New Inverse Laplace Transform Formulas

The following notation is used:

$$F(t,a,b,c) = e^{ct}\left[e^{-a\sqrt{b+c}}\ \mathrm{erfc}\left(\frac{a}{2\sqrt{t}} - \sqrt{(b+c)t}\right) + e^{a\sqrt{b+c}}\ \mathrm{erfc}\left(\frac{a}{2\sqrt{t}} + \sqrt{(b+c)t}\right)\right],$$

$$G(t,a,b,c) = \frac{e^{ct}}{2\sqrt{b+c}}\left[e^{-a\sqrt{b+c}}\ \mathrm{erfc}\left(\frac{a}{2\sqrt{t}} - \sqrt{(b+c)t}\right) - e^{a\sqrt{b+c}}\ \mathrm{erfc}\left(\frac{a}{2\sqrt{t}} + \sqrt{(b+c)t}\right)\right],$$

$$S(x,\lambda,k,m) = \frac{\sin\left(a\sqrt{x}\,\right)}{(x+\lambda+k^2)^m},$$

$$C(x,\lambda,k,m) = \frac{\cos\left(a\sqrt{x}\,\right)}{(x+\lambda+k^2)^m},$$

$$R(x,\lambda,k,m) = \frac{\sqrt{x}}{(x+k^2)^m}\, e^{-a\sqrt{\lambda-x}},$$

$$\varphi_m(t,\lambda) = \sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\frac{\lambda^j t^j}{j!}\, e^{-\lambda t},$$

$$\Psi_m(t,\lambda) = e^{-\lambda t/2}\, I_0(\lambda t/2) + \sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{\lambda^j e^{-\lambda t}}{(j-1)!}\int_0^t (t-x)^{j-1}\, e^{\lambda x/2}\, I_0(\lambda x/2)\, dx,$$

where I_0 is the modified Bessel function of order zero, and m is an integer.

$\bar f(s) = \mathcal{L}^{-1}\{f(t)\}$	$f(t)$
1. $\dfrac{e^{-a\sqrt{s+b}}}{s-c}$	$F(t,a,b,c).$
2. $\dfrac{e^{-a\sqrt{s+b}}}{(s-c)^2}$	$t\,F(t,a,b,c) - \dfrac{a}{2}\,G(t,a,b,c).$
3. $\dfrac{e^{-a\sqrt{s+b}}}{(s-c)^3}$	$\tfrac12\left\{\left[\dfrac{a^2}{4(b+c)}+t^2\right]F(t,a,b,c)+\left[\dfrac{1}{4(b+c)}-t\right]a\,G(t,a,b,c)-\dfrac{a}{2(b+c)}\sqrt{\dfrac{t}{\pi}}\,e^{-a^2/(4t)-bt}\right\}.$
4. $\dfrac{e^{-a\sqrt{s+b}}}{(s-c)^4}$	$\dfrac16\left\{\left[t^3+\dfrac{3a^2t}{4(b+c)}-\dfrac{3a^2}{8(b+c)^2}\right]F(t,a,b,c)-\left[\dfrac{3a}{8(b+c)^2}+\dfrac{a^3}{8(b+c)}-\dfrac{3at}{4(b+c)}+\dfrac{3at^2}{2}\right]\right.$ $\left.G(t,a,b,c)+\left[\dfrac{3a}{4(b+c)^2}-\dfrac{at}{b+c}\right]\sqrt{\dfrac{t}{\pi}}\,e^{-a^2/(4t)-bt}\right\}.$
5. $\dfrac{\sqrt{s+b}\,e^{-a\sqrt{s+b}}}{s-c}$	$(b+c)\,G(t,a,b,c)+\dfrac{1}{\sqrt{\pi t}}\,e^{-a^2/(4t)-bt}.$
6. $\dfrac{\sqrt{s+b}\,e^{-a\sqrt{s+b}}}{(s-c)^2}$	$-\dfrac{a}{2}\,F(t,a,b,c)+\left[\tfrac12+(b+c)\,t\right]G(t,a,b,c)+\sqrt{\dfrac{t}{\pi}}\,e^{-a^2/(4t)-bt}.$
7. $\dfrac{\sqrt{s+b}\,e^{-a\sqrt{s+b}}}{(s-c)^3}$	$\tfrac12\left\{-a\left[\dfrac{1}{4(b+c)}+t\right]F(t,a,b,c)-\left[\dfrac{1}{4(b+c)}-\dfrac{a^2}{4}-t+(b+c)\,t^2\right]G(t,a,b,c)\right.$ $\left.+\left[\dfrac{1}{2(b+c)}+t\right]\sqrt{\dfrac{t}{\pi}}\,e^{-a^2/(4t)-bt}\right\}.$
8. $\dfrac{\sqrt{s+b}\,e^{-a\sqrt{s+b}}}{(s-c)^4}$	$\dfrac16\left\{\left[\dfrac{3a}{8(b+c)^2}-\dfrac{a^3}{8(b+c)}-\dfrac{3at}{4(b+c)}-\dfrac{3a}{2}\,t^2\right]F(t,a,b,c)\right.$ $+\left[\dfrac{3}{8(b+c)^2}-\left(\dfrac{3}{4(b+c)}-\dfrac{3a^2}{4}\right)t+\dfrac32\,t^2+(b+c)\,t^3\right]G(t,a,b,c)$ $\left.+\left[t^2-\dfrac{3}{4(b+c)^2}+\dfrac{a^2}{4(b+c)}+\dfrac{1}{b+c}\right]\sqrt{\dfrac{t}{\pi}}\,e^{-a^2/(4t)-bt}\right\}.$
9. $\dfrac{(s+b)\,e^{-a\sqrt{s+b}}}{s-c}$	$(b+c)\,F(t,a,b,c)+\dfrac{a}{2t^2}\sqrt{\dfrac{t}{\pi}}\,e^{-a^2/(4t)-bt}.$
10. $\dfrac{(s+b)\,e^{-a\sqrt{s+b}}}{(s-c)^2}$	$\left[1+(b+c)\,t\right]F(t,a,b,c)-\dfrac{a(b+c)}{2}\,G(t,a,b,c).$
11. $\dfrac{(s+b)\,e^{-a\sqrt{s+b}}}{(s-c)^3}$	$\tfrac12\left\{\left[\dfrac{a^2}{4}+2t+(b+c)\,t^2\right]F(t,a,b,c)-\left[\dfrac{3a}{4}+a(b+c)\,t\right]G(t,a,b,c)\right.$ $\left.-\dfrac{a}{2}\sqrt{\dfrac{t}{\pi}}\,e^{-a^2/(4t)-bt}\right\}.$
12. $\dfrac{(s+b)\,e^{-a\sqrt{s+b}}}{(s-c)^4}$	$\dfrac16\left\{\left[\dfrac{3a^2}{8(b+c)}+\dfrac{3a^2t}{4}+3t^2+(b+c)t^3\right]F(t,a,b,c)\right.$ $+\left[\dfrac{3a}{8(b+c)}-\dfrac{a^3}{8}-\dfrac{9at}{4}-\dfrac{3a(b+c)t^2}{2}\right]G(t,a,b,c)$ $-\left[\dfrac{3a}{4(b+c)}+at\right]\sqrt{\dfrac{t}{\pi}}\,e^{-a^2/(4t)-bt}.$

$\bar{f}(s) = \mathcal{L}^{-1}\{f(t)\}$	$f(t)$
13. $\dfrac{e^{-a\sqrt{s+\lambda}}}{\sqrt{s}}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\infty \frac{\cos a\sqrt{x}}{\sqrt{\lambda+x}}\, e^{-(x+\lambda)t}\, dx + \int_0^\lambda \frac{e^{-xt-a\sqrt{\lambda-x}}}{\sqrt{x}}\, dx\right].$
14. $\dfrac{e^{-a\sqrt{s+\lambda}}}{\sqrt{s}+k}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\infty \left\{xS(x,\lambda,k,1) + \sqrt{x+\lambda}\, C(x,\lambda,k,1)\right\} e^{-(x+\lambda)t} dx + \int_0^\lambda R(x,\lambda,k,1)\, e^{-xt}\, dx\right].$
15. $\dfrac{\sqrt{s+\lambda}\, e^{-a\sqrt{s+\lambda}}}{\sqrt{s}+k}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\infty \sqrt{x}\left(\sqrt{x+\lambda}\, S(x,\lambda,k,1) - k\, C(x,\lambda,k,1)\right) e^{-(x+\lambda)t} dx + \int_0^\lambda \sqrt{\lambda-x}\, R(x,\lambda,k,1)\, e^{-xt}\, dx\right].$
16. $\dfrac{(s+\lambda)\, e^{-a\sqrt{s+\lambda}}}{\sqrt{s}+k}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\lambda (\lambda-x)\, R(x,\lambda,k,1)\, e^{-xt}\, dx - \int_0^\infty x\left\{kS(x,\lambda,k,1) + \sqrt{x+\lambda}\, C(x,\lambda,k,1)\right\} e^{-(x+\lambda)t}\, dx\right].$
17. $\dfrac{e^{-a\sqrt{s+\lambda}}}{(\sqrt{s}+k)^2}$	$\dfrac{1}{\pi}\left[2k\displaystyle\int_0^\lambda R(x,\lambda,k,2)\, e^{-xt}\, dx - \int_0^\infty \left\{S(x,\lambda,k,1) - 2k^2 S(x,\lambda,k,2) - 2k\sqrt{x+\lambda}\, C(x,\lambda,k,2)\right\} e^{-(x+\lambda)t}\, dx\right].$
18. $\dfrac{e^{-a\sqrt{s+\lambda}}}{s(\sqrt{s}+k)}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\infty \frac{kS(x,\lambda,k,1) + \sqrt{x+\lambda}\, C(x,\lambda,k,1)}{x+\lambda}\left(1 - e^{-(x+\lambda)t}\right) dx + \int_0^\lambda \frac{R(x,\lambda,k,1)}{x}\left(1 - e^{-xt}\right) dx\right].$
19. $\dfrac{e^{-a\sqrt{s+\lambda}}}{s^{3/2}}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\infty \frac{\cos a\sqrt{x}}{(x+\lambda)^{3/2}}\left(1 - e^{-(x+\lambda)t}\right) dx + \int_0^\lambda \frac{e^{-a\sqrt{\lambda-x}}}{x^{3/2}}\left(1 - e^{-xt}\right) dx\right].$
20. $\dfrac{e^{-a\sqrt{s+\lambda}}}{s^2(\sqrt{s}+k)}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\infty \left(\frac{t}{x+\lambda} - \frac{1-e^{-(x+\lambda)t}}{(x+\lambda)^2}\right)\left(kS(x,\lambda,k,1) + \sqrt{x+\lambda}\, C(x,\lambda,k,1)\right) dx + \int_0^\lambda \left(\frac{t}{x} - \frac{1-e^{-xt}}{x^2}\right) R(x,\lambda,k,1)\, dx\right].$
21. $\dfrac{e^{-a\sqrt{s+\lambda}}}{s^{5/2}}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\infty \left(\frac{t}{(x+\lambda)^{3/2}} - \frac{1-e^{-(x+\lambda)t}}{(x+\lambda)^{5/2}}\right)\cos a\sqrt{x}\, dx + \int_0^\lambda \left(\frac{t}{x^{3/2}} - \frac{1-e^{-xt}}{x^{5/2}}\right) e^{-a\sqrt{\lambda-x}}\, dx\right].$
22. $\dfrac{1}{(\sqrt{s}+\lambda+c)(\sqrt{s}+k)}$	$\dfrac{1}{\pi}\left[\displaystyle\int_0^\infty \frac{k\sqrt{x} + c\sqrt{x+\lambda}}{(x+\lambda+k^2)(x+c^2)}\, e^{-(x+\lambda)t}\, dx + \int_0^\lambda \frac{\sqrt{x}}{(x+k^2)\left(\sqrt{\lambda-x}+c\right)}\, e^{-xt}\, dx\right].$

$\bar{f}(s)=\mathcal{L}^{-1}\{f(t)\}$	$f(t)$
23. $\dfrac{e^{-a\sqrt{\lambda s/(s+\lambda)}}}{s}$	$H(t)\left[1-\dfrac{1}{\pi}\displaystyle\int_0^\lambda \dfrac{e^{-xt}}{x}\,\sin\left(a\sqrt{\dfrac{\lambda x}{\lambda-x}}\right)dx\right]$ $=H(t)\left[1-\dfrac{2\lambda}{\pi}\displaystyle\int_0^\infty \dfrac{\sin au}{u(\lambda+u^2)}\,e^{-\lambda u^2 t/(\lambda+u^2)}\,du\right]$ $=1-\displaystyle\sum_{m=0}^\infty \dfrac{\lambda^{m+1/2}}{(2m+1)!}\,\Psi_m(t,\lambda)\,a^{2m+1}+\displaystyle\sum_{m=1}^\infty \dfrac{\lambda^m}{(2m)!}\,\varphi_m(t,\lambda)\,a^{2m}.$
24. $\dfrac{1}{s}\left(\dfrac{\lambda s}{s+\lambda}\right)^m e^{-a\sqrt{\lambda s/(s+\lambda)}}$	$\dfrac{(-1)^{m+1}}{\pi}\,H(t)\displaystyle\int_0^\lambda \left(\dfrac{\lambda x}{\lambda-x}\right)^m e^{-xt}\,\sin\left(a\sqrt{\dfrac{\lambda x}{\lambda-x}}\right)dx,\ a>0.$
25. $\dfrac{1}{s}\left(\dfrac{\lambda s}{s+\lambda}\right)^{m+1/2} e^{-a\sqrt{\lambda s/(s+\lambda)}}$	$\dfrac{(-1)^{m}}{\pi}\,H(t)\displaystyle\int_0^\lambda \left(\dfrac{\lambda x}{\lambda-x}\right)^{m+1/2} e^{-xt}\,\cos\left(a\sqrt{\dfrac{\lambda x}{\lambda-x}}\right)dx,\ a>0.$
26. $\dfrac{1}{s}\left(\dfrac{\lambda s}{s+\lambda}\right)^{n/2}$	$\begin{cases}\lambda^m\,\varphi_m(t,\lambda), & n=2m,\\ \lambda^{m+1/2}\,\varphi_m(t,\lambda), & n=2m+1, \text{ where } m>0 \text{ is an integer.}\end{cases}$
27. $\dfrac{s\left(e^{-k_1x}-e^{k_2x}\right)}{k_1^2-k_2^2}$	$\dfrac{1}{\pi}\,H(t)\Bigg\{\displaystyle\int_0^\infty \dfrac{\sin[x\psi(u)]}{\sqrt{\left(u-(1+\varepsilon)\right)^2+4u}}\,e^{-ut}\,du$ $-e^{(1-\varepsilon)t}\displaystyle\int_0^{2\sqrt{\varepsilon}} \dfrac{e^{-x\alpha_1/2}\,\cos\left(x\beta_1/2-ut\right)}{\sqrt{4\varepsilon-u^2}}\,du$ $-e^{(1-\varepsilon)t}\displaystyle\int_0^{2\sqrt{\varepsilon}} \dfrac{e^{-x\alpha_2/2}\,\cos\left(x\beta_2/2-ut\right)}{\sqrt{4\varepsilon-u^2}}\,du$ $+H(t-x)\Bigg[e^{(1-\varepsilon)t}\displaystyle\int_0^{2\sqrt{\varepsilon}} \dfrac{e^{-x\alpha_1/2}\,\cos\left(x\beta_1/2-ut\right)}{\sqrt{4\varepsilon-u^2}}\,du$ $+e^{(1-\varepsilon)t}\displaystyle\int_0^{2\sqrt{\varepsilon}} \dfrac{e^{-x\alpha_2/2}\,\cos\left(x\beta_2/2-ut\right)}{\sqrt{4\varepsilon-u^2}}\,du\Bigg]\Bigg\},$

where $\varepsilon>0$ and

$$
\begin{aligned}
k_{1,2}&=\sqrt{(s/2)\left\{s+(1+\varepsilon)\pm\sqrt{(s-(1+\varepsilon))^2-4s}\,\right\}},\\
\psi(u)&=\sqrt{(u/2)\left\{-(u-(1+\varepsilon))+\sqrt{(u-(1+\varepsilon))^2+4u}\,\right\}},\\
\alpha_{1,2}&=\sqrt{(1-\varepsilon)\left\{2\pm\sqrt{4\varepsilon-u^2}\right\}-u^2+2\sqrt{((1-\varepsilon)^2+u^2)\left\{1+\varepsilon\pm\sqrt{4\varepsilon-u^2}\right\}}},\\
\beta_{1,2}&=\sqrt{-(1-\varepsilon)\left\{2\pm\sqrt{4\varepsilon-u^2}\right\}+u^2+2\sqrt{((1-\varepsilon)^2+u^2)\left\{1+\varepsilon\pm\sqrt{4\varepsilon-u^2}\right\}}}.
\end{aligned}
$$

$\bar{f}(s) = \mathcal{L}^{-1}\{f(t)\}$	$f(t)$
28. $\dfrac{e^{-r_2 x} - e^{-r_1 x}}{\sqrt{\left(sa^2+1\right)^2 - 4sb^2}}$	$-\dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{e^{-yt}\,\sin\left[xg(y)\right]}{\sqrt{a^4y^2+2y\left(2b^2-a^2\right)+1}}\,dy, \quad t>0$
29. $\dfrac{e^{-r_2 x} - e^{-r_1 x}}{s\sqrt{\left(sa^2+1\right)^2 - 4sb^2}}$	$\dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{\left(e^{-yt}-1\right)\,\sin\left[xg(y)\right]}{y\sqrt{a^4y^2+2y\left(2b^2-a^2\right)+1}}\,dy, \quad t>0$ $= e^{x/b} - 1 + \dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{e^{-yt}\,\sin\left[xg(y)\right]}{y\sqrt{a^4y^2+2y\left(2b^2-a^2\right)+1}}\,dy, \quad t>0$
30. $\dfrac{r_2^n e^{-r_2 x} - r_1^n e^{-r_1 x}}{s\sqrt{\left(sa^2+1\right)^2 - 4sb^2}}$	$\dfrac{e^{-x/b}}{b^n} - H\left(\tfrac{1}{2}-n\right) + \dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{e^{-yt}\,g^n(y)\,A\left[xg(y)\right]}{y\sqrt{a^4y^2+2y\left(2b^2-a^2\right)+1}}\,dy, \quad t>0$
31. $\dfrac{r_2^2 e^{-r_2 x} - r_1^2 e^{-r_1 x}}{s\sqrt{\left(sa^2+1\right)^2 - 4sb^2}}$	$\dfrac{e^{-x/b}}{b^2} - \displaystyle\int_0^\infty \dfrac{e^{-yt}\,g^2(y)\sin\left[xg(y)\right]}{y\sqrt{a^4y^2+2y\left(2b^2-a^2\right)+1}}\,dy, \quad t>0$
32. $\dfrac{e^{-r_2 x} + e^{-r_1 x}}{s}$	$e^{-x/b} + 1 - \dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{e^{-yt}\,\sin\left[xg(y)\right]}{y}\,dy,$
33. $e^{-r_2 x} + e^{-r_1 x}$	$\delta(t)\left[e^{-x/b} + 1 - \displaystyle\int_0^\infty \dfrac{\sin\left[xg(y)\right]}{y}\,dy\right] + \dfrac{1}{\pi}\displaystyle\int_0^\infty e^{-yt}\,\sin\left[xg(y)\right]\,dy,$

where $a \geq 0$, $b > 0$, $x \geq 0$, and

$$r_{1,2} = \frac{1}{b}\sqrt{\frac{sa^2+1 \mp \sqrt{\left(sa^2+1\right)^2 - 4sb^2}}{2}},$$

$$g(y) = \frac{1}{b}\sqrt{\frac{ya^2-1+\sqrt{a^4y^4+2y\left(2b^2-a^2\right)+1}}{2}},$$

$$A\left[xg(y)\right] = \begin{cases} (-1)^{3n/2}\,\sin\left[xg(y)\right] & \text{for } n=0,2,4,\ldots, \\ (-1)^{(3n-1)/2}\,\cos\left[xg(y)\right] & \text{for } n=1,3,5,\ldots. \end{cases}$$

$\bar{f}(s) = \mathcal{L}^{-1}\{f(t)\}$	$f(t)$
34. $\dfrac{s}{s^2+b^2}\, e^{-(x/c)\sqrt{s^2/(1+s/a)}}$	$H(t)\left[e^{-\alpha_1 x}\,\cos(bt-\alpha_2 x) - \dfrac{1}{\pi}\displaystyle\int_a^\infty \dfrac{ye^{-yt}\,\sin[xh(y)]}{y^2+b^2}\,dy\right],$
35. $\dfrac{b}{s^2+b^2}\, e^{-(x/c)\sqrt{s^2/(1+s/a)}}$	$H(t)\left[e^{-\alpha_1 x}\,\sin(bt-\alpha_2 x) + \dfrac{b}{\pi}\displaystyle\int_a^\infty \dfrac{e^{-yt}\,\sin[xh(y)]}{y^2+b^2}\,dy\right],$
36. $\dfrac{s}{s^2+b^2}\, e^{-(x/c)\sqrt{s(s+a)}}$	$H(t)H(t-x/c)\left[e^{-ax/(2c)}\,\cos[b(t-x/c)] + ax\displaystyle\int_{x/c}^t \cos[b(t-y)]\;q(x,y)\,dy\right],$
37. $\dfrac{b}{s^2+b^2}\, e^{-(x/c)\sqrt{s(s+a)}}$	$H(t)H(t-x/c)\left[e^{-ax/(2c)}\,\sin[b(t-x/c)] + ax\displaystyle\int_{x/c}^t \sin[b(t-y)]\;q(x,y)\,dy\right],$

where $a > 0$,

$$h(y) = \frac{y}{c\sqrt{y/a-1}}, \quad \alpha_{1,2} = \frac{b}{c}\sqrt{\frac{\mp 1 + \sqrt{1+b^2/a^2}}{2\,(1+b^2/a^2)}}, \quad q(x,y) = \frac{e^{-ay/2}}{2c}\left[\frac{I_1\left[\frac{b}{2}\sqrt{y^2-x^2/c^2}\,\right]}{\sqrt{y^2-x^2/c^2}}\right],$$

and I_1 is the modified Bessel function of the first kind and order 1.

These formulas are not available in Erdélyi et al. (1954), Roberts and Kaufman (1966), Oberhettinger and Badu (1973), or other tables. They are available in the following publications.

Formulas 1–12:

R. B. Hetnarski, *On inverting the Laplace transforms connected with the error function*, Zastowania Matem, **7** (1964), 399–405.

——, *An algorithm for generating some inverse Laplace transforms of exponential form*, ZAMP, **26** (1975), 249–254.

Formulas 13–26:

P. Puri, *Impulsive motion of a flat plate in a Rivlin-Ericksen fluid*, Rheol. Acta , **23** (1984), 451–453.

—— and P. K. Kulshrestha, *Rotating flow of non-Newtonian fluids*, Applicable Anal., **4** (1974), 131–140.

—— and P. K. Kythe, *Some inverse Laplace transforms of exponential form*, ZAMP, **39** (1988), 150-156.

Formulas 27–37:

P. M. Jordan, P. Puri, *Stress distribution for the coupled, one-dimensional Damilobskaya half-space problem — Exact solution*, J. Appl. Phys., **85** (1999), 1273–.

——, P. Puri, and G. Boros, *A new class of Laplace inverses and their applications*, Applied Math. Letters, **13** (2000), 97–104.

——, M. R. Meyer, and A. Puri, *Causal implications of viscous damping in compressible fluid flows*, Physical Review, E, **62** (2000), 7981–7926.