! For an efficient use of these tables, first read `HowTo.pdf`.

T2.69A. Integrands involving logarithm functions and powers of logarithm functions and rational functions on the interval $(0, 1)$.

1. $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}.$

2. $\displaystyle\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}.$

3. $\displaystyle\int_0^1 \ln\left(1-\frac{x}{2}\right)\frac{dx}{x} = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}.$

4. $\displaystyle\int_0^1 \frac{\ln\frac{1+x}{2}}{1-x}\,dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}.$

5. $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x}\,dx = \frac{1}{2}(\ln 2)^2.$

6. $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2.$

7. $\displaystyle\int_0^1 \frac{\ln(1-x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2 - \mathbf{G}.$

8. $\displaystyle\int_0^1 \frac{\ln(1+x)}{x(1+x)}\,dx = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2.$

9. $\displaystyle\int_0^1 \frac{\ln(1+x)}{(ax+b)^2}\,dx = \begin{cases} \dfrac{1}{a(a-b)}\ln\dfrac{a+b}{b} + \dfrac{2\ln 2}{b^2-a^2}, & a \neq b,\ ab > 0, \\ \dfrac{1}{2a^2}(1-\ln 2), & a = b. \end{cases}$

10. $$\int_0^1 \ln(a+x)\frac{dx}{a+x^2} = \frac{1}{2\sqrt{a}}\ \operatorname{arccot}\sqrt{a}\ \ln[(1+a)a], \quad a>0.$$

11. $$\int_0^1 \frac{\ln(1+ax)}{1+ax^2}\,dx = \frac{1}{2\sqrt{a}}\arctan\sqrt{a}\ \ln(1+a), \quad a>0.$$

12. $$\int_0^1 \frac{\ln(ax+b)}{(1+x)^2}\,dx = \frac{1}{a-b}\left[\frac{1}{2}(a+b)\ln(a+b) - b\ln\, b - a\ln\, 2\right], \quad a>0,\, b>0,\, a\neq b.$$

13. $$\int_0^1 \ln(1+x)\frac{1+x^2}{(1+x)^4}\,dx = -\frac{1}{3}\ln\, 2 + \frac{23}{72}.$$

14. $$\int_0^1 \ln(1+x)\frac{1+x^2}{a^2+x^2}\,\frac{dx}{1+a^2x^2} = \frac{1}{2a(1+a^2)}\left[\frac{\pi}{2}\ln(1+a^2) - 2\arctan a\ \ln\, a\right], \quad a>0.$$

15. $$\int_0^1 \ln(1+x)\frac{1-x^2}{(ax+b)^2}\,\frac{dx}{(bx+a)^2} = \frac{1}{a^2-b^2}\left\{\frac{1}{a-b}\left[\frac{a+b}{ab}\ \ln(a+b) - \frac{1}{a}\ln\, b - \frac{1}{b}\ln\, a\right] + \frac{4\ln\, 2}{b^2-a^2}\right\},$$
$$a>0,\, b>0,\, a^2\neq b^2.$$

16. $$\int_0^1 \ln(1+ax)\frac{1-x^2}{(1+x^2)^2}\,dx = \frac{1}{2}\frac{(1+a)^2}{1+a^2}\ln(1+a) - \frac{1}{2}\ \frac{a}{1+a^2}\ln\, 2 - \frac{\pi}{4}\ \frac{a^2}{1+a^2},$$
$$a>-1.$$

17. $$\int_0^1 \frac{\ln(1\pm x)}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\ln\, 2 \pm 2\mathbf{G}.$$

18. $$\int_0^1 \frac{x\ln(1\pm x)}{\sqrt{1-x^2}}\,dx = -1 \pm \frac{\pi}{2}.$$

19. $$\int_0^1 \frac{x\ln(1+ax)}{\sqrt{1-x^2}}\,dx = \begin{cases} -1 + \dfrac{\pi}{2}\ \dfrac{1-\sqrt{1-a^2}}{a} + \dfrac{\sqrt{1-a^2}}{a}\ \arcsin a, & |a|\le 1, \\ -1 + \dfrac{\pi}{2a} + \dfrac{\sqrt{a^2-1}}{a}\ \ln(a+\sqrt{a^2-1}), & a\ge 1. \end{cases}$$

20. $$\int_0^1 \frac{\ln(1+ax)}{x\sqrt{1-x^2}}\,dx = \frac{1}{2}\ \arcsin a(\pi - \arcsin a) = \frac{\pi^2}{8} - \frac{1}{2}(\arccos a)^2, \quad |a|\le 1.$$

21. $$\int_0^1 x^{\mu-1}\ln(1+x)\,dx = \frac{1}{\mu}[\ln\, 2 - \beta(\mu+1)], \quad \Re\{\mu\} > -1.$$

22. $\displaystyle\int_0^1 x^{2n-1} \ln(1+x)\, dx = \frac{1}{2n} \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}.$

23. $\displaystyle\int_0^1 x^{2n} \ln(1+x)\, dx = \frac{1}{2n+1} \left[\ln 4 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right].$

24. $\displaystyle\int_0^1 x^{n-1/2} \ln(1+x)\, dx = \frac{2 \ln 2}{2n+1} + (-1)^n \frac{4}{2n+1} \left[\pi - \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \right].$

25. $\displaystyle\int_0^1 x^{\mu-1} \ln(1-x)\, dx = -\frac{1}{\mu} [\psi(\mu+1) - \psi(1)], \quad \Re\{\mu\} > -1.$

26. $\displaystyle\int_0^1 \frac{\ln(1+x)}{(1+x)^{\mu+1}}\, dx = \frac{-\ln 2}{2^\mu \mu} + \frac{2^\mu - 1}{2^\mu \mu^2}.$

27. $\displaystyle\int_0^1 \frac{x^{\mu-1} \ln(1-x)}{(1-x)^{1-\nu}}\, dx = \mathrm{B}(\mu, \nu)[\psi(\nu) - \psi(\mu+\nu)], \quad \Re\{\mu\} > 0, \Re\{\nu\} > 0.$

28. $\displaystyle\int_0^1 \ln(1+x) \frac{(p-1)x^{p-1} - p x^{-p}}{x}\, dx = 2 \ln 2 - \frac{\pi}{\sin p\pi}, \quad 0 < p < 1.$

29. $\displaystyle\int_0^1 \ln(1+x) \frac{1 + x^{2n+1}}{1+x}\, dx = 2 \ln 2 \sum_{k=0}^{n} \frac{1}{2k+1} - \sum_{j=1}^{2n+1} \frac{1}{j} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k}.$

30. $\displaystyle\int_0^1 \ln(1+x) \frac{1 - x^{2n}}{1+x}\, dx = 2 \ln 2 \sum_{k=0}^{n-1} \frac{1}{2k+1} - \sum_{j=1}^{2n} \frac{1}{j} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k}.$

31. $\displaystyle\int_0^1 \ln(1+x) \frac{1 - x^{2n}}{1-x}\, dx = 2 \ln 2 \sum_{k=0}^{n-1} \frac{1}{2k+1} + \sum_{i=1}^{2n} \frac{(-1)^j}{j} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k}.$

32. $\displaystyle\int_0^1 \ln(1+x) \frac{1 - x^{2n+1}}{1-x}\, dx = 2 \ln 2 \sum_{k=0}^{n} \frac{1}{2k+1} + \sum_{j=1}^{2n+1} \frac{(-1)^j}{j} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k}.$

33. $\displaystyle\int_0^1 \ln(1-x) \frac{1 - (-1)^n x^n}{1-x}\, dx = \sum_{j=1}^{n} \frac{(-1)^j}{j} \sum_{k=1}^{j} \frac{1}{k}.$

34. $\displaystyle\int_0^1 \ln(1-x)\frac{1-x^n}{1-x}\,dx = -\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j}\frac{1}{k}.$

35. $\displaystyle\int_0^1 [\ln(1+x)]^n(1+x)^r\,dx = (-1)^{n-1}\frac{n!}{(r+1)^{n+1}} + 2^{r+1}\sum_{k=0}^{n}\frac{(-1)^k n!(\ln 2)^{n-k}}{(n-k)!(r+1)^{k+1}}.$

36. $\displaystyle\int_0^1 [\ln(1-x)]^n(1-x)^r\,dx = (-1)^n\frac{n!}{(r+1)^{n+1}}, \quad r > -1.$

37. $\displaystyle\int_0^1 \left(\ln\frac{1}{1-x^2}\right)^n x^{2q-1}\,dx = \frac{n!}{2}\zeta(n+1,\,q+1), \quad -1 < q < 0.$

38. $\displaystyle\int_0^1 (\ln x)^{2n}\ln(1-x^2)\frac{dx}{x} = -\frac{\pi^{2n+2}}{2(n+1)(2n+1)}|B_{2n+2}|.$

39. $\displaystyle\int_0^1 [\ln(1/x)]^m\ln(1-x^2)\,dx = -\sum_{n=1}^{\infty}\frac{\Gamma(m+1)}{n(2n+1)^{m+1}}, \quad m+1>0,\ n+1>0.$

40. $\displaystyle\int_0^1 \ln(1+x^2)\frac{dx}{x^2} = \frac{\pi}{2} - \ln 2.$

41. $\displaystyle\int_0^1 \ln(1+x^2)\frac{dx}{1+x^2} = \frac{\pi}{2}\ln 2 - \mathbf{G}.$

42. $\displaystyle\int_0^1 \ln\frac{1+a^2x^2}{1+a^2}\,\frac{dx}{1-x^2} = -(\arctan a)^2.$

43. $\displaystyle\int_0^1 \ln(1-x^2)\frac{dx}{x} = -\frac{\pi^2}{12}.$

44. $\displaystyle\int_0^1 \ln(1-x^2)\frac{dx}{1+x^2} = \frac{\pi}{4}\ln 2 - \mathbf{G}.$

45. $\displaystyle\int_0^1 \ln(1+x^2)\,\frac{dx}{x(1+x^2)} = \frac{1}{2}\left[\frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2\right].$

46. $\displaystyle\int_0^1 \ln(\cos^2 t + x^2\sin^2 t)\,\frac{dx}{1-x^2} = -t^2.$

47. $$\int_0^1 \ln(a^2+b^2x^2)\,\frac{dx}{(c+gx)^2}$$
$$= \frac{2}{c(c+g)}\ln\,a + \frac{b^2}{a^2g^2+b^2c^2}\left[\frac{2a}{b}\,\operatorname{arccot}\frac{a}{b} + \frac{cb^2-ga^2}{b^2(c+g)}\ln\,\frac{a^2+b^2}{a^2} - 2\frac{c}{g}\ln\,\frac{c+g}{c}\right],$$
$$a>0,\ b>0,\ c>0,\ g>0.$$

48. $$\int_0^1 \ln(1+ax^2)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\left\{\ln\,\frac{1+\sqrt{1+a}}{2} + \frac{1}{2}\,\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right\}, \quad a>0.$$

49. $$\int_0^1 \ln(1+a-ax^2)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\left\{\ln\,\frac{1+\sqrt{1+a}}{2} - \frac{1}{2}\,\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right\}, \quad a>0.$$

50. $$\int_0^1 \ln(1-a^2x^2)\,\frac{dx}{\sqrt{1-x^2}} = \pi\ln\,\frac{1+\sqrt{1-a^2}}{2}, \quad a^2<1.$$

51. $$\int_0^1 \ln(1-a^2x^2)\,\frac{dx}{x\sqrt{1-x^2}} = -\left(\arccos|a| - \frac{\pi}{2}\right)^2.$$

52. $$\int_0^1 \ln(1-x^2)\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln\,\frac{k'}{k}\mathbf{K}(k) - \frac{\pi}{2}\,\mathbf{K}(k').$$

53. $$\int_0^1 \ln(1\pm kx^2)\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2}\ln\,\frac{2\pm 2k}{\sqrt{k}}\mathbf{K}(k) - \frac{\pi}{8}\mathbf{K}(k').$$

54. $$\int_0^1 \frac{\ln(1-k^2x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx = \ln\,k'\mathbf{K}(k).$$

55. $$\int_0^1 \ln(1-k^2x^2)\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = (2-k^2)\mathbf{K}(k) - (2-\ln\,k')\mathbf{E}(k).$$

56. $$\int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}}\ln(1-k^2x^2)\,dx = \frac{1}{k^2}(1+k'^2-k'^2\,\ln\,k')\mathbf{K}(k) - (2-\ln\,k')\mathbf{E}(k).$$

57. $$\int_0^1 \ln(1-x^2)(px^{p-1}-qx^{q-1})\,dx = \psi\left(\frac{q}{2}+1\right) - \psi\left(\frac{p}{2}+1\right), \quad p>-2,\ q>-2.$$

58. $$\int_0^1 \ln(1+ax^2)\,\frac{dx}{\sqrt{1-x^2}} = \pi\ln\,\frac{1+\sqrt{1+a}}{2}, \quad a\ge -1.$$

59. $\displaystyle\int_0^1 \ln(1+x^2)x^{\mu-1}\,dx = \frac{1}{\mu}\left[\ln 2 - \beta\left(\frac{\mu}{2}+1\right)\right], \quad \Re\{\mu\} > -2.$

60. $\displaystyle\int_0^1 \ln(1+2x\cos t + x^2)\,\frac{dx}{x} = \frac{\pi^2}{6} - \frac{t^2}{2}.$

61. $\displaystyle\int_0^1 \ln\,\frac{ax+b}{bx+a}\,\frac{dx}{(1+x)^2} = \frac{1}{a-b}\left[(a+b)\ln\,\frac{a+b}{2} - a\ln\,a - b\,\ln\,b\right], \quad a>0,\, b>0.$

62. $\displaystyle\int_0^1 \ln\,\frac{1-x}{x}\,\frac{dx}{1+x^2} = \frac{\pi}{8}\ln\,2.$

63. $\displaystyle\int_0^1 \ln\,\frac{1+x}{1-x}\,\frac{dx}{1+x^2} = \mathbf{G}.$

64. $\displaystyle\int_0^1 \ln\,\frac{1+ax}{1-ax}\,\frac{dx}{x\sqrt{1-x^2}} = \pi\,\arcsin a, \quad |a|\le 1.$

65. $\displaystyle\int_0^1 \ln\,\frac{1+x^2}{x}\,x^{2n}\,dx = \frac{1}{2n+1}\left\{(-1)^n\frac{\pi}{2} + \ln\,2 - \frac{1}{2n+1} + 2\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}\right\}.$

66. $\displaystyle\int_0^1 \ln\,\frac{1+x^2}{x}\,x^{2n-1}\,dx = \frac{1}{2n}\left\{(-1)^{n+1}\ln\,2 + \ln\,2 - \frac{1}{2n} + (-1)^{n+1}\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right\}.$

67. $\displaystyle\int_0^1 \ln\,\frac{1+x^2}{x}\,\frac{dx}{1+x^2} = \frac{\pi}{2}\ln\,2.$

68. $\displaystyle\int_0^1 \ln\,\frac{1-x^2}{x}\,\frac{dx}{1+x^2} = \frac{\pi}{4}\ln\,2.$

69. $\displaystyle\int_0^1 \ln\,\frac{1+x^2}{x+1}\,\frac{dx}{1+x^2} = \frac{3\pi}{8}\ln\,2 - \mathbf{G}.$

70. $\displaystyle\int_0^1 \ln\,\frac{1+x^2}{1-x}\,\frac{dx}{1+x^2} = \frac{3\pi}{8}\ln\,2.$

71. $\displaystyle\int_0^1 \ln\left(\frac{1-x^2}{x^2}\right)^2\sqrt{1-x^2}\,dx = \pi.$

72. $$\int_0^1 \ln\,\frac{1+2x\cos t+x^2}{(1+x)^2}\,\frac{dx}{x}=\frac{1}{2}\int_0^\infty\,\ln\,\frac{1+2x\cos t+x^2}{(1+x)^2}\,\frac{dx}{x}=-\frac{t^2}{2},\quad |t|<\pi.$$

73. $$\int_0^1 \ln\,\frac{1+x^2\sin t}{1-x^2\sin t}\,\frac{dx}{\sqrt{1-x^2}}=\pi\ln\,\cot\left(\frac{\pi-t}{4}\right),\quad |t|<\pi.$$

74. $$\int_0^1 \ln\,\frac{(1-ax)(1+ax^2)}{(1-ax^2)^2}\,\frac{dx}{1+ax^2}=\frac{1}{2\sqrt{a}}\arctan\sqrt{a}\ln(1+a),\quad a>0.$$

75. $$\int_0^1 \ln\,\frac{(1-a^2x^2)(1+ax^2)}{(1-ax^2)^2}\,\frac{dx}{1+ax^2}=\frac{1}{\sqrt{a}}\arctan\sqrt{a}\ln(1+a),\quad a>0.$$

76. $$\int_0^1 \ln\,\frac{(x+1)(x+a^2)}{(x+a)^2}\,x^{\mu-1}\,dx=\frac{\pi(a^\mu-1)^2}{\mu\sin\mu\pi},\quad a>0,\,\Re\{\mu\}>0.$$

77. $$\int_0^1 \ln(1+ax)\frac{x^{p-1}-x^{q-1}}{\ln\,x}\,dx=\sum_{k=1}^\infty(-1)^{k+1}\,\frac{a^k}{k}\,\ln\,\frac{p+k}{q+k},\quad a<1,\,p>0,\,q>0.$$

78. $$\int_0^1 \frac{x\ln\,x+1-x}{x(\ln\,x)^2}\,\ln(1+x)\,dx=\ln\,\frac{4}{\pi}.$$

79. $$\int_0^1 \frac{\ln(1-x^2)dx}{x(q^2+(\ln\,x)^2)}=-\frac{\pi}{q}\ln\Gamma\left(\frac{q+\pi}{\pi}\right)+\frac{\pi}{2q}\ln\,2q+\ln\,\frac{q}{\pi}-1,\quad q>0.$$

80. $$\int_0^1 \ln(1+x)(\ln\,x)^{n-1}\,\frac{dx}{x}=(-1)^{n-1}(n-1)!\left(1-\frac{1}{2^n}\right)\,\zeta(n+1).$$

81. $$\int_0^1 \ln(1+x)(\ln\,x)^{2n}\,\frac{dx}{x}=\frac{2^{2n+1}-1}{(2n+1)(2n+2)}\pi^{2n+2}|B_{2n+2}|.$$

82. $$\int_0^1 \ln(1-x)(\ln\,x)^{n-1}\,\frac{dx}{x}=(-1)^n(n-1)!\,\zeta(n+1).$$

83. $$\int_0^1 \ln(1-x)(\ln\,x)^{2n}\,\frac{dx}{x}=-\frac{2^{2n}}{(n+1)(2n+1)}\pi^{2n+2}|B_{2n+2}|.$$

84. $$\int_0^1 \ln(1-ax^s)\left(\ln\,\frac{1}{x}\right)^p\frac{dx}{x}=-\frac{1}{s^{p+1}}\,\Gamma(p+1)\sum_{k=1}^\infty\frac{a^k}{k^{p+2}},\quad p>-1,\,a<1,\,s>0.$$

85. $$\int_0^1 \ln\left(1-2ax\cos t+a^2x^2\right)\left(\ln\frac{1}{x}\right)^p\frac{dx}{x}=-2\Gamma(p+1)\sum_{k=1}^{\infty}\frac{a^k\cos kt}{k^{p+2}}.$$

86. $$\int_0^1 \ln\frac{\sqrt{1-a^2x^2}-x\sqrt{1-a^2}}{1-x}\,\frac{dx}{x}=\frac{1}{2}(\arcsin a)^2.$$

87. $$\int_0^1 \ln\frac{1+\cos t\sqrt{1-x^2}}{1-\cos t\sqrt{1-x^2}}\,\frac{dx}{x^2+\tan^2 v}=\pi\cot t\,\frac{\cos{(v-t)/2}}{\sin{(v+t)/2}}.$$

88. $$\int_0^1 \ln\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\frac{x\,dx}{1-x^2}=\frac{\pi^2}{2}.$$

89. $$\int_0^1 \ln\{\sqrt{1+kx}+\sqrt{1-kx}\}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{1}{4}\ln{(4k)}\mathbf{K}(k)+\frac{\pi}{8}\mathbf{K}(k').$$

90. $$\int_0^1 \ln\{\sqrt{1+kx}-\sqrt{1-kx}\}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{1}{4}\,\ln{(4k)}\mathbf{K}(k)+\frac{3}{8}\pi\mathbf{K}(k').$$

91. $$\int_0^1 \ln\{1+\sqrt{1-k^2x^2}\}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{1}{2}\ln\,k\mathbf{K}(k)+\frac{\pi}{4}\mathbf{K}(k').$$

92. $$\int_0^1 \ln\{1-\sqrt{1-k^2x^2}\}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{1}{2}\ln\,k\mathbf{K}(k)-\frac{3}{4}\pi\mathbf{K}(k').$$

93. $$\int_0^1 \ln\frac{1+p\sqrt{1-x^2}}{1-p\sqrt{1-x^2}}\,\frac{dx}{1-x}=\pi\,\arcsin p,\quad p^2<1.$$

94. $$\int_0^1 \ln\frac{1+q\sqrt{1-k^2x^2}}{1-q\sqrt{1-k^2x^2}}\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\pi F(\arcsin q,\,k'),\quad q^2<1.$$

95. $$\int_0^1 \frac{\ln{(1-x^q)}}{1+(\ln\,x)^2}\,\frac{dx}{x}=\pi\left[\ln\,\Gamma\left(\frac{q}{2\pi}+1\right)-\frac{\ln\,q}{2}+\frac{q}{2\pi}\left(\ln\,\frac{q}{2\pi}-1\right)\right],\quad q>0.$$

96. $$\int_0^1 \ln\,\ln\,\left(\frac{1}{x}\right)\frac{dx}{1+x}=-\gamma_e\,\ln\,2+\sum_{k=2}^{\infty}(-1)^k\frac{\ln\,k}{k}=-\frac{1}{2}(\ln\,2)^2\approx-\gamma_e\,\ln\,2+0.159868905.$$

97. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{x+e^{i\lambda}} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}e^{-ik\lambda}(\gamma_e + \ln k).$$

98. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{(1+x)^2} = \int_1^{\infty}\ln\ln x\frac{dx}{(1+x)^2} = \frac{1}{2}\left[\psi\left(\frac{1}{2}\right)+\ln 2\pi\right] = \frac{1}{2}\left(\ln\frac{\pi}{2}-\gamma_e\right).$$

99. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x^2} = \int_1^{\infty}\ln\ln x\frac{dx}{1+x^2} = \frac{\pi}{2}\ln\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

100. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x+x^2} = \int_1^{\infty}\ln\ln x\frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}}\ln\frac{(2\pi)^{1/3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}.$$

101. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \int_1^{\infty}\ln\ln x\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right].$$

102. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+2x\cos t+x^2} = \int_1^{\infty}\ln\ln x\frac{dx}{1+2x\cos t+x^2} = \frac{\pi}{2\sin t}\ln\frac{(2\pi)^{t/\pi}\Gamma\left(\frac{1}{2}+\frac{t}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{t}{2\pi}\right)}.$$

103. $$\int_0^1 \ln\ln\frac{1}{x}x^{\mu-1}\,dx = -\frac{1}{\mu}\left(\ln\mu+\gamma_e\right), \quad \Re\{\mu\}>0.$$

104. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{(1+x^2)\sqrt{\ln\frac{1}{x}}} = \int_1^{\infty}\ln\ln x\frac{dx}{(1+x^2)\sqrt{\ln x}}$$
$$= \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\sqrt{2k+1}}\left[\ln(2k+1)+2\ln 2+\gamma_e\right].$$

105. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{x^{\mu-1}\,dx}{\sqrt{\ln\frac{1}{x}}} = -\left(\ln 4\mu+\gamma_e\right)\sqrt{\frac{\pi}{\mu}}, \quad \Re\{\mu\}>0.$$

106. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{\mu-1}x^{\nu-1}\,dx = \frac{1}{\nu^{\mu}}\Gamma(\mu)[\psi(\mu)-\ln(\nu)], \quad \Re\{\mu\}>0,\ \Re\nu\}>0.$$

107. $$\int_0^1 \ln(a-\ln x)x^{\mu-1}\,dx = \frac{1}{\mu}\left[\ln a - e^{a\mu}\mathrm{Ei}(-a\mu)\right], \quad \Re\{\mu\}>0,\ a>0.$$

108. $$\int_0^{1/e} \ln\left(2\ln\frac{1}{x}-1\right)\frac{x^{2\mu-1}}{\ln x}\,dx = -\frac{1}{2}\left[\mathrm{Ei}(-\mu)\right]^2, \quad \Re\{\mu\}>0.$$

109. $$\int_0^1 \ln\left[a^2+(\ln\, x)^2\right]\frac{dx}{1+x^2}=\pi\ln\,\frac{2\Gamma\left(\frac{2a+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2a+\pi}{4\pi}\right)}+\frac{\pi}{2}\ln\,\frac{\pi}{2},\quad a>-\frac{\pi}{2}.$$

110. $$\int_0^1 \ln\left[a^2+4(\ln\, x)^2\right]\frac{dx}{1+x^2}=\pi\ln\,\frac{2\Gamma\left(\frac{a+3\pi}{4\pi}\right)}{\Gamma\left(\frac{a+\pi}{4\pi}\right)}+\frac{\pi}{2}\ln\,\pi,\quad a>-\pi.$$