! For an efficient use of these tables, first read `HowTo.pdf`.

T3.59A. Integrands involving logarithms, trigonometric functions and rational functions on the interval $(0,\infty)$.

1. $\displaystyle\int_0^\infty \ln x \sin ax \frac{dx}{x} = -\frac{\pi}{2}\left(\gamma_e + \ln a\right), \quad a>0.$

2. $\displaystyle\int_0^\infty \ln ax \sin bx \frac{x\,dx}{\beta^2+x^2} = \frac{\pi}{2}\, e^{-b\beta'} \ln\left(a\beta'\right) - \frac{\pi}{4}\left[e^{b\beta'}\operatorname{Ei}(-b\beta') + e^{-b\beta'}\operatorname{Ei}(b\beta')\right],$

$\beta' = \beta \operatorname{sgn}\beta;\ a>0,\ b>0.$

3. $\displaystyle\int_0^\infty \ln ax \cos bx \frac{\beta'\,dx}{\beta^2+x^2} = \frac{\pi}{2}\, e^{-b\beta'} \ln\left(a\beta'\right) + \frac{\pi}{4}\left[e^{b\beta'}\operatorname{Ei}(-b\beta') - e^{-b\beta'}\operatorname{Ei}(b\beta')\right],$

$\beta' = \beta \operatorname{sgn}\beta;\ a>0,\ b>0.$

4. $\displaystyle\int_0^\infty \ln ax \sin bx \frac{x\,dx}{x^2-c^2} = \frac{\pi}{2}\{-\operatorname{Si}(bc)\sin bc + \cos bc\,[\ln ac - \operatorname{Ci}(bc)]\}, \quad a>0,\ b>0,\ c>0.$

5. $\displaystyle\int_0^\infty \ln ax \cos bx \frac{dx}{x^2-c^2} = \frac{\pi}{2c}\{\sin bc\,[\operatorname{Ci}(bc) - \ln ac] - \cos bc\,\operatorname{Si}(bc)\}, \quad a>0,\ b>0,\ c>0.$

1. $\displaystyle\int_0^\infty \ln x \sin ax\, x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu}\sin\frac{\mu\pi}{2}\left[\psi(\mu) - \ln a + \frac{\pi}{2}\cot\frac{\mu\pi}{2}\right], \quad a>0,\ |\Re\{\mu\}|<1.$

6. $\displaystyle\int_0^\infty \ln x \cos ax\, x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu}\cos\frac{\mu\pi}{2}\left[\psi(\mu) - \ln a - \frac{\pi}{2}\tan\frac{\mu\pi}{2}\right], \quad a>0,\ 0<\Re\{\mu\}<1.$

7. $\displaystyle\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x}\,dx = \ln\frac{a}{b}\left(\gamma_e + \frac{1}{2}\ln ab\right), \quad a>0,\ b>0.$

8. $\displaystyle\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}\left[(a-b)\left(\gamma_e - 1\right) + a\ln a - b\ln b\right], \quad a>0,\ b>0.$

9. $\displaystyle\int_0^\infty \ln\, x\frac{\sin^2 ax}{x^2}\,dx = -\frac{a\pi}{2}\left(\gamma_e + \ln\, 2a - 1\right), \quad a > 0.$

10. $\displaystyle\int_0^\infty (\ln\, x)^2 \sin ax \frac{dx}{x} = \frac{\pi}{2}{\gamma_e}^2 + \frac{\pi^3}{24} + \pi\gamma_e \ln\, a + \frac{\pi}{2}(\ln\, a)^2, \quad a > 0.$

11. $$\int_0^\infty (\ln\, x)^2 \sin ax\, x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu}\sin\frac{\mu\pi}{2}\left[\psi'(\mu) + \psi^2(\mu) + \pi\psi(\mu)\cot\frac{\mu\pi}{2} - 2\psi(\mu)\ln\, a \right.$$
$$\left. -\pi\ln\, a\cot\frac{\mu\pi}{2} + (\ln\, a)^2 - \frac{1}{4}\pi^2\right], \quad a > 0,\ 0 < \Re\{\mu\} < 1.$$

12. $\displaystyle\int_0^\infty \ln\,(1+x)\cos ax\frac{dx}{x} = \frac{1}{2}\left\{[\mathrm{Si}(a)]^2 + [\mathrm{Ci}(a)]^2\right\}, \quad a > 0.$

13. $\displaystyle\int_0^\infty \ln\,\left(\frac{b+x}{b-x}\right)^2 \cos ax\frac{dx}{x} = -2\pi\ \mathrm{Si}(ab), \quad a \ge 0,\, b > 0.$

14. $\displaystyle\int_0^\infty \ln\,(1+b^2x^2)\sin ax\frac{dx}{x} = -\pi\,\mathrm{Ei}\left(-\frac{a}{b}\right), \quad a > 0,\, b > 0.$

15. $\displaystyle\int_0^\infty x\, \ln\, \frac{b^2+x^2}{c^2+x^2}\sin ax\,dx = \frac{\pi}{a^2}\left[(1+ac)e^{-ac} - (1+ab)e^{-ab}\right], \quad b \ge 0,\, c \ge 0,\, a > 0.$

16. $\displaystyle\int_0^\infty \ln\, \frac{b^2x^2+p^2}{c^2x^2+p^2}\sin ax\frac{dx}{x} = \pi\left[\mathrm{Ei}\left(-\frac{ap}{c}\right) - \mathrm{Ei}\left(-\frac{ap}{b}\right)\right], \quad b > 0,\, c > 0,\, p > 0,\, a > 0.$

17. $$\int_0^\infty \ln\,(x + \sqrt{\beta^2+x^2})\frac{\sin ax}{\sqrt{\beta^2+x^2}}\,dx = \frac{\pi}{2}K_0(a\beta) + \frac{\pi}{2}\ln\,(\beta)[I_0(a\beta) - \mathbf{L}(a\beta)],$$
$$\Re\{\beta\} > 0,\, a > 0.$$

18. $\displaystyle\int_0^\infty \ln\, \cos^2\, ax\frac{\cos bx}{x^2}\,dx = \pi b\ln\, 2 - a\pi, \quad a > 0,\, b > 0.$

19. $\displaystyle\int_0^\infty \ln\,\left(4\cos^2 ax\right)\frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c}\ \cosh(bc)\ \ln\left(1+e^{-2ac}\right), \quad a < b < 2a < \frac{\pi}{c}.$

20. $\displaystyle\int_0^\infty \ln\, \cos^2\, ax\frac{\sin bx}{x(1+x^2)}\,dx = \pi\ln\,(1+e^{-2a})\sinh b - \pi\ln\, 2\left(1-e^{-b}\right), \quad a > 0,\, b > 0.$

21. $$\int_0^\infty \ln\,\cos^2\,ax\frac{\cos bx}{x^2(1+x^2)}\,dx = -\pi\ln\left(1+e^{-2a}\right)\,\cosh b + \left(b+e^{-b}\right)\pi\ln\,2 - a\pi, \quad a>0,\, b>0.$$

22. $$\int_0^\infty \ln\left(2\pm 2\cos x\right)\frac{\sin bx}{x^2+c^2}\,x\,dx = -\pi\,\sinh(bc)\,\ln\,\left(1\pm e^{-c}\right), \quad b>0,\, c>0.$$

23. $$\int_0^\infty \ln\left(2\pm 2\cos x\right)\frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c}\,\cosh(bc)\,\ln\,\left(1\pm e^{-c}\right), \quad b>0,\, c>0.$$

24. $$\int_0^\infty \ln\left(1+2a\cos x+a^2\right)\frac{\sin bx}{x}\,dx = -\frac{\pi}{2}\sum_{k=1}^{\lfloor b\rfloor}\frac{(-a)^k}{k}\left[1+\operatorname{sgn}(b-k)\right], \quad 0<a<1,\, b>0.$$

25. $$\int_0^\infty \ln\,\left(1-2a\cos x+a^2\right)\frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c}\ln\left(1-ae^{-c}\right)\,\cosh(bc) + \frac{\pi}{c}\sum_{k=1}^{\lfloor b\rfloor}\frac{a^k}{k}\,\sinh[c(b-k)],$$
$$|a|<1,\, b>0,\, c>0.$$

26. $$\int_0^\infty \ln\,\left(1-k^2\,\sin^2 x\right)\frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \int_0^\infty \ln\,\left(1-k^2\cos^2 x\right)\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \ln\,k'\mathbf{K}(k).$$

27. $$\int_0^\infty \ln\,\left(1-k^2\,\sin^2 x\right)\frac{\sin x\cos x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \frac{1}{k^2}\left\{(2-k^2-k'^2\ln\,k')\mathbf{K}(k)-(2-\ln\,k')\mathbf{E}(k)\right\}.$$

28. $$\int_0^\infty \ln\,\left(1-k^2\,\cos^2 x\right)\frac{\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \frac{1}{k^2}\left\{(k^2-2+\ln\,k')\mathbf{K}(k)+(2-\ln\,k')\mathbf{E}(k)\right\}.$$

29. $$\begin{aligned}\int_0^\infty \ln\left(1\pm k\sin^2 x\right)\frac{\sin x}{\sqrt{1-k^2\sin^2\,x}}\frac{dx}{x} &= \int_0^\infty \ln\left(1\pm k\cos^2 x\right)\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x}\\ &= \int_0^\infty \ln\left(1\pm k\sin^2 x\right)\frac{\tan x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x}\\ &= \int_0^\infty \ln\left(1\pm k\cos^2 x\right)\frac{\tan x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x}\\ &= \int_0^\infty \ln\left(1\pm k\sin^2 2x\right)\frac{\tan x}{\sqrt{1-k^2\sin^2 2x}}\frac{dx}{x}\\ &= \int_0^\infty \ln\left(1\pm k^2\cos^2 2x\right)\frac{\tan x}{\sqrt{1-k^2\cos^2 2x}}\frac{dx}{x}\\ &= \frac{1}{2}\ln\,\frac{2(1\pm k)}{\sqrt{k}}\mathbf{K}(k)-\frac{\pi}{8}\mathbf{K}(k').\end{aligned}$$

30. $\displaystyle\int_0^\infty \ln\left(1-k^2\ \sin^2 x\right)\frac{\sin^3 x}{\sqrt{1-k^2\sin^2 x}}\,\frac{dx}{x} = \frac{1}{k^2}\{(k^2-2+\ln\ k')\mathbf{K}(k)+(2-\ln\ k')\mathbf{E}(k)\}.$

31. $\displaystyle\int_0^\infty \ln\left(1-k^2\ \cos^2 x\right)\frac{\sin^3 x}{\sqrt{1-k^2\cos^2 x}}\,\frac{dx}{x} = \frac{1}{k^2}\{(2-k^2-k'^2\ln\ k')\mathbf{K}(k)-(2-\ln\ k')\mathbf{E}(k)\}.$

32. $\displaystyle\int_0^\infty \ln\left(1-k^2\ \sin^2 x\right)\frac{\sin x\cos^2 x}{\sqrt{1-k^2\sin^2 x}}\,\frac{dx}{x} = \frac{1}{k^2}\{(2-k^2-k'^2\ln\ k')\mathbf{K}(k)-(2-\ln\ k')\mathbf{E}(k)\}.$

33. $\displaystyle\int_0^\infty \ln\left(1-k^2\cos^2 x\right)\frac{\sin x\cos^2 x}{\sqrt{1-k^2\cos^2 x}}\,\frac{dx}{x} = \frac{1}{k^2}\{(k^2-2+\ln\ k')\mathbf{K}(k)+(2-\ln\ k')\mathbf{E}(k)\}.$

34. $\displaystyle\int_0^\infty \ln\left(1-k^2\sin^2 x\right)\frac{\tan x}{\sqrt{1-k^2\sin^2 x}}\,\frac{dx}{x} = \int_0^\infty \ln\left(1-k^2\cos^2 x\right)\frac{\tan x}{\sqrt{1-k^2\cos^2 x}}\,\frac{dx}{x} = \ln\ k'\mathbf{K}(k).$

35. $\displaystyle\int_0^\infty \ln\left(1-k^2\sin^2 x\right)\frac{\sin^2 x\tan x}{\sqrt{1-k^2\sin^2 x}}\,\frac{dx}{x} = \frac{1}{k^2}\{(k^2-2+\ln\ k')\mathbf{K}(k)+(2-\ln\ k')\mathbf{E}(k)\}.$

36. $\displaystyle\int_0^\infty \ln\left(1-k^2\cos^2 x\right)\frac{\sin^2 x\tan x}{\sqrt{1-k^2\cos^2 x}}\,\frac{dx}{x} = \frac{1}{k^2}\{(2-k^2-k'^2\ln\ k')\mathbf{K}(k)-(2-\ln\ k')\mathbf{E}(k)\}.$

37. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right)\frac{\sin^2 x}{\sqrt{(1-k^2\sin^2 x)^3}}\,\frac{dx}{x} = \int_0^\infty \ln\left(1-k^2\cos^2 x\right)\frac{\sin x}{\sqrt{(1-k^2\cos^2 x)^3}}\,\frac{dx}{x}$$
$$= \frac{1}{k'^2}\{(k^2-2)\mathbf{K}(k)+(2+\ln\ k')\mathbf{E}(k)\}.$$

38. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right)\frac{\sin x\cos x}{\sqrt{(1-k^2\sin^2 x)^3}}\,\frac{dx}{x} = \int_0^\infty \ln\left(1-k^2\cos^2 x\right)\frac{\sin^3 x}{\sqrt{(1-k^2\cos^2 x)^3}}\,\frac{dx}{x}$$
$$= \frac{1}{k^2}\{(2-k^2+\ln\ k')\mathbf{K}(k)-(2+\ln\ k')\mathbf{E}(k)\}.$$

39. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right)\frac{\sin^3 x}{\sqrt{(1-k^2\sin^2 x)^3}}\,\frac{dx}{x} = \int_0^\infty \ln\left(1-k^2\ \ cos^2 x\right)\frac{\sin x\cos x}{\sqrt{(1-k^2\cos^2 x)^3}}\,\frac{dx}{x}$$
$$= \frac{1}{k^2k'^2}\{(2+\ln\ k')\mathbf{E}(k)-(2-k^2+k'^2\ln\ k')\mathbf{K}(k)\}.$$

40. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\sin x\cos^2 x}{\sqrt{(1-k^2\sin^2 x)^3}}\,\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\frac{\sin^2 x\,\tan x}{\sqrt{(1-k^2\cos^2 x)^3}}\,\frac{dx}{x}$$
$$= \frac{1}{k^2}\{(2-k^2+\ln k')\mathbf{K}(k)-(2+\ln k')\mathbf{E}(k)\}.$$

41. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\sin^2 x\,\tan x}{\sqrt{(1-k^2\sin^2 x)^3}}\,\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\frac{\sin x\,\cos^2 x}{\sqrt{(1-k^2\cos^2 x)^3}}\,\frac{dx}{x}$$
$$= \frac{1}{k^2k'^2}\{(2+\ln k')\mathbf{E}(k)-(2-k^2+k'^2\ln k')\mathbf{K}(k)\}.$$

42. $$\int_0^\infty \ln(1-k^2\sin^2 x)\frac{\tan x}{\sqrt{(1-k^2\sin^2 x)^3}}\,\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\frac{\tan x}{\sqrt{(1-k^2\cos^2 x)^3}}\,\frac{dx}{x}$$
$$= \frac{1}{k'^2}\{(k^2-2)\mathbf{K}(k)+(2+\ln k')\mathbf{E}(k)\}.$$

43. $$\int_0^\infty \ln(1-k^2\sin^2 x)\sqrt{1-k^2\sin^2 x}\,\sin x\,\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\sqrt{1-k^2\cos^2 x}\,\sin x\frac{dx}{x}$$
$$= (2-k^2)\mathbf{K}(k)-(2-\ln k')\mathbf{E}(k).$$

44. $$\int_0^\infty \ln(1-k^2\sin^2 x)\sqrt{1-k^2\sin^2 x}\,\tan x\,\frac{dx}{x} = \int_0^\infty \ln(1-k^2\cos^2 x)\sqrt{1-k^2\cos^2 x}\,\tan x\frac{dx}{x}$$
$$= (2-k^2)\mathbf{K}(k)-(2-\ln k')\mathbf{E}(k).$$

45. $$\int_0^\infty \ln(\sin^2 x+k'\cos^2 x)\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\,\frac{dx}{x} = \int_0^\infty \ln(\sin^2 x+k'\cos^2 x)\frac{\tan x}{\sqrt{1-k^2\cos^2 x}}\,\frac{dx}{x}$$
$$= \int_0^\infty \ln(\sin^2 2x+k'\cos^2 2x)\frac{\tan x}{\sqrt{1-k^2\cos^2 2x}}\,\frac{dx}{x}$$
$$= \frac{1}{2}\ln\left[\frac{2(\sqrt{k'})^3}{1+k'}\right]\mathbf{K}(k).$$