T1.37. Integrand involving inverse Trigonometric Functions.

1. $\displaystyle\int \arcsin ax\,dx = x\arcsin ax + \frac{\sqrt{1-a^2x^2}}{a}.$
2. $\displaystyle\int \arccos ax\,dx = x\arccos ax - \frac{\sqrt{1-a^2x^2}}{a}.$
3. $\displaystyle\int \arctan ax\,dx = x\arctan ax - \frac{1}{2a}\ln\left(1+a^2x^2\right).$
4. $\displaystyle\int \operatorname{arccot} ax\,dx = x\operatorname{arccot} ax + \frac{1}{2a}\ln\left(1+a^2x^2\right).$
5. $\displaystyle\int \operatorname{arcsec} ax\,dx = x\operatorname{arcsec} ax - \frac{1}{a}\ln\left(ax+\sqrt{a^2x^2-1}\right).$
6. $\displaystyle\int \operatorname{arccsc} ax\,dx = x\operatorname{arccsc} ax + \frac{1}{a}\ln\left(ax+\sqrt{a^2x^2-1}\right).$
7. $\displaystyle\int x\arcsin ax\,dx = \frac{1}{4a^2}\left[\left(2a^2x^2-1\right)\arcsin ax + ax\sqrt{1-a^2x^2}\right].$
8. $\displaystyle\int x\arccos ax\,dx = \frac{1}{4a^2}\left[\left(2a^2x^2-1\right)\arccos ax - ax\sqrt{1-a^2x^2}\right].$
9. $\displaystyle\int x^n\arcsin ax\,dx = \frac{x^{n+1}}{n+1}\arcsin ax - \frac{a}{n+1}\int\frac{x^{n+1}}{\sqrt{1-a^2x^2}}\,dx, \quad n\neq -1.$
10. $\displaystyle\int x^n\arccos ax\,dx = \frac{x^{n+1}}{n+1}\arccos ax + \frac{a}{n+1}\int\frac{x^{n+1}}{\sqrt{1-a^2x^2}}\,dx, \quad n\neq -1.$
11. $\displaystyle\int x\arctan ax\,dx = \frac{1+a^2x^2}{2a^2}\arctan ax - \frac{x}{2a}.$
12. $\displaystyle\int x^n\arctan ax\,dx = \frac{x^{n+1}}{n+1}\arctan ax - \frac{a}{n+1}\int\frac{x^{n+1}}{1+a^2x^2}\,dx, \quad n\neq -1.$
13. $\displaystyle\int x\operatorname{arccot} ax\,dx = \frac{1+a^2x^2}{2a^2}\operatorname{arccot} ax + \frac{x}{2a}.$
14. $\displaystyle\int x^n\operatorname{arccot} ax\,dx = \frac{x^{n+1}}{n+1}\arctan ax + \frac{a}{n+1}\int\frac{x^{n+1}}{1+a^2x^2}\,dx, \quad n\neq -1.$
15. $\displaystyle\int \frac{\arcsin ax}{x^2}\,dx = a\ln\left(\frac{1-\sqrt{1-a^2x^2}}{x}\right) - \frac{\arcsin ax}{x}.$

16. $\displaystyle\int \frac{\arccos ax}{x^2}\,dx = \frac{1}{x}\,\arccos ax + a\,\ln\left(\frac{1+\sqrt{1-a^2x^2}}{x}\right).$

17. $\displaystyle\int \frac{\arctan ax}{x^2}\,dx = -\frac{1}{x}\,\arctan ax - \frac{a}{2}\,\ln\left(\frac{1+a^2x^2}{x^2}\right).$

18. $\displaystyle\int \frac{\operatorname{arccot} ax}{x^2}\,dx = -\frac{1}{x}\,\operatorname{arccot} ax - \frac{a}{2}\,\ln\left(\frac{x^2}{1+a^2x^2}\right).$

19. $\displaystyle\int (\arcsin ax)^2\,dx = x\,(\arcsin ax)^2 - 2x + \frac{2\sqrt{1-a^2x^2}}{a}\,\arcsin ax.$

20. $\displaystyle\int (\arccos ax)^2\,dx = x\,(\arccos ax)^2 - 2x - \frac{2\sqrt{1-a^2x^2}}{a}\,\arccos ax.$

21. $\displaystyle\int (\arcsin ax)^n\,dx$

$$= \begin{cases} x\,(\arcsin ax)^n + \dfrac{n\sqrt{1-a^2x^2}}{a}\,(\arcsin ax)^{n-1} - n(n-1)\displaystyle\int (\arcsin ax)^{n-2}\,dx, \\ \qquad \text{or} \\ \displaystyle\sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k\,n!}{(n-2k)!}x\,(\arcsin ax)^{n-2k} + \sum_{k=0}^{\lfloor (n-1)/2\rfloor} (-1)^k \frac{n!\,\sqrt{1-a^2x^2}}{(n-2k-1)!\,a}\,(\arcsin ax)^{n-2k-1}. \end{cases}$$

22. $\displaystyle\int (\arccos ax)^n\,dx$

$$= \begin{cases} x\,(\arccos ax)^n - \dfrac{n\sqrt{1-a^2x^2}}{a}\,(\arccos ax)^{n-1} - n(n-1)\displaystyle\int (\arccos ax)^{n-2}\,dx, \\ \qquad \text{or} \\ \displaystyle\sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k\,n!}{(n-2k)!}x\,(\arccos ax)^{n-2k} - \sum_{k=0}^{\lfloor (n-1)/2\rfloor} (-1)^k \frac{n!\,\sqrt{1-a^2x^2}}{(n-2k-1)!\,a}\,(\arccos ax)^{n-2k-1}. \end{cases}$$

23. $\displaystyle\int \arcsin\frac{x}{a}\,dx = \operatorname{sgn}(a)\left[x\,\arcsin\frac{x}{|a|} + \sqrt{a^2-x^2}\right].$

24. $\displaystyle\int \left(\arcsin\frac{x}{a}\right)^2\,dx = x\left(\arcsin\frac{x}{|a|}\right)^2 + 2\sqrt{a^2-x^2}\,\arcsin\frac{x}{|a|} - 2x.$

25. $\displaystyle\int \left(\arcsin\frac{x}{a}\right)^3\,dx = \operatorname{sgn}(a)\left[x\left(\arcsin\frac{x}{|a|}\right)^3 + 3\sqrt{a^2-x^2}\left(\arcsin\frac{x}{|a|}\right)^2 - 6x\,\arcsin\frac{x}{|a|} - 6\sqrt{a^2-x^2}\right].$

26. $\displaystyle\int \arccos\frac{x}{a}\,dx = x\,\arccos\frac{x}{a} - \sqrt{a^2-x^2}.$

27. $\displaystyle\int \left(\arccos\frac{x}{a}\right)^2\,dx = x\left(\arccos\frac{x}{a}\right)^2 - 2\sqrt{a^2-x^2}\,\arccos\frac{x}{a} - 2x.$

28. $\int \left(\arccos\frac{x}{a}\right)^3 dx = x\left(\arccos\frac{x}{a}\right)^3 - 3\sqrt{a^2-x^2}\left(\arccos\frac{x}{a}\right)^2 - 6x\,\arccos\frac{x}{a} + 6\sqrt{a^2-x^2}.$

29. $\int \left(\arcsin\frac{x}{a}\right)^n dx = x\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}(2k)!\left(\arcsin\frac{x}{a}\right)^{n-2k}$
$$+\sqrt{a^2-x^2}\sum_{k=1}^{[(n+1)/2]}(-1)^{k-1}\binom{n}{2k-1}(2k-1)!\left(\arcsin\frac{x}{a}\right)^{n-2k+1}.$$

30. $\int \left(\arccos\frac{x}{a}\right)^n dx = x\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}(2k)!\left(\arccos\frac{x}{a}\right)^{n-2k}$
$$+\sqrt{a^2-x^2}\sum_{k=1}^{[(n+1)/2]}(-1)^k\binom{n}{2k-1}(2k-1)!\left(\arccos\frac{x}{a}\right)^{n-2k+1}.$$

31. $\int \operatorname{arccsc}\frac{x}{a}\,dx = \begin{cases} \int \arcsin\frac{a}{x}\,dx = x\,\arcsin\frac{x}{2} + a\,\ln(x+\sqrt{x^2-a^2}), & 0 < \arcsin\frac{a}{x} < \frac{\pi}{2}, \\ x\,\arcsin\frac{a}{x} - a\,\ln(x+\sqrt{x^2-a^2}), & -\frac{\pi}{2} < \arcsin\frac{a}{x} < 0. \end{cases}$

32. $\int \operatorname{arcsec}\frac{x}{a}\,dx = \begin{cases} \int \arccos\frac{a}{x}\,dx = x\,\arccos\frac{a}{x} - a\,\ln(x+\sqrt{x^2-a^2}), & 0 < \arccos\frac{a}{x} < \frac{\pi}{2}, \\ x\,\arccos\frac{a}{x} - a\,\ln(x+\sqrt{x^2-a^2}), & -\frac{\pi}{2} < \arccos\frac{a}{x} < 0. \end{cases}$

33. $\int \arctan\frac{x}{q}\,dx = x\,\arctan\frac{x}{a} - \frac{a}{2}\,\ln(a^2+x^2).$

34. $\int \operatorname{arccot}\frac{x}{a}\,dx = x\,\operatorname{arccot}\frac{x}{a} - \frac{a}{2}\,\ln(a^2+x^2).$

35. $\int x\,\arctan\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\,\arctan\frac{x}{a} - \frac{ax}{2}.$

36. $\int x\,\operatorname{arccot}\frac{x}{a}\,dx = \frac{ax}{2} + \frac{\pi x^2}{4} - \frac{1}{2}(x^2+a^2)\,\arctan\frac{x}{a}.$

37. $\int x^2\,\arctan\frac{x}{a}\,dx = \frac{1}{3}x^3\,\arctan\frac{x}{a} + \frac{1}{6}a^3\,\ln(x^2+a^2) - \frac{ax^2}{6}.$

38. $\int x^2\,\operatorname{arccot}\frac{x}{a}\,dx = -\frac{1}{3}x^3\,\arctan\frac{x}{a} - \frac{1}{6}a^3\,\ln(x^2+a^2) + \frac{\pi x^3}{6} + \frac{ax^2}{6}.$

39. $\int x\,\arcsin\frac{x}{a}\,dx = \operatorname{sgn}(a)\,\left[\left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arcsin\frac{x}{|a|} + \frac{x}{4}\sqrt{a^2-x^2}\right].$

40. $\int x\,\arccos\frac{x}{a}\,dx = \frac{\pi x^2}{4} - \operatorname{sgn}(a)\,\left[\frac{1}{4}(2x^2-a^2)\arcsin\frac{x}{|a|} + \frac{x}{4}\sqrt{a^2-x^2}\right].$

41. $\int x^2\,\arcsin\frac{x}{a}\,dx = \operatorname{sgn}(a)\,\left[\frac{x^3}{3}\,\arcsin\frac{x}{|a|} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}\right].$

42. $\int x^2 \arccos\frac{x}{a}\,dx = \frac{\pi x^3}{6} - \operatorname{sgn}(a)\left[\frac{x^3}{3}\arcsin\frac{x}{|a|} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}\right].$

43. $\int x^3 \arcsin\frac{x}{a}\,dx = \operatorname{sgn}(a)\left[\left(\frac{x^4}{4} - \frac{3a^4}{32}\right)\arcsin\frac{x}{|a|} + \frac{1}{32}x(2x^2+3a^2)\sqrt{a^2-x^2}\right].$

44. $\int x^3 \arccos\frac{x}{a}\,dx = \frac{\pi x^4}{8} - \operatorname{sgn}(a)\left[\frac{(8x^4-3a^4)}{32}\arcsin\frac{x}{|a|} + \frac{1}{32}x(2x^2+3a^2)\sqrt{a^2-x^2}\right].$

45. $\int x^n \arcsin\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arcsin\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{a^2-x^2}}, \quad n \neq -1.$

46. $\int x^n \arccos\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arccos\frac{x}{a} + \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{a^2-x^2}}, \quad n \neq -1.$

47. $\int\frac{\arccos x}{x}\,dx = -\frac{\pi}{2}\ln\frac{1}{x} - \int\frac{\arcsin x}{x}\,dx.$

48. $\int\frac{1}{x^2}\arcsin\frac{x}{a}\,dx = -\frac{1}{x}\arcsin\frac{x}{a} - \frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}.$

49. $\int\frac{1}{x^2}\arccos\frac{x}{a}\,dx = -\frac{1}{x}\arccos\frac{x}{a} - \frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}.$

50. $$\int\frac{\arcsin x}{(a+bx)^2}\,dx = \begin{cases} \dfrac{\arcsin x}{b(a+bx)} - \dfrac{2}{b\sqrt{a^2-b^2}}\arctan\sqrt{\dfrac{(a-b)(1-x)}{(a+b)(1+x)}}, & a^2 > b^2, \\ -\dfrac{\arcsin x}{b(a+bx)} - \dfrac{1}{b\sqrt{b^2-a^2}}\ln\dfrac{\sqrt{(a+b)(1+x)}+\sqrt{(b-a)(1-x)}}{\sqrt{(a+b)(1+x)}-\sqrt{(b-a)(1-x)}}, & a^2 < b^2. \end{cases}$$

51. $$\int\frac{x\arcsin x}{(1+cx^2)^2}\,dx = \begin{cases} -\dfrac{\arcsin x}{2c(1+cx^2)} + \dfrac{1}{2c\sqrt{c+1}}\arctan\dfrac{\sqrt{c+1}x}{\sqrt{1-x^2}}, & c > -1, \\ -\dfrac{\arcsin x}{2c(1+cx^2)} + \dfrac{1}{4c\sqrt{-(c+1)}}\ln\dfrac{\sqrt{1-x^2}+x\sqrt{-(c+1)}}{\sqrt{1-x^2}-x\sqrt{-(c+1)}}, & c < -1. \end{cases}$$

52. $\int\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = x - \sqrt{1-x^2}\arcsin x.$

53. $\int\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{x^2}{4} - \frac{x}{2}\sqrt{1-x^2}\arcsin x + \frac{1}{4}(\arcsin x)^2.$

54. $\int\frac{x^3\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{x^3}{9} + \frac{2x}{3} - \frac{1}{3}(x^2+2)\sqrt{1-x^2}\arcsin x.$

55. $\int\frac{\arcsin x}{\sqrt{(1-x^2)^3}}\,dx = \frac{x\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\ln(1-x^2).$

56. $\int\frac{x\arcsin x}{\sqrt{(1-x^2)^3}}\,dx = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\ln\frac{1-x}{1+x}.$

57. $$\int x \operatorname{arcsec}\frac{x}{a}\,dx = \begin{cases} \displaystyle\int \arccos\frac{a}{x}\,dx = \frac{1}{2}\left\{x^2 \arccos\frac{a}{x} - a\sqrt{x^2-a^2}\right\}, & 0 < \arccos\frac{a}{x} < \frac{\pi}{2}, \\ \dfrac{1}{2}\left\{x^2 \arccos\dfrac{a}{x} + a\sqrt{x^2-a^2}\right\}, & \dfrac{\pi}{2} < \arccos\dfrac{a}{x} < \pi. \end{cases}$$

58. $$\int x^2 \operatorname{arcsec}\frac{x}{a}\,dx = \begin{cases} \displaystyle\int \arccos\frac{a}{x}\,dx = \frac{1}{3}\left\{x^3 \arccos\frac{a}{x} - \frac{a}{2}\,x\sqrt{x^2-a^2} - \frac{a^3}{2}\,\ln(x+\sqrt{x^2-a^2})\right\}, & \\ & 0 < \arccos\dfrac{a}{x} < \dfrac{\pi}{2}, \\ \dfrac{1}{3}\left\{x^3 \arccos\dfrac{a}{x} + \dfrac{a}{2}x\sqrt{x^2-a^2} + \dfrac{a^3}{2}\,\ln(x+\sqrt{x^2-z^2})\right\}, & \dfrac{\pi}{2} < \arccos\dfrac{a}{x} < \pi. \end{cases}$$

59. $$\int x \operatorname{arccsc}\frac{x}{a}\,dx = \begin{cases} \displaystyle\int \arcsin\frac{a}{x}\,dx = \frac{1}{2}\left\{x^2 \arcsin\frac{a}{x} + a\sqrt{x^2-a^2}\right\}, & 0 < \arcsin\frac{a}{x} < \frac{\pi}{2}, \\ \dfrac{1}{2}\left\{x^2 \arcsin\dfrac{a}{x} - a\sqrt{x^2-a^2}\right\}, & -\dfrac{\pi}{2} < \arcsin\dfrac{a}{x} < 0. \end{cases}$$

60. $$\int x \arctan\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\arctan\frac{x}{a} - \frac{ax}{2}.$$

61. $$\int x \operatorname{arccot}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\operatorname{arccot}\frac{x}{a} + \frac{ax}{2}.$$

62. $$\int x^2 \arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} + \frac{a^3}{6}\ln(x^2+a^2) - \frac{ax^2}{6}.$$

63. $$\int x^2 \operatorname{arccot}\frac{x}{a}\,dx = -\frac{x^3}{3}\arctan\frac{x}{a} - \frac{a^3}{6}\ln(x^2+a^2) + \frac{\pi x^3}{6} + \frac{ax^2}{6}.$$

64. $$\int x^n \operatorname{arccot}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\operatorname{arccot}\frac{x}{a} + \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2+x^2}, \quad n \neq -1.$$

65. $$\int \frac{\operatorname{arccot} x}{x}\,dx = \frac{\pi}{2}\ln x - \int\frac{\arctan x}{x}\,dx.$$

66. $$\int x^n \arctan\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arctan\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2+x^2}, \quad n \neq -1.$$

67. $$\int \frac{1}{x^2}\arctan\frac{x}{a}\,dx = -\frac{1}{x}\arctan\frac{x}{a} - \frac{1}{2a}\ln\frac{a^2+x^2}{x^2}.$$

68. $$\int \frac{\arctan x}{(\alpha+\beta x)^2}\,dx = \frac{1}{\alpha^2+\beta^2}\left\{\ln\frac{\alpha+\beta x}{\sqrt{1+x^2}} - \frac{\beta-\alpha x}{\alpha+\beta x}\arctan x\right\}.$$

69. $$\int \frac{x\arctan x}{1+x^2}\,dx = \frac{1}{2}\arctan x\,\ln(1+x^2) - \frac{1}{2}\int\frac{\ln(1+x^2)\,dx}{1+x^2}.$$

70. $$\int \frac{x^2\arctan x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}(\arctan x)^2.$$

71. $$\int \frac{x^3\arctan x}{1+x^2}\,dx = -\frac{1}{2}x + \frac{1}{2}(1+x^2)\arctan x - \int\frac{x\arctan x}{1+x^2}\,dx.$$

72. $\displaystyle\int \frac{x^4 \arctan x}{1+x^2}\,dx = -\frac{1}{6}x^2 + \frac{2}{3}\ln(1+x^2) + \left(\frac{x^3}{3} - x\right)\arctan x + \frac{1}{2}(\arctan x)^2.$

73. $\displaystyle\int \frac{x \arctan x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}\arctan x + \sqrt{2}\arctan\frac{x\sqrt{2}}{\sqrt{1-x^2}} - \arcsin x.$

74. $\displaystyle\int \frac{\arctan x}{\sqrt{(a+bx^2)^3}}\,dx = \begin{cases} \dfrac{x\arctan x}{a\sqrt{a+bx^2}} - \dfrac{1}{a\sqrt{b-a}}\arctan\sqrt{\dfrac{a+bx^2}{b-a}}, & a<b, \\ \dfrac{x\arctan x}{a\sqrt{a+bx^2}} + \dfrac{1}{2a\sqrt{a-b}}\ln\dfrac{\sqrt{a+bx^2}-\sqrt{a-b}}{\sqrt{a+bx^2}+\sqrt{a-b}}, & a>b. \end{cases}$

75. $\displaystyle\int \frac{\arctan x\,dx}{(1+x^2)^{n+1}} = \left[\sum_{k=1}^{n}\frac{(2n-2k)!!(2n-1)!!}{(2n)!!(2n-2k+1)!!}\frac{x}{(1+x^2)^{n-k+1}} + \frac{1}{2}\frac{(2n-1)!!}{(2)!!}\arctan x\right]\arctan x$
$\displaystyle + \frac{1}{2}\sum_{k=1}^{n}\frac{(2n-1)!!(2n-2k)!!}{(2n)!!(2n-2k+1)!!(n-k+1)}\frac{1}{(1+x^2)^{n-k+1}}.$

76. $\displaystyle\int x \operatorname{arcsec} ax\,dx = \frac{x^2}{2}\operatorname{arcsec} ax - \frac{\sqrt{a^2x^2-1}}{2a^2}.$

77. $\displaystyle\int x^n \operatorname{arcsec} ax\,dx = \frac{x^{n+1}}{n+1}\operatorname{arcsec} ax - \frac{1}{n+1}\int \frac{x^n}{\sqrt{a^2x^2-1}}\,dx.$

78. $\displaystyle\int \frac{\operatorname{arcsec} ax}{x^2}\,dx = -\frac{\operatorname{arcsec} ax}{x} + \frac{\sqrt{a^2x^2-1}}{x}.$

79. $\displaystyle\int x \operatorname{arccsc} ax\,dx = \frac{x^2}{2}\operatorname{arccsc} ax + \frac{\sqrt{a^2x^2-1}}{2a^2}.$

80. $\displaystyle\int x^n \operatorname{arccsc} ax\,dx = \frac{x^{n+1}}{n+1}\operatorname{arccsc} ax + \frac{1}{n+1}\int \frac{x^n}{\sqrt{a^2x^2-1}}\,dx.$

81. $\displaystyle\int \frac{\operatorname{arccsc} ax}{x^2}\,dx = -\frac{\operatorname{arccsc} ax}{x} - \frac{\sqrt{a^2x^2-1}}{x}.$