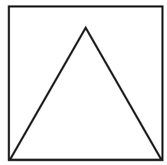
HANDOUT

How to Fold an Equilateral Triangle

The goal of this activity is to fold an equilateral triangle from a square piece of paper.



Question 1: First fold your square to produce a 30°-60°-90° triangle inside it. Hint: You want your folds to make the hypothenuse twice as long as one of the sides. Keep trying! Explain why your method works in the space below.

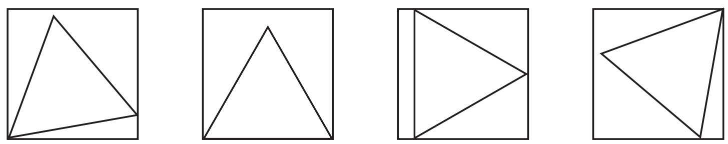
Question 2: Now use what you did in Question 1 to fold an equilateral triangle inside a square.

Follow-up: If the side length of your original square is 1, what is the length of a side of your equilateral triangle? Would it be possible to make the triangle's side length bigger?

HANDOUT

What’s the Biggest Equilateral Triangle in a Square?

If we are going to turn a square piece of paper into an equilateral triangle, we’d like to make the **biggest possible** triangle. In this activity your task is to make a mathematical model to find the equilateral triangle with the **maximum area** that we can fit inside a square. Follow the steps below to help set up the model.



Question 1: If such a triangle is maximal, then can we assume that one of its corners will coincide with a corner of the square? Why?

Question 2: Assuming Question 1, draw a picture of what your triangle-in-the-square might look like, where the “common corner” of the triangle and square is in the lower left. Now you’ll need to create your model by introducing some variables. What might they be? (Hint: One will be the angle between the bottom of the square and the bottom of the triangle. Call this one θ.)

Question 3: One of your variables will be your *parameter* that you’ll change until you get the maximum area of the triangle. Pick one variable (and try to pick wisely—a bad choice may make the problem harder) and then come up with a formula for the area of the triangle in terms of your variable.

Question 4: With your formula in hand, use techniques you know to find the value of your variable that gives you the maximum area for the equilateral triangle. Be sure to pay attention to the proper range of your parameter.

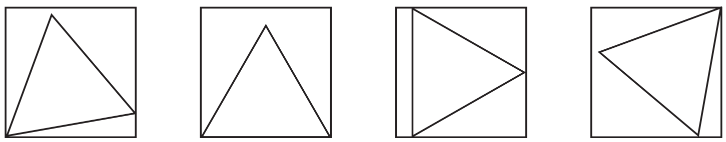
Question 5: So, what is your answer? What triangle gives the biggest area? Find a folding method that produces this triangle.

Follow-up: Your answer to Question 5 can also give a way to fold the largest *regular hexagon* inside a square piece of paper. Can you see how this would work?

What's the Biggest Equilateral Triangle in a Square?

In this activity your task is to find the biggest equilateral triangle that can fit inside a square of side length 1. (Note: An equilateral triangle is the triangle with all sides of equal length and all three angles measuring 60°.) The step-by-step procedure will help you find a mathematical model for this problem, and then to solve the optimization problem of finding the triangle's position and maximum area.

Here are some random examples:



Question 1: If such a triangle is maximal, then can we assume that one of its corners will coincide with a corner of the square? (Hint: The answer is yes. Explain why.)

Question 2: Assuming Step 1 above, draw a picture of what your triangle-in-the-square might look like, where the common corner of the two figures is in the lower left. (Hint: See one of the four examples above.) Now you'll need to create your model by labeling your picture with some variables. (Hint: Let θ be the angle between the bottom of the square and the bottom of the triangle. Let x be the side length of the triangle.)

Question 3: Come up with the formula for the area of the triangle in terms of one variable, x. Then, find an equation that relates your two variables, x and θ. Combine the two to get the formula for the area of the triangle in terms of only one variable, θ. (Hint: Your last formula will be $A = \frac{\sqrt{3}}{4}\sec^2\theta$.)

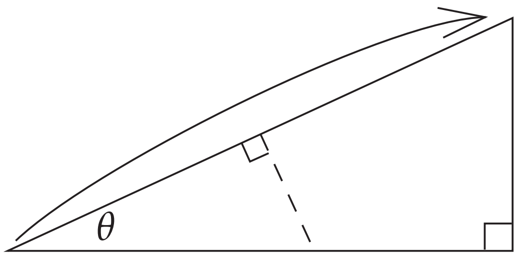
Question 4: What is the range of your variable θ? Explain. (Hint: The range should be $0^\circ \leq \theta \leq 15^\circ$.)

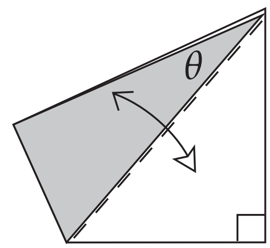
Question 5: Most important part: With your formula and the range for θ in hand, use techniques of optimization to find the value of θ that gives you the maximum area for the equilateral triangle. Also, find the value of this maximum area. (Hint: For simplicity, you may want to express all trigonometric functions in terms of sin and cos).

HANDOUT

Proving the Double-Angle Formulas

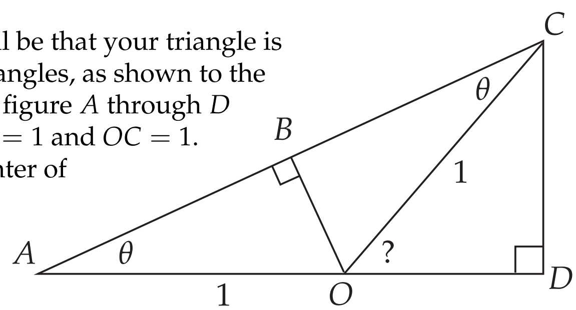
Make a piece of paper shaped like a right triangle with smallest angle θ.





Fold the corner of the smallest angle to the other corner, as shown above. Then fold along the edge of the flap that you just made. Unfold everything.

The result of your folding will be that your triangle is divided into three smaller triangles, as shown to the right. Label the points of this figure A through D and O, as shown, and let $AO = 1$ and $OC = 1$. (You can think of O as the center of a circle of radius 1.)



What is $\angle COD$ in terms of θ? $\angle COD =$ ________

Write the following lengths in terms of trigonometric functions of the angle θ:

$AB =$ ______________ $BC =$ ______________

$CD =$ ______________ $OD =$ ______________

Question 1: Looking at the big triangle ACD, what is $\sin\theta$ equal to? Use this to generate the double-angle formula for $\sin 2\theta$.

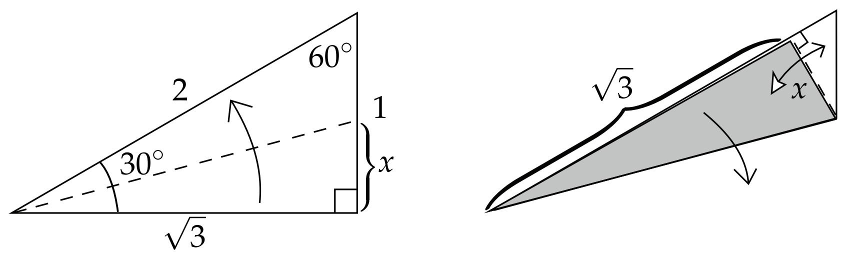
Question 2: Looking at triangle ACD again, what is $\cos\theta$? Use this to find the double-angle formula for $\cos 2\theta$.

HANDOUT

Trigonometry on Other Triangles

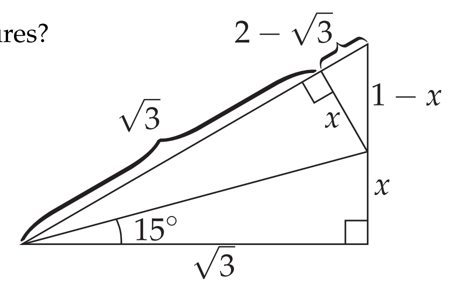
In high school you learn the side lengths of 45°-45°-90° triangles and 30°-60°-90° triangles, and this allows you to know precisely what sine, cosine, and tangent are for these angles. For example, you know that $\sin 60° = \sqrt{3}/2$ because of the 1, 2, and $\sqrt{3}$ sides of a 30°-60°-90° triangle.

But what about other triangles? We can find exact side lengths for other triangles too if we fold up triangles that we already know!

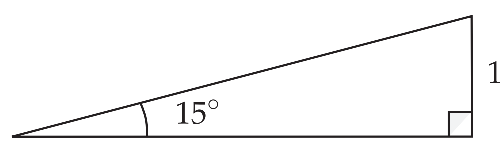


Exercise 1: Take a 30°-60°-90° triangle and fold the 30° leg up to the hypothenuse making a 15° angle. Then fold the rest of the triangle over this flap, as shown above, and unfold.

What is the length labeled x in these figures?
(Hint: Do you see any similar triangles?)



Use your answer from above to find the best exact lengths for a 15°-75°-90° triangle, where we scale the lengths to make the short side length 1. (Try to make your lengths *as simple as possible*.)



Fill in the blanks: $\sin 15° =$ __________, $\cos 15° =$ __________, $\tan 15° =$ __________.

Exercise 2: Do the same thing with a 45°-45°-90° triangle to find exact lengths of a 22.5°-67.5°-90° triangle.

HANDOUT

How Do You Divide a Strip into Nths?

Oftentimes in origami we are asked to fold the side of a square piece of paper into an equal number of pieces. If the instructions say to fold it in half or into fourths, then it's easy to do. But if they ask for equal fifths, it's a lot harder. Here you'll learn a popular origami way of doing this, called **Fujimoto's approximation method.**

(1) Make a **guess pinch** where you think a $1/5$ mark might be, say on the left side of the paper.

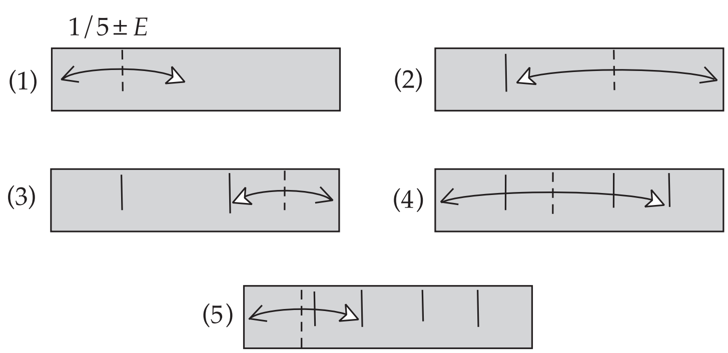
(2) To the right of this guess pinch is $\approx 4/5$ of the paper. Pinch this side **in half**.

(3) That last pinch is near the $3/5$ mark. To the right of this is $\approx 2/5$ of the paper. Pinch this right side **in half**.

(4) Now we have a $1/5$ mark on the right. To the left of this is $\approx 4/5$. Pinch this side **in half**.

(5) This gives a pinch nearby the $2/5$ mark. Pinch the left side of this **in half**.

(6) This last pinch will be **very close** to the actual $1/5$ mark!



Once you do this you can repeat the above steps starting with the last pinch made, except this time **make all your creases sharp and go all the way through the paper**. You should end up with very accurate 1/5ths divisions of your paper.

Question: Why does this work?

Tip: If the strip is one unit length, then your first "guess pinch" can be thought of as being at $1/5 \pm E$ on the x-axis, where E represents the error you have. In the above picture, write in the x-position of the other pinch marks you made. What would their coordinates be?

Explain: Seeing what you did in the tip, write, in a complete sentence or two, an explanation of why Fujimoto's approximation method works.

HANDOUT

Details of Fujimoto's Approximation Method

(1) Binary decimals?

Recall how our base 10 decimals work: We say that 1/8 = 0.125 because

$$\frac{1}{8} = \frac{1}{10} + \frac{2}{10^2} + \frac{5}{10^3}.$$

If we were to write $1/8$ as a **base 2 decimal**, we would use powers of 2 in the denominators instead of powers of 10. So we'd get $\frac{1}{8} = \frac{0}{2} + \frac{0}{2^2} + \frac{1}{2^3}$. We write this as $1/8 = (0.001)_2$.

Question 1: What is $1/5$ written as a base 2 decimal?

Question 2: When we did Fujimoto's approximation method to make 1/5ths, what was the sequence of left and right folds that we made? What's the connection between this and Question 1?

Question 3: Take a new strip of paper and use Fujimoto to divide it into equal 1/7ths. How is this different from the way 1/5ths worked? Find the base 2 decimal for $1/7$ and check your observations made in Question 2.

(2) A discrete dynamics approach... (courtesy of Jim Tanton)

We've been assuming that our strip of paper lies on the x-axis with the left end being at 0 and the right end at 1. Let's define two functions on this interval $[0, 1]$:

$$T_0(x) = \frac{x}{2} \text{ and } T_1(x) = \frac{x+1}{2}.$$

Question 4: What do these two functions mean in terms of Fujimoto's method?

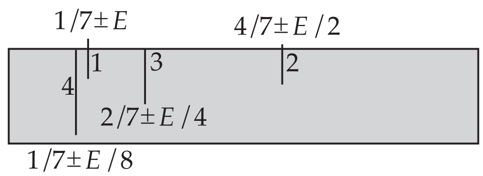
Question 5: Let $x \in [0, 1]$ be our initial guess in Fujimoto's method for approximating $1/5$ths. (So x will be something like $1/5 \pm E$.) Write x as a binary decimal, $x = (0.i_1 i_2 i_3 \ldots)_2$.

What will $T_0(x)$ be? How about $T_1(x)$? Proofs?

Question 6: As we perform Fujimoto's method on our initial guess x, we can think of it as performing T_0 and T_1 over and over again to x. When approximating $1/5$ths, what happens to the binary decimal of x as we do this? Use this to prove the observation that you made in Question 2.

(3) A number theory question... (courtesy of Tamara Veenstra)

In Question 3 you were asked to use Fujimoto to approximate 1/7ths, and you should have noticed that in doing so you do not make pinch marks at every multiple of 1/7, unlike when approximating 1/5ths. Indeed, only pinch marks at 1/7, 4/7, and 2/7 are made.



We can keep track of what's going on in a table, like the one to the right. The first line shows how many 1/7ths are on the left of the first pinch and how many are on the right. The second line does the same for the second pinch, and so on. As you can see, the right side starts at 6 and comes back to 6 after only 3 lines. So it doesn't make all 1/7ths pinch marks.

7ths left	7ths right
1	6
4	3
2	5
1	6

Assignment: Make similar tables for 1/5ths, 1/9ths, 1/11ths, and 1/19ths:

5ths left	5ths right
1	4

9ths left	9ths right
1	8

11ths left	11ths right
1	10

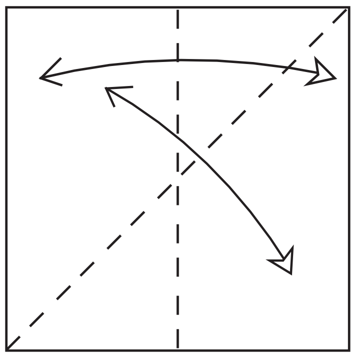
19ths left	19ths right
1	18

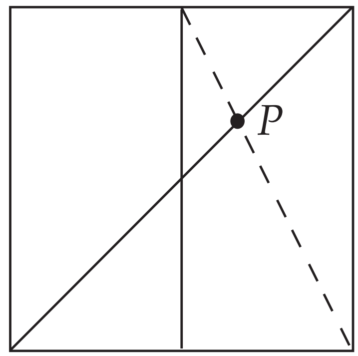
Question 7: Think about what these tables are telling you in the number system $\mathbb{Z}_n$ (the integers mod n) under multiplication, where n is the number of divisions. Then answer the question: How can we tell whether or not Fujimoto will give us pinch marks at every multiple of $1/n$ when approximating $1/n$ths?

HANDOUT

What's This Fold Doing?

Below are some origami instructions. Take a square and make creases by folding it in half vertically and folding one diagonal, as shown. Then make a crease that connects the midpoint of the top edge and the bottom right-hand corner.





Question 1: Find the coordinates of the point P, where the diagonal creases meet. (Assume that the lower left corner is the origin and that the square has side length 1.)

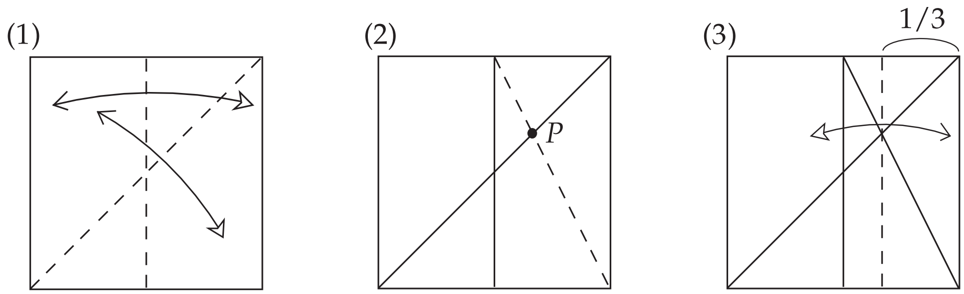
Question 2: Why is this interesting? What could this be used for?

Question 3: How could you generalize this method, say, to make perfect 5ths or $n$ths (for n odd)?

HANDOUT

Folding Perfect Thirds

It is easy to fold the side of a square into halves, or fourths, or eighths, etc. But folding odd divisions, like thirds, exactly is more difficult. The below procedure is one was to fold thirds.



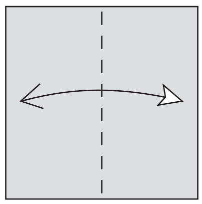
Question 1: Prove that this method actually works.

Question 2: How could you generalize this method, say, to make perfect 5ths or $n$ths (for n odd)?

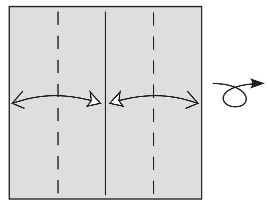
HANDOUT

Folding a Helix

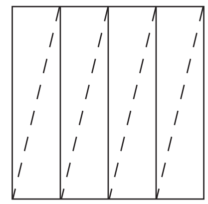
This model pleats the paper so that it twists. When made from a long strip the result is a helix.



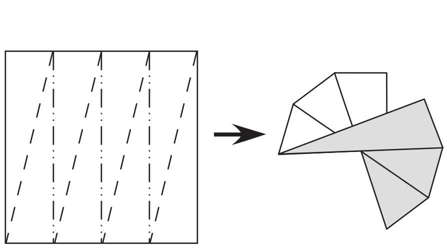
(1) Fold and unfold in half, from side to side.



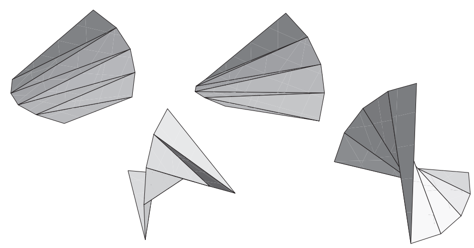
(2) Fold the sides to the center and unfold. Turn over.



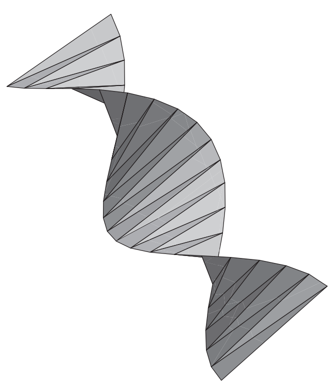
(3) Now carefully fold diagonal creases in each rectangle.



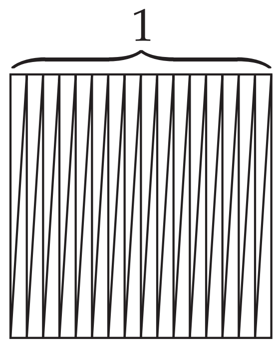
(4) Now fold all the creases at the same time. The result will be a square that has been twisted.



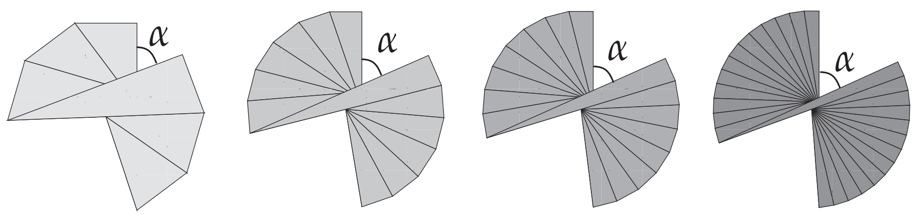
If you let the model be 3D, it makes an interesting shape!



Folding this model from a strip of paper makes a twisted helix shape, as shown on the left. (You need to make a lot more divisions along the strip for this to work.)



Question: If we made more divisions in steps (1)–(2) in the above instructions, we would get more of a twist from our square. Below is a row of examples made with only 3 divisions as in steps (1)–(2), with 6 divisions, with 8 divisions, and with 13 divisions. In each the angle α is slowly getting smaller!



So the question is, what happens to this angle α as we make more and more divisions?

Or, putting it another way, how much does the square twist as we make more and more divisions? Will it keep twisting more and more, or does it approach a limit?

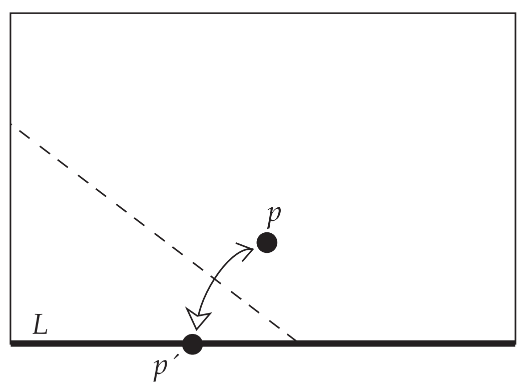
Exploring a Basic Origami Move

Origami books display many different folding moves that can be made with paper. One common move, especially in geometric folding, is the following:

> Given two points p_1 and p_2 and a line L, fold p_1 onto L so that the resulting crease line passes through p_2.

Let's explore this basic origami operation by seeing exactly what is happening when we fold a point to a line.

Activity: Take a sheet of regular writing paper, and let one side of it be the line L. Choose a point p somewhere on the paper, perhaps like below. Your task is to fold p onto L over and over again.



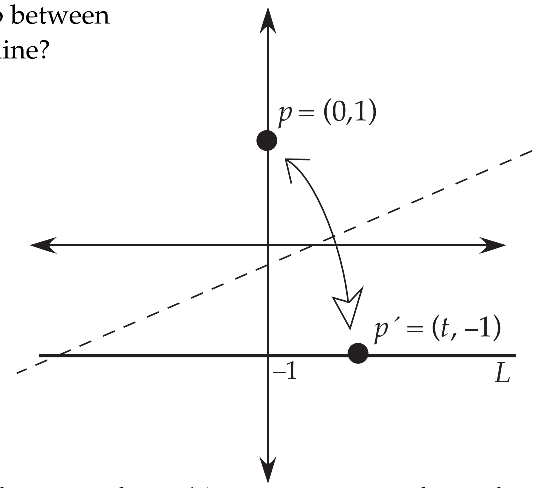
It is easier, actually, to fold L to p, by bending the paper until L touches p and then flattening the crease. Do this many times—as many as you can stand!—choosing different points p' where p lands on L.

Question 1: Describe, as clearly as you can, exactly what you see happening. What are the crease lines forming? How does your choice of the point p and the line L fit into this? Prove it.

Now we'll try to find the equation for the curve you discovered.

First, let's define where things lie on the xy-plane. Let the point $p = (0, 1)$ and let L be the line $y = -1$. Now suppose that we fold p to a point $p' = (t, -1)$ on the line L, where t can be any number.

Question 2: What is the relationship between the line segment $\overline{pp'}$ and the crease line? What is the slope of the crease line?



Question 3: Find an equation for the crease line. (Write it in terms of x and y, although it will have the t variable in it as well.)

Question 4: Your answer to Question 3 should give you a **parameterized family** of lines. That is, for each value of t that you plug in, you'll get a different crease line. For a fixed value of t, find the point on the crease line that is **tangent** to your curve from Question 1.

Question 5: Now find the equation for the curve from Question 1.

HANDOUT

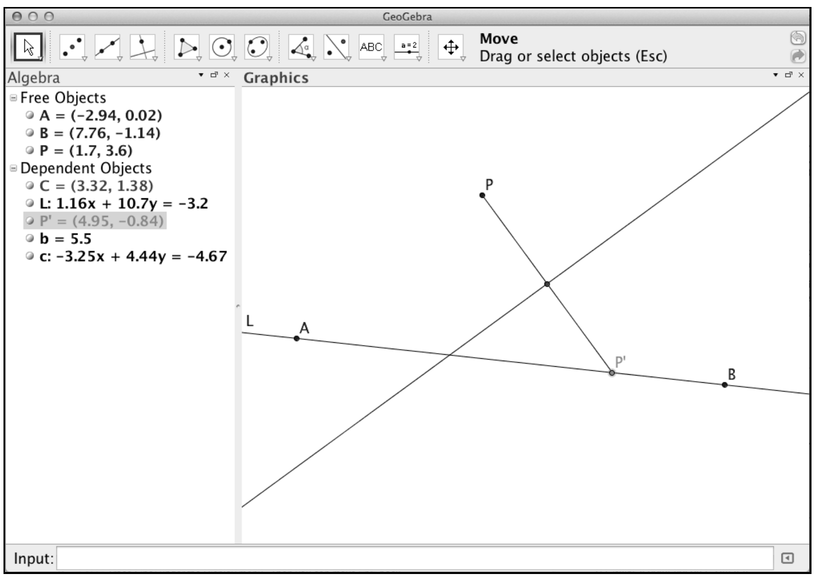
Origami with Geometry Software

In this activity we'll use geometry software, like Geogebra or Geometer's Sketchpad, to explore a basic origami move:

> Given two points p_1 and p_2 and a line L, fold p_1 onto L so that the resulting crease line passes through p_2.

We'll explore this basic origami operation by modeling on the software what happens when we fold a point to a line. We'll make use of a key observation:

When we fold a point p to a point p', the crease line we make will be the ______________________________ of the line segment ______________.



Instructions: Open a new worksheet in your software (above is shown Geogebra).

(1) Draw a line AB and label it L.

(2) Make a point not on L, call it p.

(3) Make a point on L, call it p'.

Then, with the key observation above, use the software's tools to draw the crease line made when folding p to p'.

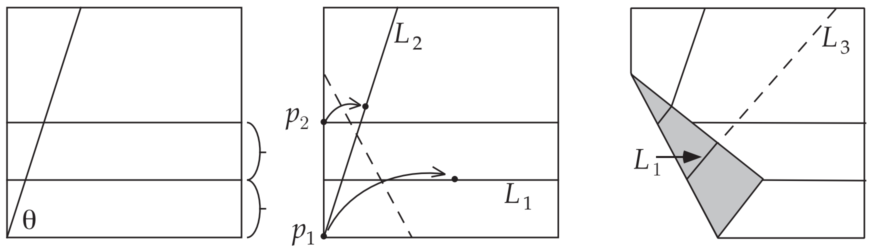
Once you've done this, select the crease line and turn on **Trace** of the line (in Geogebra, CTRL-click or right-click on the line to do this). Then you can move p' back and forth across L and make many different crease lines. In this way you can make software do the "folding" for you! (Plus, it looks cool.)

Follow-up: What happens if we use a circle instead of the line L?

HANDOUT

What's This Doing?

Take a square piece of paper and fold a line from the lower-left corner going up at some angle, θ. Then fold the paper in half from top to bottom and unfold. Then fold the bottom 1/4 crease line. That should give you something like the left figure below.



Then do the operation in the middle figure: Make a fold that places point p_1 onto line L_1 **and at the same time** places point p_2 onto line L_2. You will have to curl the paper over, line up the points, and then flatten.

Lastly, with the flap folded, extend the L_1 crease line shown in the right-most figure. Call this crease line L_3.

Question 1: Unfold everything. Prove that we if we extend L_3 then it will hit the lower-left corner, p_1.

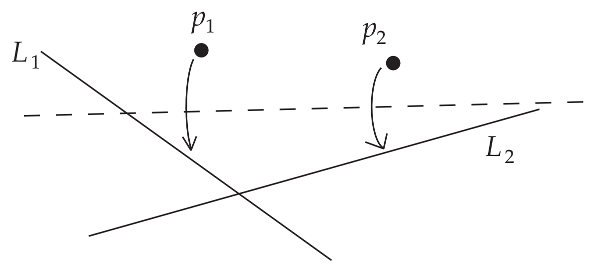
Question 2: What is crease line L_3 in relation to the other lines in the paper? Can you prove it, or is this just a coincidence?

HANDOUT

A More Complicated Fold

The origami angle trisection method is able to do what it does by using a rather complex origami move:

> Given two points p_1 and p_2 and two lines L_1 and L_2, we can make a crease that simultaneously places p_1 onto L_1 and p_2 onto L_2.

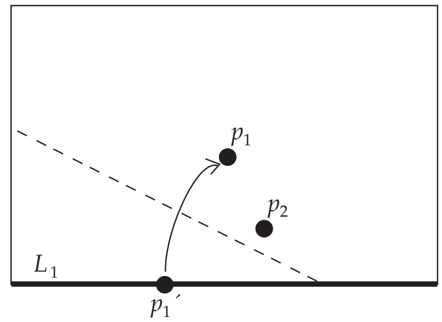


Question 1: Will this operation always be possible to do, no matter what the choice of the points and lines are?

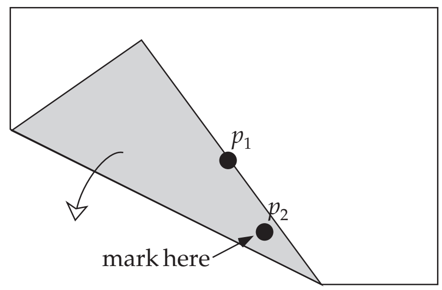
Question 2: Remember that when we fold a point p to a line L over and over again, we can interpret the creases as being tangent to a parabola with focus p and directrix L. What does this tell us about this more complex folding operation? How can we interpret it geometrically? Draw a picture of this.

Activity: Let's explore what this operation is doing in a different way. Take a sheet of paper and mark a point p_1 (somewhere near the center is usually best) and let the bottom edge be the line L_1.

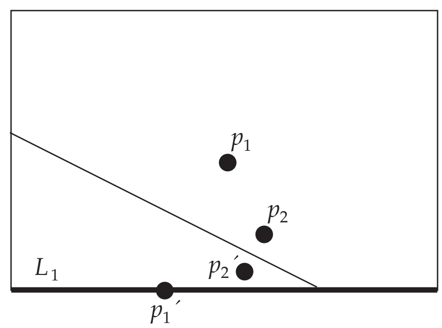
Pick a second point p_2 to be anywhere else on the paper. Our objective is to see where p_2 goes as we fold p_1 onto L_1 over and over again.



So pick a spot on L_1 (call it p_1') and fold it up to p_1. Using a marker or pen, draw a point where the folded part of the paper touches p_2. (If no other parts of the paper touch p_2, try a different choice of p_1'.) Then unfold. You should see a dot (which we could call p_2') that represents where p_2 went as we make the fold.



Now choose a different p_1' and do this over and over again. Make enough p_2' points so that you can connect the dots and see what kind of curve you get.



Question 3: What does this curve look like? Look at other people's work in the class. Do their curves look like yours? Do you know what kind of equation would generate such a curve?

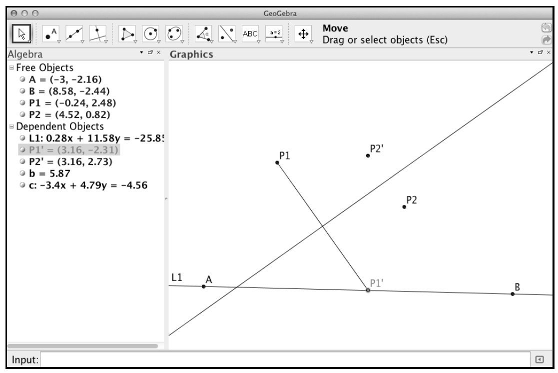
HANDOUT

Simulating This Curve with Software

We're still considering this unusual origami maneuver:

> Given two points p_1 and p_2 and two lines L_1 and L_2, we can make a crease that simultaneously places p_1 onto L_1 and p_2 onto L_2.

So that you don't have to keep folding paper over and over again, let's model our folding activity using geometry software, like Geogebra. This will allow us to look at many examples of the curve this operation generates and do so very quickly.



Here's how to set it up:

(1) Make the line L_1 and the point p_1.

(2) Make a point p_1' on L_1 and construct a line segment from p_1 to p_1'.

(3) Construct the perpendicular bisector of $\overline{p_1 p_1'}$. This makes the crease line.

(4) Now make a new point, p_2.

(5) Reflect the point p_2 about the crease line made in step (3). In Geogebra, this is done using the **Reflect Object about Line** tool. The new point should be labeled p_2'.

Then when you move p_1' back and forth along L_1, the software will trace out how p_2' changes. You can either draw this curve by turning on the **Trace** of p_2' (CTRL-click or right-click on p_2' to turn this on in Geogebra) or use a **Locus** tool to plot the locus of p_2' as p_1 changes.

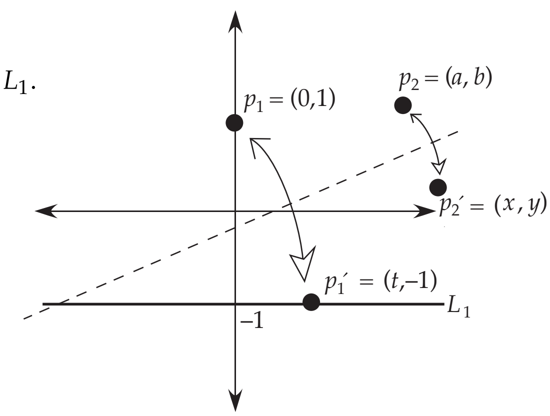
Activity: Move p_2 to different places on the screen and see how the curve changes. How many different basic shapes can this curve take on? Describe them in words.

HANDOUT

What Kind of Curve Is It?

To see what type of curve this operation is giving us, make a model of the fold.

Let $p_1 = (0, 1)$.
Let L_1 be the line $y = -1$.
We'll fold p_1 to $p_1' = (t, -1)$ on L_1.
Let $p_2 = (a, b)$ be fixed.
Then, we want to find the coordinates of $p_2' = (x, y)$, the image of p_2 under the folding. This will give us an equation in terms of x and y that should describe the curve that we got in our folding activity.



Instructions: Find the equation of the crease line that we get when folding p_1 onto p_1'. Use this and the geometry of the fold to get equations involving x and y. Combine these to get a single equation in terms of x and y (with the constants a and b in it as well, but no t variables). What kind of equation is this?

HANDOUT

Lill's Method for Solving Cubics

In this activity you'll learn Lill's Method for using geometry to solve cubic equations. Lill's Method is cool because we can do it via origami!

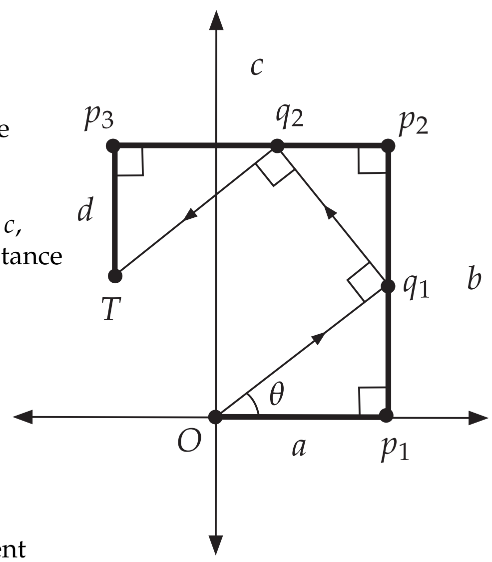
Imagine we want to solve (find a real root) of the following cubic:

$$ax^3 + bx^2 + cx + d = 0.$$

Setup: Start at the origin, point O, and draw a line segment of length a along the positive x-axis. Then rotate 90° counterclockwise and go up a length of b. Repeat: Turn 90° counterclockwise and go a length of c, then turn once more and travel a distance d, ending at a point T.

Note: If any of the coefficients are negative, then go backwards. If any are zero, then rotate but do not travel.

Then imagine that we stand at the point O and try to "shoot" T with a bullet that bounces off the coefficient path at right angles, as shown.



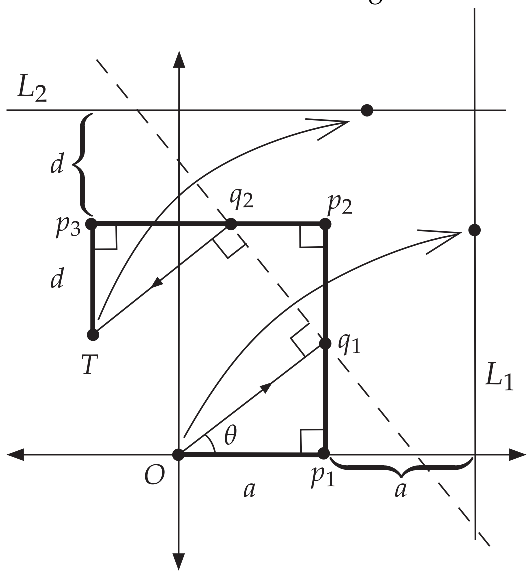
Lill's Method states that if we can successfully hit the point T with such a bullet path, and θ is the angle the path makes at O, then $x = -\tan\theta$ will be a root of our cubic equation!

Your task: Prove that Lill's Method for solving a cubic equation is correct. (Hint: What do you notice about the triangles in the figure? And what is $\tan\theta$ equal to?)

HANDOUT

Lill's Method via Origami

We can use origami to solve any cubic equation by using Lill's Method. This idea was discovered by the Italian mathematician Margherita Beloch in 1933.



Here is how:

(1) Draw (or fold) the coefficient path from Lill's Method on your paper.

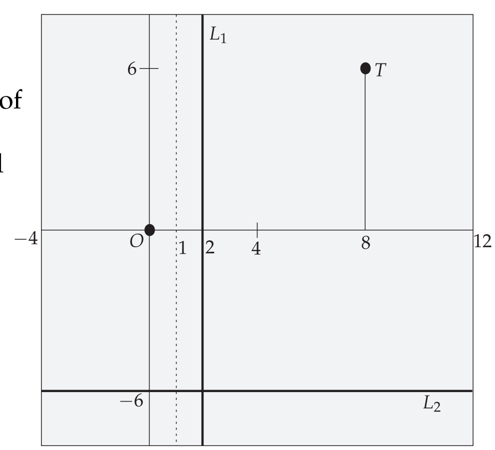
(2) Then fold a line perpendicular to the x-axis at a distance of a from $\overline{p_1 p_2}$ (on the side opposite of O). Call this line L_1.

(3) Then fold a line perpendicular to the y-axis at a distance d from $\overline{p_2 p_3}$ (on the side opposite of T). Call this line L_2.

(4) Then fold O onto L_1 while at the same time folding T onto L_2. This crease line will form one side of the bullet path needed for Lill's Method (and contain the angle θ that we need).

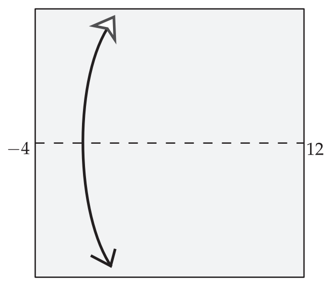
Activity: Study the above instructions, and then try it yourself to find the roots of the polynomial $x^3 - 7x - 6$ only using paper folding.

Doing this will require you to think of the paper as the xy-plane, decide where the origin should be, and fold the coefficient path. Try setting this up as shown to the right, and then fold the point O onto the line L_1 while also placing T onto L_2.

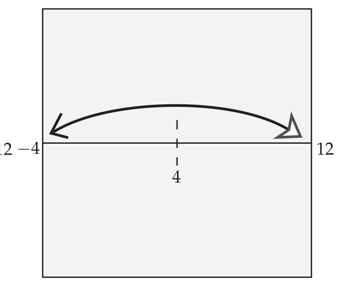


Lill’s Method Example: Step-by-Step Folds

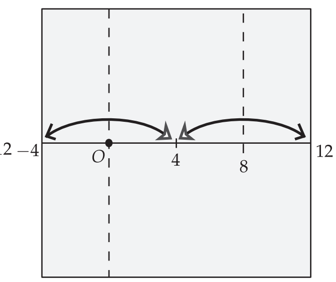
This handout provides step-by-step folds for making the crease lines needed to solve $x^3 - 7x - 6 = 0$ using Lill’s Method. Begin with a large square piece of paper, and imagine that it goes from $-4 \le x \le 12$ and $-8 \le y \le 8$ in the xy-plane.



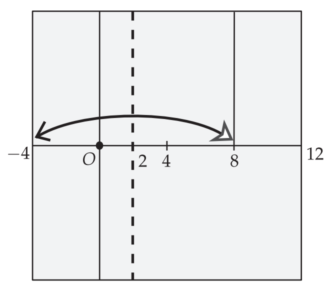
(1) Crease in half to make the x-axis.



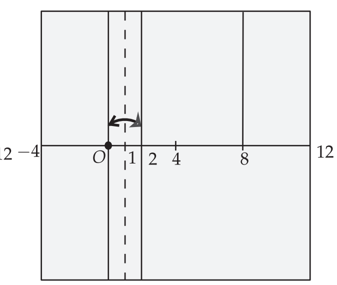
(2) Pinch in half at $x = 4$.



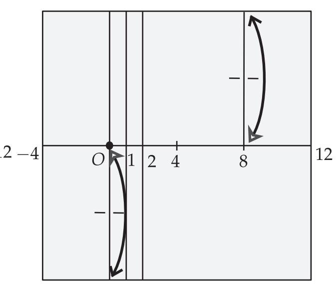
(3) Fold the y-axis and the $x = 8$ line.



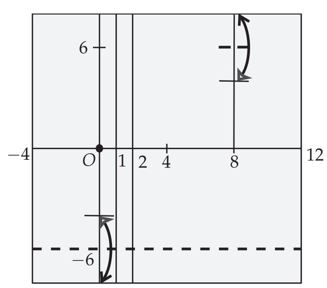
(4) Fold the $x = 2$ line.



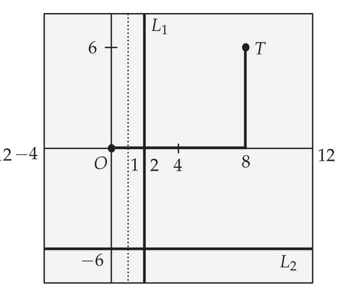
(5) Fold the $x = 1$ line.



(6) Make these pinches.



(4) Fold the $y = -6$ line and a pinch at $y = 6$.

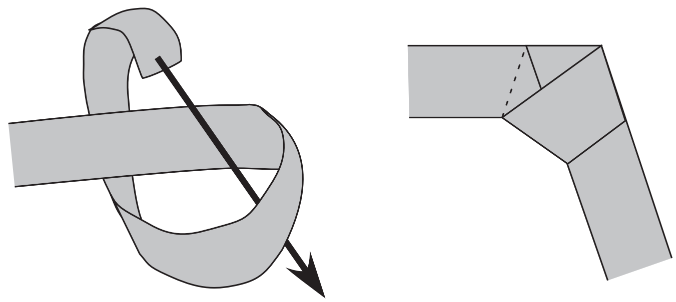


(5) Ready for Lill’s Method!

Draw points $O = (0, 0)$ and $T = (8, 6)$ and lines $L_1 : y = -6$ and $L_2 : x = 2$. The coefficient path is drawn from O to T in bold. Do you see why this is the path?

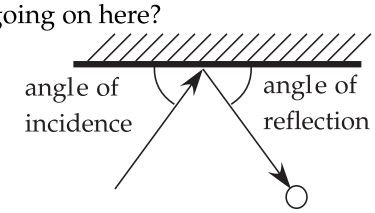
Knotting a Strip of Paper

Activity: Take a long strip of paper and tie it into a tight, flat knot. That may sound weird, so the below picture might help.



Question 1: Prove that this pentagon is regular (all sides have the same length).

Tip: When bouncing a billiard ball off a wall, the "angle of incidence" equals the "angle of reflection." Is anything like that going on here?

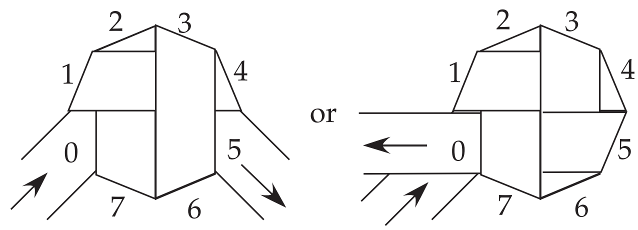


Question 2: Can you tie a strip of paper into any other knots? Hexagon, heptagon (7 sides), or octagon? How about triangle or square? Explore this and make a conjecture about what you think is going on.

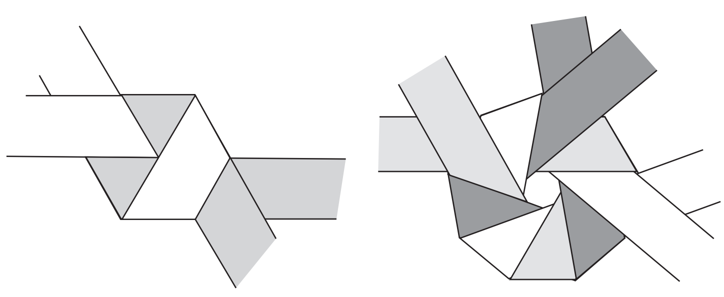
Question 3: In the previous question, you should have been able to make some other knots. For example, it is possible to make an octagon knot in a number of different ways. Below is shown one way, finished off in two different fashions.

Think of each side of the octagon as being a number, starting with 0 as the side the strip entered. Then the strip weaves around and then either exits once the polygon is finished or when you get back to 0.

In what order does the paper hit the sides? Does this remind you of anything about the cyclic group $\mathbb{Z}_8$ (the integers mod 8)? Use this concept to prove the conjecture that you made in Question 2.

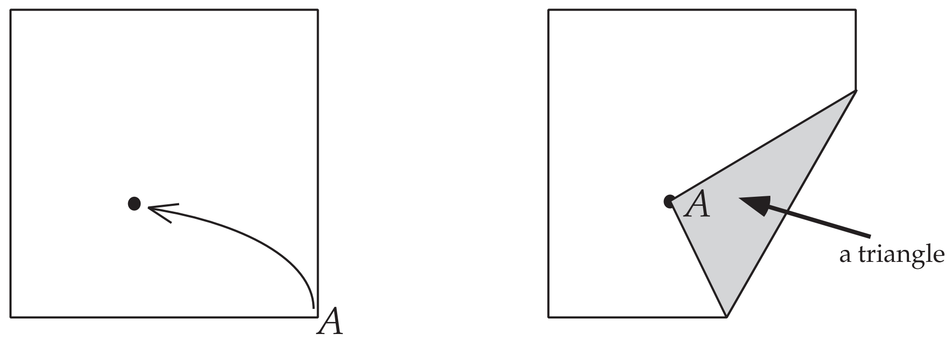


Question 4: What if we allowed ourselves to use more than one strip of paper? It turns out that then we can make just about any knot. Below are shown ways a hexagonal knot and a nonagonal (9 sides) knot can be made from 2 and 3 strips, respectively. How can the group $\mathbb{Z}_n$ be used to analyze what these knots are doing? What do the individual strips represent?



Folding TUPs

Take a square piece of paper and label the lower right-hand corner A. Pick a random point on the paper and fold A to that point. This creates a flap of paper, called the Turned-Up Part (or TUP for short).



How many sides does your TUP have? Three? Four? Five?

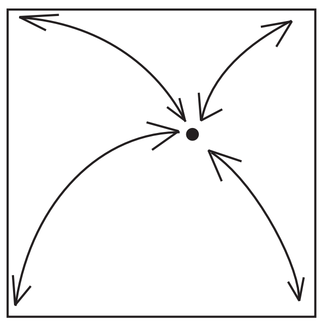
Your task: Experiment with many TUPs to find an answer to the question, “How can we tell how many sides a TUP will have?”

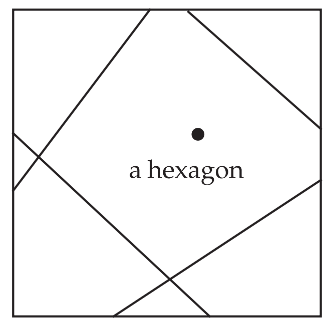
Follow-up: What if we allowed the point to be outside the square? Then what are the possibilities?

HANDOUT

Haga's Origamics: All Four Corners to a Point

Take a square piece of paper and pick a point on it at random. Fold and unfold each corner, in turn, to this point. The crease lines should make a polygon on the square. (Some sides of the square may be sides of this polygon.)





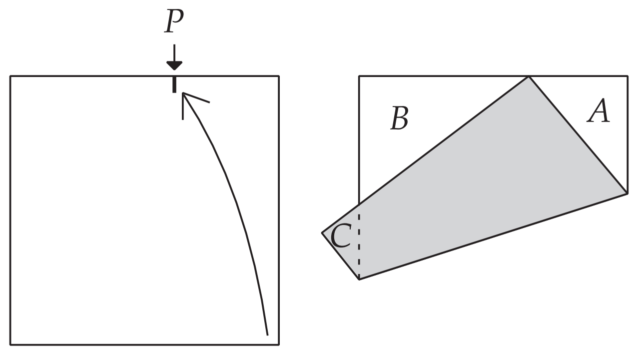
How many sides does your polygon have? Five? Six? Could it have three, four, or seven?

Your task: Do this "all four corners to a point" exercise on many squares of paper. How can you tell how many sides your polygon will have?

Follow-up: What if we used a rectangle instead of a square? Then what are the possibilities?

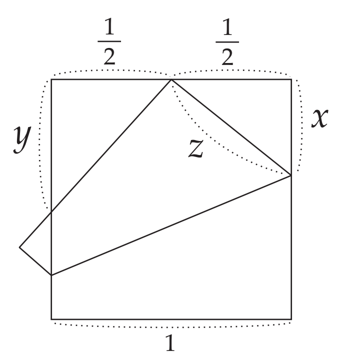
Haga's Origamics: Haga's Theorem

Take a square piece of paper and mark a point P at random along the top edge of the paper. Then fold the lower right corner to this point.



Question 1: What nice relationship must be true about the triangles A, B, and C? Proof? (This is known as Haga's Theorem.)

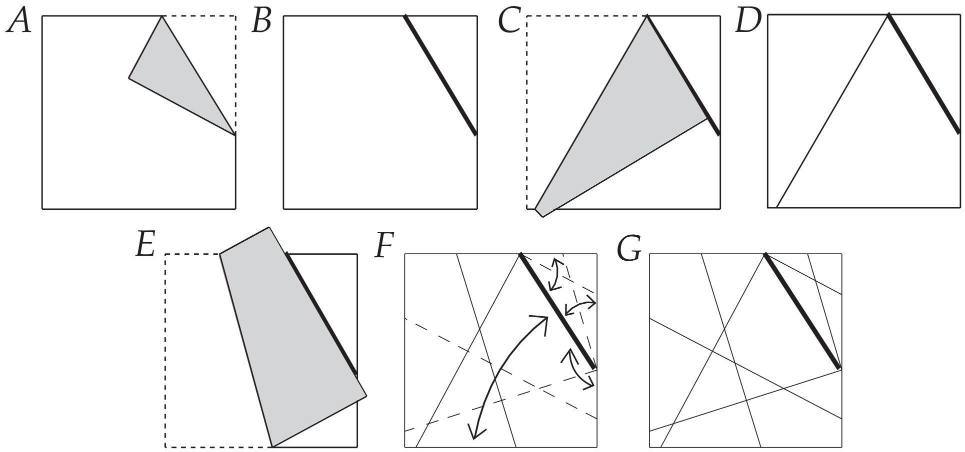
Question 2: Suppose that you took the point P to be the midpoint of the top edge. Use Haga's Theorem to find out what the lengths x and y must be in the below figure.



HANDOUT

Haga's Origamics: Mother and Baby Lines

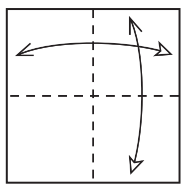
Take a square piece of paper and make a random crease through it. (Like in figures A and B below. This is called the **mother line**.) Then fold and unfold all the other sides of the paper to this line. (Like in figures C–F below. These are called **baby lines**.) You'll see a bunch of crease lines (figure G).



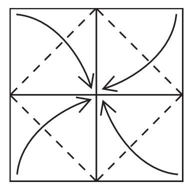
Your task: Experiment with various mother lines on separate sheets of paper and compare your results. What conjectures can you make about the intersections of the baby lines? Prove it/them.

Modular Star Ring

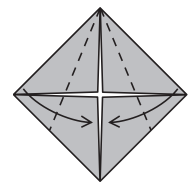
This unit makes a star-shaped ring. You will need about 12–20 squares of paper.



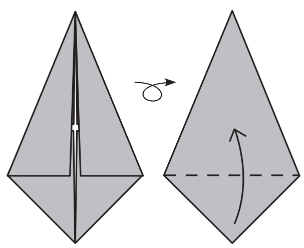
(1) Fold and unfold in half in both directions.



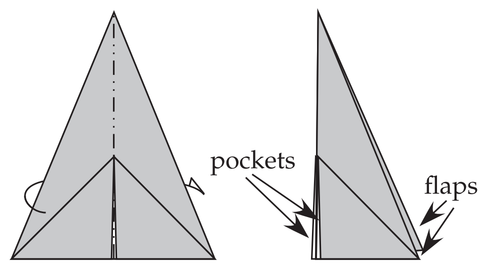
(2) Fold all four corners to the center.



(3) Fold the top edges to the center vertical line.

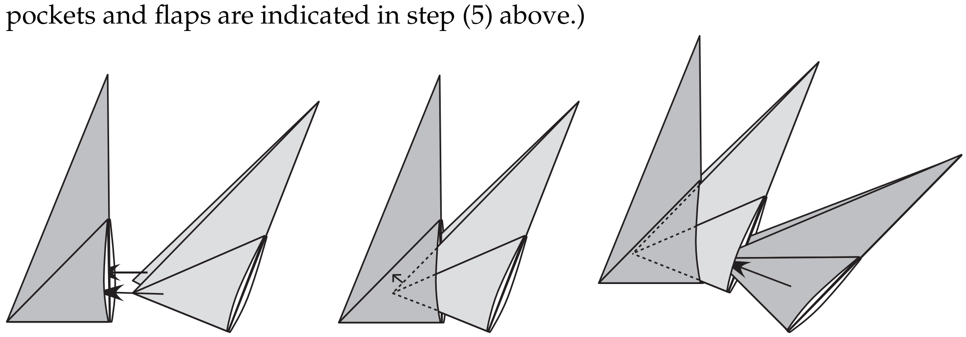


(4) Turn over and fold the bottom point up as shown.



(5) Fold in half **away from you** and you're done! Make a bunch more.

Putting them together: Slide the flaps of one unit into the pockets of another. (The pockets and flaps are indicated in step (5) above.)

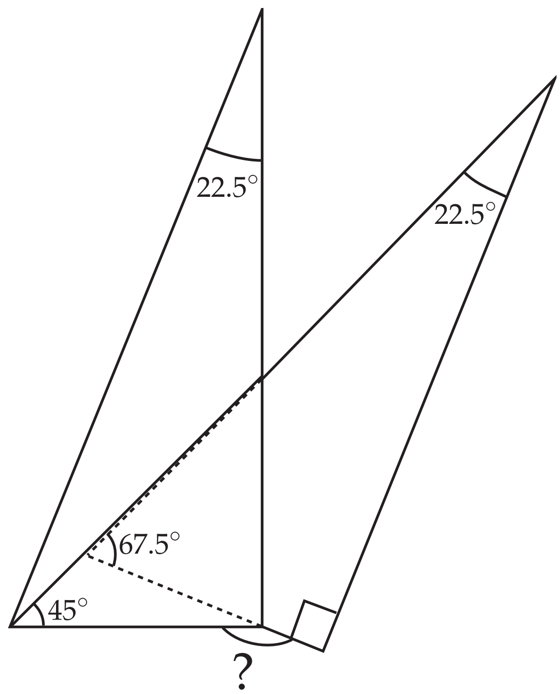


Keep adding more units until it comes back to the first unit and forms a ring!

Question: You may have noticed that you can make this ring close up with 12, 13, 14, or even more units, but some of these feel pretty loose. What is the best number of units you should use to make a tight, perfectly-fitting ring?

Additional hint: To make the units fit perfectly, you want each unit to slide in as far as it can, with the top edge of each unit's flaps "flush" against the top edge of its neighbor's pockets.

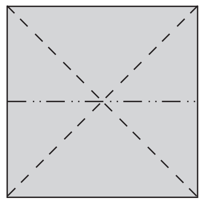
The below picture might also help you see what the proper angles should be if the units are perfectly fitted together.

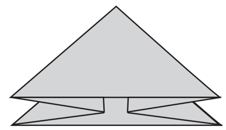


HANDOUT

Making a Butterfly Bomb (invented by Kenneth Kawamura)

You'll need 12 pieces of stiff, square paper. Use 3 colors (4 sheets per color).





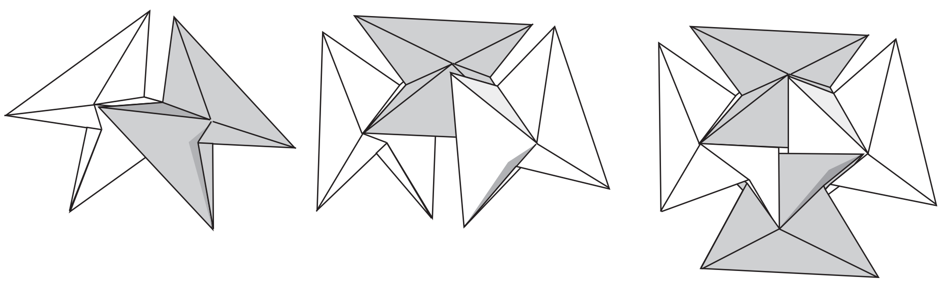
(1) Take a sheet and fold both diagonals (with valley folds). Fold in half horizontally with a **mountain** fold.

(2) Collapse all these creases at the same time to get the above figure. Press flat and score the creases firmly. Then open it up again.

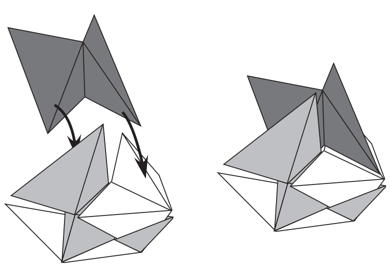
Repeat with the other 11 squares.

Putting it together: The object is to make a **cuboctahedron**, which has 6 square faces and 8 triangle faces.

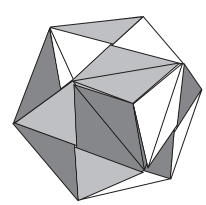
First form a square base using four units as shown. The units should be layered over-under-over-under to weave together.



Then use a unit to make a triangle-shaped cavity to the side of the square base. Again, the units should weave. It will be **hard** to make them stay together. Working in pairs (with more hands) will help. Do this on each side of the square base.



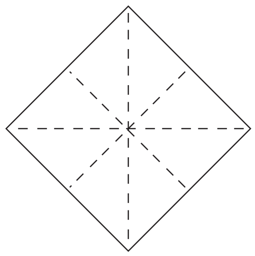
Keep adding units, making square faces and triangle cavities. It won't stay together until the last one is in place.
Why is it a bomb? Toss the finished model in the air and smack it underneath with an open palm to see!



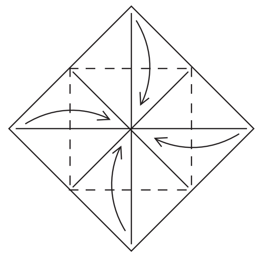
HANDOUT

The Classic Masu Box

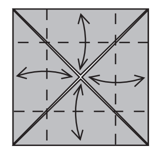
This box is a classic Japanese model. It also can be a big help for making the Butterfly Bomb. If making a Butterfly Bomb from 3 in to 3.5 in paper, then make your Masu Box out of a 10 in square.



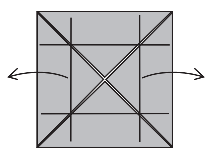
(1) Crease both diagonals and both horizontals.



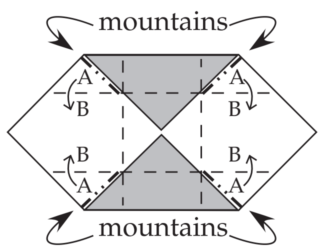
(2) Fold all four corners to the center.



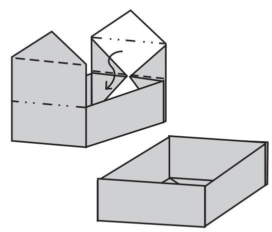
(3) Fold each side to the center, crease, and unfold.



(4) Unfold the left and right sides.

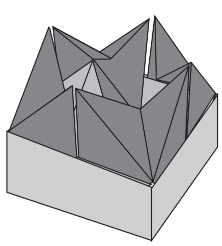


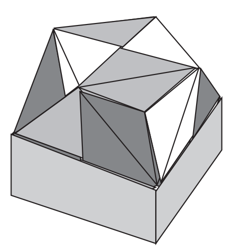
(5) Use the mountain creases shown to form a 3D box. The A regions should land on top of the B regions as shown…



(6) …here. Then fold the other sides inside, making them line up with the other tabs, to finish the box!

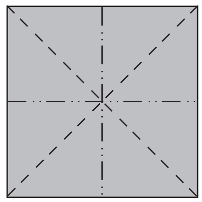
How this can help with the Butterfly Bomb: Use the Masu Box as a holder for the Butterfly Bomb units as you make it. The square sides of the Butterfly Bomb should be flat against the Masu Box sides.

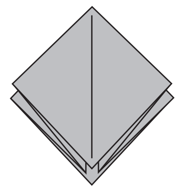




Making a Butterfly Bomb Dual

You'll need 6 pieces of square paper. Use 3 colors (2 sheets per color).



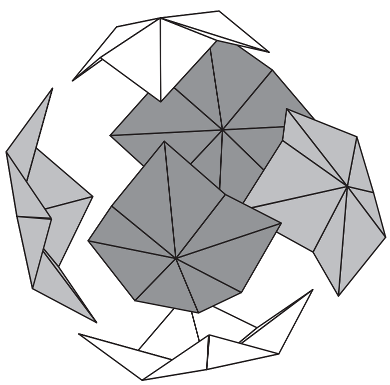


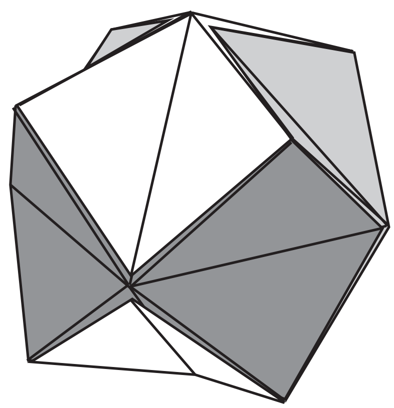
(1) Take a sheet and fold both diagonals (with valley folds). Fold in half both ways with **mountain** folds.

(2) Collapse all these creases at the same time to get the above figure. Press flat and score the creases firmly. Then open it up again.

Repeat with the other 5 squares.

Putting it together: The object is to arrange the units like the 6 faces of a cube. They should weave together to form eight pyramids





The units will not want to stay together until the last one is in place. If you have trouble, work with someone else to help. (The more hands the better!)

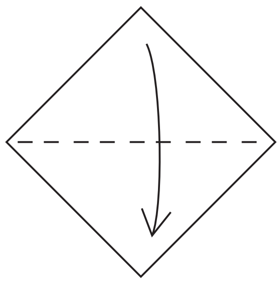
This model is also a "bomb." Toss it in the air and smack it from underneath with an open palm to make it explode!

Question: What does this shape remind you of? How would you describe it?

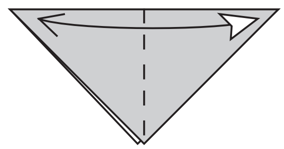
HANDOUT

Molly’s Hexahedron

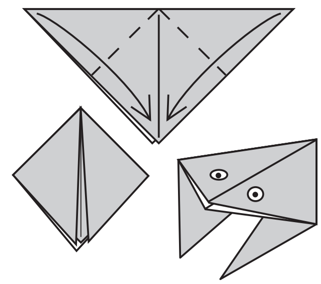
This model, invented by Molly Kahn, requires 3 squares of paper. Fold each square into a unit that kind of looks like a frog. Then we will put these units together to make an interesting object!



(1) Fold a diagonal.

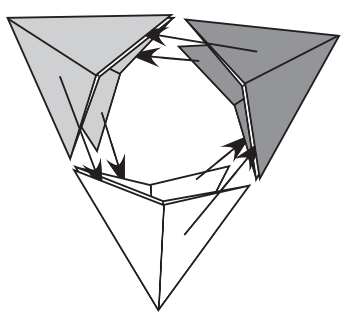


(2) Fold in half. Unfold.

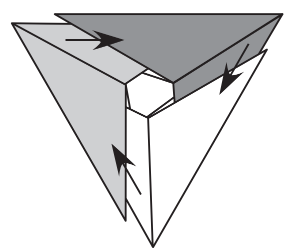


(3) Fold the corners to the bottom and you are done! Make 2 more.

Putting them together:



Slide the “legs” of one frog unit into the “mouth” of another one. To make this work, the frogs need to be positioned properly, like in the left drawing. Add a third frog to complete a triangle, and squeeze them all together!



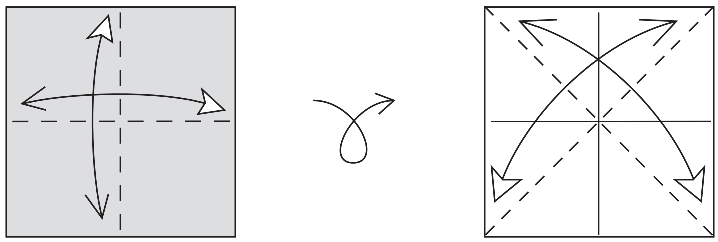
Question 1: How would you describe this object? What is the shape of its faces and how many are there?

Question 2: Suppose the side length of your original squares is 1. Then what is the volume of the finished object? Hint: Use the fact that the volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height.

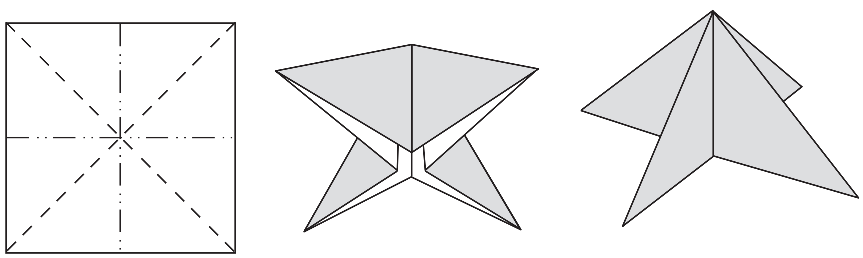
HANDOUT

The Octahedron Skeleton

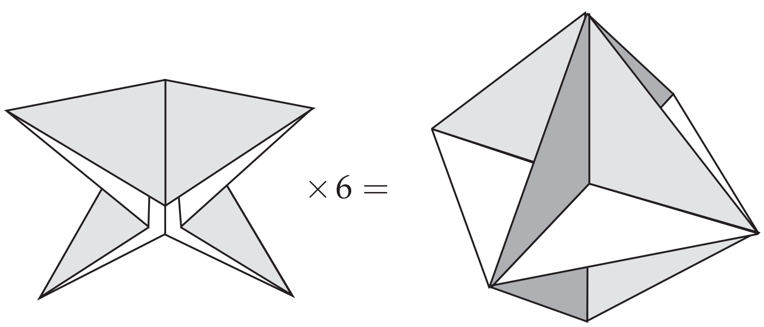
This is a classic modular origami model. It was invented by Bob Neale in the 1960s and requires 6 sheets of square paper.



For each piece of paper, valley fold in half from left-to-right and from top-to-bottom. Then **turn the paper over** and valley fold both diagonals. It is important to turn the paper over in between doing the "horizontal-vertical" folds and the diagonal folds.



Then collapse the paper into a star shape, as shown above. The shape that results, which is called the *waterbomb base* by origamists, should have the four original corners of the square becoming long, triangular flaps.



Make 6 of these waterbomb bases, 2 each of 3 colors. Then the **puzzle** is to lock them together to make an octahedron skeleton!

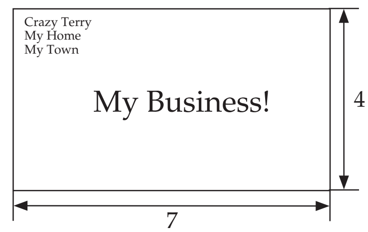
Hint: The triangle flaps will insert into the triangle flaps of other units in an over-under-over-under pattern.

HANDOUT

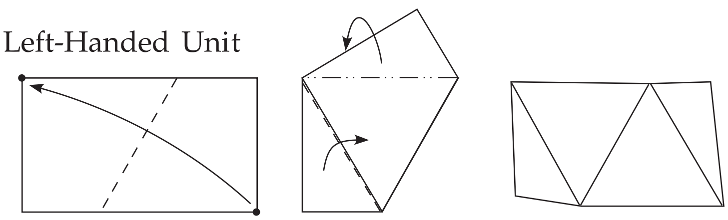
Business Card Polyhedra

Business cards are a very popular medium in **modular origami**, where pieces of paper are folded into **units** and then combined, without tape or glue, to make various shapes. Standard business cards are 2 inch $\times$ 3.5 inch rectangles, or have dimensions 4×7.

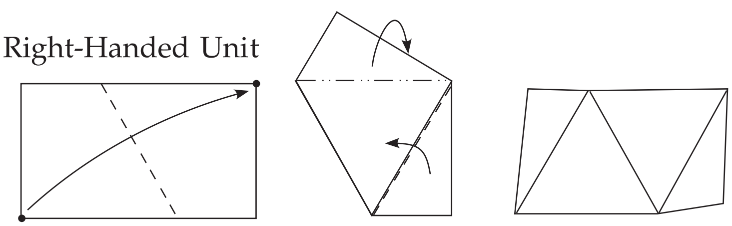
Below are instructions for making a very simple unit from business cards that can make many different polyhedra. **Make the creases sharp!** This unit was originally invented by Jeannine Mosely and Kenneth Kawamura.



Left-Handed Unit

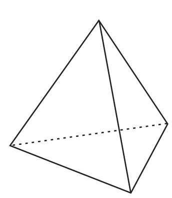


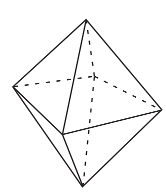
Right-Handed Unit



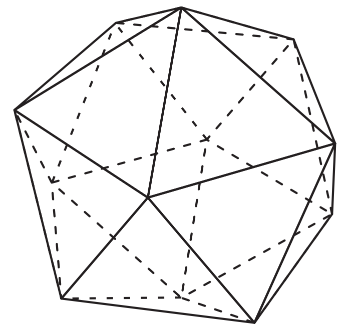
Question 1: Notice that these simple folds on a business card give us, it seems, equilateral triangles. Are they **really** equilateral? How can we tell?

Task 1: Make one left- and one right-handed unit and find a way to lock them together to make a **tetrahedron** (shown below left). After you do that, use 4 units to make an **octahedron** (shown below right). We're not telling you how many left and right units you need—you figure it out!





Task 2: Now make 10 units (5 left and 5 right) and make an **icosahedron** with them. An icosahedron has 20 triangle faces. (See the below figure.) Putting this together is quite hard—an extra pair of hands (or temporary tape) might help.

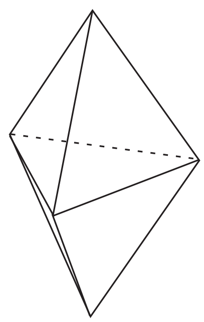


Task 3: What other polyhedra can you make with this unit? Hint: There are lots more. Try making something using only 6 units. How about 8 units? Try to describe the polyhedra that you discover in words.

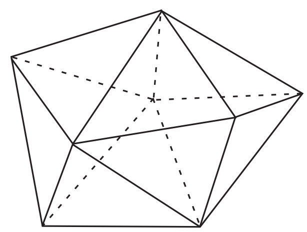
HANDOUT

Johnson Solids with Triangle Faces

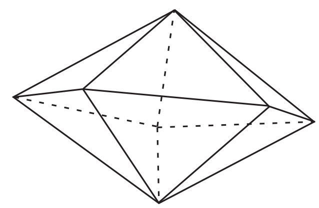
Try making these strange polyhedra using the business card unit. You'll have to figure out how many units you'll need and whether they should be left- or right-handed, or a combination of both!



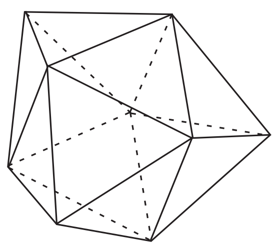
triangular dipyramid



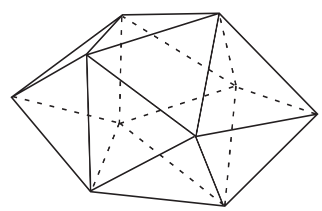
snub disphenoid



pentagonal dipyramid



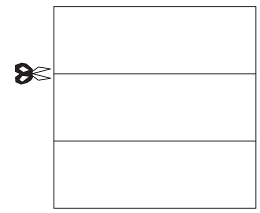
triaugmented triangular prism



gyroelongated square dipyramid

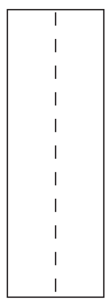
Five Intersecting Tetrahedra

This origami model is a real puzzle! But first we'll start with the **one tetrahedron** made from Francis Ow's 60° unit [**?**].

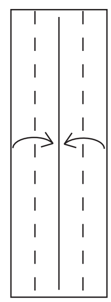


Francis Ow's 60° unit

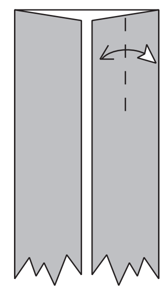
This requires 1×3 paper. So fold a square sheet into thirds and cut along the creases.



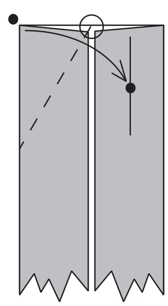
(1) Crease in half lengthwise.



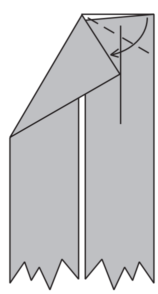
(2) Fold the sides to the center.



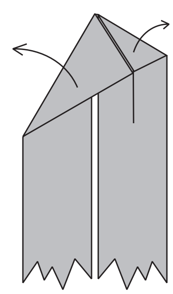
(3) At the top end, make a short crease along the half-way line of the right side.



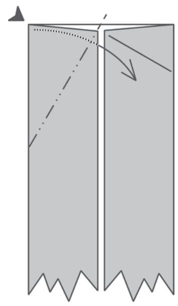
(4) Fold the top left corner to the pinch mark just made **and at the same time** make sure the crease hits the midpoint of the top…



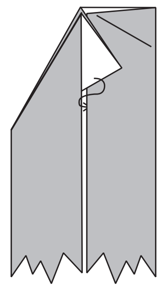
(5) …like this. Fold the top right side to meet the flap you just folded.



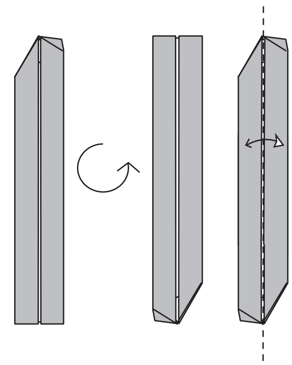
(6) Undo the last two steps.



(7) Now use the creases made in step (4) to **reverse** the top left corner through to the right. This should make a white flap appear. . .

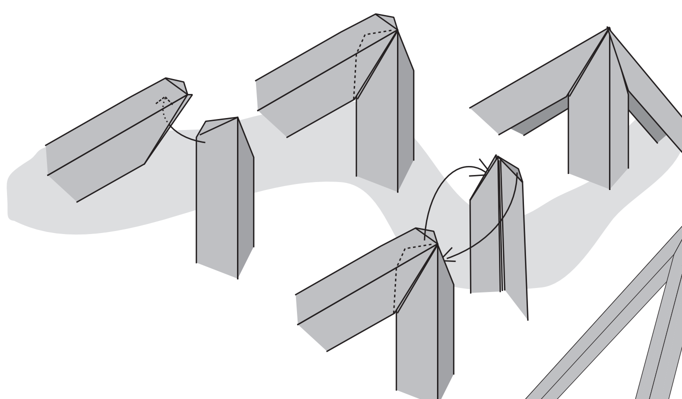


(8) . . . like this. Tuck the white flap underneath the right side paper.

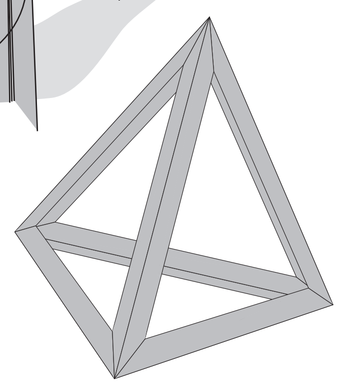


(9) Now rotate the unit 180° and repeat steps (3)–(8) on the other end. Then fold the whole unit in half lengthwise (to strengthen the spine of the unit) and you're done!

Locking the units together: Three units make one corner. **Make sure** to have the flap of one unit **hook** around the spine of the other!

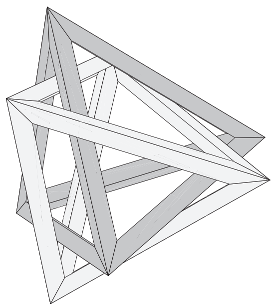


The lock should be fairly strong.
Make 6 units and form them into a tetrahedron!



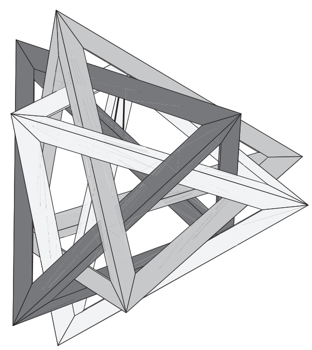
Linking the Tetrahedra Together

The five tetrahedra must be woven together, one at a time. The second tetrahedron must be woven into the first one as it's constructed. That is, it's not very practical to make two completed tetrahedra and *then* try to get them to weave together. Instead, make one corner of the second tetrahedron, weave this into the first one, then lock the other three units into the second tetrahedron.

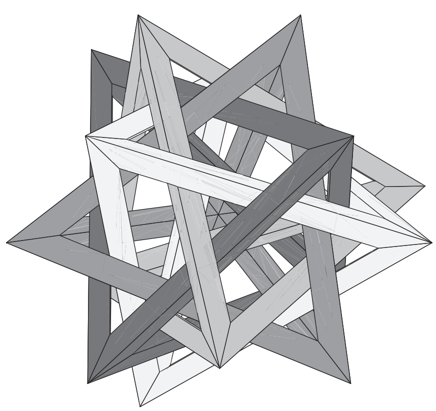


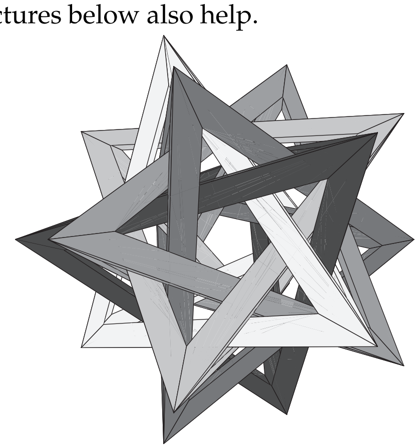
The first two tetrahedra make a sort-of 3D Star of David, with a corner of one tetrahedron poking through the side of the other, and a corner of the other poking through a side of the one. In fact, when the whole model is done **every** pair of tetrahedra should form such a 3D Star of David form.
The third tetrahedron is the most difficult one to weave into the model.

The figure to the right is drawn at a specific angle to help you do this. Notice how in the center of the picture there are three struts weaving together in a triangle pattern. If you look carefully, the same thing is happening on the opposite side of the model. As you insert your units for the third tetrahedron, try to form these triangular weaves and use them as a guide. In the finished model, there will be one of these triangular weave points under *every* tetrahedron corner.



These two types of symmetry—two tetrahedra making a 3D Star of David and the triangular weave points—are the best visual tools to use when inserting the units for the fourth and fifth tetrahedra. The pictures below also help.





HANDOUT

What Is the Optimal Strut Width?

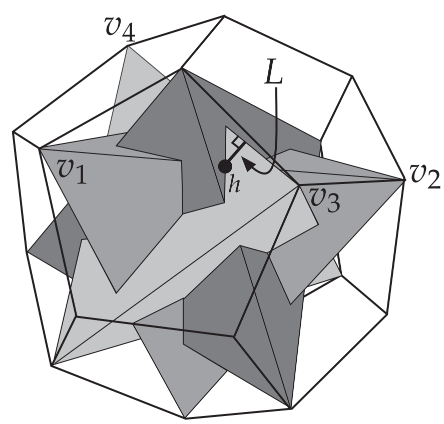
The instructions for Francis Ow's 60° unit have us start with 1×3 sized paper, which gives us a unit that is $1 \times 1/12$ in dimensions. In other words, if the side of one of the tetrahedra is 1, then the width of the strut in the tetrahedral frame that we make is $1/12$.

Is this the optimal strut width, or should we be using a wider or thinner strut for a more ideal fit? In this activity you'll use vector geometry and calculus to approximate the ideal strut width. This calculation is hard to do by hand, so you're better off using a computer algebra system to help.

The ideal strut width is given by the line segment L, shown below. It marks the shortest distance between the tetrahedron edge $\overline{v_3v_4}$ and the point h which is the midpoint of the $\overline{v_1v_2}$ edge from another tetrahedron.

We can use nice coordinates for v_1 and v_2 so that h will be the point $(0,0,1)$ on the z-axis. Then, since the tetrahedra fit inside a dodecahedron, the coordinates of v_3 and v_4 can be found to be as follows:

$$v_1 = (-1,1,1)$$
$$v_2 = (1,-1,1)$$
$$v_3 = \left(0, \frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$
$$v_4 = \left(\frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, 0\right)$$



Our goal is to find $L =$ the minimum distance between the point $h = (0,0,1)$ and the line segment $\overline{v_3v_4}$ (as shown above).

Question 1: Find a parameterization $F(t) = \{x(t), y(t), z(t)\}$ for the line in $\mathbb{R}^3$ that contains $\overline{v_3v_4}$.

Question 2: Now find a formula for the distance between an arbitrary point $F(t)$ on the $\overline{v_3 v_4}$ line and the point $h = (0,0,1)$.

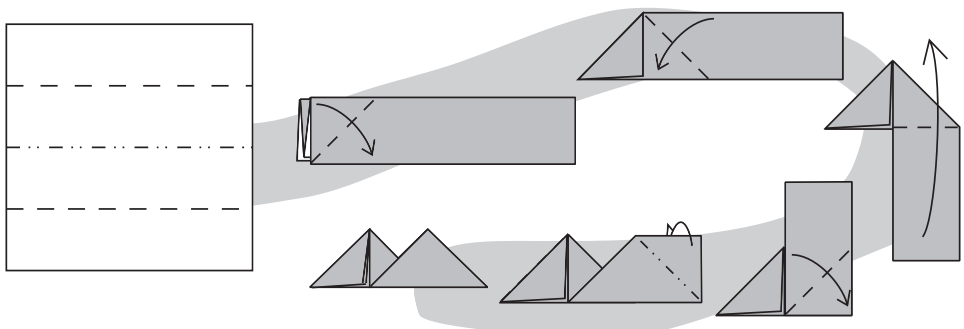
Question 3: Now minimize the distance function you found in Question 2 to find the length L. Hints: It might be easier to minimize the square of the distance function to get L^2.

Question 4: So what is the ideal strut width L? How does it compare to our use of struts that were $1/12$ the side of a triangle?

The PHiZZ Unit

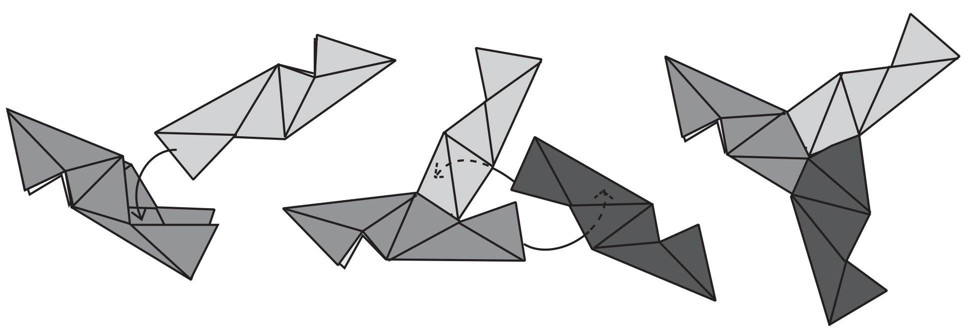
This modular origami unit (created by Tom Hull in 1993) can make a large number of different polyhedra. The name stands for **P**entagon **H**exagon **Z**ig-**Z**ag unit. It is especially good for making large objects, since the locking mechanism is strong.

Making a unit: The first step is to fold the square into a 1/4 zig-zag.



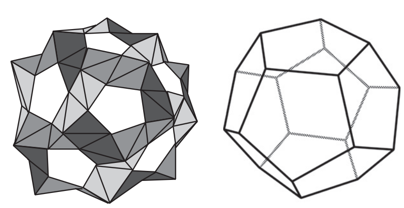
When making these units, it's important to make all your units **exactly** the same. It's possible to do the second step backwards and thus make a unit that's a **mirror image** and won't fit into the others. Beware!

Locking them together: In these pictures, we're looking at the unit "from above." The first one has been "opened" a little so that the other unit can be slid inside.



Be sure to insert one unit **in-between** layers of paper of the other. Also, make sure that the flap of the "inserted" unit hooks over a crease of the "opened" unit. That forms the lock.

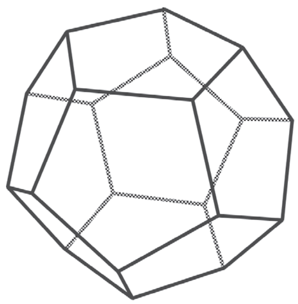
Assignment: Make **30 units** and put them together to form a **dodecahedron** (shown to the right), which has all pentagon faces. **Also** use only 3 colors (10 sheets of each color) and try to have no two units of the same color touching.

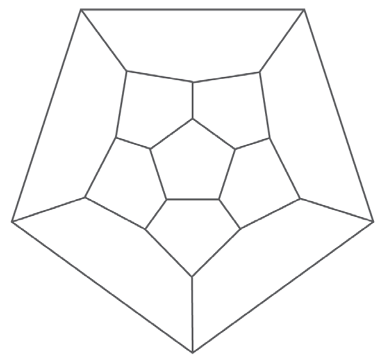


HANDOUT

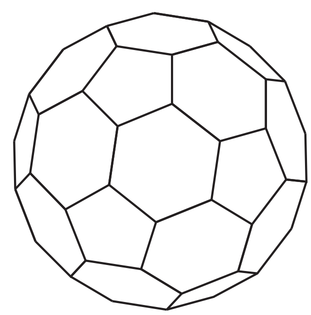
Planar Graphs and Coloring

Drawing the **planar graph** of the polyhedron can be a great way to plan a coloring when using PHiZZ units. To make the planar graph of a polyhedron, imagine putting it on a table, stretching the top, and pushing it down onto the tabletop so that none of the edges cross. Below is shown the dodecahedron and its planar graph.





Task 1: Draw the planar graph of a soccer ball. Make sure it has 12 pentagons and 20 hexagons.



Task 2: A **Hamilton circuit** is a path in a graph that starts at a vertex, visits every other vertex, and comes back to where it started without visiting the same vertex twice. Find a Hamilton circuit in the planar graph of the dodecahedron.

When making objects using PHiZZ units, it's always a puzzle to try to make it using only 3 colors of paper with no two units of the same color touching. Each unit corresponds to an edge of the planar graph, so this is equivalent to a **proper 3-edge-coloring** of the graph.

Question: How could we use our Hamilton circuit in the graph of the dodecahedron to get a proper 3-edge-coloring of the dodecahedron?

Task 3: Find a Hamilton circuit in your planar graph of the soccer ball and use it to plan a proper 3-edge-coloring of a PHiZZ unit soccer ball. (It requires 90 units. Feel free to do this in teams!)

HANDOUT

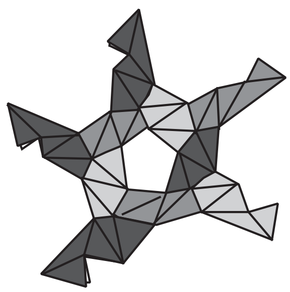
Making PHiZZ Buckyballs

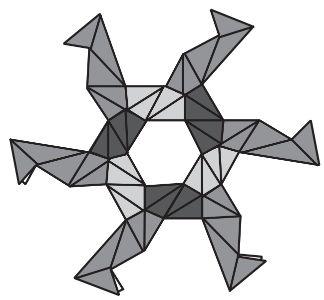
Buckyballs are polyhedra with the following two properties:

(a) each vertex has degree 3 (3 edges coming out of it), and

(b) they have only pentagon and hexagon faces.

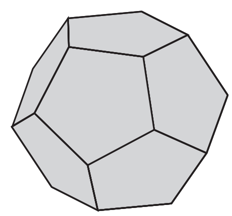
The PHiZZ unit is great for making Buckyballs because you can make pentagon and hexagon rings:

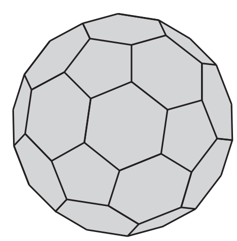


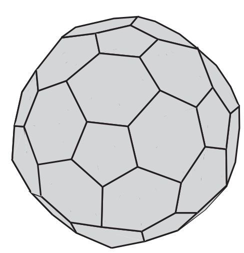


These represent the faces of the Buckyball. But when making these things, it helps to know how many pentagons and hexagons we'll need!

To the right are shown three Buckyballs: The dodecahedron (12 pentagons, no hexagons), the soccer ball (12 pentagons, 20 hexagons), and a different one. (Can you see why?)







Question 1: How many vertices and edges does the dodecahedron have? How about the soccer ball? Find a formula relating the number of vertices V and the number of edges E of a Buckyball.

Question 2: Let F_5 = the number of pentagon faces in a given Buckyball. Let F_6 = the number of hexagon faces. Find formulas relating

(a) F_5, F_6, and F (the total number of faces). (Easy!)

(b) F_5, F_6, and E. (Harder.)

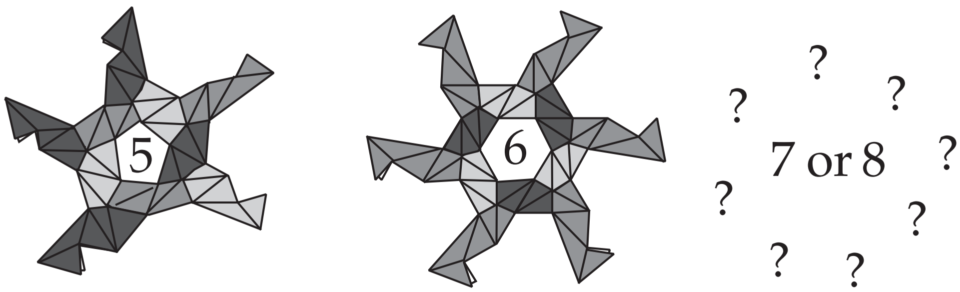
Question 3: Now use Euler's formula for polyhedra, $V - E + F = 2$, together with your answers to Questions 1 and 2, to find a formula relating F_5 and F_6, the number of pentagons and hexagons.

Question 4: What can you conclude about all Buckyballs?

HANDOUT

Bigger PHiZZ Unit Rings

This handout asks you to experiment with making larger “rings” using PHiZZ units.



Activity: Make a heptagon or octagon ring out of PHiZZ units (it’ll require 14 or 16 units, so feel free to do it in groups). This will be challenging: How can you make the ring close up? Do not force any extra creases in the units! They should go together just like normal.

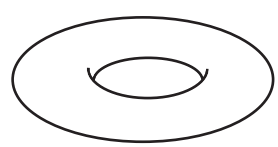
Question: Compare what a pentagon ring, a hexagon ring, and a bigger ring (like a heptagon or octagon ring) look like.

Specifically, imagine these rings lying on a surface. What kind of surface would the pentagon ring be lying on?

How about a hexagon ring?

How about a heptagon or octagon ring?

So, if you were to make a **torus** (i.e., a doughnut) using PHiZZ units, where on the torus might you place your pentagons, your hexagons, and your bigger-gons?

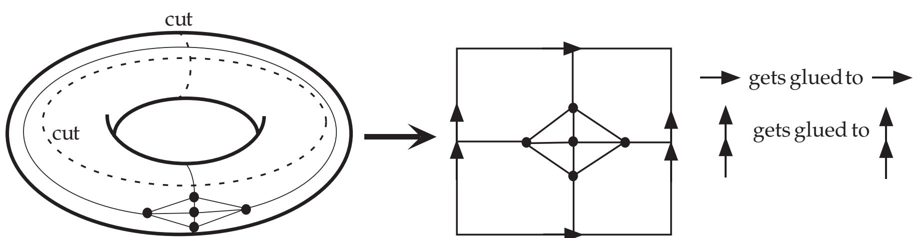


HANDOUT

Drawing Toroidal Graphs

When planning a PHiZZ unit torus model, it can be hard to visualize what you're doing because you can't just draw the planar graph of the structure like you can with, say, Buckyballs.

But there is a way to **flatten** a torus so that we can draw graphs on the torus using pen and paper. The idea is shown in the picture below. You imagine making two perpendicular cuts on the torus surface and then "unroll" the torus into a rectangle. This is called the **fundamental domain of the torus**.

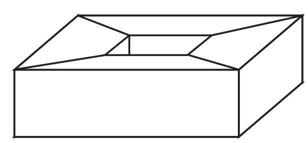


A torus
cut along the dotted lines

The fundamental domain of the torus

The idea in the fundamental domain is that any edge you draw that hits the boundary must come back on the other side. Thus a graph drawn on the torus, like the one shown above, can be represented on the fundamental domain by making some edges "wrap around" from top to bottom and from left to right.

Activity: Draw the graph of the **square torus** (shown below right) on a fundamental domain.



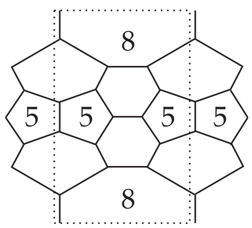
You now have what you need to start designing your own PHiZZ unit torus. Just start with the fundamental domain of a torus and try to draw a graph on it that has

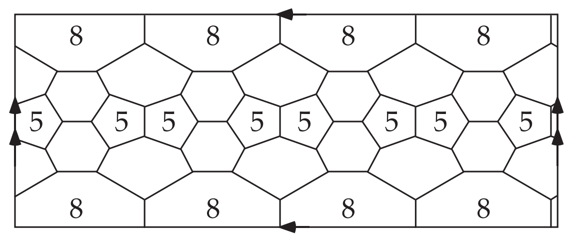
(1) all vertices of degree 3 and

(2) only pentagon, hexagon, or higher faces.

(Square and triangle faces don't work very well with the PHiZZ unit.)

Unfortunately, making PHiZZ unit tori can take a lot of units. People have made ones using hundreds of units. But, they can be made with a more reasonable number. Below is a torus, designed by mathematician sarah-marie belcastro, that requires 84 units. It's made from a small pattern (below left, in the dotted box) that is repeated four times on the fundamental domain (below right). It uses only pentagon, hexagon, and octagon faces.





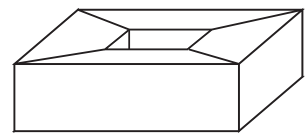
You can make the above torus or try designing your own. You might be able to design a smaller one by using larger polygons, like 10-gons, instead of octagons.

Advice: When making such a torus, make the larger, negative curvature polygons on the inside rim **first**. This may seem hard, but it's a lot easier to do them at the beginning than waiting until the end. Once you have the inner rim in place, it's a lot easier to then make the hexagons and pentagons.

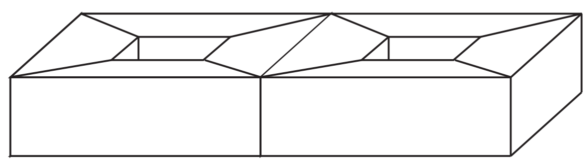
HANDOUT

Euler's Formula on the Torus

Question 1: Below is shown a **square torus**. What does Euler's Formula, $V - E + F$, give for this polyhedron?



Question 2: How about for a **2-holed torus**?



Question 3: We define the **genus** of a polyhedron to be the number of "holes" it has. (So a torus has genus 1, a two-holed torus has genus 2, an icosahedron has genus 0, etc.) Find a **generalized Euler's formula** for a polyhedron with genus g.

Properties of Toroidal "Buckyballs"

Now that you know Euler's Formula for the torus, we can learn some things that will help you plan making tori using PHiZZ units.

Question 4: Suppose that we make a torus using PHiZZ units and only making **pentagon**, **hexagon**, and **heptagon** (7-sided) faces. Find a formula relating F_5 (the number of pentagon faces) and F_7 (the number of heptagon faces).

Hint: Remember that we still have $3V = 2E$. Use the same techniques that we used to prove that all Buckyballs have only 12 pentagon faces.

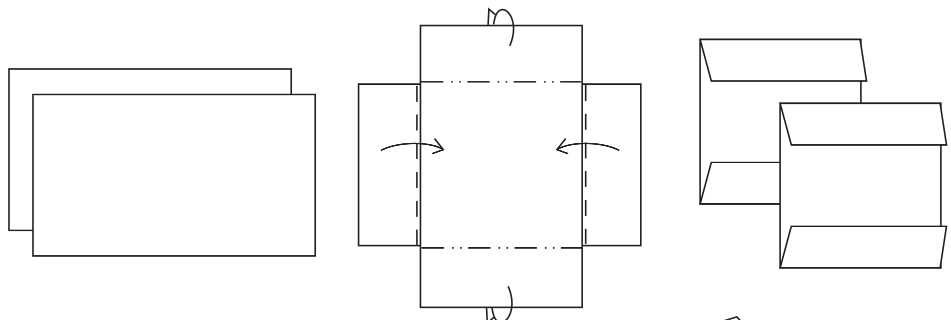
Question 5: Suppose that we made a PHiZZ unit torus using only **pentagon**, **hexagon**, and **octagon** faces. Find a formula relating the number of pentagon and octagon faces.

Question 6: Can you generalize these results?

HANDOUT

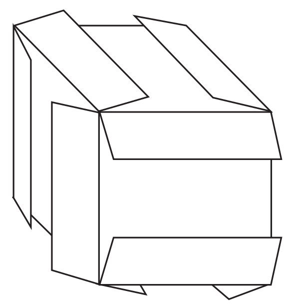
Business Card Cubes and the Menger Sponge

One of the easiest modular origami things to make from standard business cards is a cube. It takes 6 cards. To make a unit, make a “plus” sign with two cards and bend them around each other. Separate them, and you’ll have just made two units!



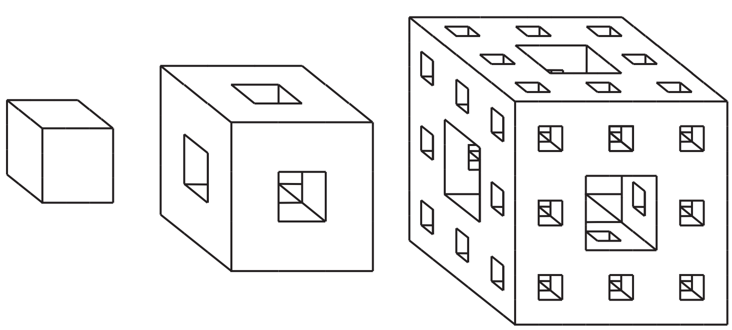
Make six units and use them to form a cube. Each unit is a face of the cube, and the folded flaps have to grip the other units. When you’re done, you’ll still see these folded flaps on the outside, gripping it all together.

It’s possible to take 6 more units and use them to “panel” the cube so that its faces are smooth. Do you see how this would work?



Two (unpaneled) cubes can be locked together along a face by making the folded flaps grip into each other. This allows you to build structures with these cubes.

Activity: Working in groups, make a “Level 1” **Menger Sponge**. A Menger Sponge is a fractal object made by starting with a cube (Level 0), then taking 20 cubes and making a cube frame with them (Level 1), and then taking 20 of these frames and making a bigger cube frame with them (Level 2), and so on. If we scale the model down after each iteration (so it remains the same size throughout), in the infinite case we’ll get what is known as Menger’s Sponge.



How many business cards will it take to make a Level 1 Sponge? With paneling?

Question 1: Let U_n = the number of business cards needed to make an unpaneled Level n Menger's Sponge. So $U_0 = 6$.

Compute values for U_1, U_2, and U_3. Find a closed formula for U_n in terms of n.

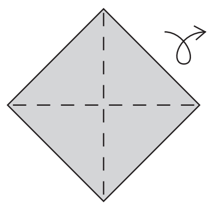
Question 2: Let P_n = the number of business cards needed to make a **paneled** Level n Menger's Sponge. So $P_0 = 12$.

Find P_1, P_2, and P_3. Can you find a formula (not necessarily closed) for P_n in general? How about a closed formula?

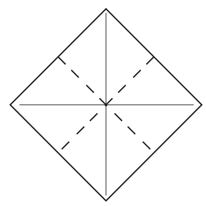
HANDOUT

Folding a Flapping Bird (Crane)

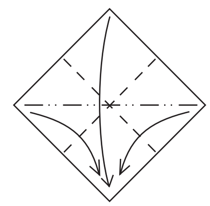
Begin with a square piece of paper.



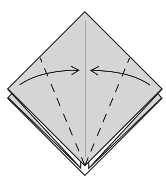
(1) Crease both diagonals. Then **turn over**.



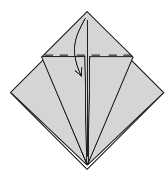
(2) Fold in half both ways.



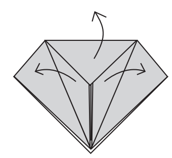
(3) Now bring all corners down to the bottom, using the creases just made,. . .



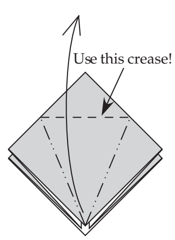
(4) . . . like this. This is called the **preliminary base**. Bisect the two angles at the open end.



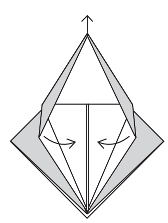
(5) Then fold the top point down.



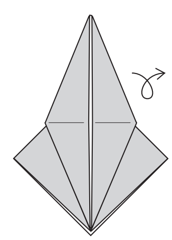
(6) Undo the last two steps.



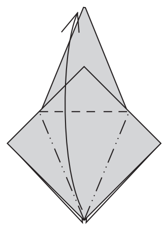
(7) Now do a **petal fold**: lift one layer of paper up, using the indicated crease as a hinge,. . .



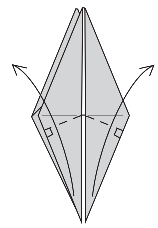
(8) . . . like this. Bring the point all the way up. The sides will come to the center. Flatten. . .



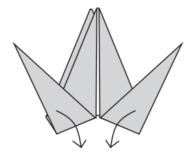
(9) . . . like this. Turn over.



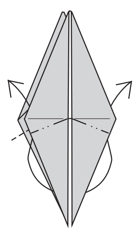
(10) Now do the same petal fold on this side.



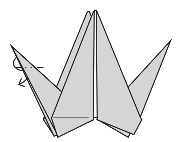
(11) This is called the **bird base**. Fold the bottom two flaps up. (These will become the head and tail.)



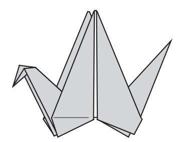
(12) **Crease firmly.** Then unfold.



(13) Now refold the last creases, but this time make them **reverse fold** through the layers. (See the next…



(14) …picture.) Lastly, reverse fold the head.



(15) You're done with the flapping bird!

This is an example of a **flat origami model**, since the finished result can be pressed in a book without crumpling.

Activity 1: Carefully **unfold** your bird and draw with a pen the crease pattern for this model. Make sure to draw **only** those creases that are actually used in the finished model, not auxiliary creases made along the way.

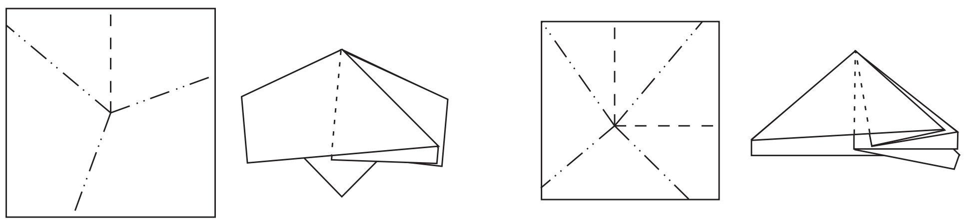
Activity 2: Then take your crease pattern and **color the faces** with as few colors as possible. That is, color the regions in between crease lines following the rule that no two regions that border the same crease line can get the same color (just like when coloring countries on a map). What's the fewest number of colors that you can use?

Activity 3: What will the coloring look like when you refold the model? Make a conjecture before you fold it back up to see what happens. Will this happen for **every** flat origami model? Proof?

HANDOUT

Exploring Flat Vertex Folds

Activity: Take a square piece of paper and make, at random, a single vertex crease pattern that folds flat. Place the vertex near the center of the paper (not on the paper's boundary—that doesn't count), make some crease lines coming out of it, and then add more to make the whole thing fold flat. Some examples are shown below. Make lots of your own.



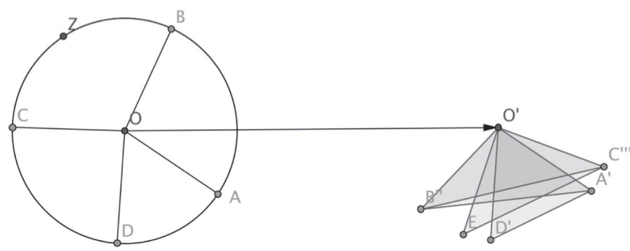
The question is, "What's going on here?" Are there any rules that such flat vertex folds follow? **Your task** is to formulate as many conjectures as you can about how such folds work.

If you come up with a conjecture, write it on the board to see if others in the class agree or if anyone can find a counterexample. Or, better yet, see if anyone can actually give proofs of your conjectures!

HANDOUT

Flat Vertex Folds in Geogebra

To simulate a flat vertex fold using Geogebra, do the following:



(1) Make a circle on the left side of your worksheet. Label the center O.

(2) Make four points on the circle, A, B, C, and D.

(3) Construct segments between these points and O. This circle and these lines will be your unfolded piece of paper.

(4) On the right side of your worksheet make a point O'. Use the **Vector between Two Points** tool to make a vector from O to O'.

(5) Now use the **Translate Object by Vector** tool to translate A to a new point A' in the direction of the vector from step (4). Do the same with points B, C, and D to make points B', C', and D'.

(6) Use the **Polygon** tool to make $\triangle O'A'D'$. This triangle will be the start of our folded paper.

(7) We now will reflect (fold) the points B', C', and D' about the crease line $O'A'$. Use the **Reflect Object about Line** tool to reflect each point, one at a time, to get new points B'', C'', and D''.

(8) Now use the **Polygon** tool to make $\triangle O'A'B''$.

(9) Now reflect (fold!) the points C'' and D'' about the line OB'' to do the third fold. This will make new points C''' and D'''.

(10) Use the **Polygon** tool to make $\triangle O'B''C'''$.

(11) Now reflect D''' about the crease line $O'C'''$ to make a new point E. (Geogebra will use a new letter because it doesn't like D''''.)

(12) Use the **Polygon** tool to make the last triangle of the folded paper, $\triangle O'C'''E$.

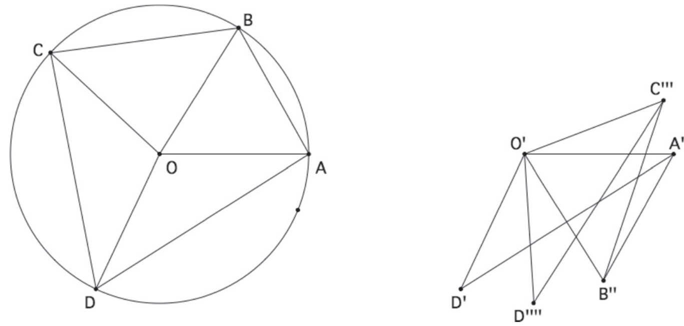
(13) Now use the **Show / Hide Object** tool to hide the points B', C', C'', D'', and D''' because we no longer need them.

Exercise: Does the last point you made, E line up with point D'? If so, then the crease lines you made on the left can fold flat. If they do not, then move the points on the left circle until they do. Use Geogebra to measure the angles $\angle AOB$, $\angle BOC$, $\angle COD$, and $\angle DOA$. What can you conjecture about these angles when the creases fold flat?

HANDOUT

Flat Vertex Folds on Geometer's Sketchpad

To simulate a flat vertex fold on Geometer's Sketchpad, do the following:



(1) Make a circle on the left side of your worksheet. Label the center O.

(2) Make four points on the circle, A, B, C, and D.

(3) Construct segments between these points and O. Also construct segments between A, B, C, and D in order to make a quadrilateral (as shown above).

(4) Select the quadrilateral, points A, B, C, D, and O, and the segments at O, and select **Translate** from the **Transform** menu. Choose Rectangular coordinates and make the horizontal and vertical distance be 12 cm and 0 cm, respectively.

(5) You now have a second copy of the quadrilateral "paper" with creases. Select the text tool and click on all the points of this copy to see what they are (A', B', C', D', O').

(6) We now will reflect parts of this copy about the creases to make it fold up. Select segment $O'A'$ and choose **Mark Mirror** from the **Transform** menu.

(7) Now select segments $A'B'$, $B'C'$, $C'D'$, $O'B'$, $O'C'$, and $O'D'$ and points B', C', and D'. With all this selected, choose **Reflect** from the **Transform** menu.

(8) You've just make $\triangle O'A'D'$ fixed and reflected the rest of the paper about crease $O'A'$! Now we want to hide the parts that we had previously selected. Under the **Edit** menu choose **Select Parents** and then *unselect* segments $O'A'$ and $O'D'$ and point D'. Then, under the **Display** menu choose **Hide Objects**.

(9) Use the text tool to click on the new points to see what they are (B'', C'', D'').

(10) Now select segment $O'B''$ and do **Mark Mirror**.

(11) Select segments $B''C''$, $C''D''$, $O'C''$, and $O'D''$ and points C'' and D''. Then do **Reflect**.

(12) Again, do **Select Parents**, *unselect* segment $O'B''$, and then **Hide Objects**.

(13) Label the points again, select segment $O'C'''$, and do **Mark Mirror**.

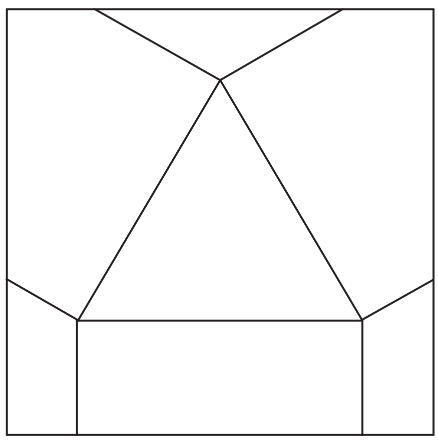
(14) Select $C'''D'''$ and $O'D'''$ and **Reflect**. Then **Hide** $C'''D'''$, $O'D'''$, and D'''.

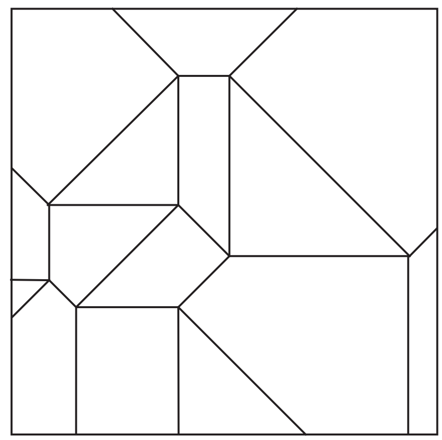
Exercise: Does the last point you made, D'''' line up with point D'? If so, then the crease lines you made on the left can fold flat. If they do not, then move the points on the left circle until they do. Use Geometer's Sketchpad to measure the angles $\angle AOB$, $\angle BOC$, $\angle COD$, and $\angle DOA$. What can you conjecture about these angles when the creases fold flat?

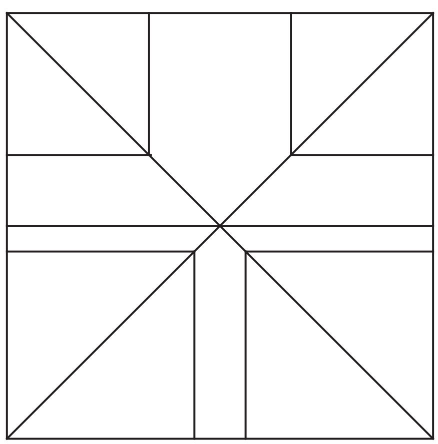
HANDOUT

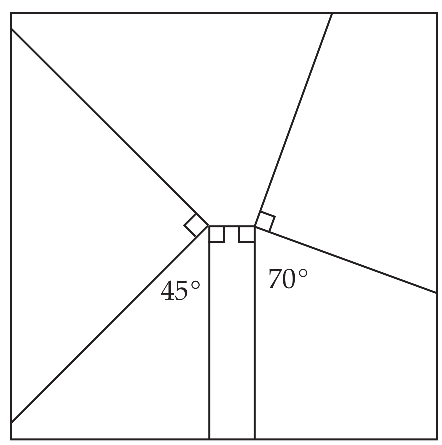
Fold Me Up

Activity: Below are some origami crease patterns. Your task is to cut them out and try to see what they can fold into. Note: You're only allowed to fold along the indicated crease lines. Adding more creases is breaking the rules. You get to decide, however, whether to make them mountains or valleys.









HANDOUT

Folding a Square Twist

Activity: Below is shown a crease pattern. The creases are all on the 1/4 lines of the square, but the center diamond needs to be "pinched" in place. Take a square piece of paper and reproduce this crease pattern to see how it folds up.

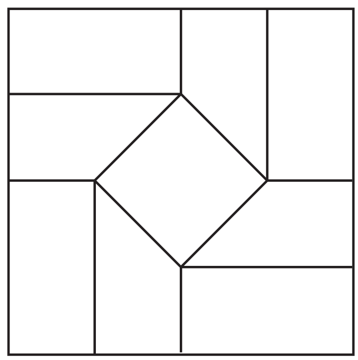
To help you fold this, follow these instructions:

(1) Fold a 4×4 grid of creases on your square.

(2) Pinch the four crease segments that make the diamond in the middle.

(3) Draw the crease pattern below on your creases with a pen.

Then you can try to fold it up.

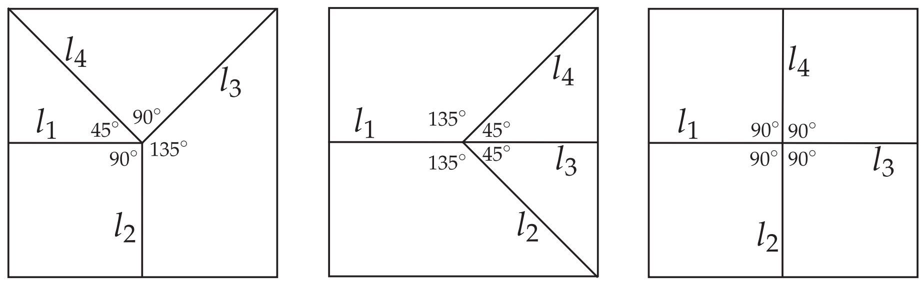


This origami maneuver is called a **square twist** and is one of the less obvious ways in which paper can be folded flat.

Question: Look at your classmates' square twists. Do they look the same as yours? Are you sure? Work together to count how many different ways there are to fold up this crease pattern (without making any new creases).

Counting Flat Vertex Folds

Below are shown three different degree-4 origami vertices, v_1, v_2, and v_3.



$C(v_1) =$ _______ $C(v_2) =$ _______ $C(v_3) =$ _______

For each of these flat vertex folds, we want to compute

$$C(v) = \text{ the number of ways that } v \text{ can fold flat.}$$

For example, in the third one above, v_3, we could have that l_1, l_2, and l_3 are all valley creases and l_4 is a mountain. That would be one way that v_3 could fold flat.

So fold these vertices using small squares of paper and experiment to compute $C(v)$ for each of them. Then try to answer the following questions.

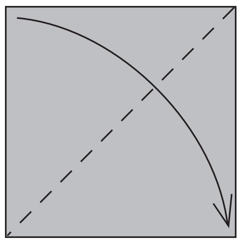
Question 1: Are there any other values that you think $C(v)$ can take for a degree-4 flat vertex fold than the values you found above?

Question 2: If you had a degree-$2n$ flat vertex fold v, what is the **largest** value that you think $C(v)$ could be? (This would be an *upper bound* on $C(v)$.)

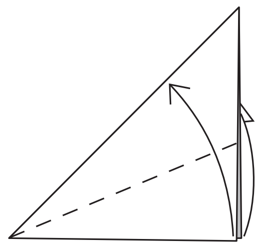
How about the **smallest** (a *lower bound*) value for $C(v)$ that you could get?

The Self-Similar Wave

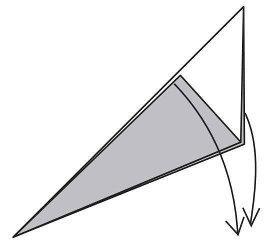
This wave model requires one square piece of paper. The diagrams assume that the paper is white on one side and has a color on the other side.



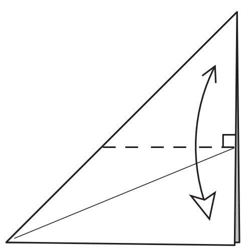
(1) Looking at the color side, fold a diagonal.



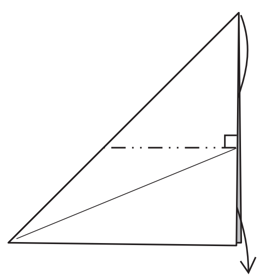
(2) Fold one layer up to the diagonal. Repeat behind.



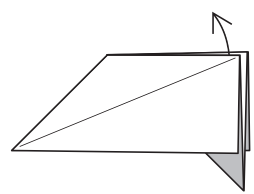
(3) Unfold step (2).



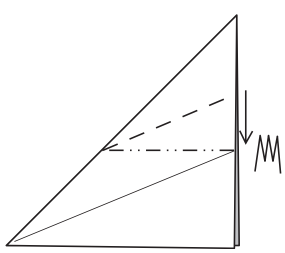
(4) Fold perpendicularly to the right side at the indicated spot.



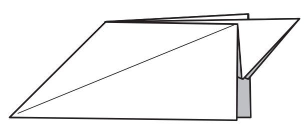
(5) Now use the creases from (4) to reverse the point inside…



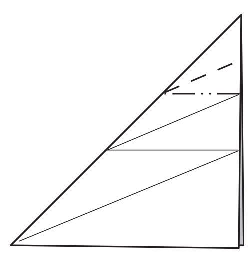
(6) …like this. Crease sharply and unfold step (5).



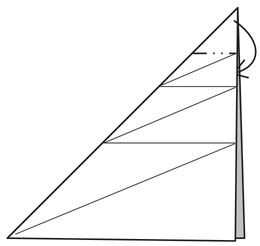
(7) Refold step (5), but this time add angle bisectors with the diagonal to crimp the paper.

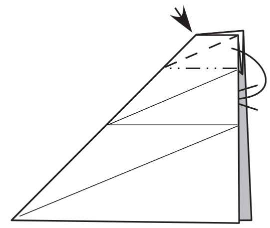


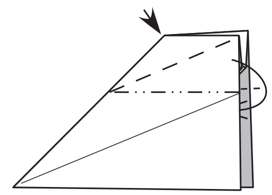
(8) This should be the result. Crease firmly and unfold.



(9) Now repeat steps (4)–(8) to make the next "level."



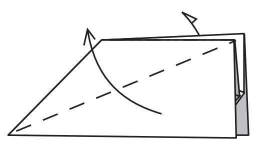


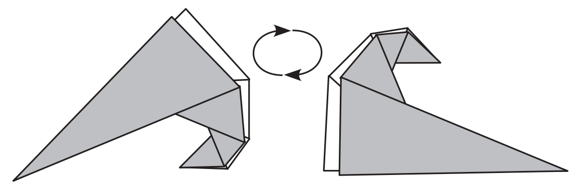


(10) You could keep going, but for the first time, stop after 3 levels by performing steps (4)–(5) one last time.

(11) Then use the creases of the third level to swivel the paper inside.

(12) Do it again with the second level creases. The wave spiral will be forming inside.

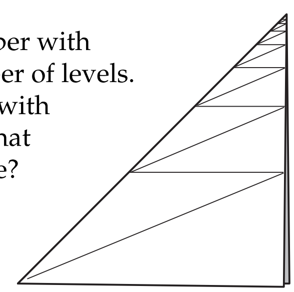


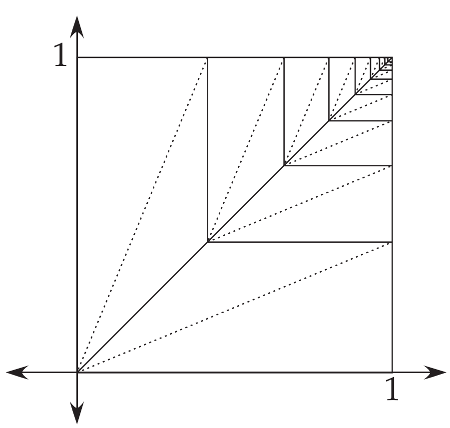


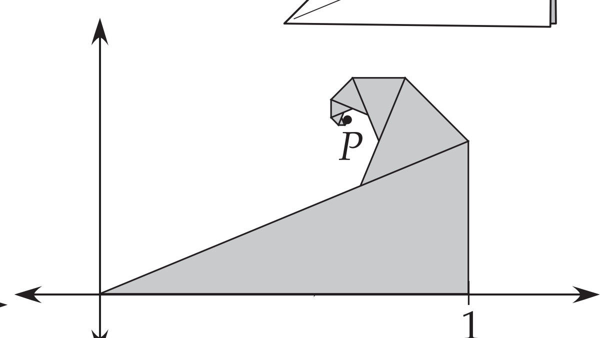
(13) For the first level, all you need to do is refold the creases from step (2).

(14) This reveals the wave! Of course, you can, and should, do more levels to make the wave curl more.

Question: Suppose we started with a square piece of paper with side length 1 and folded this wave with an infinite number of levels. If we put the finished model on a set of coordinate axes, with the tip of the base at the origin, as shown below right, what would the coordinates of the limit point P of the spiral be?



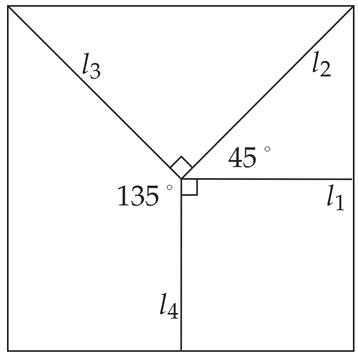




HANDOUT

Matrices and Flat Origami

Idea: When we fold a piece of paper flat, we're really **reflecting** one half of the paper onto the other half. We can use this to model flat origami using matrices.



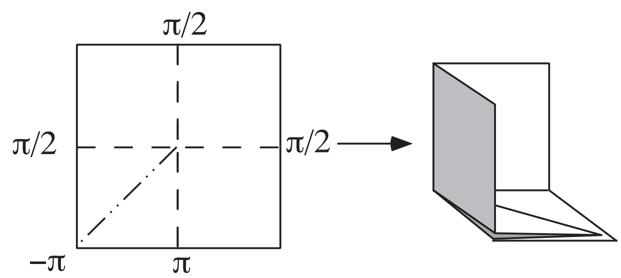
Activity: Above is shown the creases of a flat vertex fold. Assume that the vertex is located at the origin of the xy-plane.

Question 1: Find a 2×2 matrix $R(l_1)$ that reflects the plane about crease line l_1. Do the same thing for the other crease lines.

Question 2: What happens when you multiply these matrices together? Explain what's going on.

Matrices and 3D Origami

Take a square piece of paper and make the below creases to form the 3D **corner of a cube** fold.



The angles at each crease are the **folding angles**, which is the amount each crease needs to be folded by to make the model.

Question 1: Let χ_i be the 3D, 3×3 rotation matrix that rotates $\mathbb{R}^3$ about the crease line l_i by an angle equal to the folding angle at that crease. Find the five 3×3 matrices $\chi_1, \ldots, \chi_5$ for the above 3D fold. (Assume that the vertex is at the origin and the paper lies in the xy-plane.)

Question 2: What happens when you multiply these matrices together?

Question 3: In the previous question you should have gotten that the product $\chi_1\chi_2\chi_3\chi_4\chi_5 = I$, the identity matrix. Why is this the case?

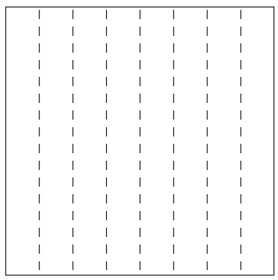
Be careful with your answer. Remember that the χ_i matrices are rotations about the crease line in the **unfolded** paper.

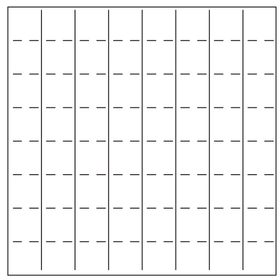
Question 4: Prove in general that if we are given a 3D single vertex fold with folding matrices $\chi_1, \chi_2, \ldots, \chi_n$, then the product of these matrices, in order, is the identity. Hint: Think of a bug crawling in a circle around the vertex on the folded paper. What rotations would the bug make when it crosses a crease line?

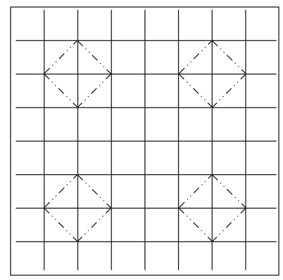
HANDOUT

Folding a Square Twist Tessellation

These instructions show how to tile the classic square twist in square piece of paper. We first look at making a 4×4 tiling, and we begin by making a lot of pre-creases!





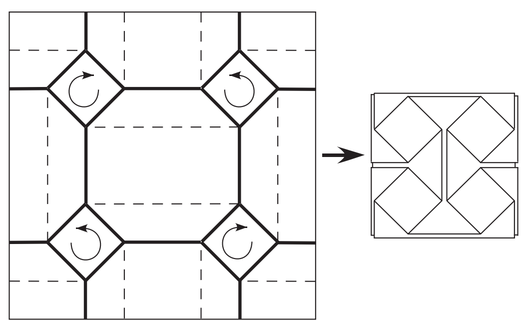


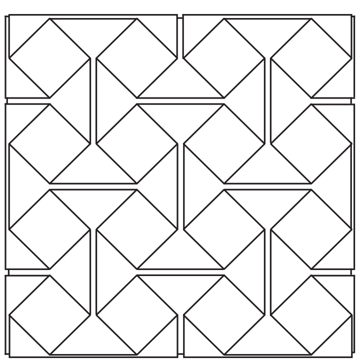
(1) Valley crease the square into 8ths in one direction.

(2) Then valley crease it into 8ths in the other direction.

(3) **Mountain crease** 4 diamond-shaped squares carefully, as shown.

(4) You now have all the creases you need to fold the four square twists. Use the creases shown to the right, where the bold lines are mountains and the dashed lines valleys. Adjacent square twists will rotate in opposite directions. Be persistent!



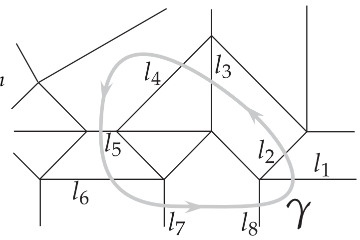


If you succeeded in making a 4×4 square twist tessellation, try shooting for an 8×8 tessellation! You would need to start by pre-creasing your square into 16ths, and using a larger sheet of paper is recommended.

HANDOUT

The Flat-folding Homomorphism

Suppose you have a crease pattern that folds flat. Let γ be a closed, vertex-avoiding curve drawn on the crease pattern that crosses crease lines $l_1, \ldots, l_{2n}$ in order. Let $R(l_i) =$ the transformation that reflects the plane about the line l_i. Since each fold is reflecting part of the paper about the crease, and the paper cannot rip in origami, we have that



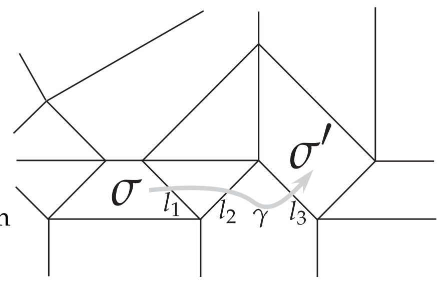
$$R(l_1)R(l_2)R(l_3)\cdots R(l_{2n}) = I,$$

where I is the identity transformation.

Now let σ and σ' be any two faces of the crease pattern C. Define the transformation

$$[\sigma, \sigma'] = R(l_1)R(l_2)\cdots R(l_k),$$

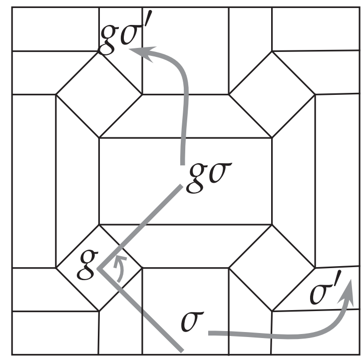
where $l_1, \ldots, l_k$ are the creases, in order, that a vertex-avoiding curve γ crosses going from σ to σ'.



Question 1: Explain why the transformation $[\sigma, \sigma']$ is independent of the choice of the curve γ.

Question 2: Explain why $[\sigma, \sigma''] = [\sigma, \sigma'][\sigma', \sigma'']$ for all faces σ, σ', and σ'' in the crease pattern C.

Question 3: Explain why $[g\sigma, g\sigma'] = g[\sigma, \sigma']g^{-1}$ for all faces $\sigma, \sigma' \in C$ and for any symmetry g of the crease pattern. (In the example shown to the right below, g is a 90° rotation about one of the square twists.)



Now let $\mathrm{Isom}(\mathbb{R}^2)$ denote the group of isometries of the plane. Let C be our flat-foldable origami crease pattern, and let $\Gamma \leq \mathrm{Isom}(\mathbb{R}^2)$ be the symmetry group of C. (That is, Γ is the subgroup of isometries that leave C invariant.)

For a fixed face $\sigma \in C$, define a mapping $\varphi_\sigma : \Gamma \to \mathrm{Isom}(\mathbb{R}^2)$ by

$$\varphi_\sigma(g) = [\sigma, g\sigma]g \text{ for all } g \in \Gamma.$$

Question 4: Prove that φ_σ is a homomorphism. (That is, prove that $\varphi_\sigma(gh) = \varphi_\sigma(g)\varphi_\sigma(h)$ for all $g, h \in \Gamma$.)

Question 5: Since φ_σ is a homomorphism, what simple fact can we conclude about the image set $\varphi_\sigma(\Gamma)$?

For a fixed face σ of C, we can also define the **folding map** $[\sigma]$ **of** C **toward** σ by

$$[\sigma](x) = [\sigma, \sigma'](x) \text{ for } x \in \sigma' \in C.$$

Question 6: Prove that for any symmetry $g \in \Gamma$, we have that $\varphi_\sigma(g)[\sigma] = [\sigma]g$.

(That is, you want to show these products of transformations are equal, so $\varphi_\sigma(g)[\sigma](x) = [\sigma]g(x)$ for all points x in the crease pattern. Hint: Any point $x \in C$ must lie in a face of the crease pattern, so call this face σ'.)

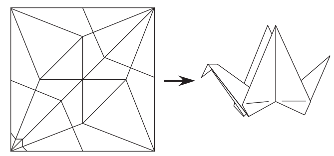
Question 7: Why does Question 6 imply that $\varphi_\sigma(g) = [\sigma]g[\sigma]^{-1}$ for all $g \in \Gamma$?

What Question 7 says is that the action of any $\varphi_\sigma(g)$ on a flat-folded origami model is equivalent to unfolding it ($[\sigma]^{-1}$), doing an isometry that leaves the crease pattern invariant (g), and then refolding the paper ($[\sigma]$).

Question 8: Explain why this proves that $\varphi_\sigma(\Gamma)$ is the symmetry group of the folded paper!

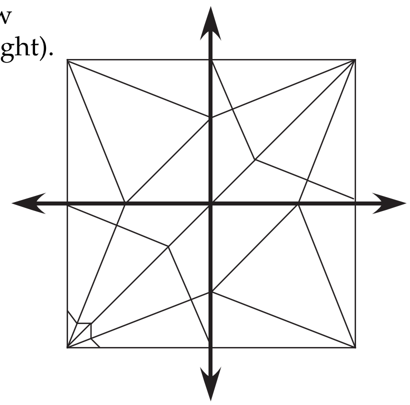
HANDOUT

Finding the Symmetry Group of Origami



Example 1: The classic flapping bird (crane)

What is the symmetry group Γ of the crease pattern of the flapping bird? It might be helpful to view the crease pattern in a set of coordinate axes (right).



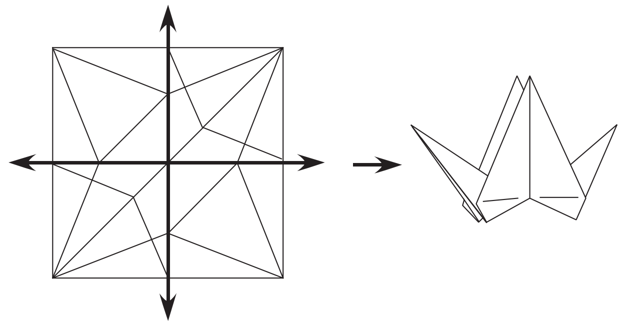
You should get that the symmetry group Γ of this crease pattern has only two elements. For each of these two elements, call them a and b, determine what $\varphi_\sigma(a)$ and $\varphi_\sigma(b)$ are, for a fixed face σ of the crease pattern.

Conclusion: What does this mean the group $\varphi_\sigma(\Gamma)$ is? Is this the symmetry group of the folded crane model?

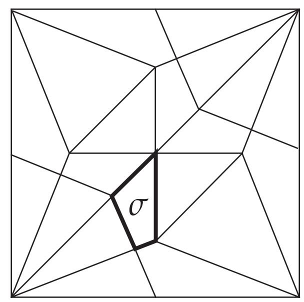
(Note: You need to think of the folded crane as being flattened into the plane $\mathbb{R}^2$, not as a 3D model.)

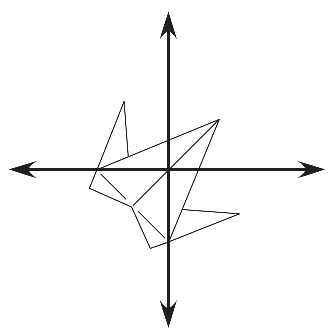
Example 2: The headless crane

Find the symmetry group Γ of this crease pattern.



To the right we have labeled a face σ. For each element $g \in \Gamma$, compute $\varphi_\sigma(g)$, and thereby determine the group $\varphi_\sigma(\Gamma)$.





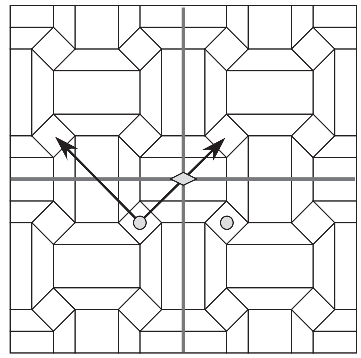
Conclusion: Does your calculation of the group $\varphi_\sigma(\Gamma)$ match the symmetry group of the folded headless crane? Note that the exact orientation of your headless crane in the plane under the folding map is determined by our choice of σ.

Example 3: Origami tessellations
Let C be a flat origami crease pattern on the infinite plane $\mathbb{R}^2$ whose symmetry group Γ is one of the wallpaper groups.

For example, the square twist tessellation has symmetry group $\Gamma = \text{p4g}$.

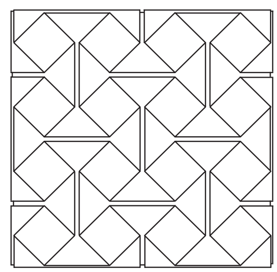
This is an infinite group, generated by

- two centers of 90° rotation (circles),
- two lines of reflection (in grey),
- one center of 180° rotation (diamond) at the intersection of the reflection lines,
- two translation vectors.



Facts:

- Every wallpaper group contains two linearly independent translations.
- Wallpaper groups have no finite normal subgroups.



Problem: Prove that if C is a flat origami crease pattern whose symmetry group Γ is a wallpaper group and if the image $\varphi_\sigma(\Gamma)$ is also a wallpaper group, then

$$\varphi_\sigma(\Gamma) \cong \Gamma.$$

That is, the symmetry group of the folded paper will be isomorphic to the symmetry group of the crease pattern.

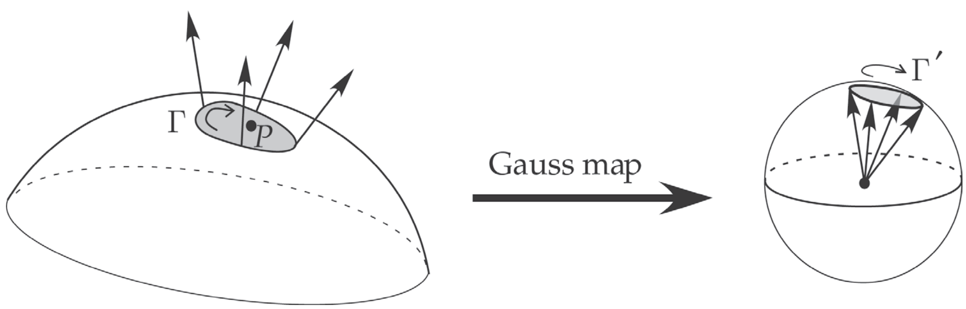
Follow-up: Can you think of an example of an origami model whose crease pattern is a tessellation but where the folded model is not a wallpaper group? Why does this not contradict the above problem?

HANDOUT

An Introduction to Gaussian Curvature

Definition: The **Gaussian curvature at a point** P on a surface is a real number κ that can be computed as follows: Draw a closed curve Γ on the surface going clockwise around P. Draw unit vectors on the points of Γ that are normal to the surface. Then translate these vectors to the center of a sphere of radius 1 and consider the curve Γ' that they trace on the sphere. (This mapping from Γ to Γ' is called the **Gauss map**.) Then, letting Γ shrink around P, we define the Gaussian curvature at P to be

$$\kappa = \lim_{\Gamma \to P} \frac{\text{Area}(\Gamma')}{\text{Area}(\Gamma)}.$$

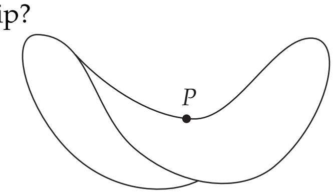


This can be difficult to compute, but not always. . . .

Question 1: What is the Gaussian curvature of a random point on a sphere of radius 1? Radius 2? Radius 1/2?

Question 2: What is the Gaussian curvature of a flat plane?

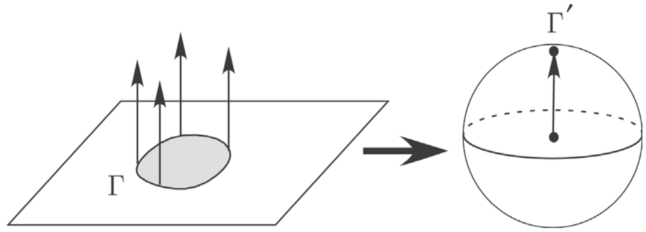
Question 3: What would happen if you tried to find the Gaussian curvature of a **saddle point**, i.e., the center of a Pringles$^{\text{TM}}$potato chip?



HANDOUT

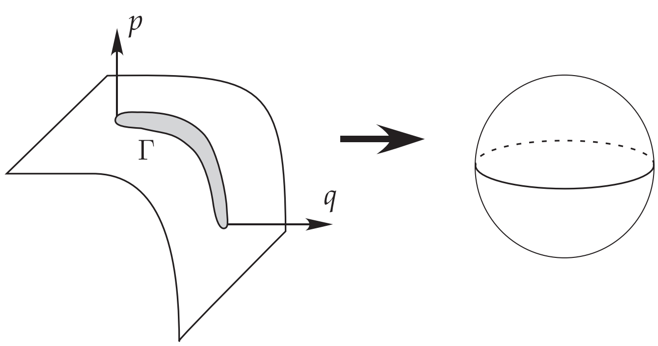
Gaussian Curvature and Origami

In the previous handout, you saw how a flat piece of paper will have zero Gaussian curvature. This is because no matter what our choice of Γ is, the normal vectors along the curve will all be pointing in the same direction, so $\text{Area}(\Gamma') = 0$.

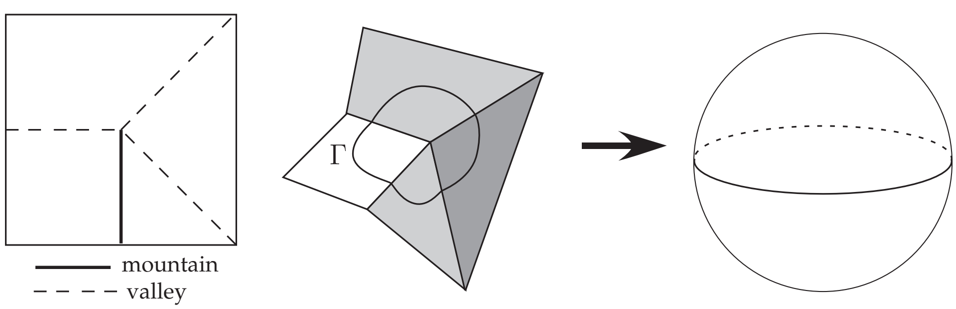


This means that we get zero in the numerator of our Gaussian curvature limit equation no matter what Γ is. Therefore, when determining curvature on a piece of paper, we don't need to worry about the limit part of the equation—one choice for Γ should always give us $\text{Area}(\Gamma') = 0$. This will be very useful later on.

Question 1: Suppose that we take a sheet of paper and bend it. Should this change the paper's curvature or not? Explore this by determining the Gauss map of a curve Γ that straddles such a bend, as pictured below.



Question 2: Suppose that we make more than one fold, like in an origami model? Draw what the Gauss map should be for the curve Γ shown on the vertex fold below. What should the curvature generated by Γ be? Does this make sense?



Question 3: The claim that you should have made in Question 2 is this: The Gaussian curvature is zero at every point on a folded piece of paper. Use the Gauss map that you made in Question 2 to prove that this is true for any curve Γ around a 4-valent vertex. (You'll need to use the fact that the area of a triangle on the unit sphere is (the sum of the angles) $- \pi$.)

Question 4: What is the connection between this Gaussian curvature stuff and **rigid origami** (where we pretend that the regions of paper between creases are made of metal and thus are rigid)?

Putting the rigidity criterion to the test

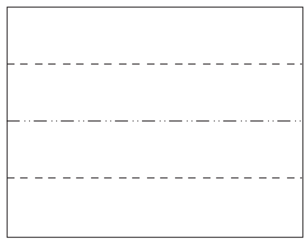
Question 5: Use your conclusions from Question 4 to prove that it is impossible to have a 3-valent folded vertex in a rigid origami model. Draw the Gauss map for such a vertex to back up your argument.

Question 6: Now prove that it is impossible to have a 4-valent vertex in a rigid origami model where **all** of the creases are mountains.

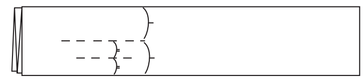
HANDOUT

The Miura Map Fold

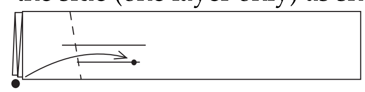
Japanese astrophysicist Koryo Miura wanted a way to unfold large solar panels in outer space. His fold also makes a great way to fold maps.



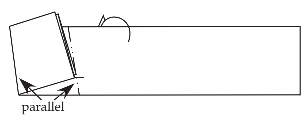
(1) Take a rectangle of paper and mountain-valley-mountain fold it into 1/4ths lengthwise.



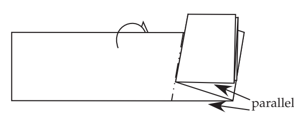
(2) Make 1/2 and 1/4 pinch marks on the side (one layer only) as shown.



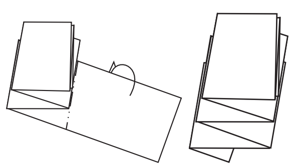
(3) Folding **all layers**, bring the lower left corner to the 1/4 line, as in the picture.



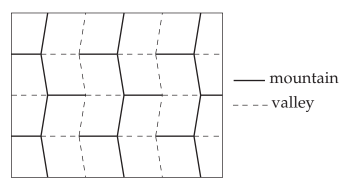
(4) Fold the remainder of the strip behind, making the crease parallel to the previous crease.



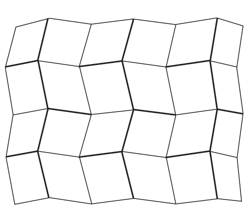
(5) Repeat, but this time use the fold from step (3) as a guide.



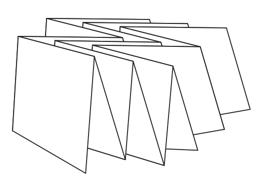
(6) Repeat this process until the strip is all used up. Then **unfold everything**.



(7) Now re-collapse the model, but change some of the mountains and valleys. Note how the zig-zag creases alternate from all-mountain to all-valley. Use these as a guide as you collapse it…

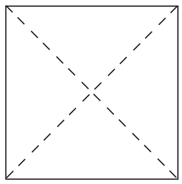


…In the end the paper should fold up neatly as shown to the right. You can then pull apart two opposite corners to easily open and close the model.

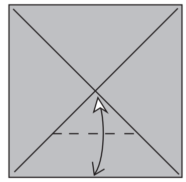


The Hyperbolic Paraboloid

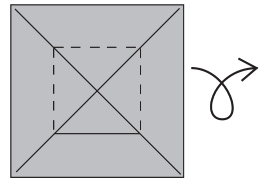
This unusual fold has been rediscovered by numerous people over the years. It resembles a 3D surface that you may recall from Multivariable Calculus.



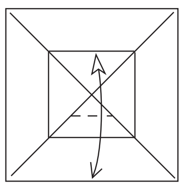
(1) Take a square and crease both diagonals. Turn over.



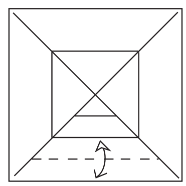
(2) Fold the bottom to the center, but **only** crease in the middle.



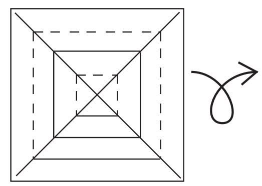
(3) Repeat step (2) on the other three sides. Turn over.



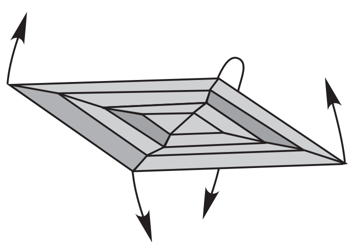
(4) Bring the bottom to the top crease line, creasing **only** between the diagonals.



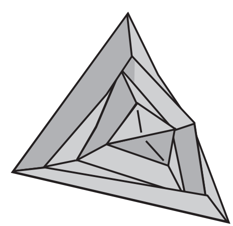
(5) Then bring the bottom to the nearest crease line. Again, do not crease all the way across.



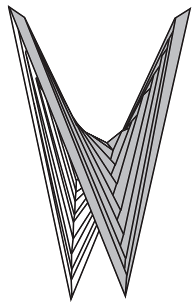
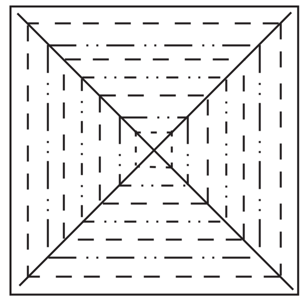
(6) Repeat steps (4) and (5) on the other three sides. Turn over.



(7) Now make all the creases at once. It may help to fold the creases on the outer ring first and work your way in.



(8) Once the creases are folded, the paper will twist into this shape, and you're done!

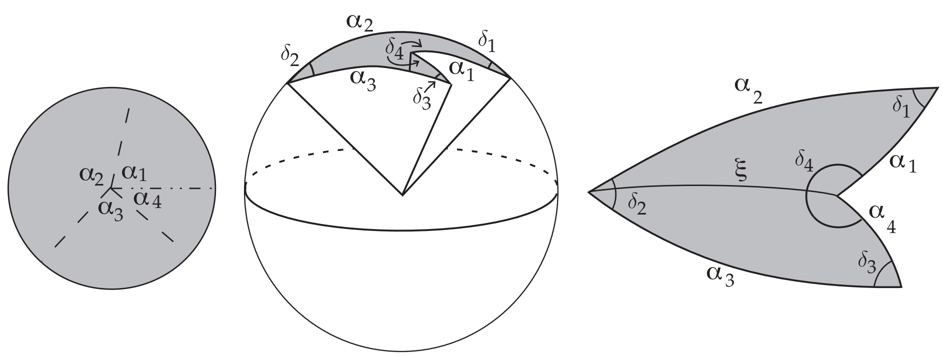
 

(9) You can make a larger one by folding more divisions in the paper. The key is to have the concentric squares alternate mountain-valley-mountain in the end. You can do steps (1)–(3), do not turn the paper over, then do 1/4 divisions in steps (4)–(6), then turn it over and make 1/8 divisions. Or you could shoot for 1/16ths!

Question: Is the hyperbolic parabola a **rigid origami** model or not? (Could it be made out of rigid sheet metal, with hinges at the creases?) Proof?

HANDOUT

Spherical Trigonometry and Rigid Flat Origami 1



Consider a degree-4 flat vertex fold, as shown above with the angles on the crease pattern $\alpha_1, \ldots, \alpha_4$ and the **dihedral angles** between the regions of folded paper $\delta_1, \ldots, \delta_4$. This is easy to visualize if you imagine the vertex being at the center of a sphere and look at the **spherical polygon** the paper cuts out on the sphere's surface.

If δ_4 is the lone mountain crease, let ξ be an arc on the sphere connecting the δ_4 and the δ_2 corners of this polygon, which divides it into two spherical triangles. Then, we can use the **spherical law of cosines**:

$$\cos\xi = \cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2 \cos\delta_1, \tag{1}$$

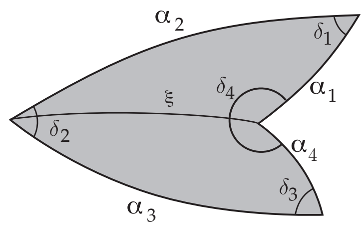
$$\cos\xi = \cos\alpha_3 \cos\alpha_4 + \sin\alpha_3 \sin\alpha_4 \cos\delta_3. \tag{2}$$

Question 1: Remember that since this vertex folds flat, Kawasaki's Theorem says that $\alpha_3 = \pi - \alpha_1$ and $\alpha_4 = \pi - \alpha_2$. What do you get when you plug these into equation (2) and simplify?

Question 2: Subtract this new equation from equation (1). Use this to find an equation relating the dihedral angles δ_1 and δ_3. What about δ_2 and δ_4?

HANDOUT

Spherical Trigonometry and Rigid Flat Origami 2



When studying this subject, origami master Robert Lang used spherical trigonometry and the picture above to derive the following equation:

$$\cos \delta_2 = \cos \delta_1 - \frac{\sin^2 \delta_1 \sin \alpha_1 \sin \alpha_2}{1 - \cos \xi}.$$

Question 3: What does this equation tell us about the relationship between the dihedral angles δ_1 and δ_2?

Question 4: Remember that these results assume that the paper is *rigid* between the creases (for otherwise our spherical polygons would not have straight sides). So use your answers to Questions 2 and 3 to prove that the square twist, shown below, **cannot** be folded rigidly. (Bold creases are mountains, non-bold are valleys.)