! For an efficient use of these tables, first read `HowTo.pdf`.

T2.41B. Integrands involving sine and cosine of single and multiple arguments on the interval $(0, \pi)$.

1. $\displaystyle \int_0^{\pi} (\cos t + i\, \sin t\, \cos x)^n\, dx = \int_0^{\pi} (\cos t + i\, \sin t\, \cos x)^{-n-1}\, dx = \pi P_n(\cos t).$

2. $\displaystyle \int_0^{\pi} \frac{\sin nx\, \cos mx}{\sin x}\, dx = \begin{cases} 0 & \text{for } n \le m, \\ \pi & \text{for } n > m, \quad \text{if } m+n \text{ is odd and positive}, \\ 0 & \text{for } n > m, \quad \text{if } m+n \text{ is even}. \end{cases}$

3. $\displaystyle \int_0^{\pi} \frac{\sin nx}{\sin x}\, dx = \begin{cases} 0 & \text{for } n \text{ even}, \\ \pi & \text{for } n \text{ odd}. \end{cases}$

4. $\displaystyle \int_0^{\pi} \frac{\sin 2nx}{\cos x}\, dx = 2\int_0^{\pi/2} \frac{\sin 2nx}{\cos x}\, dx = (-1)^{n-1} 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1}\right).$

5. $\displaystyle \int_0^{\pi} \frac{\cos(2n+1)x}{\cos x}\, dx = 2\int_0^{\pi/2} \frac{\cos(2n+1)x}{\cos x}\, dx = (-1)^n\, \pi.$

6. $\displaystyle \int_0^{\pi} \frac{\cos nx\, dx}{1 + a\, \cos x} = \frac{\pi}{\sqrt{1-a^2}} \left(\frac{\sqrt{1-a^2} - 1}{a}\right)^n, \quad a^2 < 1,\ n \ge 0.$

7. $\displaystyle \int_0^{\pi} \frac{\cos nx\, dx}{1 - 2a\, \cos x + a^2} = \begin{cases} \dfrac{\pi a^n}{1-a^2}, & a^2 < 1,\ n \ge 0, \\ \dfrac{\pi}{(a^2-1)\, a^n}, & a^2 > 1,\ n \ge 0. \end{cases}$

8. $\displaystyle \int_0^{\pi} \frac{\sin nx\, \sin x\, dx}{1 - 2a\, \cos x + a^2} = \begin{cases} \dfrac{\pi}{2} a^{n-1}, & a^2 < 1,\ n \ge 1, \\ \dfrac{\pi}{2a^{n+1}}, & a^2 > 1,\ n \ge 1. \end{cases}$

9. $$\int_0^\pi \frac{\cos nx\,\cos x\,dx}{1-2a\,\cos x+a^2} = \begin{cases} \dfrac{\pi}{2}\cdot\dfrac{1+a^2}{1-a^2}a^{n-1}, & a^2<1,\ n\geq 1, \\ \dfrac{\pi}{2a^{n+1}}\cdot\dfrac{a^2+1}{a^2-1}, & a^2>1,\ n\geq 1, \\ \dfrac{\pi a}{1-a^2}, & n=0,\ a^2<1, \\ \dfrac{\pi}{a(a^2-1)}, & n=0,\ a^2>1. \end{cases}$$

10. $$\int_0^\pi \frac{\cos(2n-1)x\,dx}{1-2a\,\cos 2x+a^2} = \int_0^\pi \frac{\cos 2nx\,\cos x\,dx}{1-2a\,\cos 2x+a^2} = 0, \quad a^2\neq 1.$$

11. $$\int_0^\pi \frac{\cos(2n-1)x\,\cos 2x\,dx}{1-2a\,\cos 2x+a^2} = 0, \quad a^2\neq 1.$$

12. $$\int_0^\pi \frac{\sin 2nx\,\sin x\,dx}{1-2a\,\cos 2x+a^2} = \int_0^\pi \frac{\sin(2n-1)x\,\sin 2x\,dx}{1-2a\,\cos 2x+a^2} = 0, \quad a^2\neq 1.$$

13. $$\int_0^\pi \frac{\sin(2n-1)x\,\sin x\,dx}{1-2a\,\cos 2x+a^2} = \begin{cases} \dfrac{\pi}{2}\cdot\dfrac{a^{n-1}}{1+a}, & a^2<1, \\ \dfrac{\pi}{2}\cdot\dfrac{1}{(1+a)a^n}, & a^2>1. \end{cases}$$

14. $$\int_0^\pi \frac{\cos(2n-1)x\,\cos x\,dx}{1-2a\,\cos 2x+a^2} = \begin{cases} \dfrac{\pi}{2}\,\dfrac{a^{n-1}}{1-a}, & a^2<1, \\ \dfrac{\pi}{2}\,\dfrac{1}{(a-1)a^n}, & a^2>1. \end{cases}$$

15. $$\int_0^\pi \frac{\sin nx - a\,\sin(n-1)x}{1-2a\,\cos x+a^2}\,\sin mx\,dx = \begin{cases} 0, & \text{for } m<n, \\ \dfrac{\pi}{2}a^{m-n}, & \text{for } m\geq n; \end{cases} \quad a^2<1.$$

16. $$\int_0^\pi \frac{\cos nx - a\,\cos(n-1)x}{1-2a\,\cos x+a^2}\,\cos mx\,dx = \frac{\pi}{2}\left(a^{|m|-n}-1\right), \quad a^2<1.$$

17. $$\int_0^\pi \frac{\sin nx - a\,\sin[(n+1)x]}{1-2a\,\cos x+a^2}\,dx = 0, \quad a^2<1.$$

18. $$\int_0^\pi \frac{\cos nx - a\,\cos[(n+1)x]}{1-2a\,\cos x+a^2}\,dx = \pi a^n, \quad a^2<1.$$

19. $$\int_0^\pi \frac{\sin x}{a^2-2ab\,\cos x+b^2}\cdot\frac{\sin px\cdot dx}{1-2a^p\,\cos px+a^{2p}}$$
$$=\begin{cases}\dfrac{\pi b^{p-1}}{2a^{p+1}(1-b^p)}, & 0<b\le a\le 1,\ p=1,2,3,\ldots,\\ \dfrac{\pi a^{p-1}}{2b(b^p-a^{2p})}, & 0<a\le 1,\ a^2<b, a^2\ p=1,2,3,\ldots.\end{cases}$$

20. $$\int_0^\pi \frac{\cos x\,\sin 2nx\,dx}{1+(a+b\,\sin x)^2}=-\frac{\pi}{b}\,\sin\left\{2n\,\arctan\sqrt{\frac{s}{2}}\right\}\tan^{2n}\left(\frac{1}{2}\,\arccos\sqrt{\frac{s}{2a^2}}\right).$$

21. $$\int_0^\pi \frac{\cos x\,\cos(2n+1)x\,dx}{1+(a+b\,\sin x)^2}=\frac{\pi}{b}\,\cos\left\{(2n+1)\,\arctan\sqrt{\frac{s}{2}}\right\}\tan^{2n+1}\left(\frac{1}{2}\,\arccos\sqrt{\frac{s}{2a^2}}\right),$$
where $s=-(1+b^2-a^2)+\sqrt{(1+b^2-a^2)^2+4a^2}$.

22. $$\int_0^\pi (1-2a\,\cos x+a^2)^n\,dx=\pi\sum_{k=0}^{n}\binom{n}{k}^2 a^{2k}.$$

23. $$\int_0^\pi \frac{dx}{(1-2a\,\cos x+a^2)^n}$$
$$=\begin{cases}\dfrac{\pi}{(1-a^2)^n}\displaystyle\sum_{k=0}^{n-1}\frac{(n+k-1)!}{(k!)^2(n-k-1)!}\left(\frac{a^2}{1-a^2}\right)^k, & a^2<1,\\ \dfrac{\pi}{(a^2-1)^n}\displaystyle\sum_{k=0}^{n-1}\frac{(n+k-1)!}{(k!)^2(n-k-1)!}\,\frac{1}{(a^2-1)^k}, & a^2>1.\end{cases}$$

24. $$\int_0^\pi (1-2a\,\cos x+a^2)^n\,\cos nx\,dx=(-1)^n\pi a^n.$$

25. $$\int_0^\pi (1-2a\,\cos x+a^2)^n\,\cos mx\,dx\frac{1}{2}\int_0^{2\pi}(1-2a\,\cos x+a^2)^n\,\cos mx\,dx$$
$$=\begin{cases}0, & n<m,\\ \pi(-a)^m(1+a^2)^{n-m}\displaystyle\sum_{k=0}^{[(n-m)/2]}\binom{n}{k}\binom{n-k}{m+k}\left(\frac{a}{1+a^2}\right)^{2k}, & n\ge m.\end{cases}$$

26. $$\int_0^\pi \frac{\sin x\,dx}{(1-2a\,\cos 2x+a^2)^m}=\frac{1}{2(m-1)a}\left[\frac{1}{(1-a)^{2m-2}}-\frac{1}{(1+a)^{2m-2}}\right],\quad a\ne 0,\ \pm 1.$$

27. $$\int_0^{\pi} \frac{\cos nx\,dx}{(1-2a\,\cos x+a^2)^m} = \frac{1}{2}\int_0^{2\pi} \frac{\cos nx\,dx}{(1-2a\,\cos x+a^2)^m}$$
$$= \begin{cases} \dfrac{a^{2m+n-2}\pi}{(1-a^2)^{2m-1}} \displaystyle\sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} \left(\frac{1-a^2}{a^2}\right)^k, & a^2<1, \\ \dfrac{\pi}{a^n(a^2-1)^{2m-1}} \displaystyle\sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} (a^2-1)^k, & a^2>1. \end{cases}$$

28. $$\int_0^{\pi} \frac{dx}{(1-2a\,\cos x+a^2)^{n+1/2}} = \frac{2}{|1+a|^{2n+1}} F_n\left(\frac{2\sqrt{|a|}}{|1+a|}\right), \quad |a|\neq 1,$$

where $F_n(k) = \int_0^{\pi/2} \frac{dx}{(1-k^2\,\sin^2\,x)^{n+1/2}}$, such that $F_n(k)$ satisfies the recurrence relation

$F_{n+1}(k) = F_n(k) + \frac{k}{2n+1}\,\frac{dF_n(k)}{dk},\ n=0,1,2,\ldots;$ and $F_0(k) = K(k) \equiv \int_0^{\pi/2} \frac{dx}{(1-k^2\,\sin^2\,x)^{1/2}}.$