \Re\{y\} > 0.$

6. $\mathrm{B}\,(x,\,y) = z^y\,(1+z)^x\displaystyle\int_0^1 \frac{t^{x-1}(1-t)^{y-1}}{(t+z)^{x+y}}\,dt,$
$\Re\{x\} > 0,\ \Re\{y\} > 0,\ 0\} > z\} > -1,\ \Re\{(x+y) < 1.$

7. $\mathrm{B}\,(x,\,y) = z^y\,(1+z)^x\displaystyle\int_0^{\pi/2} \frac{\cos^{2x-1}\,\theta\,\sin^{2y-1}\,\theta}{(z+\cos^2\,\theta)^{x+y}}\,d\theta,$
$\Re\{x\} > 0,\ \Re\{y\} > 0,\ 0\} > z\} > -1,\ \Re\{(x+y) < 1,$

8. $\mathrm{B}\,(x,\,x) = \displaystyle\frac{1}{2^{2x-2}}\int_0^1 (1-t^2)^{x-1}\,dt = \frac{1}{2^{2x-1}}\int_0^1 \frac{(1-t)^{x-1}}{\sqrt{t}}\,dt.$

9. $\mathrm{B}\,(x+y,\,x-y) = 4^{1-x}\displaystyle\int_0^\infty \frac{\cosh 2yt}{\cosh^{2x} t}\,dt, \quad \Re\{x\} > |\Re\{y|,\, \Re\{x\} > 0.$

10. $\mathrm{B}\,\left(x,\,\dfrac{y}{z}\right) = z\displaystyle\int_0^1 (1-t^z)^{x-1}t^{y-1}\,dt, \quad \Re\{z\} > 0,\, \Re\{y/z\} > 0,\, \Re\{x\} > 0.$

The series representations are:

1. $\mathrm{B}\,(x,\,y) = \displaystyle\frac{1}{y}\sum_{n=0}^\infty (-1)^n y\frac{(y-1)\ldots(y-n)}{n!(x+n)}, \quad y > 0.$

2. $\mathrm{B}\,\left(z,\,\tfrac{1}{2}\right) = \displaystyle\sum_{k=1}^\infty \frac{(2k-1)!!}{2^k k!}\,\frac{1}{z+k} + \frac{1}{z}.$

Functional relations involving the beta function are:

1. $\mathrm{B}\,(x,\,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \mathrm{B}\,(y,\,x)$.
2. $\mathrm{B}\,(x,\,y)\,\mathrm{B}\,(x+y,\,z) = \mathrm{B}\,(y,\,z)\,\mathrm{B}\,(y+z,\,x)$.
3. $\displaystyle\sum_{k=0}^{\infty} \mathrm{B}\,(x,\,y+k) = \mathrm{B}\,(x-1,\,y)$.
4. $\mathrm{B}\,(x,\,x) = 2^{1-2x}\,\mathrm{B}\,\left(\frac{1}{2},\,x\right)$.
5. $\mathrm{B}\,(x,\,x)\,\mathrm{B}\,\left(x+\frac{1}{2},\,x+\frac{1}{2}\right) = \dfrac{\pi}{2^{4x-1}x}$.
6. $\dfrac{1}{\mathrm{B}\,(n,\,m)} = m\dbinom{n+m-1}{n-1} = n\dbinom{n+m-1}{m-1}, \quad m, n \in \mathrm{N}$.

7. Incomplete Beta Function $\mathrm{B}_x(p,q)$ is defined by

$$\mathrm{B}_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt = \frac{x^p}{p}\,{}_2F_1(p,\,1-q;\,p+1;\,x).$$

8. Bernoulli Numbers. The Bernoulli numbers B_n represent the coefficients of $t^n/n!$ in the expansion of the function

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \quad 0 < |t| < 2\pi,$$

and the function $\dfrac{t}{e^t-1}$ is a generating function for the Bernoulli numbers. These numbers have the following integral representations:

1. $B_{2n} = (-1)^{n-1}4n\displaystyle\int_0^{\infty} \frac{x^{2n-1}}{e^{2\pi x}-1}\,dx, \quad n = 1, 2, \dots.$
2. $B_{2n} = (-1)^{n-1}\pi^{-2n}\displaystyle\int_0^{\infty} \frac{x^{2n}}{\sinh^2 x}\,dx, \quad n = 1, 2, \dots.$
3. $B_{2n} = (-1)^{n-1}\dfrac{2n(1-2n)}{\pi}\displaystyle\int_0^{\infty} x^{2n-2}\ln(1-e^{-2\pi x})\,dx, \quad n = 1, 2, \dots.$

The Bernoulli numbers have the following properties:

$$B_n = -n!\sum_{k=0}^{n-1} \frac{B_k}{k!(n+1-k)!}, \quad n \geq 2.$$

All the Bernoulli numbers are rational numbers; those with odd index are equal to zero except that $B_1 = -\frac{1}{2}$. Thus, $B_{2n+1} = 0$ for $n \in \mathrm{N}$, and

$$B_{2n} = -\frac{1}{2n+1} + \frac{1}{2} - \sum_{k=1}^{n-1}\frac{2n(2n-1)\dots(2n-k+2)}{(2k)!}B^{2k}, \quad n \geq 1.$$

9. Bernoulli Polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

The generating function is

$$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!}, \quad 0 < |t| < 2\pi.$$

These polynomials have the following series representations:

1. $B_n(x) = -2\dfrac{n!}{(2\pi)^n}\displaystyle\sum_{k=1}^{\infty}\frac{\cos\left(2\pi kx - \frac{1}{2}\pi n\right)}{k^n}, \quad 0 \le x \le 1;\; n = 1, 2, \dots$

2. $B_{2n-1}(x) = 2\dfrac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}}\displaystyle\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{k^{2n-1}}, \quad 0 \le x \le 1;\; n = 1, 2, \dots$

3. $B_{2n}(x) = \dfrac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}}\displaystyle\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n}}, \quad 0 \le x \le 1;\; n = 1, 2, \dots$

These polynomials have the following properties:

1. $B_{m+1}(n) = B_{m+1} + (m+1)\displaystyle\sum_{k=1}^{n-1} k^m, \quad m, n \in \mathrm{N}.$
2. $B_n(x+1) - B_n(x) = nx^{n-1}.$
3. $B_n'(x) = nB_{n-1}(x), \quad n = 1, 2, \dots.$
4. $B_n(1-x) = (-1)^n B_n(x).$
5. $(-1)^n B_n(-x) = B_n(x) + nx^{n-1}, \quad n = 0, 1, \dots.$

For odd n, the differences $B_n(x) - B_n$ vanish on the interval $[0, 1]$ only at the points $0, \frac{1}{2}$, and 1. They change sign at the point $x = \frac{1}{2}$. For even n, these differences vanish at the end points of the interval $[0, 1]$, and inside this interval they do not change sign and their greatest absolute value occurs at the point $\frac{1}{2}$. The polynomials $B_{2n}(x) - B_{2n}$ and $B_{2n+2}(x) - B_{2n+2}$ have opposite signs in the interval $(0, 1)$ Some special values are as follows:

1. $B_1(x) = x - \frac{1}{2},$
2. $B_2(x) = x^2 - x + \frac{1}{6},$
3. $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$
4. $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$
5. $B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.$

Some particular values are:

$B_n(0) = B_n, \quad B_1(1) = -B_1 = \frac{1}{2}$, and $B_n(1) = B_n$ for $n \neq 1$.

10. Fresnel Integrals $C(x)$ and $S(x)$ are defined by

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}\,t^2\right) dt, \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}\,t^2\right) dt.$$

Their series representations are

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n}}{(2n)!\,(4n+1)}\, x^{4n+1}$$
$$= \cos\left(\frac{\pi}{2}\,x^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1\cdot 3\cdots(4n+1)}\, x^{4n+1} + \sin\left(\frac{\pi}{2}\,x^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1\cdot 3\cdots(4n+3)}\, x^{4n+3},$$

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!\,(4n+3)}\, x^{4n+3}$$
$$= \sin\left(\frac{\pi}{2}\,x^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1\cdot 3\cdots(4n+1)}\, x^{4n+1} - \cos\left(\frac{\pi}{2}\,x^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1\cdot 3\cdots(4n+3)}\, x^{4n+3}.$$

Other relations are:

$$C(-x) = -C(x), \quad S(-x) = -S(x), \quad C(i\,x) = i\,C(x), \quad S(i\,x) = -i\,S(x),$$
$$C(\infty) = \frac{1}{2}, \quad S(\infty) = \frac{1}{2}.$$

11. Young's Function $C_\nu(x)$ is defined by

$$C_\nu(a) = \frac{a^\nu}{2\Gamma(\nu+1)}\left[\,{}_1F_1(1;\,\nu+1;\,i\,a) + {}_1F_1(1;\,\nu+1;\,-i\,a)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n a^{\nu+2n}}{\Gamma(\nu+2n+1)}.$$

12. Sine and Cosine Integrals $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ are defined, respectively, by

$$\mathrm{Si}\,(x) = -\int_x^\infty \frac{\sin t}{t}\,dt = -\frac{\pi}{2} + \mathrm{Si}(x), \text{ where } \mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt = \gamma_e + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt.$$

Other results are:

1. $\mathrm{Si}\,(xy) = -\int_x^\infty \frac{\sin ty}{t}\,dt.$
2. $\mathrm{Ci}(xy) = -\int_x^\infty \frac{\cos ty}{t}\,dt.$
3. $\mathrm{Si}\,(x) = -\int_0^{\pi/2} e^{-x\,\cos t} \cos(x\,\sin t)\,dt.$

4. $\mathrm{Si}\,(x) = -\dfrac{\pi}{2} + \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)(2k-1)!}.$

5. $\mathrm{Ci}(x) = \gamma_e + \ln x + \displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!}.$

6. $\mathrm{Ci}(x) \pm i\ \mathrm{Si}\,(x) = \mathrm{Ei}(\pm i\, x).$

7. $\mathrm{Ci}(x) - \mathrm{Ci}(xe^{\pm \pi i}) = \mp \pi i.$

8. $\mathrm{Si}\,(x) + \mathrm{Si}\,(-x) = -\pi.$

9. $[\mathrm{Ci}(x)]^2 + [\mathrm{Si}\,(x)]^2 = -2 \displaystyle\int_0^{\frac{\pi}{2}} \frac{\exp(-x\ \tan\theta)\ \ln\cos\theta}{\sin\theta\ \cos\theta}\, d\theta, \quad \Re\{x\} > 0.$

13. Hyperbolic Sine and Hyperbolic Cosine Integrals $\mathrm{shi}(x)$ and $\mathrm{chi}(x)$ are defined, respectively by

$$\mathrm{shi} x = \int_0^x \frac{\sinh t}{t}\, dt = -i\left[\frac{\pi}{2} + \mathrm{Si}\,(ix)\right],$$
$$\mathrm{chi} x = \gamma_e + \ln x + \int_0^x \frac{\cosh t - 1}{t}\, dt.$$

14. Parabolic Cylinder Functions $D_p(z)$ are defined by

$$D_p(z) = 2^{1/4+p/2} W_{1/4+p/2,-1/4}\left(\frac{z^2}{2}\right) z^{-1/2}$$
$$= 2^{p/2} e^{-z^2/4} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \Phi\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi} z}{\Gamma\left(-\frac{p}{2}\right)} \Phi\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \right\}.$$

Their integral representations are:

1. $D_p(z) = \dfrac{1}{\sqrt{\pi}} 2^{p+1/2} e^{-i\,p\pi/2}\, e^{z^2/4} \displaystyle\int_{-\infty}^{\infty} x^p e^{-2x^2+2i\,xz}\, dx,$

   $\Re\{p\} > -1;\ \arg x^p = i\,p\pi$ for $x < 0$.

2. $D_p(z) = \dfrac{e^{-z^2/4}}{\Gamma(-p)} \displaystyle\int_0^{\infty} e^{-xz-x^2/2} x^{-p-1}\, dx, \quad \Re\{p\} < 0.$

3. $D_p(z) = -\dfrac{\Gamma(p+1)}{2\pi i}\, e^{-z^2/4} \displaystyle\int_{\infty}^{(0+)} e^{-zt-t^2/2} (-t)^{-p-1}\, dt, \quad |\arg(-t)| \le \pi.$

4. $D_p(z) = 2^{(p-1)/2} \dfrac{\Gamma\left(\frac{p}{2}+1\right)}{i\pi} \displaystyle\int_{-\infty}^{(-1+)} e^{z^2 t/4} (1+t)^{-p/2-1} (1-t)^{(p-1)/2}\, dt,$

   $|\arg\ z| < \dfrac{\pi}{4};\ |\arg(1+t)| \le \pi.$

5. $D_p(z) = \dfrac{1}{2\pi i} e^{-z^2/4} \displaystyle\int_{-i\infty}^{+i\infty} \Gamma\left(\frac{t}{2} - \frac{p}{2}\right) \quad |\arg z| < \frac{3}{4}\pi;$ p is not a positive integer.

6. $D_p(z) = \dfrac{1}{2\pi i} e^{-z^2/4} \displaystyle\int_{\infty}^{(0-)} \frac{\Gamma\left(\frac{t}{2} - \frac{p}{2}\right)\Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t \, dt,$

for all values of $\arg z$, where the contours encircle the poles of the function $\Gamma(-t)$ but not those of the function $\Gamma\left(\dfrac{t}{2} - \dfrac{p}{2}\right)$. If $|z| \gg 1$, $|z| \gg |p|$, then the asymptotic expansions are given by

1. $D_p(z) \sim e^{-z^2/4} z^p \left(1 - \dfrac{p(p-1)}{2z^2} + \dfrac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right),$

$$|\arg z| < \tfrac{3}{4}\pi,$$

2. $$D_p(z) \sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$$
$$-\frac{\sqrt{2\pi}}{\Gamma(-p)} e^{i p\pi} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right),$$
$$\frac{\pi}{4} < \arg z < \frac{5}{4}\pi.$$

3. $$D_p(z) \sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$$
$$-\frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-i p\pi} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right),$$
$$-\frac{\pi}{4} > \arg z > \tfrac{5}{4}\pi.$$

Connections with other functions are given by

1. $D_n(z) = -2^{-n/2} e^{-z^2/4} H_n\left(\dfrac{z}{\sqrt{2}}\right).$

2. $D_{-1}(z) = e^{z^2/4} \sqrt{\dfrac{\pi}{2}} \left[1 - \Phi\left(\dfrac{z}{\sqrt{2}}\right)\right].$

3. $D_{-2}(z) = e^{z^2/4} \sqrt{\dfrac{\pi}{2}} \left\{\sqrt{\dfrac{\pi}{2}} e^{-\frac{z^2}{2}} - z\left[1 - \Phi\left(\dfrac{z}{\sqrt{2}}\right)\right]\right\}.$

15. Euler Numbers E_n are the coefficients of $t^n/n!$ in the expansion of the function

$$\operatorname{sech} t = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \frac{\pi}{2}.$$

The function $\operatorname{sech} t$ is a generating function for the Euler numbers. A recursion formula is:

$$(E+1)^{[n]} + (E-1)^{[n]} = 0, \quad n \geq 1, \quad E_0 = 1.$$

Properties of the Euler numbers are: (i) They are integers; (ii) The Euler numbers of odd index are equal to zero, and the signs of two adjacent numbers of even indices are opposite, that is, $E_{2n+1} = 0,\ E_{4n} > 0,\ E_{4n+2} < 0$; and (iii) If $\alpha, \beta\gamma, \dots$ are the divisors of the number $n - m$, the difference $E_{2n} - E_{2m}$ is divisible by those of the numbers $2\alpha + 1, 2\beta + 1, 2\gamma + 1, \dots$, that are primes.

Euler numbers are connected to the Bernoulli numbers by the following relations:

1. $E_{n-1} + 4(-1)^n(3^{n-1} - 1)B_1 = \dfrac{(4B-1)^{[n]}(4B-3)^{[n]}}{2n} + 4(-1)^{n+1}(3^{n-1} - 1)B_1.$

2. $B_n = \dfrac{n(E+1)^{[n-1]}}{2^n(2^n - 1)}, \quad n \geq 2.$

3. $\left(B + \frac{1}{4}\right)^{[2n+1]} = -4^{-2n-1}(2n+1)E_{2n}, \quad n \geq 0.$

Some Euler numbers are:

$E_0 = 1,\quad E_2 = -1,\quad E_4 = 5,\quad E_6 = -61,\quad E_8 = 1385,\quad E_{10} = -50521,\quad E_{12} = 2702765,$

$E_{14} = -199360981,\quad E_{16} = 19391512145,\quad E_{18} = -2404879675441,\quad E_{20} = 370371188237525.$

Bernoulli and Euler numbers of odd index (with the exception of B_1) are equal to zero.

16. Elliptic Integrals

The following three integrals

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx, \quad \int \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}},$$

are, respectively, called the *elliptic integrals of the first, second, and third kind* in the Legendre normal form. Using the substitution $x = \sin\varphi$, these elliptic integrals are reduced to the trigonometric form:

$$\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \quad \int \sqrt{1-k^2\sin^2\varphi}\,d\varphi, \quad \int \frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}},$$

respectively. Elliptic integrals from 0 to 1 in the former definition, or from 0 to $\pi/2$ in the latter definition, are called *complete elliptic integrals*.

The following notation is used: $\Delta\varphi = \sqrt{1-k^2\sin^2\varphi};\quad k' = \sqrt{1-k^2};\quad k^2 < 1.$

Elliptic Integral of the First Kind $F(\varphi, k)$ is defined by

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Elliptic Integral of the Second Kind $E(\varphi, k)$ is defined by

$$E(\varphi,\, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2\, \alpha}\, d\alpha = \int_0^{\sin\varphi} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\, dx$$

Elliptic Integral of the Third Kind $\Pi(\varphi,\, n,\, k)$ is defined by

$$\Pi(\varphi,\, n,\, k) = \int_0^{\varphi} \frac{d\alpha}{(1 - n\, \sin^2\, \alpha)\sqrt{1-k^2 \sin^2\, \alpha}} = \frac{\int_0^{\sin\varphi} dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}.$$

17. Complete Elliptic Integrals are defined by

1. $\mathbf{K}(k) = F\left(\frac{\pi}{2},\, k\right) = \mathbf{K}'(k')$.
2. $\mathbf{E}(k) = E\left(\frac{\pi}{2},\, k\right) = \mathbf{E}'(k')$.
3. $\mathbf{K}'(k) = F\left(\frac{\pi}{2},\, k'\right) = \mathbf{K}(k')$.
4. $\mathbf{E}'(k) = E\left(\frac{\pi}{2},\, k'\right) = \mathbf{E}(k')$.

Often the modulus k is omitted and we write

$\mathbf{K}\,(\equiv \mathbf{K}(k))$, $\quad \mathbf{K}'\,(\equiv \mathbf{K}'(k))$, $\quad \mathbf{E}\,(\equiv \mathbf{E}(k))$, $\quad \mathbf{E}'\,(\equiv \mathbf{E}'(k))$. The series representations are as follows:

1. $\mathbf{K} = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n} + \cdots\right\} = \frac{\pi}{2} F\left(\frac{1}{2},\, \frac{1}{2};\, 1;\, k^2\right)$.
2. $\mathbf{K} = \frac{\pi}{1+k'}\left\{1 + \left(\frac{1}{2}\right)^2 \left(\frac{1-k'}{1+k'}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \left(\frac{1-k'}{1+k'}\right)^4 + \cdots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \cdots\right\}$
3. $\mathbf{E} = \frac{\pi}{2}\left\{1 - \frac{1}{2^2}k^2 - \frac{1^2\cdot 3}{2^2\cdot 4^2}k^4 - \cdots - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1} - \cdots\right\} = \frac{\pi}{2} F\left(-\frac{1}{2},\, \frac{1}{2};\, 1;\, k^2\right)$.
4. $\mathbf{E} = \frac{(1+k')\pi}{4}\left\{1 + \frac{1}{2^2}\left(\frac{1-k'}{1+k'}\right)^2 + \frac{1^2}{2^2\cdot 4^2}\left(\frac{1-k'}{1+k'}\right)^4 + \cdots + \left[\frac{(2n-3)!!}{2^n n!}\right]^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \cdots\right\}$.

Other relations are:

1, $\mathbf{E}(k)\mathbf{K}'(k) + \mathbf{E}'(k)\mathbf{K}(k) - \mathbf{K}(k)\mathbf{K}'(k) = \frac{\pi}{2}$

2. $\mathbf{K}\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2}\mathbf{K}(k)$.

3. $\mathbf{E}\left(\frac{1-k'}{1+k'}\right) = \frac{1}{1+k'}[\mathbf{E}(k) + k'\mathbf{K}(k)]$.

4. $\mathbf{K}\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\mathbf{K}(k)$.

5. $\mathbf{E}\left(\dfrac{2\sqrt{k}}{1+k}\right) = \dfrac{1}{1+k}[2\mathbf{E}(k) - k'^2\mathbf{K}(k)]$.

6. $\mathbf{K}\left(i\,\dfrac{k}{k'}\right) = k'\mathbf{K}(k), \quad \Im\{k)\} < 0$.

7. $\mathbf{K}'\left(i\,\dfrac{k}{k'}\right) = k'[\mathbf{K}(k') - i\,\mathbf{K}(k)], \quad \Im\{k\} < 0$.

8. $\mathbf{K}\left(\dfrac{1}{k}\right) = k[\mathbf{K}(k) + i\,\mathbf{K}'(k)], \quad \Im\{k\} < 0$.

Some special values are:

1. $\mathbf{K}\left(\sin\dfrac{\pi}{4}\right) = \mathbf{K}\left(\dfrac{1}{\sqrt{2}}\right) = \mathbf{K}'\left(\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}\displaystyle\int_0^1 \dfrac{dt}{\sqrt{1-t^4}} = \dfrac{1}{4\sqrt{\pi}}\left[\Gamma\left(\dfrac{1}{4}\right)\right]^2$.
2. $\mathbf{K}'(\sqrt{2}-1) = \sqrt{2}\mathbf{K}(\sqrt{2}-1)$.
3. $\mathbf{K}'\left(\sin\dfrac{\pi}{12}\right) = \sqrt{3}\mathbf{K}\left(\sin\dfrac{\pi}{12}\right)$.
4. $\mathbf{K}'\left(\tan^2\dfrac{\pi}{8}\right) = \mathbf{K}'\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right) = 2\mathbf{K}\left(\tan^2\dfrac{\pi}{8}\right)$.

18. $\mathrm{Ei}(-x) = -x\displaystyle\int_1^\infty e^{-tx}\,\ln t\,dt, \quad x > 0$.

Note the following two approximate values:

1. $\mathrm{Ei}(-1) =\approx -0.219383934395520273665$.
2. $\mathrm{Ei}(1) \approx 1.89511781635593676 5478$.

18. Anger Function $\mathbf{J}_\nu(z)$ and Weber Functions $\mathbf{E}_\nu(z)$ are defined by

$$\mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)d\theta.$$

$$\mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(\nu\theta - z\sin\theta)d\theta.$$

Their series representations are:

1. $\mathbf{J}_\nu(z) = \cos\dfrac{\nu\pi}{2}\displaystyle\sum_{n=0}^\infty \dfrac{(-1)^n\,(z/2)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)} + \sin\dfrac{\nu\pi}{2}\displaystyle\sum_{n=0}^\infty \dfrac{(-1)^n\,(z/2)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right)\Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)}$.

2. $\mathbf{E}_\nu(z) = \sin\dfrac{\nu\pi}{2}\displaystyle\sum_{n=0}^\infty \dfrac{(-1)^n\,(z/2)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)}$

$$-\cos\frac{\nu\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\,(z/2)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right)\Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)}.$$

Their functional relations are given by

1. $2\mathbf{J}'_\nu(z)=\mathbf{J}_{\nu-1}(z)-\mathbf{J}_{\nu+1}(z)$.
2. $2\mathbf{E}'_\nu(z)=\mathbf{E}_{\nu-1}(z)-\mathbf{E}_{\nu+1}(z)$.
3. $\mathbf{J}_{\nu-1}(z)+\mathbf{J}_{\nu+1}(z)=2\nu z^{-1}\mathbf{J}_\nu(z)-2(\pi z)^{-1}\sin(\nu\pi)$.
4. $\mathbf{E}_{\nu-1}(z)+\mathbf{E}_{\nu+1}(z)=2\nu z^{-1}\mathbf{E}_\nu(z)-2(\pi z)^{-1}(1-\cos\nu\pi)$,

and their asymptotic expansions by

1. $$\mathbf{J}_\nu(z)=J_\nu(z)+\frac{\sin\nu\pi}{\pi z}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+\frac{1+\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)}\frac{\Gamma\left(n+\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}z^{-2n}\right.$$
$$\left.+O\left(|z|^{-2p}\right)-\nu\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\nu\right)}z^{-2n-1}+\nu O\left(|z|^{-2p-1}\right)\right],$$
$$|\arg z|<\pi.$$
2. $$\mathbf{E}_\nu(z)=-Y_\nu(z)$$
$$-\frac{1+\cos(\nu\pi)}{\pi z}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+\frac{1+\nu}{2}\right)\Gamma\left(n+\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}z^{-2n}+O\left(|z|^{-2p}\right)\right]$$
$$-\frac{\nu(1-\cos\nu\pi)}{z\pi}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\nu\right)}z^{-2n-1}+O\left(|z|^{-2p-1}\right)\right].$$

19. Exponential Elliptic Function $\mathrm{Ei}(x)$ is defined by

1. $\mathrm{Ei}(x)=-\int_{-x}^{\infty}\frac{e^{-t}}{t}\,dt=\int_{-\infty}^{x}\frac{e^t}{t}\,dt=\mathrm{li}(e^x),\quad x<0.$
2. $\mathrm{Ei}(x)=-\lim_{e\to+0}\left[\int_{-x}^{-\varepsilon}\frac{e^{-t}}{t}\,dt+\int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}\,dt\right]p.v.\int_{-\infty}^{x}\frac{e^t}{t}dt,\quad x>0$
3. $\mathrm{Ei}(x)=\frac{1}{2}\{\mathrm{Ei}(x+i0)+\mathrm{Ei}(x-i0)\},\quad x>0.$
4. $$\mathrm{Ei}(-x)=\gamma_e+\ln x+\int_0^x\frac{e^{-t}-1}{t}\,dt,\quad x>0$$
$$=\gamma_e+e^{-x}\ln x+\int_0^x e^{-t}\ln t\,dt,\quad x>0.$$
5. $\mathrm{Ei}(x)=e^x\left[\frac{1}{x}+\int_0^\infty\frac{e^{-t}dt}{(x-t)^2}\right],$ with a divergent integral.

6. $\mathrm{Ei}(-x) = e^{-x}\left[-\frac{1}{x} + \int_0^\infty \frac{e^{-t}dt}{(x+t)^2}\right], \quad x > 0.$

7. $\mathrm{Ei}(\pm x) = \pm e^{\pm x}\int_0^1 \frac{dt}{x \pm \ln t}, \quad x > 0.$

8. $\mathrm{Ei}(\pm xy) = \pm e^{\pm xy}\int_0^\infty \frac{e^{-xt}}{y \mp t}\,dt, \quad \Re\{y\} > 0,\ x > 0.$

9. $\mathrm{Ei}(\pm x) = -e^{\pm x}\int_0^\infty \frac{e^{-it}}{t \pm ix}\,dt, \quad x > 0.$

7. $\mathrm{Ei}(xy) = e^{xy}\int_0^1 \frac{t^{y-1}}{x + \ln t}\,dt.$

10. $$\mathrm{Ei}(-xy) = -e^{-xy}\int_0^1 \frac{t^{y-1}}{x - \ln t}\,dt$$
$$= x^{-1}e^{-xy}\left[\int_0^1 \frac{t^{x-1}}{(y-\ln t)^2}\,dt - y^{-1}\right], \quad x > 0,\ y > 0.$$

11. $\mathrm{Ei}(x) = e^{x}\int_1^\infty \frac{1}{x - \ln t}\frac{dt}{t^2}, \quad x > 0.$

12. $\mathrm{Ei}(-x) = -e^{-x}\int_1^\infty \frac{1}{x + \ln t}\frac{dt}{t^2}, \quad x > 0.$

13. $\mathrm{Ei}(-x) = -e^{-x}\int_0^\infty \frac{t\,\cos t + x\,\sin t}{t^2 + x^2}\,dt, \quad x > 0.$

14. $\mathrm{Ei}(-x) = -e^{-x}\int_0^\infty \frac{t\,\cos t - x\,\sin t}{t^2 + x^2}\,dt, \quad x < 0.$

15. $\mathrm{Ei}(-x) = \frac{2}{\pi}\int_0^\infty \frac{\cos t}{t}\arctan\frac{t}{x}\,dt, \quad \Re\{x\} > 0.$

16. $\mathrm{Ei}(-x) = \frac{2e^{-x}}{\pi}\int_0^\infty \frac{x\,\cos t - t\,\sin t}{t^2 + x^2}\ln t\,dt, \quad x > 0.$

17. $\mathrm{Ei}(x) = 2\,\ln x - \frac{2e^{x}}{\pi}\int_0^\infty \frac{x\,\cos t + t\,\sin t}{t^2 + x^2}\ln t\,dt, \quad x > 0.$

20. Error Function $\mathrm{erf}(x)$**, the Fresnel Integrals** $S(x)$ **and** $C(x)$ are defined by

1. $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$

2. $S(x) = \frac{2}{\sqrt{2\pi}}\int_0^x \sin t^2\,dt.$

3. $C(x) = \frac{2}{\sqrt{2\pi}}\int_0^x \cos t^2\,dt.$

The function $\operatorname{erf}(x)$ is also known as the probability integral. The complementary error function is defined by $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. Their integral representations are:

1. $\operatorname{erf}(x) = \dfrac{1}{\sqrt{\pi}} \displaystyle\int_0^{x^2} \frac{e^{-t}}{\sqrt{t}}\, dt.$

2. $S(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{x^2} \frac{\sin t}{\sqrt{t}}\, dt.$

3. $C(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{x^2} \frac{\cos t}{\sqrt{t}}\, dt.$

4. $\operatorname{erf}(xy) = \dfrac{2y}{\sqrt{\pi}} \displaystyle\int_0^{x} e^{-t^2 y^2}\, dt.$

5. $S(xy) = \dfrac{2y}{\sqrt{2\pi}} \displaystyle\int_0^{x} \sin(t^2 y^2)\, dt.$

6. $C(xy) = \dfrac{2y}{\sqrt{2\pi}} \displaystyle\int_0^{x} \cos(t^2 y^2)\, dt.$

7. $$\begin{aligned} \operatorname{erf}(xy) &= 1 - \frac{2}{\sqrt{\pi}} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2}\, ty\, dt}{\sqrt{t^2 + x^2}}, \quad \Re\{y^2\} > 0 \\ &= 1 - \frac{2}{\sqrt{\pi}} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2}\, ty\, dt}{\sqrt{t^2 + x^2}}, \quad \Re\{y^2\} > 0. \end{aligned}$$

Their series representations are given by

1. $$\begin{aligned} \operatorname{erf}(x) &= \sum_{k=1}^\infty (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!} = \frac{2}{\sqrt{\pi}}\, e^{-x^2} \sum_{k=0}^\infty \frac{2^k x^{2k+1}}{(2k+1)!!} \\ &= \frac{2}{\sqrt{\pi}}\, e^{-x^2}\, x\ {}_1F_1\left(1; \tfrac{3}{2}; x^2\right). \end{aligned}$$

2. $$\begin{aligned} S(x) &= \sum_{k=0}^\infty \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)} \\ &= \frac{2}{\sqrt{2\pi}} \left\{ \sin x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} - \cos x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} \right\}. \end{aligned}$$

3. $$\begin{aligned} C(x) &= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)} \\ &= \frac{2}{\sqrt{2\pi}} \left\{ \sin x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} + \cos x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} \right\}. \end{aligned}$$

Asymptotic expansions for these functions are:

$$\operatorname{erf}(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi}z}\left[\sum_{k=0}^{n}(-1)^k\frac{(2k-1)!!}{(2z^z)k} + O(|z|^{-2n-2})\right],$$

$$z \to \infty,\ |\arg -z)| \le \pi - \delta;\ \delta > 0 \text{ small},$$

where $|R_n| < \dfrac{\Gamma\left(n+\frac{1}{2}\right)}{|x|^{n+1/2}} \cos\dfrac{\theta}{2}$, $x = |x|e^{i\theta}$, and $\theta^2 < \pi^2$.

$$S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi x}}\cos x^2 + O\left(\frac{1}{x^2}\right), \quad x \to \infty.$$

$$C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi x}}\sin x^2 + O\left(\frac{1}{x^2}\right), \quad x \to \infty.$$

The following relations hold:

1. $C(z) + iS(z) = \sqrt{\dfrac{i}{2}}\Phi\left(\dfrac{z}{\sqrt{i}}\right) = \dfrac{2}{\sqrt{2\pi}}\displaystyle\int_0^z e^{it^2}\,dt.$
2. $C(z) - iS(z) = \dfrac{1}{\sqrt{2i}}\Phi(z\sqrt{i}) = \dfrac{2}{\sqrt{2\pi}}\displaystyle\int_0^z e^{-it^2}\,dt.$
3. $[\cos u^2 C(u) + \sin u^2 S(u)] = \dfrac{1}{2}[\cos u^2 + \sin u^2] + \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty e^{-2ut}\sin t^2\,dt,$
$\Re\{u\} \ge 0.$
4. $[\cos u^2 S(u) - \sin u^2 C(u)] = \dfrac{1}{2}[\cos u^2 - \sin u^2] - \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty e^{-2ut}\cos t^2\,dt,$
$\Re\{u\} \ge 0.$
5. $\left[C(x) - \frac{1}{2}\right]^2 + \left[S(x) - \frac{1}{2}\right]^2 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\dfrac{\exp(-x^2\tan\theta)\,\sin\frac{\theta}{2}\sqrt{\cos\theta}}{\sin 2\theta}\,d\theta.$

21. Meijer's G-Function is defined by

$$G_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) = \frac{1}{2\pi i}\int\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,x^s\,ds,$$

$$0 \le m \le q,\ 0 \le n \le p,$$

such that the poles of $\Gamma(b_j - s)$ do not coincide with those of $\Gamma(1 - a_k + s)$ for any j and k ($j = 1,\dots,m; k = 1,\dots,n$). The following notations are also used:

$$G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right), \quad G_{pq}^{mn}(x), \quad G(x).$$

The function $G_{pq}^{mn}\left(x\,\middle|\,\begin{smallmatrix}a_r\\ b_s\end{smallmatrix}\right)$ is analytic with respect to x ; it is symmetric with respect to the parameters $a_1,\dots,a_n$ and also with respect to $a_{n+1},\dots,a_p$; $b_1,\dots,b_m$; $b_{m+1},\dots,b_q$.

If no two b_j (for $j = 1, 2, \ldots, n$) differ by an integer, then either $p < q$ or $p = q$ and $|x| < 1$, and

$$G_{pq}^{mn}\left(x\middle|\begin{matrix}a_r\\ b_s\end{matrix}\right) = \sum_{k=1}^{m} \frac{\prod_{j=1}^{m}{}' \,\Gamma(b_j - b_k) \prod_{j=1}^{n}{}' \,\Gamma(1 + b_k - a_j)}{\prod_{j=m+1}^{q}{}' \,\Gamma(1 + b_k - b_j) \prod_{j=n+1}^{p}{}' \,\Gamma(a_j - b_k)} \, x^{b_k}$$

$\times {}_pF_{q-1}\big[1 + b_k - a_1, \ldots, 1 + b_k - a_p;\ 1 + b_k - b_1, \ldots\ldots, *, \ldots, 1 + b_k - b_q;\ (-1)^{p-m-n}\, x\big]$,

where $\prod'$ denotes the omission of the product when $j = k$; and the asterisk in the argument of the function ${}_pF_{q-1}$ denotes the omission of the k-th parameter. If no two a_k (for $k = 1, 2, \ldots, n$) differ by an integer, then $q < p$ or $q = p$ and $|x| > 1$, and

$$G_{pq}^{mn}\left(x\middle|\begin{matrix}a_r\\ b_s\end{matrix}\right) = \sum_{k=1}^{n} \frac{\prod_{j=1}^{n}{}' \,\Gamma(a_k - a_j) \prod_{j=1}^{m} \Gamma(b_j - a_k + 1)}{\prod_{j=n+1}^{p} \Gamma(a_j - a_k + 1) \prod_{j=m+1}^{q} \Gamma(a_k - b_j)} \, x^{a_k - 1}$$

$\times {}_qF_{p-1}\big[1+b_1-a_k, \ldots, 1+b_q-a_k;\ 1+a_1-a_k, \ldots, *, \ldots, 1+a_p-a_k;\ (-1)^{q-m-n}\, x^{-1}\big].$

22. Catalan's Constant $\mathbf{G} = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \approx 0.915965594.$

23. Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$ are defined by

$$\mathbf{H}_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{(z/2)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right)\Gamma\left(\nu + m + \frac{3}{2}\right)}.$$

$$\mathbf{L}_\nu(z) = -i\, e^{-i\,\nu\pi/2}\, \mathbf{H}_\nu\left(z\, e^{i\,\pi/2}\right) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right)\Gamma\left(\nu + m + \frac{3}{2}\right)}.$$

Their integral representations are:

$$\mathbf{H}_\nu(z) = \frac{2\,(z/2)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 (1 - t^2)^{\nu-1/2} \sin zt\, dt$$

$$= \frac{2\,(z/2)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi/2} \sin(z\cos\theta)(\sin\theta)^{2\nu} d\theta, \quad \Re\{\nu\} > -\tfrac{1}{2}.$$

$$\mathbf{L}_\nu(z) = \frac{2\,(z/2)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi/2} \sinh(z\cos\theta)(\sin\theta)^{2\nu} d\theta, \quad \Re\{\nu\} > -\tfrac{1}{2}.$$

Some special cases of these functions are:

1. $\mathbf{H}_n(z) = \dfrac{1}{\pi}\displaystyle\sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \frac{\Gamma\left(m+\frac{1}{2}\right)(z/2)^{n-2m-1}}{\Gamma\left(n+\frac{1}{2}-m\right)} - \mathbf{E}_n(z), \quad n = 1, 2, \ldots.$
2. $\mathbf{H}_{-n}(z) = (-1)^{n+1}\dfrac{1}{\pi}\displaystyle\sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \frac{\Gamma\left(n-m-\frac{1}{2}\right)(z/2)^{-n+2m+1}}{\Gamma\left(m+\frac{3}{2}\right)} - \mathbf{E}_{-n}(z),$
$n = 1, 2, \ldots.$
3. $\mathbf{H}_{n+1/2}(z) = Y_{n+1/2}(z) + \dfrac{1}{\pi}\displaystyle\sum_{m=0}^{n} \frac{\Gamma\left(m+\frac{1}{2}\right)(z/2)^{-2m+n-1/2}}{\Gamma(n+1-m)}, \quad n = 0, 1, \ldots.$
4. $\mathbf{H}_{(-n+1/2)}(z) = (-1)^n J_{n+1/2}(z), \quad n = 0, 1, \ldots.$
5. $\mathbf{L}_{(-n+1/2)}(z) = I_{n+1/2}(z), \quad n = 0, 1, \ldots.$
6. $\mathbf{H}_{1/2}(z) = \dfrac{\sqrt{2}}{\sqrt{\pi z}}(1 - \cos z).$
7. $\mathbf{H}_{3/2}(z) = \sqrt{\dfrac{z}{2\pi}}\left(1 + \dfrac{2}{z^2}\right) - \sqrt{\dfrac{2}{\pi z}}\left(\sin z + \dfrac{\cos z}{z}\right).$

Their functional relations are:

1. $\mathbf{H}_\nu(ze^{im\pi}) = e^{i\pi(\nu+1)m}\mathbf{H}_\nu(z), \quad m = 1, 2, 3, \ldots.$
2. $\dfrac{d}{dz}[z^\nu \mathbf{H}_\nu(z)] = z^\nu \mathbf{H}_{\nu-1}(z).$
3. $\dfrac{d}{dz}[z^{-\nu} \mathbf{H}_\nu(z)] = 2^{-\nu}\pi^{-1/2}\left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1} - z^{-\nu}\mathbf{H}_{\nu+1}(z).$
4. $\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\nu z^{-1}\mathbf{H}_\nu(z) + \pi^{-1/2}\left(\dfrac{z}{2}\right)^\nu \left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1}.$
5. $\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}'_\nu(z) - \pi^{-1/2}\left(\dfrac{z}{2}\right)^\nu \left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1}.$

24. Bateman's Function $k_\nu(x)$ is defined by

$$k_\nu(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\tan\theta - \nu\theta)\, d\theta.$$

25. Lobachevskiy's Function $L(x)$ is defined by

$$L(x) = -\int_0^x \ln\, \cos t\, dt.$$

A series representation of this function is

$$L(x) = x\, \ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin 2kx}{k^2}.$$

Some functional relationships are:

1. $L(-x) = -L(x), \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}.$
2. $L(\pi - x) = \pi \ln 2 - L(x).$
3. $L(\pi + x) = \pi \ln 2 + L(x).$
4. $L(x) - L\left(\dfrac{\pi}{2} - x\right) = \left(x - \dfrac{\pi}{4}\right) \ln 2 - \dfrac{1}{2} L\left(\dfrac{\pi}{2} - 2x\right), \quad 0 \le x < \dfrac{\pi}{4}.$

26. Logarithm Integral $\mathrm{li}(x)$ is defined by

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t} = \mathrm{Ei}(\ln x), \quad x < 1.$$

Other definitions are:

1. $\mathrm{li}(x) = \mathrm{li}m_{\to 0}\left[\displaystyle\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t}\right] = \mathrm{Ei}(\ln x), \quad x > 1.$
2. $\mathrm{li}\{\exp(-xe^{\pm\pi i})\} = \mathrm{Ei}(-xe^{\pm i\pi}) = \mathrm{Ei}(x \mp i0) = \mathrm{Ei}(x) \pm i\pi = \mathrm{li}(e^x) \pm i\pi, \quad x > 0.$

This function has the following integral representations:

1. $\mathrm{li}(x) = \displaystyle\int_{-\infty}^{\ln x} \frac{e^t}{t}\, dt = x \ln\ln\frac{1}{x} - \int_{-\ln x}^{\infty} e^{-t} \ln t\, dt, \quad [x < 1].$

2, $\mathrm{li}(x) = x\displaystyle\int_0^1 \frac{dt}{\ln x + \ln t} = \frac{x}{\ln x} + x\int_0^1 \frac{dt}{(\ln x + \ln t)^2} = x\int_1^\infty \frac{1}{\ln x - \ln t}\frac{dt}{t^2}, \quad x <$ 1.

3. $\mathrm{li}(a^x) = \dfrac{1}{\ln a}\displaystyle\int_{-\infty}^x \frac{a^t}{t}\, dt, \quad x > 0.$

27. Whittaker Functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$ are defined by

$$M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-z/2} \Phi\left(\mu - \lambda + \frac{1}{2},\, 2\mu + 1;\, z\right).$$

$$M_{\lambda,-\mu}(z) = z^{-\mu+\frac{1}{2}} e^{-z/2} \Phi\left(-\mu - \lambda + \frac{1}{2},\, 2\mu + 1;\, z\right).$$

We also have the Whittaker function $W_{\lambda,\mu}(z)$ which holds for $2\mu = \pm 1, \pm 2, \dots$:

$$W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \lambda\right)} M_{\lambda,-\mu}(z).$$

These functions have the integral representations:

$$M_{\lambda,\mu}(z) = \frac{z^{\mu+1/2}}{2^{2\mu}\mathrm{B}\left(\mu + \lambda + \frac{1}{2},\, \mu - \lambda + \frac{1}{2}\right)} \int_{-1}^1 (1+t)^{\mu-\lambda-1/2}(1-t)^{\mu+\lambda-1/2}\, e^{zt/2}\, dt,$$

if the integral converges. Other results are:

1. $W_{\lambda,\mu}(z) = \dfrac{z^{\mu+1/2}e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)}\displaystyle\int_0^\infty e^{-zt}t^{\mu-\lambda-1/2}(1+t)^{\mu-\lambda-1/2}\,dt.,$

$\Re\{\mu-\lambda\} > -\frac{1}{2},\ |\arg z| < \frac{\pi}{2}.$

2. $W_{\lambda,\mu}(z) = \dfrac{z^{\lambda}e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)}\displaystyle\int_0^\infty t^{\mu-\lambda-1/2}e^{-t}\left(1+\frac{t}{z}\right)^{\mu+\lambda-1/2}\,dt,$

$\Re\{\mu-\lambda\} > -\frac{1}{2},\ |\arg z| < \pi.$

3. $W_{\lambda,\mu}(z) = \dfrac{e^{-z/2}}{2i\,\pi}\displaystyle\int_{-i\,\infty}^{+i\,\infty}\frac{\Gamma(u-\lambda)\Gamma\left(-u-\mu+\frac{1}{2}\right)\Gamma\left(-u+\mu+\frac{1}{2}\right)}{\Gamma\left(-\lambda+\mu+\frac{1}{2}\right)\Gamma\left(-\lambda-\mu+\frac{1}{2}\right)}z^u\,du.$

4. $W_{\mu,1/2+\mu}(z) = z^{\mu+1}e^{-z/2}\displaystyle\int_0^\infty (1+t)^{2\mu}\,e^{-zt}\,dt = z^{-\mu}\,e^{z/2}\int_z^\infty t^{2\mu}e^{-t}\,dt,$

$\Re\{z\} > 0.$

5. $W_{\lambda,\mu}(x)W_{-\lambda,\mu}(x) = -x\displaystyle\int_0^\infty \tanh\,^{2\lambda}\frac{t}{2}\left\{J_{2\mu}(x\,\sinh\,t)\sin(\mu-\lambda)\pi\right.$

$\left.+Y_{2\mu}(x\,\sinh\,t)\cos(\mu-\lambda)\pi\right\}dt,\quad |\Re\{\mu\}|-\Re\{\lambda\} < \frac{1}{2};\ x > 0.$

6. $$W_{\kappa,\mu}(z_1)W_{\lambda,\mu}(z_2) = \frac{(z_1z_2)^{\mu+1/2}\exp\left[-\frac{1}{2}(z_1+z_2)\right]}{\Gamma(1-\kappa-\lambda)}$$

$$\times\int_0^\infty e^{-t}t^{-\kappa-\lambda}(z_1+t)^{-1/r2+\kappa-\mu}(z_2+t)^{-1/2+\lambda-\mu}$$

$$\times F\left(\frac{1}{2}-\kappa+\mu,\ \frac{1}{2}-\lambda+\mu;\ 1-\kappa-\lambda;\ \eta\right)\,dt,$$

$\eta = \dfrac{t(z_1+z_2+t)}{(z_1+t)(z_2+t)};\quad z_1 \neq 0,\ z_2 \neq 0,\ |\arg\,z_1| < \pi,\ |\arg\,z_2| < \pi,\ \Re\{\kappa+\lambda\} < 1.$

The series representations are:

$$M_{0,\mu}(z) = z^{\mu+1/2}\left\{1+\sum_{k=1}^\infty\frac{z^{2k}}{2^{4k}\,k!\,(\mu+1)(\mu+2)\ldots(\mu+k)}\right\},$$

and the asymptotic representations are given by

For large values of $|z|$:

$$W_{\lambda,\mu}(z) \sim e^{-z/2}z^{\lambda}\left(1+\sum_{k=1}^\infty\frac{\left[\mu^2-\left(\lambda-\frac{1}{2}\right)^2\right]\left[\mu^2-\left(\lambda-\frac{3}{2}\right)^2\right]\ldots\left[\mu^2-\left(\lambda-k+\frac{1}{2}\right)^2\right]}{k!\,z^k}\right),$$

$|\arg\,z| \leq \pi-\alpha < \pi;$

For large values of $|\lambda|$:

$$M_{\lambda,\mu}(z) \sim \frac{1}{\sqrt{\pi}}\Gamma(2\mu+1)\lambda^{-\mu-1/4}\,z^{1/4}\cos\left(2\sqrt{\lambda z}-\mu\pi-\tfrac{1}{4}\,\pi\right).$$

Also,

1. $W_{\lambda,\mu} \sim -\left(\frac{4z}{\lambda}\right)^{1/4} e^{-\lambda+\lambda\ln\lambda} \sin\left(2\sqrt{\lambda z} - \lambda\pi - \frac{\pi}{4}\right).$

2. $W_{-\lambda,\mu} \sim \left(\frac{z}{4\lambda}\right)^{1/4} e^{\lambda-\lambda\ \ln\ \lambda-2\sqrt{\lambda z}}.$

The above three formulas are valid for $|\lambda| \gg 1,\ |\lambda| \gg |z|,\ |\lambda| \gg |\mu|,\ z \neq 0,\ |\arg\sqrt{z}| < 3\pi/4$, and $|\arg\lambda| < \pi/2;\ x,\ \nu$ real.

28. Neumann's Polynomials $O_n(z)$ are defined by

1. $O_n(z) = \frac{1}{4}\sum_{m=0}^{\lfloor n/2\rfloor} \frac{n(n-m-1)!}{m!}\left(\frac{z}{2}\right)^{2m-n-1}, \quad n \geq 1.$

2. $O_{-n}(z) = (-1)^n O_n(z), \quad n \geq 1.$

3. $O_0(z) = \frac{1}{z}.$

4. $O_1(z) = \frac{1}{z^2}.$

5. $O_2(z) = \frac{1}{z} + \frac{4}{z^3}.$

In general, $O_n(z)$ is a polynomial in z^{-1} of degree $n+1$. The functional relations are given by

1. $O_0'(z) = -O_1(z).$

2. $2O_n'(z) = O_{n-1}(z) - O_{n+1}(z), \quad n \geq 1.$

3. $(n-1)O_{n+1}(z) + (n+1)O_{n-1}(z) - 2z^{-1}(n^2-1)O_n(z) = 2nz^{-1}\left(\sin n\frac{\pi}{2}\right)^2,$
$n \geq 1.$

4. $nzO_{n-1}(z) - (n^2-1)O_n(z) = (n-1)zO_n'(z) + n\left(\sin n\frac{\pi}{2}\right)^2.$

5. $nzO_{n+1}(z) - (n^2-1)O_n(z) = -(n+1)zO_n'(z) + n\left(\sin n\frac{\pi}{2}\right)^2.$

The generating function is

$$\frac{1}{z-\xi} = J_0(\xi)z^{-1} + 2\sum_{n=1}^{\infty} J_n(\xi)O_n(z), \quad |\xi| < |z|,$$

and the integral representation is

$$O_n(z) = \int_0^{\infty} \frac{[u+\sqrt{u^2+z^2}]^n + [u-\sqrt{u^2+z^2}]^n}{2z^{n+1}} e^{-u}du.$$

29. Associated Legendre Functions of the first kind, $P_\nu^{(\alpha)}(z)$, and of the second kind, $Q_\nu^{(\alpha)}(z)$, are single valued for $\Re\{z\} > 1$, and uniquely determined for $z = x$. They are defined, respectively, by

$$P_\nu^{(\alpha)}(z) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{z+1}{z-1}\right)^{\alpha/2} F\left(-\nu, \nu+1; \quad 1-\alpha; \quad \frac{1-z}{2}\right),$$

where $\arg \dfrac{z+1}{z-1} = 0$ if z is real and greater than 1; and

$$Q_\nu^{(\alpha)}(z) = \frac{e^{i\,\alpha\pi}\,\Gamma(\nu+\alpha+1)\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)} \left(z^2-1\right)^{\alpha/2} z^{-\nu-\alpha-1} F\left(\frac{\nu+\alpha+2}{2}, \frac{\nu+\alpha+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right),$$

where $\arg(z^2-1) = 0$ when z is real and greater than 1, and $\arg z = 0$ when z is real and positive.

For $z = x$ real in the interval $[-1, 1]$, set $z = x = \cos\theta$. Then

$$\begin{aligned} P_\nu^{(\alpha)}(x) &= \frac{1}{2}\left[e^{i\,\alpha\pi/2}\,P_\nu^{(\alpha)}(\cos\theta + i\,0) + e^{-i\,\alpha\pi/2}\,P_\nu^{(\alpha)}(\cos\theta - i\,0)\right] \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{1+x}{1-x}\right)^{\alpha/2} F\left(-\nu, \nu+1; 1-\alpha; \frac{1-x}{2}\right). \\ Q_\nu^{(\alpha)}(x) &= \frac{1}{2}e^{-i\,\alpha\pi}\left[e^{-i\,\alpha\pi/2}\,Q_\nu^{(\alpha)}(x+i\,0) + e^{i\,\alpha\pi/2}\,Q_\nu^{(\alpha)}(x-i\,0)\right] \\ &= \frac{\pi}{2\sin\alpha\pi}\left[P_\nu^{(\alpha)}(x)\cos\alpha\pi - \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu-\alpha+1)}\,P_\nu^{(-\alpha)}(x)\right]. \end{aligned}$$

When $\alpha = \pm m$ is an integer, the following formulas are obtained by passing to the limit:

1. $Q_\nu^{(m)}(x) = (-1)^m(1-x^2)^{m/2}\dfrac{d^m}{dx^m}Q_\nu(x)$.
2. $Q_\nu^{(-m)}(x) = (-1)^m\dfrac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}Q_\nu^{(m)}(x)$.

The functions $Q_\nu^{(\alpha)}(z)$ are not defined when $\nu+\alpha$ is a negative integer. Hence, we exclude $\nu+\alpha = -1, -2, -3, \dots$ for these formulas. For $\alpha = 0$, we will denote $P_\nu^{(0)}(x)$ and $Q_\nu^{(0)}(x)$ simply by $P_\nu(x)$ and $Q_\nu(x)$, respectively. Some special cases and particular values are:

1. $P_\nu^{(m)}(x) = (-1)^m\dfrac{\Gamma(\nu+m+1)(1-x^2)^{m/2}}{2^m\Gamma(\nu-m+1)m!}F\left(m-\nu, m+\nu+1; m+1; \dfrac{1-x}{2}\right)$.
2. $P_\nu^{(m)}(z) = \dfrac{\Gamma(\nu+m+1)(z^2-1)^{m/2}}{2^m m!\Gamma(\nu-m+1)}F\left(m-\nu, m+\nu+1; m+1; \dfrac{1-z}{2}\right)$.
3. $Q_{n-1/2}^{(\alpha)}(z) = \dfrac{e^{i\,\mu\pi}\,\Gamma\left(\mu+n+\frac{3}{2}\right)}{2^{n+\frac{3}{2}}(n+1)!}(z^2-1)^{\mu/2}\pi^{\frac{1}{2}}z^{-n-\mu-3/2}F\left(\dfrac{\mu+n+\frac{5}{2}}{2}, \dfrac{\mu+n+\frac{3}{2}}{2}; n+2; \dfrac{1}{z^2}\right)$.
4. $P_\nu^{(m)}(x) = (-1)^m\left(1-x^2\right)^{m/2}\dfrac{d^m}{dx^m}P_\nu(x)$.

5. $P_\nu^{(-m)}(x) = (-1)^m \dfrac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} P_\nu^m(x)$

$$= \left(1 - x^2\right)^{-m/2} \int_x^1 \ldots \int_x^1 P_\nu(x)(dx)^m, \quad m \geq 1$$

6. $P_\nu^{(-m)}(z) = \left(z^2 - 1\right)^{-m/2} \displaystyle\int_1^z \ldots \int_1^z P_\nu(z)(dz)^m, \quad m \geq 1.$

7. $Q_\nu^{(m)}(z) = \left(z^2 - 1\right)^{m/2} \dfrac{d^m}{dz^m} Q_\nu(z).$

8. $Q_\nu^{(-m)}(z) = (-1)^m \left(z^2 - 1\right)^{-m/2} \displaystyle\int_z^\infty \ldots \int_z^\infty Q_\nu(z)(dz)^m, \quad m \geq 1.$

9. $P_0^{(\alpha)}(\cos\theta) = \dfrac{1}{\Gamma(1 - \alpha)} \cot^\alpha \dfrac{\theta}{2}.$

10. $P_\nu^{(-1)}(\cos\theta) = -\dfrac{1}{\nu(\nu + 1)} \dfrac{dP_\nu(\cos\theta)}{d\theta}.$

11. $P_n^{(m)}(z) \equiv 0, \quad P_n^{(m)}(x) \equiv 0$ for $m > n$.

12. $P_{\nu-1/2}^{(1/2)}(\cosh\alpha) = \sqrt{\dfrac{2}{\pi \sinh\alpha}} \cosh\nu\alpha.$

13. $P_{\nu-1/2}^{(1/2)}(\cos\theta) = \sqrt{\dfrac{2}{\pi \sin\theta}} \cos\nu\theta.$

14. $P_{\nu-1/2}^{-(1/2)}(\cos\theta) = \sqrt{\dfrac{2}{\pi \sin\theta}} \dfrac{\sin\nu\theta}{\nu}.$

15. $Q_{\nu-1/2}^{(1/2)}(\cosh\alpha) = i \sqrt{\dfrac{\pi}{2 \sinh\alpha}}\, e^{-\nu\alpha}.$

16. $P_\nu^{(-\nu)}(\cos\theta) = \dfrac{1}{\Gamma(1 + \nu)} \left(\dfrac{\sin\theta}{2}\right)^\nu.$

17. $P_\nu^{(-\nu)}(\cosh\alpha) = \dfrac{1}{\Gamma(1 + \nu)} \left(\dfrac{\sinh\alpha}{2}\right)^\nu.$

18. $P_\nu^{(\alpha)}(0) = \dfrac{2^\alpha \sqrt{\pi}}{\Gamma\left(\frac{\nu-\alpha}{2} + 1\right) \Gamma\left(\frac{-\nu-\alpha+1}{2}\right)}.$

19. $\dfrac{dP_\nu^{(\alpha)}(0)}{dx} = \dfrac{2^{\alpha+1} \sin \frac{1}{2}(\nu + \alpha)\pi \Gamma\left(\frac{\nu+\alpha}{2} + 1\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-\alpha+1}{2}\right)}.$

20. $Q_\nu^{(\alpha)}(0) = -2^{\alpha-1} \sqrt{\pi} \sin \dfrac{1}{2}(\nu + \alpha)\pi \dfrac{\Gamma\left(\frac{\nu+\alpha+1}{2}\right)}{\Gamma\left(\frac{\nu-\alpha}{2} + 1\right)}.$

21. $\dfrac{dQ_\nu^{(\alpha)}(0)}{dx} = 2^\alpha \sqrt{\pi} \cos \dfrac{1}{2}(\nu + \alpha)\pi \dfrac{\Gamma\left(\frac{\nu+\alpha}{2} + 1\right)}{\Gamma\left(\frac{\nu-\alpha+1}{2}\right)}.$

30. Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$ are defined by

$$s_{\mu,\nu}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{\mu+1+2m}}{[(\mu+1)^2-\nu^2][(\mu+3)^2-\nu^2]\ldots[(\mu+2m+1)^2-\nu^2]}$$

$$= z^{\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+2} \Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+m+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+m+\frac{3}{2}\right)},$$

$\mu \pm \nu$ is not a negative odd integer.

2. $$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + \left[2^{\mu-1}\Gamma\left(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2}\right)\right] \times \frac{\cos\left[\frac{1}{2}(\mu-\nu)\pi\right] J_{-\nu}(z) - \cos\left[\frac{1}{2}(\mu+\nu)\pi\right] J_{\nu}(z)}{\sin\nu\pi},$$

$\mu\pm\nu$ is a positive odd integer, ν is an odd integer

$$= s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2}\right) \times \left\{\sin\left[\tfrac{1}{2}(\mu-\nu)\pi\right] J_{\nu}(z) - \cos\left[\tfrac{1}{2}(\mu-\nu)\pi\right] Y_{\nu}(z)\right\},$$

$\mu \pm \nu$ is a positive odd integer, ν is an integer

Integral representations are:

$$s_{\mu,\nu}(z) = \frac{\pi}{2}\left[Y_{\nu}(z)\int_0^z z^{\mu} J_{\nu}(z)\,dz - J_{\nu}(z)\int_0^z z^{\mu} Y_{\nu}(z) dz\right].$$

$$s_{\mu,\nu}(z) = 2^{\mu}\left(\frac{z}{2}\right)^{(1+\nu+\mu)/2}\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu\right) \times \int_0^{\pi/2} J_{(1+\mu-\nu)/2}(z\sin\theta)(\sin\theta)^{(1+\nu-\mu)/2}(\cos\theta)^{\nu+\mu}d\theta, \qquad \Re\{\nu+\mu+1\} > 0.$$

Some special cases are:

1. $S_{1,2n}(z) = zO_{2n}(z)$.
2. $S_{0,2n+1}(z) = \dfrac{z}{2n+1}\, O_{2n+1}(z)$.
3. $S_{-1,2n}(z) = \dfrac{1}{4n} S_{2n}(z)$.
4. $S_{0,2n+1}(z) = \frac{1}{2}\, S_{2n+1}(z)$.
5. $s_{\nu,\nu}(z) = \Gamma\left(\nu+\dfrac{1}{2}\right)\, 2^{\nu-1}\sqrt{\pi}\,\mathbf{H}_{\nu}(z)$.
6. $S_{\nu,\nu}(z) = [\mathbf{H}_{\nu}(z) - Y_{\nu}(z)]\, 2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)$.

Functional relations are as follows:

1. $s_{\mu+2,\nu}(z) = z^{\mu+1} - \left[(\mu+1)^2-\nu^2\right] s_{\mu,\nu}(z)$.

2. $s_{\mu,\nu}(z) + \left(\dfrac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu+\nu-1)s_{\mu-1,\nu-1}(z)$.

3. $s'_{\mu,\nu}(z) - \left(\dfrac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu-\nu-1)s_{\mu-1,\nu+1}(z)$.

4. $\left(2\dfrac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu+\nu-1)s_{\mu-1,\nu-1}(z) - (\mu-\nu-1)s_{\mu-1,\nu+1}(z)$.

5. $2s_{\mu,\nu}(z) = (\mu+\nu-1)s_{\mu-1,\nu-1}(z) + (\mu-\nu-1)s_{\mu-1,\nu+1}(z)$.

In these relations $s_{\mu,\nu}(z)$ can be replaced by $S_{\mu,\nu}(z)$.

31. Lommel Functions of Two Variables $U_\nu(w,z)$ and $V_\nu(w,z)$. These functions are defined by

$$U_\nu(w,z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z).$$

$$V_\nu(w,z) = \cos\left[\frac{1}{2}\left(w + \frac{z^2}{w} + \nu\pi\right)\right] + U_{-\nu+2}(w,z).$$

Some of their particular values are:

1. $U_0(z,z) = V_0(z,z) = \frac{1}{2}\left\{J_0(z) + \cos z\right\}$.

2. $U_1(z,z) = -V_1(z,z) = \frac{1}{2}\sin z$.

3. $$U_{2n}(z,z) = V_{2n}(z,z) = \frac{(-1)^n}{2}\left\{\cos z - \sum_{m=0}^{n-1}(-1)^m \varepsilon_{2m} J_{2m}(z)\right\},$$
$$n \ge 1, \quad \text{and} \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0. \end{cases}$$

4. $$U_{2n+1}(z,z) = -V_{2n+1}(z,z) = \frac{(-1)^n}{2}\left\{\sin z - \sum_{m=0}^{n-1}(-1)^m \varepsilon_{2m+1} J_{2m+1}(z)\right\},$$
$$n \ge 0, \quad \text{and} \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0. \end{cases}$$

5. $V_n(w,z) = (-1)^n U_n\left(\dfrac{z^2}{w}, z\right)$.

6. $U_\nu(w,0) = \dfrac{(w/2)^{1/2}}{\Gamma(\nu-1)} s_{\nu-3/2,1/2}\left(\dfrac{w}{2}\right)$.

7. $V_{-\nu+2}(w,0) = \dfrac{(w/2)^{1/2}}{\Gamma(\nu-1)} S_{\nu-3/2,1/2}\left(\dfrac{w}{2}\right)$.

32. Schläfli's Polynomials $S_n(z)$ are defined by

$$S_0(z) = 0, \quad S_n(z) = \frac{1}{n}\left[2zO_n(z) - 2\left(\cos n\frac{\pi}{2}\right)^2\right], \quad n \geq 1$$

$$= \sum_{m=0}^{[n/2]} \frac{(n-m-1)!}{m!}\left(\frac{z}{2}\right)^{2m-n}, \quad n \geq 1$$

such that $S_{-n}(z) = (-1)^{n+1}S_n(z)$. Their functional relations are given by

$$S_{n-1}(z) + S_{n+1}(z) = 4O_n(z).$$

33. Lerch Function $\Phi(z, s, v)$ is defined by

$$\Phi(z,\, s,\, v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n, \quad |z| < 1,\ v \neq 0, -1, \ldots .$$

It has the following functional relations and series representations:

1. $\Phi(z,\, s,\, v) = z^m\Phi(z,\, s,\, m+v) + \sum_{n=0}^{m-1}(v+n)^{-s}z^n,$
$$m = 1, 2, 3, \ldots,\ v \neq 0, -1, -2, \ldots .$$

2. $\Phi(z,\, s,\, v)$
$$= iz^{-v}(2\pi)^{s-1}\Gamma(1-s)\left[e^{-i\pi s/2}\Phi\left(e^{-2\pi iv},\, 1-s,\, \frac{\ln\, z}{2\pi i}\right) - e^{i\pi(s/2-2v)}\Phi\left(e^{2\pi iv},\, 1-s,\, 1-\frac{\ln\, z}{2\pi i}\right)\right].$$

3. $\Phi(z,\, s,\, v) = z^{-v}\Gamma(1-s)\sum_{n=-\infty}^{\infty}(-\ln\, z + 2\pi ni)^{s-1}e^{2\pi nvi},$
$$0 < v \leq 1,\ \Re\{\, s\} < 0,\ |\arg(-\ln\, z + 2\pi ni)| \leq \pi.$$

4. $\Phi(z,\, m,\, v) = z^{-v}\left\{\sum_{n=0}^{\infty}{}'\,\zeta(m-n,\, v)\frac{(\ln\, z)^n}{n!} + \frac{(\ln\, z)^{m-1}}{(m-1)!}\left[\psi(m) - \psi(v) - \ln\left(\ln\, \frac{1}{z}\right)\right]\right\},$
$$m = 2, 3, 4, \ldots,\ |\ln\, z| < 2\pi,\ v \neq 0, -1, -2, \ldots,$$
where $\sum{}'$ means that the term corresponding to $n = m-1$ is omitted.

5. $\Phi(z,\, -m,\, v) = \frac{m!}{z^v}\left(\ln\, \frac{1}{z}\right)^{-m-1} - \frac{1}{z^v}\sum_{r=0}^{\infty}\frac{B_{m+r+1}(v)(\ln\, z)^r}{r!(m+r+1)}, \quad |\ln\, z| < 2\pi.$

Other relations are:

1. $\Phi(z,\, s,\, v) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}\, dt = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}\, dt}{e^t - z},$
$$\Re\{v\} > 0, \text{ or } |z| \leq 1,\ z \neq 1,\ \Re\{s\} > 0, \text{ or } z = 1,\ \Re\{s\} > 1.$$

2. $\lim_{z\to 1}(1-z)^{1-s}\Phi(z,\, s,\, v) = \Gamma(1-s), \quad \Re\{s\} < 1.$

3. $\lim_{z\to 1} \dfrac{\Phi(z,\, 1,\, v)}{-\ln(1-z)} = 1.$

4. $\Phi(z,\, 1,\, v) = v^{-1}{}_2F_1(1,\, v;\, 1+v;\, z), \quad |z| < 1.$

34. Lobachevskiy's Angle of Parallelism $\Pi(x)$ is defined by

1. $\Pi(x) = 2\ \arctan\ e^{x} = 2\ \arctan\ e^{-x}, \quad x \geq 0.$
2. $\Pi(x) = \pi - \Pi(-x), \quad x < 0.$

The functional relations are:

1. $\sin\ \Pi(x) = \dfrac{1}{\cosh\ x}.$
2. $\cos\ \Pi(x) = \tanh\ x.$
3. $\tan\ \Pi(x) = \dfrac{1}{\sinh\ x}.$
4. $\cot\ \Pi(x) = \sinh\ x.$
5. $\sin\ \Pi(x+y) = \dfrac{\sin\ \Pi(x)\ \sin\ \Pi(y)}{1+\cos\ \Pi(x)\ \cos\ \Pi(y)}.$
6. $\cos\ \Pi(x+y) = \dfrac{\cos\ \Pi(x) + \cos\ \Pi(y)}{1+\cos\ \Pi(x)\ \cos\ \Pi(y)}.$

35. Weierstrass Sigma Function $\sigma(u)$ is defined by

$$\sigma(u) = u\prod{}' \left(1 - \frac{u}{2m\omega_1 + 2n\omega_2}\right)\ \exp\left\{\frac{u}{2m\omega_1 + 2n\omega_2} + \frac{u^2}{2(2m\omega_1 + 2n\omega_2)^2}\right\}.$$

Its series and infinite-product representation is given by

$$\sigma(u) = u - \frac{g_2 u^5}{2^4 \cdot 3 \cdot 5} - \frac{g_3 u^7}{2^3 \cdot 3 \cdot 5 \cdot 7} - \frac{g_2^2 u^9}{2^9 \cdot 3^2 \cdot 5 \cdot 7} - \frac{3 g_2 g_3 u^{11}}{2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11} - \cdots.$$

Functional relations and properties are:

1. $\sigma(u + 2\omega_1) = -\sigma(u)\ \exp\{2(u+\omega_1)\zeta(\omega_1)\}.$
2. $\sigma(u + 2\omega_2) = -\sigma(u)\ \exp\{2(u+\omega_2)\zeta(\omega_2)\}.$
3. $\sigma(u;\, \omega_1,\, \omega_2) = t\,\sigma(tu;\, t\omega_1,\, t\omega_2).$

36. Weierstrass Zeta Function $\zeta(u)$ is defined by

$$\zeta(u) = \frac{1}{u} - \int_0^u \left(\wp(z) - \frac{1}{z^2}\right) dz.$$

Its series and infinite-product representations are

1. $\zeta(u) = \dfrac{1}{u} + \sum{}' \left(\dfrac{1}{u - 2m\omega_1 - 2n\omega_2} + \dfrac{1}{2m\omega_1 + 2n\omega_2} + \dfrac{u}{(2m\omega_1 - 2n\omega_2)^2}\right).$

2. $\zeta(u) = \dfrac{1}{u} + \sum{}' \left(\dfrac{1}{u - 2m\omega_1 - 2n\omega_2} + \dfrac{1}{2m\omega_1 + 2n\omega_2} + \dfrac{u}{(2m\omega_1 - 2n\omega_2)^2} \right).$

3. $\zeta(u) = u - \dfrac{g_2 u^3}{2^2 \cdot 3 \cdot 5} - \dfrac{g_3 u^5}{2^2 \cdot 5 \cdot 7} - \dfrac{g_2^2 u^7}{2^4 \cdot 3 \cdot 5^2 \cdot 7} - \dfrac{3 g_2 g_3 u^9}{2^4 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \cdots.$

37. Riemann's Zeta Functions $\zeta(z, q)$, and $\zeta(z)$ are defined by

$$\zeta(z,\, q) = \int_0^\infty \frac{t^{z-1} e^{-qt}}{1 - e^{-t}}\, dt$$
$$= \frac{1}{2} q^{-z} + \frac{q^{1-z}}{z-1} + 2\int_0^\infty (q^2 + t^2)^{-z/2} \left[\sin\left(z \text{ arctan } \frac{t}{q} \right) \right] \frac{dt}{e^{2\pi t} - 1},$$
$$0 < q < 1,\ \Re\{z >\}1.$$

Some integral representations are:

1. $\zeta(z,\, q) = -\dfrac{\Gamma(1-z)}{2\pi i} \displaystyle\int_\infty^{(0+)} \dfrac{(-\theta)^{z-1} e^{-q\theta}}{1 - e^{-\theta}}\, d\theta,$

which is valid for all z except $z = 1, 2, 3, \ldots.$

2. $\zeta(z) = \dfrac{1}{(1 - 2^{1-z})\Gamma(z)} \displaystyle\int_0^\infty \dfrac{t^{z-1}}{e^t + 1}\, dt, \quad \Re\{z\} > 0.$

3. $\zeta(z) = \dfrac{2^z}{(2^z - 1)\Gamma(z)} \displaystyle\int_0^\infty \dfrac{t^{z-1} e^t}{e^{2t} - 1}\, dt, \quad \Re\{z\} > 1.$

4. $\zeta(z) = \dfrac{\pi^{z/2}}{\Gamma\left(\frac{z}{2}\right)} \left[\dfrac{1}{z(z-1)} + \displaystyle\int_1^\infty \left(t^{(1-z)/2} + t^{z/2} \right) t^{-1} \sum_{k=1}^\infty e^{-i\, k^2 \pi}\, dt \right].$

5. $\zeta(z) = \dfrac{2^{z-1}}{z-1} - 2^z \displaystyle\int_0^\infty (1 + t^2)^{-z/2} \dfrac{\sin(z \text{ arctan } t)}{e^{\pi t} + 1}\, dt.$

6. $\zeta(z) = \dfrac{2^{z-1}}{2^z - 1} \dfrac{z}{z-1} + \dfrac{2}{2^z - 1} \displaystyle\int_0^\infty \left(\dfrac{1}{4} + t^2 \right)^{-z/2} \dfrac{\sin(z \text{ arctan } 2t)}{e^{2\pi t} - 1}\, dt.$

Representation in terms of infinite series are given by

1. $\zeta(z,\, q) = \displaystyle\sum_{n=0}^\infty \dfrac{1}{(q+n)^z}, \quad \Re\{z\}1,\ q \neq 0,\ -1, -2, \ldots.$

2. $\zeta(z,\, q) = \dfrac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin \dfrac{z\pi}{2} \displaystyle\sum_{n=1}^\infty \dfrac{\cos 2\pi q n}{n^{1-z}} + \cos \dfrac{z\pi}{2} \sum_{n=1}^\infty \dfrac{\sin 2\pi q n}{n^{1-z}} \right],$

$$\Re\{z\} < 0,\ 0 < q \leq 1.$$

3. $\zeta(z,\, q) = \displaystyle\sum_{n=0}^N \dfrac{1}{(q+n)^z} - \dfrac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^\infty F_n(z),$

where

$$F_n(z) = \frac{1}{1-z}\left(\frac{1}{(n+1+q)^{z-1}} - \frac{1}{(n+q)^{z-1}}\right) - \frac{1}{(n+1+q)^z}$$
$$= z\int_n^{n+1} \frac{(t-n)\,dt}{(t+q)^{z+1}}, \quad \Re\{z\} > 1.$$

4. $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \Re\{z\} > 1.$

5. $\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z}, \quad \Re\{z\} > 0.$

The functional relations are given by

1. $\zeta(-n,\, q) = -\frac{B'_{n+2}(q)}{(n+1)(n+2)} = \frac{-B_{n+1}(q)}{n+1}, \quad n$ a nonnegative integer.

2. $\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} z^k \zeta(k,\, q) = \ln\, \frac{\exp(-\gamma_e\, z)\Gamma(q)}{\Gamma(z+q)} - \frac{z}{q} + \sum_{k=1}^{\infty} \frac{qz}{k(q+k)}, \quad |z| < q.$

Other useful relations are:

1. $\lim_{z\to 1} \frac{\zeta(z,\, q)}{\Gamma(1-z)} = -1.$

2. $\lim_{z\to 1} \left\{\zeta(z,\, q) - \frac{1}{z-1}\right\} = -\psi(q).$

3. $\left[\frac{d}{dz}\zeta(z,\, q)\right]_{z=0} = \ln\, \Gamma(q) - \frac{1}{2}\ln 2\pi.$

4. $\zeta(z,\, 1) = \zeta(z).$

5. $\zeta(z) = \frac{1}{2^z - 1}\zeta\left(z,\, \frac{1}{2}\right), \quad \Re\{z\} > 1.$

6. $2^z\, \Gamma(1-z)\, \zeta(1-z)\, \sin\frac{z\pi}{2} \pi^{1-z}\zeta(z).$

7. $2^{1-z}\Gamma(z)\, \zeta(z) \cos\frac{z\pi}{2} = \pi^z \zeta(1-z).$

8. $\Gamma\left(\frac{z}{2}\right) \pi^{-z/2}\zeta(z) = \Gamma\left(\frac{1-z}{2}\right) \pi^{(z-1)/2}\, \zeta(1-z).$

38. The Function $\xi(s)$ is defined by

$$\xi(s) = \frac{1}{2}s(s-1)\frac{\Gamma\left(\frac{1}{2}s\right)}{\pi^{\frac{1}{2}s}}\zeta(s).$$

where $\xi(1-s) = \xi(s)$.

39. The Functions $\nu(x)$, and $\nu(x, \alpha)$ are defined by

1. $\nu(x) = \displaystyle\int_0^\infty \frac{x^t\,dt}{\Gamma(t+1)}$.
2. $\nu(x,\,\alpha) = \displaystyle\int_0^\infty \frac{x^{\alpha+t}\,dt}{\Gamma(\alpha+t+1)}$.
3. $\mu(x,\,\beta) = \displaystyle\int_0^\infty \frac{x^t t^\beta\,dt}{\Gamma(\beta+1)\Gamma(t+1)}$.
4. $\mu(x,\,\beta,\,\alpha) = \displaystyle\int_0^\infty \frac{x^{\alpha+t} t^\beta\,dt}{\Gamma(\beta+1)\Gamma(\alpha+t+1)}$.
5. $\lambda(x,\,y) = \displaystyle\int_0^y \frac{\Gamma(u+1)\,du}{x^u}$.

40. Confluent Hypergeometric Series in Two Variables. These functions, denoted by $\Phi_{1,2,3}(\alpha,\beta,\gamma,x,y)$, are defined in terms of infinite series as

1. $\Phi_1(\alpha,\,\beta,\,\gamma,\,x,\,y) = \displaystyle\sum_{m,n=0}^\infty \frac{(\alpha)_{m+n}\,(\beta)_m}{m!\,n!\,(\gamma)_{m+n}}\,x^m y^n, \quad |x|<1.$
2. $\Phi_2(\beta,\,\beta',\,\gamma,\,x,\,y) = \displaystyle\sum_{m,n=0}^\infty \frac{(\beta)_m\,(\beta')_m}{m!\,n!\,(\gamma)_{m+n}}\,x^m\,y^n.$
3. $\Phi_3(\beta,\,\gamma,\,x,\,y) = \displaystyle\sum_{m,n=0}^\infty \frac{(\beta)_m}{m!\,n!\,(\gamma)_{m+n}}\,x^m\,y^n.$

41. Kummer's Confluent Hypergeometric Functions $M(a,b,z)$ and $U(a,b,z)$ are defined by

$$M(a,b,z) = 1 + \frac{a\,z}{b} + \frac{(a)_2\,z^2}{(b)_2\,2!} + \cdots + \frac{(a)_n\,z^n}{(b)_n\,n!} + \cdots\,,$$

where $(a)_n$ denotes the Pochhammer's symbol, $(a)_0 = 1$, and

$$M(a,b,z) = 1 + \frac{a\,z}{b} + \frac{(a)_2\,z^2}{(b)_2\,2!} + \cdots + \frac{(a)_n\,z^n}{(b)_n\,n!} + \cdots\,,$$

$$U(a,b,z) = \frac{\pi}{\sin\pi b} + \left\{\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right\}.$$

For m,n positive integers, we have:

1. $M(a,b,z)$ is a convergent series for all values of z and $a \neq -m$ and $b \neq -n$;
2. $M(a,b,z) \in \mathcal{P}_m$ in z for $a \neq -m$ and $b \neq -n$;
3. $M(a,b,z)$ has a simple pole at $b = -n$ for $a \neq -m$ and $b = -n$ or $a = -m$ and $b \neq -n$, $m > n$;

4. $M(a,b,z)$ is undefined for $a=-m$ and $b=-n$, $m \leq n$; and

5. $U(a,b,z)$ is defined even when $b \to \pm n$; it is a many-valued function and its principal branch is given by $-\pi < \arg\{z\} \leq \pi$.

Asymptotic Expansions as $|z| \to \infty$:

$$M(a,b,z) = \begin{cases} \dfrac{\Gamma(b)}{\Gamma(a)}\, e^z\, z^{a-b}\, \left[1 + O\left(|z|^{-1}\right]\right. & \text{for } \Re\{z\} > 0, \\ \dfrac{\Gamma(b)}{\Gamma(b-a)}\, (-z)^{-a}\, \left[1 + O\left(|z|^{-1}\right]\right. & \text{for } \Re\{z\} < 0. \end{cases}$$