

**T1.32.** Integrand involving algebraic functions of hyperbolic functions.

Notation used:  $\alpha = \arccos \frac{1 - \sinh 2ax}{1 + \sinh 2ax}, \quad \beta = \arcsin \sqrt{\frac{\cosh 2ax - 1}{\cosh 2ax}},$

$$\gamma = \arccos \frac{\sqrt{a^2 + b^2} - a - b \sinh x}{\sqrt{a^2 + b^2} + a + b \sinh x}, \quad \delta = \arcsin \left( \tanh \frac{x}{2} \right),$$

$$\alpha' = \arcsin \sqrt{\frac{a - b \cosh x}{a - b}}, \quad \beta' = \arcsin \sqrt{\frac{b(\cosh x - 1)}{b \cosh x - a}},$$

$$\gamma' = \arcsin \sqrt{\frac{b \cosh x - a}{b(\cosh x - 1)}}, \quad \delta' = \arccos \frac{(b^2 - a^2)^{1/4}}{\sqrt{a \sinh x + b \cosh x}},$$

$$p = \frac{1}{\sqrt{2}} \quad \text{for } ax > 0; \quad q = \frac{1}{\sqrt{2}} \quad \text{for } x \neq 0;$$

$$r = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2\sqrt{a^2 + b^2}}}, \quad \text{for } a > 0, b > 0, x > -\operatorname{arcsinh} \left( \frac{a}{b} \right);$$

$$s = \sqrt{\frac{a - b}{a + b}} \quad \text{for } 0 < b < a, x > 0;$$

$$t = \sqrt{\frac{a - b}{a + b}} \quad \text{for } 0 < b < a, 0 < x < \operatorname{arccosh} \left( \frac{a}{b} \right);$$

$$u = \sqrt{\frac{a + b}{2b}} \quad \text{for } 0 < a < b, x > 0;$$

$$v = \sqrt{\frac{2b}{a + b}} \quad \text{for } 0 < b < a, x > \operatorname{arccosh} \left( \frac{a}{b} \right);$$

$$w = \frac{1}{\sqrt{2}} \quad \text{for } 0 < a < b, -\operatorname{arcsinh} \frac{a}{\sqrt{b^2 - a^2}} < x.$$

$$1. \int \frac{\sinh x \, dx}{\sqrt{a^2 + \sinh^2 x}} = \begin{cases} \operatorname{arcsinh} \frac{\cosh x}{\sqrt{a^2 - 1}} = \ln (\cosh x + \sqrt{a^2 + \sinh^2 x}), & a^2 > 1, \\ \operatorname{arccosh} \frac{\cosh x}{\sqrt{1 - a^2}} = \ln (\cosh x + \sqrt{a^2 + \sinh^2 x}), & a^2 < 1, \\ \ln \cosh x, & a^2 = 1. \end{cases}$$

2.  $\int \frac{\sinh x \, dx}{\sqrt{a^2 - \sinh^2 x}} = \arcsin \frac{\cosh x}{\sqrt{a^2 + 1}}, \quad \sinh^2 x < a^2.$
3.  $\int \frac{\sinh x \, dx}{\sqrt{\sinh^2 x - a^2}} = \operatorname{arccosh} \frac{\cosh x}{\sqrt{a^2 + 1}} = \ln (\cosh x + \sqrt{\sinh^2 x - a^2}), \quad \sinh^2 x > a^2.$
4.  $\int \frac{\cosh x \, dx}{\sqrt{a^2 + \sinh^2 x}} = \operatorname{arcsinh} \frac{\sinh x}{a} = \ln (\sinh x + \sqrt{a^2 + \sinh^2 x}).$
5.  $\int \frac{\cosh x \, dx}{\sqrt{a^2 - \sinh^2 x}} = \arcsin \frac{\sinh x}{a}, \quad \sinh^2 x < a^2.$
6.  $\int \frac{\cosh x \, dx}{\sqrt{\sinh^2 x - a^2}} = \operatorname{arccosh} \frac{\sinh x}{a} = \ln (\sinh x + \sqrt{\sinh^2 x - a^2}), \quad \sinh^2 x > a^2.$
7.  $\int \frac{\sinh x \, dx}{\sqrt{a^2 + \cosh^2 x}} = \operatorname{arcsinh} \frac{\cosh x}{a} = \ln (\cosh x + \sqrt{a^2 + \cosh^2 x}).$
8.  $\int \frac{\sinh x \, dx}{\sqrt{a^2 - \cosh^2 x}} = \arcsin \frac{\cosh x}{a}, \quad \cosh^2 x < a^2.$
9.  $\int \frac{\sinh x \, dx}{\sqrt{\cosh^2 x - a^2}} = \operatorname{arccosh} \frac{\cosh x}{a} = \ln (\cosh x + \sqrt{\cosh^2 x - a^2}), \quad \cosh^2 x > a^2.$
10.  $\int \frac{\cosh x \, dx}{\sqrt{a^2 + \cosh^2 x}} = \operatorname{arcsinh} \frac{\sinh x}{\sqrt{a^2 + 1}} = \ln (\sinh x + \sqrt{a^2 + \cosh^2 x}).$
11.  $\int \frac{\cosh x \, dx}{\sqrt{a^2 - \cosh^2 x}} = \arcsin \frac{\sinh x}{\sqrt{a^2 - 1}}, \quad \cosh^2 x < a^2.$
12.  $\int \frac{\cosh x \, dx}{\sqrt{\cosh^2 x - a^2}} = \begin{cases} \operatorname{arccosh} \frac{\sinh x}{\sqrt{a^2 - 1}}, & a^2 > 1, \\ \ln \sinh x, & a^2 = 1. \end{cases}$
13.  $\int \frac{\coth x \, dx}{\sqrt{a + b \sinh x}} = \begin{cases} 2\sqrt{a} \operatorname{arccoth} \sqrt{1 + \frac{b}{a} \sinh x}, & b \sinh x > 0, a > 0, \\ 2\sqrt{a} \operatorname{arctanh} \sqrt{1 + \frac{b}{a} \sinh x}, & b \sinh x < 0, a > 0, \\ 2\sqrt{-a} \operatorname{arctanh} \sqrt{-\left(1 + \frac{b}{a} \sinh x\right)}, & a < 0. \end{cases}$
14.  $\int \frac{\tanh x \, dx}{\sqrt{a + b \cosh x}} = \begin{cases} 2\sqrt{a} \operatorname{arccoth} \sqrt{1 + \frac{b}{a} \cosh x}, & b \cosh x > 0, a > 0, \\ 2\sqrt{a} \operatorname{arctanh} \sqrt{1 + \frac{b}{a} \cosh x}, & b \cosh x < 0, a > 0, \\ = 2\sqrt{-a} \operatorname{arctanh} \sqrt{-\left(1 + \frac{b}{a} \cosh x\right)}, & a < 0. \end{cases}$

- $$15. \int \frac{\sinh x \sqrt{a + b \cosh x}}{p + q \cosh x} dx = \begin{cases} 2\sqrt{\frac{aq - bp}{q}} \operatorname{arccoth} \sqrt{\frac{q(a + b \cosh x)}{aq - bp}}, & b \cosh x > 0, \frac{aq - bp}{q} > 0, \\ 2\sqrt{\frac{aq - bp}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \cosh x)}{aq - bp}}, & b \cosh x < 0, \frac{aq - bp}{q} > 0, \\ 2\sqrt{\frac{bp - aq}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \cosh x)}{bp - aq}}, & \frac{aq - bp}{q} < 0. \end{cases}$$
- $$16. \int \frac{\cosh x \sqrt{a + b \sinh x}}{p + q \sinh x} dx = 2 \begin{cases} \sqrt{\frac{aq - bp}{q}} \operatorname{arccoth} \sqrt{\frac{q(a + b \sinh x)}{aq - bp}}, & b \sinh x > 0, \frac{aq - bp}{q} > 0, \\ 2\sqrt{\frac{aq - bp}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \sinh x)}{aq - bp}}, & b \sinh x < 0, \frac{aq - bp}{q} > 0, \\ 2\sqrt{\frac{bp - aq}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \sinh x)}{bp - aq}}, & \frac{aq - bp}{q} < 0. \end{cases}$$
- $$17. \int \frac{dx}{\sqrt{k^2 + (k')^2 \cosh^2 x}} = \int \frac{dx}{\sqrt{1 + (k')^2 \sinh^2 x}} = F(\arcsin(\tanh x), k), \quad x > 0.$$
- $$18. \int \frac{dx}{\sqrt{\cosh^2 x - k^2}} = \int \frac{dx}{\sqrt{\sinh^2 x + (k')^2}} = F\left(\arcsin\left(\frac{1}{\cosh x}\right), k\right) \quad x > 0.$$
- $$19. \int \frac{dx}{\sqrt{1 - (k')^2 \cosh^2 x}} = F\left(\arcsin\left(\frac{\tanh x}{k}\right), k\right) \quad 0 < x < \operatorname{arccosh} \frac{1}{k'}.$$
- $$20. \int \frac{dx}{\sqrt{\sinh 2ax}} = \frac{1}{2a} F(\alpha, p).$$
- $$21. \int \sqrt{\sinh 2ax} dx = \frac{1}{2a} [F(\alpha, p) - 2E(\alpha, p)] + \frac{1}{a} \frac{\sqrt{\sinh 2ax(1 + \sinh^2 2ax)}}{1 + \sinh 2ax}.$$
- $$22. \int \frac{\cosh^2 2ax dx}{(1 + \sinh 2ax)^2 \sqrt{\sinh 2ax}} = \frac{1}{2a} E(\alpha, p).$$
- $$23. \int \frac{(1 - \sinh 2ax)^2 dx}{(1 + \sinh 2ax)^2 \sqrt{\sinh 2ax}} = \frac{1}{2a} [2E(\alpha, p) - F(\alpha, p)].$$
- $$24. \int \frac{\sqrt{\sinh 2ax} dx}{(1 + \sinh 2ax)^2} = \frac{1}{4a} [F(\alpha, p) - E(\alpha, p)].$$
- $$25. \int \frac{dx}{\sqrt{\cosh 2ax}} = \frac{1}{a\sqrt{2}} F(\beta, q).$$
- $$26. \int \sqrt{\cosh 2ax} dx = \frac{1}{a\sqrt{2}} [F(\beta, q) - 2E(\beta, q)] + \frac{\sinh 2ax}{a\sqrt{\cosh 2ax}}.$$

27.  $\int \frac{dx}{\sqrt{\cosh^3 2ax}} = \frac{1}{a\sqrt{2}} [2E(\beta, q) - F(\beta, q)].$
28.  $\int \frac{dx}{\sqrt{\cosh^5 2ax}} = \frac{1}{3\sqrt{2}a} F(\beta, q) + \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}}.$
29.  $\int \frac{\sinh^2 2ax dx}{\sqrt{\cosh 2ax}} = -\frac{\sqrt{2}}{3a} F(\beta, q) + \frac{1}{3a} \sinh 2ax \sqrt{\cosh 2ax}.$
30.  $\int \frac{\tanh^2 2ax dx}{\sqrt{\cosh 2ax}} = \frac{\sqrt{2}}{3a} F(\beta, q) - \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}}.$
31.  $\int \frac{\sqrt{\cosh 2ax} dx}{sp^2 + (1 - p^2) \cosh 2ax} = \frac{1}{a\sqrt{2}} \Pi(\beta, p^2, q).$
32.  $\int \frac{dx}{\sqrt{a + b \sinh x}} = \frac{1}{(a^2 + b^2)^{1/4}} F(\gamma, r).$
33.  $\int \sqrt{a + b \sinh x} dx = (a^2 + b^2)^{1/4} [F(\gamma, r) - 2E(\gamma, r)] + \frac{2b \cosh x \sqrt{a + b \sinh x}}{\sqrt{a^2 + b^2} + a + b \sinh x}.$
34.  $\int \frac{\sqrt{a + b \sinh x}}{\cosh^2 x} dx = (a^2 + b^2)^{1/4} E(\gamma, r) - \frac{\sqrt{a^2 + b^2} - a}{2(a^2 + b^2)^{1/4}} F(\gamma, r)$   
 $- \frac{a + \sqrt{a^2 + b^2}}{b} \frac{\sqrt{a^2 + b^2} - a - b \sinh x}{\sqrt{a^2 + b^2} + a + b \sinh x} \frac{\sqrt{a + b \sinh x}}{\cosh x}.$
35.  $\int \frac{\cosh^2 x dx}{[\sqrt{a^2 + b^2} + a + b \sinh x]^2 \sqrt{a + b \sinh x}} = \frac{1}{b^2 (a^2 + b^2)^{1/4}} E(\gamma, r).$
36.  $\int \frac{\sqrt{a + b \sinh x} dx}{[\sqrt{a^2 + b^2} - a - b \sinh x]^2} = -\frac{1}{(a^2 + b^2)^{1/4} (\sqrt{a^2 + b^2} - a)} E(\gamma, r)$   
 $+ \frac{b}{\sqrt{a^2 + b^2} - a} \frac{\cosh x \sqrt{a + b \sinh x}}{a^2 + b^2 - (a + b \sinh x)^2}.$
37.  $\int \frac{dx}{\sqrt{a + b \cosh x}} = \frac{2}{\sqrt{a + b}} F(\delta, s).$
38.  $\int \sqrt{a + b \cosh x} dx = 2\sqrt{a + b} [F(\delta, s) - E(\delta, s)] + 2 \tanh \frac{x}{2} \sqrt{a + b \cosh x}.$
39.  $\int \frac{\cosh x dx}{\sqrt{a + b \cosh x}} = \frac{2}{\sqrt{a + b}} F(\delta, r) - \frac{2\sqrt{a + b}}{b} E(\delta, s) + \frac{2}{b} \tanh \frac{x}{2} \sqrt{a + b \cosh x}.$
40.  $\int \frac{\tanh^2 \frac{x}{2}}{\sqrt{a + b \cosh x}} dx = \frac{2\sqrt{a + b}}{a - b} [F(\delta, s) - E(\delta, s)].$
41.  $\int \frac{\tanh^4 \frac{x}{2}}{\sqrt{a + b \cosh x}} dx = \frac{2\sqrt{a + b}}{3(a - b)^2} [(3a + b)F(\delta, s) - 4aE(\delta, s)] + \frac{2}{3(a - b)} \frac{\sinh \frac{x}{2} \sqrt{a + b \cosh x}}{\cosh^3 \frac{x}{2}}.$

42.  $\int \frac{\cosh x - 1}{\sqrt{a+b} \cosh x} dx = \frac{2}{b} \left[ \tanh \frac{x}{2} \sqrt{a+b} \cosh x - \sqrt{a+b} E(\delta, s) \right].$
43.  $\int \frac{(\cosh x - 1)^2}{\sqrt{a+b} \cosh x} dx = \frac{4\sqrt{a+b}}{3b^2} [(a+3b)E(\delta, s) - bF(\delta, s)]$   
 $+ \frac{4}{3b^2} \left[ b \cosh^2 \frac{x}{2} - (a+3b) \right] \tanh \frac{x}{2} \sqrt{a+b} \cosh x.$
44.  $\int \frac{\sqrt{a+b} \cosh x}{\cosh x + 1} dx = \sqrt{a+b} E(\delta, s).$
45.  $\int \frac{dx}{(\cosh x + 1)\sqrt{a+b} \cosh x} = \frac{\sqrt{a+b}}{a-b} E(\delta, s) - \frac{2b}{(a-b)\sqrt{a+b}} F(\delta, s).$
46.  $\int \frac{dx}{(\cosh x + 1)^2 \sqrt{a+b} \cosh x} = \frac{1}{3(a-b)^2 \sqrt{a+b}} [b(5b-a)F(\delta, s)$   
 $+ (a-3b)(a+b)E(\delta, s)] + \frac{1}{6(a-b)} \frac{\sinh \frac{x}{2}}{\cosh^3(x/2)} \sqrt{a+b} \cosh x.$
47.  $\int \frac{(1 + \cosh x) dx}{[1 + p^2 + (1 - p^2) \cosh x] \sqrt{a+b} \cosh x} = \frac{2}{\sqrt{a+b}} \Pi(\delta, p^2, s).$
48.  $\int \frac{dx}{\sqrt{a-b} \cosh x} = \frac{2}{\sqrt{a+b}} F(\alpha', t).$
49.  $\int \sqrt{a-b} \cosh x dx = 2\sqrt{a+b} [F(\alpha', t) - E(\alpha', t)].$
50.  $\int \frac{\cosh x dx}{\sqrt{a-b} \cosh x} = \frac{2\sqrt{a+b}}{b} E(\alpha', t) - \frac{2}{\sqrt{a+b}} F(\alpha', t).$
51.  $\int \frac{\cosh^2 x dx}{\sqrt{a-b} \cosh x} = \frac{2(b-2a)}{3b\sqrt{a+b}} F(\alpha', t) + \frac{4a\sqrt{a+b}}{3b^2} E(\alpha', t) + \frac{2}{3b} \sinh x \sqrt{a-b} \cosh x.$
52.  $\int \frac{(1 + \cosh x) dx}{\sqrt{a-b} \cosh x} = \frac{2\sqrt{a+b}}{b} E(\alpha', t).$
53.  $\int \frac{dx}{\cosh x \sqrt{a-b} \cosh x} = \frac{2b}{a\sqrt{a+b}} \Pi \left( \alpha', \frac{a-b}{a}, t \right).$
54.  $\int \frac{dx}{(1 + \cosh x) \sqrt{a-b} \cosh x} = \frac{1}{\sqrt{a+b}} E(\alpha', t) - \frac{1}{a+b} \tanh \frac{x}{2} \sqrt{a-b} \cosh x.$
55.  $\int \frac{dx}{(1 + \cosh x)^2 \sqrt{a-b} \cosh x} = \frac{1}{3\sqrt{(a+b)^3}} [(a+3b)E(\alpha', t) - bF(\alpha', t)]$   
 $- \frac{1}{3(a+b)^2} \frac{\tanh \frac{x}{2} \sqrt{a-b} \cosh x}{\cosh x + 1} [2a + 4b + (a+3b) \cosh x].$
56.  $\int \frac{dx}{(a-b-ap^2+bp^2 \cosh x) \sqrt{a-b} \cosh x} = \frac{2}{(a-b)\sqrt{a+b}} \Pi(\alpha', p^2, t).$

- $$57. \int \frac{dx}{\sqrt{b \cosh x - a}} = \sqrt{\frac{2}{b}} F(\beta', u).$$
- $$58. \int \sqrt{b \cosh x - a} dx = (b - a) \sqrt{\frac{2}{b}} F(\beta', u) - 2\sqrt{2b} E(\beta', u) + \frac{2b \sinh x}{\sqrt{b \cosh x - a}}.$$
- $$59. \int \frac{dx}{\sqrt{(b \cosh x - a)^3}} = \frac{1}{b^2 - a^2} \sqrt{\frac{2}{b}} [2bE(\beta', u) - (b - a)F(\beta', u)].$$
- $$60. \int \frac{dx}{\sqrt{(b \cosh x - a)^5}} = \frac{1}{3(b^2 - a^2)^2} \sqrt{\frac{2}{b}} [(b - 3a)(b - a)F(\beta', u) + 8abE(\beta', u)] \\ + \frac{2b}{3(b^2 - a^2)} \frac{\sinh x}{\sqrt{(b \cosh x - a)^3}}.$$
- $$61. \int \frac{\cosh x dx}{\sqrt{b \cosh x - a}} = \sqrt{\frac{2}{b}} [F(\beta', u) - 2E(\beta', u)] + \frac{2 \sinh x}{\sqrt{b \cosh x - a}}.$$
- $$62. \int \frac{(\cosh x + 1) dx}{\sqrt{(b \cosh x - a)^3}} = \frac{2}{b - a} \sqrt{\frac{2}{b}} E(\beta', u).$$
- $$63. \int \frac{\sqrt{b \cosh x - a} dx}{p^2 b - a + b(1 - p^2) \cosh x} = \sqrt{\frac{2}{b}} \Pi(\beta', p^2, u).$$
- $$64. \int \frac{dx}{\sqrt{b \cosh x - a}} = \frac{2}{\sqrt{a + b}} F(\gamma', v).$$
- $$65. \int \sqrt{b \cosh x - a} dx = -2\sqrt{a + b} E(\gamma', v) + 2 \operatorname{arccoth} \frac{x}{2} \sqrt{b \cosh x - a}.$$
- $$66. \int \frac{\operatorname{arccoth}^2 \frac{x}{2} dx}{\sqrt{b \cosh x - a}} = \frac{2\sqrt{a + b}}{a - b} E(\gamma', v).$$
- $$67. \int \frac{\sqrt{b \cosh x - a}}{\cosh x - 1} dx = \sqrt{a + b} [F(\gamma', v) - E(\gamma', v)].$$
- $$68. \int \frac{dx}{(\cosh x - 1) \sqrt{b \cosh x - a}} = \frac{\sqrt{a + b}}{a - b} E(\gamma', v) - \frac{1}{\sqrt{a + b}} F(\gamma', v).$$
- $$69. \int \frac{dx}{(\cosh x - 1)^2 \sqrt{b \cosh x - a}} = \frac{1}{3(a - b)^2 \sqrt{a + b}} [(a - 2b)(a - b)F(\gamma', v) \\ + \frac{a + b}{6b(a - b)} \frac{\cosh \frac{x}{2}}{\sinh^3 \frac{x}{2}} \sqrt{b \cosh x - a}].$$
- $$70. \int \frac{dx}{(\cosh x + 1) \sqrt{b \cosh x - a}} = \frac{1}{\sqrt{a + b}} [F(\gamma', v) - E(\gamma', v)] + \frac{2\sqrt{b \cosh x - a}}{(a + b) \sinh x}.$$

71. 
$$\int \frac{dx}{(\cosh x + 1)^2 \sqrt{b \cosh x - a}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+b)F(\gamma', v) - (a+3b)E(\gamma', v)] + \frac{\sqrt{b \cosh x - a}}{3(a+b) \sinh x} \left( 2\frac{a+3b}{a+b} - \tanh^2 \frac{x}{2} \right).$$
72. 
$$\int \frac{dx}{\sqrt{a \sinh x + b \cosh x}} = \left( \frac{4}{b^2 - a^2} \right)^{1/4} F(\delta', w).$$
73. 
$$\int \sqrt{a \sinh x + b \cosh x} dx = (4(b^2 - a^2))^{1/4} [F(\delta', w) - 2E(\delta', w)] + \frac{2(a \cosh x + b \sinh x)}{\sqrt{a \sinh x + b \cosh x}}.$$
74. 
$$\int \frac{dx}{\sqrt{(a \sinh x + b \cosh x)^3}} = \left( \frac{4}{(b^2 - a^2)^3} \right)^{1/4} [2E(\delta', w) - F(\delta', w)].$$
75. 
$$\int \frac{dx}{\sqrt{(a \sinh x + b \cosh x)^5}} = \frac{1}{3} \left( \frac{4}{(b^2 - a^2)^5} \right)^{1/4} F(\delta', w) + \frac{2}{3(b^2 - a^2)} \frac{a \cosh x + b \sinh x}{\sqrt{(a \sinh x + b \cosh x)^3}}.$$
76. 
$$\int \frac{(\sqrt{b^2 - a^2} + a \sinh x + b \cosh x) dx}{\sqrt{(a \sinh x + b \cosh x)^3}} = 2 \left( \frac{4}{b^2 - a^2} \right)^{1/4} E(\delta', w).$$
77. 
$$\int \sqrt{\tanh x} dx = \operatorname{arctanh} \sqrt{\tanh x} - \arctan \sqrt{\tanh x}.$$
78. 
$$\int \sqrt{\operatorname{arccoth} x} dx = \operatorname{arccoth} \sqrt{\operatorname{arccoth} x} - \arctan \sqrt{\operatorname{arccoth} x}.$$
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