! For an efficient use of these tables, first read `HowTo.pdf`.

T3.33A. Integrands involving trigonometric functions and powers of trigonometric functions with quadratic, cubic, biquadratic and p-th degree polynomials on the interval $(0,\infty)$.

1. $$\int_0^\infty \sin(ax^2)\,dx = \int_0^\infty \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a>0.$$

2. $$\int_0^\infty \sin(ax^2)\,\sin 2bx\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}C\left(\frac{b}{\sqrt{a}}\right)+\sin\frac{b^2}{a}S\left(\frac{b}{\sqrt{a}}\right)\right\}, \quad a>0,\ b>0.$$

3. $$\int_0^\infty \sin(ax^2)\,\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}-\sin\frac{b^2}{a}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\cos\left(\frac{b^2}{a}+\frac{\pi}{4}\right),$$ $$a>0,\ b>0.$$

4. $$\int_0^\infty \cos ax^2\,\sin 2bx\,dx = \sqrt{\frac{\pi}{2a}}\left\{\sin\frac{b^2}{a}C\left(\frac{b}{\sqrt{a}}\right)-\cos\frac{b^2}{a}S\left(\frac{b}{\sqrt{a}}\right)\right\}, \quad a>0,\ b>0.$$

5. $$\int_0^\infty \cos ax^2\,\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}+\sin\frac{b^2}{a}\right\}, \quad a>0,\ b>0.$$

6. $$\int_0^\infty (\cos ax+\sin ax)\,\sin(b^2x^2)\,dx$$ $$= \frac{1}{2b}\sqrt{\frac{\pi}{2}}\left\{\left(1+2C\left(\frac{a}{2b}\right)\right)\cos\left(\frac{a^2}{4b^2}\right)-\left(1-2S\left(\frac{a}{2b}\right)\right)\sin\left(\frac{a^2}{4b^2}\right)\right\}, \quad a>0,\ b>0.$$

7. $$\int_0^\infty (\cos ax+\sin ax)\,\cos(b^2x^2)\,dx$$ $$= \frac{1}{2b}\sqrt{\frac{\pi}{2}}\left\{\left(1+2C\left(\frac{a}{2b}\right)\right)\sin\left(\frac{a^2}{4b^2}\right)+\left(1-2S\left(\frac{a}{2b}\right)\right)\cos\left(\frac{a^2}{4b^2}\right)\right\}, \quad a>0,\ b>0.$$

8. $$\int_0^\infty \sin(a^2x^2)\,\sin 2bx\,\sin 2cx\,dx = \frac{\sqrt{\pi}}{2a}\sin\frac{2bc}{a^2}\cos\left(\frac{b^2+c^2}{a^2}-\frac{\pi}{4}\right), \quad a>0,\ b>0,\ c>0.$$

9. $$\int_0^\infty \sin(a^2x^2)\,\cos 2bx\,\cos 2cx\,dx = \frac{\sqrt{\pi}}{2a}\,\cos\frac{2bc}{a^2}\,\cos\left(\frac{b^2+c^2}{a^2}+\frac{\pi}{4}\right), \quad a>0,\ b>0,\ c>0.$$

10. $$\int_0^\infty \cos(a^2x^2)\,\sin 2bx\,\sin 2cx\,dx = \frac{\sqrt{\pi}}{2a}\,\sin\frac{2bc}{a^2}\,\sin\left(\frac{b^2+c^2}{a^2}-\frac{\pi}{4}\right), \quad a>0,\ b>0,\ c>0.$$

11. $$\int_0^\infty \sin(ax^2)\,\cos(bx^2)\,dx = \begin{cases} \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{a-b}}\right), & a>b>0, \\ \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{b+a}}-\frac{1}{\sqrt{b-a}}\right), & b>a>0. \end{cases}$$

12. $$\int_0^\infty (\sin^2\ ax^2 - \sin^2\ bx^2)\,dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{b}}-\sqrt{\frac{\pi}{a}}\right), \quad a>0,\ b>0.$$

13. $$\int_0^\infty (\cos^2\ ax^2 - \sin^2\ bx^2)\,dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{b}}+\sqrt{\frac{\pi}{a}}\right), \quad a>0,\ b>0.$$

14. $$\int_0^\infty (\cos^2\ ax^2 - \cos^2\ bx^2)\,dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{a}}-\sqrt{\frac{\pi}{b}}\right), \quad a>0,\ b>0.$$

15. $$\int_0^\infty (\sin^4\ ax^2 - \sin^4\ bx^2)\,x = \frac{1}{64}(8-\sqrt{2})\left(\sqrt{\frac{\pi}{b}}-\sqrt{\frac{\pi}{a}}\right), \quad a>0,\ b>0.$$

16. $$\int_0^\infty (\cos^4\ ax^2 - \sin^4\ bx^2)\,dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{a}}+\sqrt{\frac{\pi}{b}}\right)+\frac{1}{32}\left(\sqrt{\frac{\pi}{2a}}-\sqrt{\frac{\pi}{2b}}\right), \quad a>0,\ b>0.$$

17. $$\int_0^\infty (\cos^4\ ax^2 - \cos^4\ bx^2)\,dx = \frac{1}{64}(8+\sqrt{2})\left(\sqrt{\frac{\pi}{a}}-\sqrt{\frac{\pi}{b}}\right), \quad a>0,\ b>0.$$

18. $$\int_0^\infty \sin^{2n}\ ax^2\,dx = \int_0^\infty \cos^{2n}\ ax^2\,dx = \infty.$$

19. $$\int_0^\infty \sin^{2n+1}(ax^2)\,dx = \frac{1}{2^{2n+1}}\sum_{k=0}^{n}(-1)^{n+k}\binom{2n+1}{k}\sqrt{\frac{\pi}{2(2n-2k+1)a}}, \quad a>0.$$

20. $$\int_0^\infty \cos^{2n+1}(ax^2)\,dx = \frac{1}{2^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}\sqrt{\frac{\pi}{2(2n-2k+1)a}}, \quad a>0.$$

21. $$\int_0^\infty \left[\sin(a-x^2)+\cos(a-x^2)\right]\,dx = \sqrt{\frac{\pi}{a}}\,\sin a.$$

22. $$\int_0^\infty \cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\cos ax\,dx = \sqrt{\frac{\pi}{2}}\cos\left(\frac{a^2}{2}-\frac{\pi}{8}\right), \quad a>0.$$

23. $$\int_0^\infty \sin\left[a(1-x^2)\right]\cos bx\,dx = -\frac{1}{2}\sqrt{\frac{\pi}{a}}\cos\left(a+\frac{b^2}{4a}+\frac{\pi}{4}\right), \quad a>0.$$

24. $$\int_0^\infty \cos\left[a(1-x^2)\right]\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\sin\left(a+\frac{b^2}{4a}+\frac{\pi}{4}\right), \quad a>0.$$

25. $$\int_0^\infty \sin\left(ax^2+\frac{b^2}{a}\right)\cos 2bx\,dx = \int_0^\infty \cos\left(ax^2+\frac{b^2}{a}\right)\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a>0.$$

26. $$\int_0^\infty \sin(ax^2+2bx)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)-\sin\frac{b^2}{a}\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\right\}, \quad a>0.$$

27. $$\int_0^\infty \cos(ax^2+2bx)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)+\sin\frac{b^2}{a}\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\right\}, \quad a>0.$$

28. $$\int_0^\infty \sin(ax^2+2bx+c)\,dx = \sqrt{\frac{\pi}{2a}}\cos\frac{b^2}{a}\left\{\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\sin c+\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\cos c\right\}$$
$$+\sqrt{\frac{\pi}{2a}}\sin\frac{b^2}{a}\left\{\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\sin c-\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\cos c\right\}, \quad a>0.$$

29. $$\int_0^\infty \cos(ax^2+2bx+c)\,dx = \sqrt{\frac{\pi}{2a}}\cos\frac{b^2}{a}\left\{\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\cos c-\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\sin c\right\}$$
$$+\sqrt{\frac{\pi}{2a}}\sin\frac{b^2}{a}\left\{\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\cos c+\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\sin c\right\}, \quad a>0.$$

30. $$\int_0^\infty \sin(a^3x^3)\,\sin(bx)\,dx = \frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{1/3}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+J_{-1/3}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)-\frac{\sqrt{3}}{\pi}K_{1/3}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}, \quad a>0,\ b>0.$$

31. $$\int_0^\infty \cos(a^3x^3)\,\cos(bx)\,dx = \frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{1/3}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right) + J_{-1/3}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right) + \frac{\sqrt{3}}{\pi}K_{1/3}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}, \quad a>0,\ b>0.$$

32. $$\int_0^\infty \sin(ax^4)\,\sin(bx^2)\,dx = -\frac{\pi}{4}\sqrt{\frac{b}{2a}}\,\sin\left(\frac{b^2}{8a} - \frac{3}{8}\pi\right) J_{1/4}\left(\frac{b^2}{8a}\right), \quad a>0,\, b>0.$$

33. $$\int_0^\infty \sin(ax^4)\,\cos(bx^2)\,dx = -\frac{\pi}{4}\sqrt{\frac{b}{2a}}\,\sin\left(\frac{b^2}{8a} - \frac{\pi}{8}\right) J_{-1/4}\left(\frac{b^2}{8a}\right), \quad a>0,\, b>0.$$

34. $$\int_0^\infty \cos(ax^4)\,\sin(bx^2)\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2a}}\,\cos\left(\frac{b^2}{8a} - \frac{3}{8}\pi\right) J_{1/4}\left(\frac{b^2}{8a}\right), \quad a>0,\, b>0.$$

35. $$\int_0^\infty \cos(ax^4)\,\cos(bx^2)\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2a}}\,\cos\left(\frac{b^2}{8a} - \frac{\pi}{8}\right) J_{-1/4}\left(\frac{b^2}{8a}\right), \quad a>0,\, b>0.$$

36. $$\int_0^\infty \sin\left(\frac{a^2}{x}\right)\sin(bx)\,dx = \frac{a\pi}{2\sqrt{b}}J_1(2a\sqrt{b}), \quad a>0,\ b>0.$$

37. $$\int_0^\infty \sin\left(\frac{a^2}{x^2}\right)\sin(b^2x^2)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab - \cos 2ab + e^{-2ab}\right], \quad a>0,\, b>0.$$

38. $$\int_0^\infty \sin\left(\frac{a^2}{x^2}\right)\cos(b^2x^2)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab + \cos 2ab - e^{-2ab}\right].$$

39. $$\int_0^\infty \cos\left(\frac{a^2}{x^2}\right)\sin(b^2x^2)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab + \cos 2ab + e^{-2ab}\right], \quad a>0,\ b>0.$$

40. $$\int_0^\infty \cos\left(\frac{a^2}{x^2}\right)\cos(b^2x^2)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\cos 2ab - \sin 2ab + e^{-2ab}\right], \quad a>0,\ b>0.$$

41. $$\int_0^\infty \sin\left(a^2x^2 + \frac{b^2}{x^2}\right)dx = \frac{\sqrt{2\pi}}{4a}(\cos 2ab + \sin 2ab), \quad a>0,\ b>0.$$

42. $$\int_0^\infty \cos\left(a^2x^2 + \frac{b^2}{x^2}\right)dx = \frac{\sqrt{2\pi}}{4a}(\cos 2ab - \sin 2ab), \quad a>0,\ b>0.$$

43. $$\int_0^\infty \sin\left(a^2x^2 - 2ab + \frac{b^2}{x^2}\right)dx = \int_0^\infty \cos\left(a^2x^2 - 2ab + \frac{b^2}{x^2}\right)dx = \frac{\sqrt{2\pi}}{4a}, \quad a>0,\, b>0.$$

44. $$\int_0^\infty \sin\left(a^2x^2 - \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a}\, e^{-2ab}, \quad a > 0,\ b > 0.$$

45. $$\int_0^\infty \cos\left(a^2x^2 - \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a}\, e^{-2ab}, \quad a > 0,\ b > 0.$$

46. $$\int_0^\infty \sin(ax^p)\, dx = \frac{\Gamma\left(\frac{1}{p}\right) \sin\frac{\pi}{2p}}{p\, a^{1/p}}, \quad a > 0,\ p > 1.$$

47. $$\int_0^\infty \cos(ax^p)\, dx = \frac{\Gamma\left(\frac{1}{p}\right) \cos\frac{\pi}{2p}}{p\, a^{1/p}}, \quad a > 0,\ p > 1.$$

48. $$\int_0^\infty \sin(ax^p + bx^q)\, dx = \frac{1}{p}\sum_{k=0}^{\infty} \frac{(-b)^k}{k!} a^{-(kq+1)/p} \Gamma\left(\frac{kq+1}{p}\right) \sin\left[\frac{k(q-p)+1}{2p}\pi\right],$$
$$a > 0,\ b > 0,\ p > 0,\ q > 0.$$

49. $$\int_0^\infty \cos(ax^p + bx^q)\, dx = \frac{1}{p}\sum_{k=0}^{\infty} \frac{(-b)^k}{k!} a^{-(kq+1)/p} \Gamma\left(\frac{kq+1}{p}\right) \cos\left[\frac{k(q-p)+1}{2p}\pi\right],$$
$$a > 0,\ b > 0,\ p > 0,\ q > 0.$$
